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THE INTEGRAL ON QUANTUM SUPERGROUPS OF TYPE $A_{R|S}$*

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Abstract. Quantum groups of type $A_{r|s}$ generalize the general linear supergroups $GL(r|s)$. We compute the integral on these quantum supergroups and whence derive a quantum analogue of (super) HCIZ integral formula.

Introduction. A quantum supergroup of type $A_{r|s}$ is a Hopf superalgebra, which, as an algebra, is a quotient of a free non-commutative tensor algebra on a certain finite dimensional vector superspace V, by certain relations, determined in terms of a Hecke symmetry of birank $(r+1, s+1)$ acting on $V \otimes V$. If the Hecke symmetry reduces to the ordinary super-permuting operator in the category of vector superspaces, we recover the quantum general linear supergroup $GL(r+1, s+1)$, (Section 1).

In this work, we compute the integral on quantum supergroups of type $A_{r|s}$. The notion of integral on Hopf algebras was introduced by Sweedler [24]. This is an analogue of the notion of invariant integrals on compact groups (see, e.g., [21]). In fact, the algebra of representative functions on a compact group is dense in the algebra of all continuous complex valued functions by Peter-Weyl's theorem. The former algebra is a Hopf algebra and the definition of the invariant integral can be given in a purely algebraic way using the coalgebra structure, namely, a (left) integral on a Hopf algebra H, defined over a field k, is a linear form $\int : H \longrightarrow k$, satisfying the following condition:

$$\int(a) = m(\int \otimes \mathrm{id}_H)\Delta(a),$$

where m, Δ denote the product and coproduct on H and we identify the field k with a subspace of H by means of the unit of H. This definition can also be extended for Hopf superalgebras. On the other hand, the existence of integral on compact supergroups was established by Berezin [3]. Unlike the case of compact group, the volume of a compact supergroup may vanish, as it happens to the supergroups $U(r|s)$ of unitary supermatrices.

A formula for the integral on the group $U(d)$ was obtained by Itzykson and Zuber [4]. This formula computes the integral at the function of the form $\mathrm{tr}_{M,N,n}(U) := \left[\mathrm{tr}(MUNU^{-1})\right]^n$, where M, N are hermitian matrices. It can be given in the following form

$$\int_{U(d)} \left[\mathrm{tr}(MUNU^{-1})\right]^n [dU] = \sum_{\lambda \in \mathcal{P}_n^d} \frac{d_\lambda}{r_\lambda} \Phi_\lambda(M)\Phi_\lambda(N) \tag{0.1}$$

for any hermitian matrices M and N. Here, $\mathcal{P}_n^d$ is the set of partitions of n of length d. Φ_λ is the irreducible character of $U(d)$, corresponding to the partition λ. If $\xi_1, \xi_2, ..., \xi_d$

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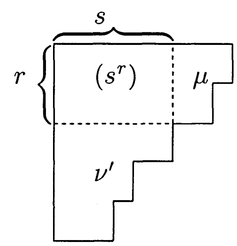
are the eigenvalues of M then $\Phi_\lambda(M) = s_\lambda(\xi_1, \xi_2, ..., \xi_d)$, s_λ are the Schur functions [19, Chapter I]. d_λ is the dimension of the irreducible module of the symmetric group $\mathfrak{S}_d$, r_λ is the dimension of the irreducible representation of $U(d)$, corresponding to partition λ. Explicitly, $d_\lambda = d!\prod_{x\in[\lambda]} h(x)^{-1}$, $r_\lambda = \prod_{x\in[\lambda]}(c(x)+d)h(x)^{-1}$, where $c(x)$ is the content, $h(x)$ is the hook-length of the box x in the diagram $[\lambda]$ (see [loc.cit.]).

This formula turns out to be a special case of a formula obtained by Harish-Chandra [16]. The integral on the left-hand side of (0.1) is therefore referred as Harish-Chandra-Itzykson-Zuber (HCIZ) integral.

A super analogue of the HCIZ formula was obtained by Alfaro, Medina and Urrutia [1, 2], it reads

$$(0.2)\quad \int_{U(r|s)} \mathsf{str}(MUNU^{-1})^n[dU] = \sum_{\substack{\mu\in\mathcal{P}^m,\nu\in\mathcal{P}^n\\ |\mu|+|\nu|=n-rs}} \frac{(-1)^{|\nu|}n!}{|\mu|!|\nu|!}\frac{d_\mu d_\nu}{r_\mu r_\nu}\Phi_\lambda(M)\Phi_\lambda(N),$$

where $\lambda = (s^r) + \mu \cup \nu'$:



Our problem of finding integrals on the function algebra of quantum linear supergroup $GL_q(r|s)$ is thus motivated by the HCIZ integral. On the other hand, it is an interesting problem from the point of view of Hopf algebra theory. Integrals on Hopf algebras were studied by several authors since the pioneering work of Sweedler [24], see e.g. [23, 18, 8]. For finite dimensional Hopf algebra is is known that the integrals exist uniquely up to a scalar. However, few examples of infinite dimensional Hopf algebras with integral, except cosemisimple Hopf algebras, are known.

Let R be a Hecke symmetry on a finite dimensional vector super space V of dimension d. Define an algebra H_R as a quotient of the free non-commutative algebra $k\langle\{z_i^j, t_i^j, 1\le i,j\le d\}\rangle$ by relations in (1.2), (1.3). Then H_R has a structure of a Hopf superalgebra and is called the algebra of functions on a quantum supergroup of type $A_{r-1|s-1}$, where (r,s) is the birank of R, see Section 1.

The Hopf algebra H_R is cosemisimple if and only if $r=0$ or $s=0$. In this particular case, the integral was explicitly computed in [11] and an analogue of the HCIZ integral formula was given in [12], where the notion of characters of coribbon Hopf algebras was introduced. The case of arbitrary (r,s) will be treated in the present paper. Since our object is considered to generalize the supergroup $GL(r|s)$, it is natural that we are working in the category of vector superspaces, so that our algebra H_R will be a Hopf superalgebras. Nevertheless, as we shall see in the course of the paper, the ground category does not plays any essential role.

Using the commutation rule on H_R, one can show that an element of H_R can be represented as a linear combination of monomials on z_i^j, t_k^l of the form $z_{i_1}^{j_1}\cdots z_{i_p}^{j_p}t_{k_1}^{l_1}\cdots t_{k_q}^{l_q}$. From the linearity of the integral, we see that it is sufficient to find the integral on the set of monomials of the form $z_{i_1}^{j_1}\cdots z_{i_p}^{j_p}t_{k_1}^{l_1}\cdots t_{k_q}^{l_q}$. It turns out that any integral should vanish on those monomials with $p\ne q$.

The formula of the integral on $z_{i_1}^{j_1}\cdots z_{i_n}^{j_n}t_{k_1}^{l_1}\cdots t_{k_n}^{l_n}$ is based on an operator P_n :

$V^{\otimes n} \longrightarrow V^{\otimes n}$. In Section 2 we show that the axiom for the integral is equivalent to certain conditions on P_n. In Section 3 we construct P_n. In Section 4 we derive a quantum analogue of HCIZ integral from our formula for the integral. To do this we first introduce the notion of character of H_R-comodules and recall the notion of points of a quantum group. The quantum super HCIZ integral formula in Theorem 4.1 then follows immediately from the formula in (3.6). In the last section we discuss the orthogonal relation of some simple H_R-comodules.

In the case of the compact group $SU_q(2)$, the orthogonal relations for matrix elements of the coefficient matrix of simple comodules over was given by Soibelman and Vaskman [26], the case of arbitrary compact quantum groups was treated by Woronowicz [25]. Their basic idea was that any simple comodules over a compact quantum groups should be isomorphic to its double dual and the formula could be given in terms of the intertwiner of these two comodules. This method can not be applied directly for quantum supergroups since in many case, the trace of the intertwiner may vanish.

Notations and Conventions. We work in the category of vector superspaces over an algebraically closed field k of characteristic zero. All algebraic objects, like algebras, morphisms, linear forms etc..., will then be considered as objects in this category, for instance, morphisms are always even, i.e. map even (odd) elements to even (odd) elements. An element from a vector superspace usually means a homogeneous one. The dual space to a vector spaces of finite dimension is defined by an even form. If $x_1, x_2, \ldots, x_d$ is a homogeneous basis for a vector superspace V, then the dual basis $\xi^1, \xi^2, \ldots \xi^d$ on V^* has the same parity as $x_1, x_2, \ldots, x_d$ and satisfies $\xi^k(x_i) = \delta_i^k$. To avoid any signs appearing, the dual to a tensor product $V \otimes W$ of two vector superspaces V and W is canonically identified with $W^* \otimes V^*$. The standard permuting operator on the tensor product $V \otimes W$ is given by the rule $a \otimes b \longmapsto (-1)^{\hat{a}\hat{b}} b \otimes a$, for homogeneous elements a, b, where $\hat{a}$ denotes the parity of a.

A matrix C of entries from the field k will have the entry on the i-th row and j-th column denoted by C_j^i. For a matrix Z of entries from an algebra, its (i, j) entry will be denoted by z_j^i. We adopt the convention of summing up after indices that apper both in lower and upper places, for example, $a_j^i b_k^j = \sum_j a_j^i b_k^j$.

For the notion of partitions, diagrams, standard tableaux, contents, hook-length, etc..., the reader is referred to [19, Chapter I]. Let $\mathcal{P}$ denote the set of all partitions, $\mathcal{P}_n$ denote the set of partitions of n, for $\lambda \in \mathcal{P}_n$ we shall write $\lambda \vdash n$ or $|\lambda| = n$. Let $\Gamma^{r,s}$ denote the set of (r, s)-hook-partitions: $\{\lambda \in \mathcal{P} | \lambda_{r+1} \leq s\}$, $\Omega^{r,s}$ denote the subset of $\Gamma^{r,s}$: $\{\lambda \in \Gamma^{r,s} | \lambda_r = s\}$, finally, let $\Gamma_n^{r,s} = \mathcal{P}_n \cap \Gamma^{r,s}$ and $\Omega_n^{r,s} = \Omega_n^{r,s} \cap \mathcal{P}_n$, $\mathcal{P}_n^d = \Gamma_n^{d,0}$.

For a partition λ, $[\lambda]$ denotes the corresponding diagram. The diagram $[\lambda]$, filled with the numbers $1, 2, \ldots, |\lambda|$ in such a way that they increase in each column and each row, is called a standard λ-tableau. The number of standard λ-tableaux is precisely the dimension of the irreducible representation of the symmetric group $\mathfrak{S}_n$, $n = |\lambda|$. The content of the node on the i-th row and j-th column is $j - i$, its hook-length is the cardinal number of nodes lying on the same row and to the right and lying on the same column and below. For a λ-tableau $t(\lambda)$, $c_{t(\lambda)}(m)$ is the content of the node containing m.

EXAMPLE. Let $\lambda = (3,2,2)$. Then

$$[\lambda] = \begin{array}{|c|c|c|}\hline & & \\ \hline & \\ \cline{1-2} & \\ \cline{1-2}\end{array} \quad \text{some standard } \lambda\text{-tableaux are: } t_1(\lambda) = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 6 \\ \cline{1-2} 5 & 7 \\ \cline{1-2}\end{array} \quad t_2(\lambda) = \begin{array}{|c|c|c|}\hline 1 & 3 & 7 \\ \hline 2 & 4 \\ \cline{1-2} 5 & 6 \\ \cline{1-2}\end{array}$$

we have $c_{t_1(\lambda)}(5) = -2, c_{t_1(\lambda)}(6) = -1$. λ can be considered as element of $\Omega^{1,2}$ or $\Omega^{3,2}$. However, if consider λ as an element of $\Omega^{1,2}$, $t_1(\lambda)$ and $t_2(\lambda)$ are both essential while, if consider λ as an element of $\Omega^{3,2}$, $t_2(\lambda)$ is still essential but $t_1(\lambda)$ isn't (see Lemma 3.1).

1. Quantum Groups Associated to Hecke Symmetries. Let V be a super vector space over k, a fixed field of characteristic zero. Fix a homogeneous basis $x_1, x_2, \ldots, x_d$ of V. We denote the parity of the basis element x_i by $\hat{i}$. An even operator R on $V \otimes V$ can be given by a matrix R^{kl}_{ij}: $R(x_i \otimes x_j) = x_k \otimes x_l R^{kl}_{ij}$. Since R is an even operator, the matrix element R^{ij}_{kl} is zero if $\hat{i} + \hat{j} \neq \hat{k} + \hat{l}$. R is called *Hecke symmetry* if the following conditions are satisfied:

i) R satisfies the Yang-Baxter equation $R_1 R_2 R_1 = R_2 R_1 R_2$, where $R_1 := R \otimes I$, $R_2 := I \otimes R$, and I is the identity matrix of degree d.
ii) R satisfies the Hecke equation $(R - q)(R + 1) = 0$ for some q which will be assumed *not to be a root of unity of degree greater then 1.*
iii) There exists a matrix P^{kl}_{ij} such that $P^{im}_{jn} R^{nk}_{ml} = \delta^i_l \delta^k_j$. A matrix satisfying this condition is called *closed.*

EXAMPLE. The following main example of Hecke symmetries was first considered by Manin [20]. Assume that the variable x_i, $i \leq r$ are even and the rest s variables are odd. Define, for $1 \leq i, j, k, l \leq r + s$,

$$R_{r|s}{}^{kl}_{ij} := \begin{cases} q^2 & \text{if} \quad i = j = k = l, \hat{i} = 0 \\ -1 & \text{if} \quad i = j = k = l, \hat{i} = 1 \\ q^2 - 1 & \text{if} \quad k = i < j = l \\ (-1)^{\hat{i}\hat{j}} q & \text{if} \quad k = j \neq i = l \\ 0 & \text{otherwise.} \end{cases}$$

The Hecke equation for $R_{r|s}$ is $(x - q^2)(x + 1) = 0$. When $q = 1$, $R_{r|s}$ reduces to the super-permuting operator on $V \otimes V$.

Let R be a Hecke symmetry. We define the matrix bialgebra E_R and the matrix Hopf algebra H_R as follows. Let $\{z^i_j, t^i_j | 1 \leq i, j \leq d\}$ be a set of indeterminates, with parities $\hat{x}^i_j = \hat{t}^i_j = \hat{i} + \hat{j}$. We define E_R as the quotient algebra of the free non-commutative algebra, generated by $\{z^i_j | 1 \leq i, j \leq d\}$, by the relations

$$(1.1) \qquad (-1)^{\hat{s}(\hat{i}+\hat{p})} R^{kl}_{ps} z^p_i z^s_j = (-1)^{\hat{l}(\hat{q}+\hat{k})} z^k_q z^l_n R^{qn}_{ij}, \quad 1 \leq i, j, k, l \leq d.$$

Here, we use the convention of summing up by the indices that appear in both lower and upper places. We define the algebra H_R as the quotient of the free non-commutative algebra generated by $\{z^i_j, t^i_j | 1 \leq i, j \leq d\}$, by the relations

$$(1.2) \qquad (-1)^{\hat{s}(\hat{i}+\hat{p})} R^{kl}_{ps} z^p_i z^s_j = (-1)^{\hat{l}(\hat{q}+\hat{k})} z^k_q z^l_n R^{qn}_{ij}, \quad 1 \leq i, j, k, l \leq d,$$

$$(1.3) \qquad (-1)^{\hat{j}(\hat{j}+\hat{k})} z^i_j t^j_k = (-1)^{\hat{l}(\hat{l}+\hat{i})} t^i_l z^l_k = \delta^i_k, \quad 1 \leq i, k \leq d.$$

The relations in (1.1) can be considered as the commuting rule for elements of E_R. On H_R the relations (1.2) and (1.3) also imply the following relations:

$$(1.4)\qquad (-1)^{\hat{k}(\hat{\imath}+\hat{\jmath})}R^{pj}_{ql}z^i_jt^l_k = (-1)^{\hat{m}(\hat{n}+\hat{p})}t^p_nz^m_qR^{ni}_{mk}, \quad 1\le p,q,i,k\le d,$$

$$(1.5)\qquad (-1)^{\hat{s}(\hat{\imath}+\hat{p})}R^{kl}_{ps}t^s_jt^p_i = (-1)^{\hat{l}(\hat{q}+\hat{k})}t^l_nt^k_qR^{qn}_{ij}, \quad 1\le i,j,k,l\le d.$$

By the closedness of R, (1.4) is equivalent to

$$(1.6)\qquad (-1)^{\hat{k}(\hat{\imath}+\hat{\jmath})}P^{km}_{in}z^i_jt^l_k = (-1)^{\hat{m}(\hat{n}+\hat{p})}t^p_nz^m_qP^{lq}_{jp}, \quad 1\le k,l,m,n\le d.$$

Setting $m=n$ in (1.6) and summing up after this index, and using (1.3) we get

$$(1.7)\qquad (-1)^{\hat{k}(\hat{\imath}+\hat{\jmath})}C^k_iz^i_jt^l_k = C^l_j, \text{ where } C^k_i := P^{km}_{im}.$$

It is easy to show that E_R and H_R are bialgebras with respect to the coproduct given by

$$\Delta(z^i_j) = z^i_k\otimes z^k_j, \quad \Delta(t^i_j) = t^k_j\otimes t^i_k.$$

H_R is, in fact, a Hopf algebra with the antipode on H_R is given by

$$S(z^i_j) = (-1)^{\hat{\jmath}(\hat{\imath}+\hat{\jmath})}t^i_j, \quad S(t^i_j) = (-1)^{\hat{\imath}(\hat{\imath}+\hat{\jmath})}C^i_kz^k_l{C^{-1}}^l_j.$$

We also define $D^i_j := P^{li}_{lj}$. The matrices C and D, called reflection operators, play important roles in this work. Using the Hecke equation we can show that (cf [11])

$$(1.8)\qquad CD = DC = q^{-1} - (q^{-1}-1)\mathrm{tr}(C).$$

Since we are working in the category of vector superspaces, the rule of sign effects on the coproduct. More precisely, the compatibility of product and coproduct of a super bialgebra reads

$$\Delta(ab) = \sum_{(a)(b)}(-1)^{\hat{a}_2\hat{b}_1}a_1b_1\otimes a_2b_2,$$

for homogeneous a,b. Analogously, the antipode satisfies

$$S(ab) = (-1)^{\hat{a}\hat{b}}S(b)S(a), \Delta(S(a)) = (-1)^{\hat{a}_1\hat{a}_2}S(a_2)\otimes S(a_1)$$

for homogeneous a,b. Consequently, though the matrix Z is multiplicative, i.e. $\Delta(z^i_j) = z^i_k\otimes z^k_j$, its tensor powers are not.

To compute the coproduct on the tensor powers of Z, we introduce a sign function $\mathsf{sign}(I,J)$, where I,J are multi-indices of the same length. For any fixed n, the matrix $(\mathsf{sign}(I,J))$ where I,J run through the set of multi-indices of length n is the matrix of the standard isomorphism

$$\theta_n : (V^*\otimes V)^{\otimes n} \longrightarrow (V^{\otimes n})^*\otimes V^{\otimes n},$$

which is defined in the standard way using the super-permuting operator, with respect to the basis $x_1,x_2,\ldots,x_d$ and its dual $\xi^1,\xi^2,\ldots,\xi^d$. Explicitly, θ moves the elements of V^* to the left of the elements of V without reshuffling them and then reverses the order of the elements from V^*. The signs that appear define the function sign. In other words, θ_n is given by

$$\theta(\xi^{i_1} \otimes x_{j_1}) \otimes \cdots \otimes (\xi^{i_n} \otimes x_{j_n}) = \mathsf{sign}(I, J)(\xi^{i_n} \otimes \cdots \otimes \xi^{i_1} \otimes x_{j_1} \otimes \cdots \otimes x_{j_n}).$$

We thus have the following recurrent formulae for sign:

$$\mathsf{sign}(Ii, Jj) = (-1)^{\hat{i}(|\hat{I}|+|\hat{J}|)}\mathsf{sign}(I, J), \mathsf{sign}(iI, jJ) = (-1)^{|\hat{I}|(\hat{i}+\hat{j})}\mathsf{sign}(I, J),$$

where $|\hat{I}| := \hat{i}_1 + \hat{i}_2 + \cdots + \hat{i}_n$ if $I = (i_1, i_2, \ldots, i_n)$. For single indices, we have $\mathsf{sign}(i, j) = 1$ for any i, j.

Then, using induction we have

$$\Delta(z^{i_1}_{j_1} z^{i_2}_{j_2} \cdots z^{i_2}_{j_2}) = \mathsf{sign}(I, K)\mathsf{sign}(K, J)\mathsf{sign}(I, J) z^{i_1}_{k_1} z^{i_2}_{k_2} \cdots z^{i_n}_{k_n} \otimes z^{k_1}_{j_1} z^{k_2}_{j_2} \cdots z^{k_2}_{j_n}.$$

Thus, setting $Z^I_J = \mathsf{sign}(I, J) z^{i_1}_{j_1} z^{i_2}_{j_2} \cdots z^{i_2}_{j_2}$, we have

$$\Delta(Z^I_J) = Z^I_K \otimes Z^K_L.$$

Analogously, setting $T^I_J := \mathsf{sign}(J, I) z^{i_1}_{j_1} z^{i_2}_{j_2} \cdots z^{i_2}_{j_2}$, we have

$$\Delta(T^I_J) = T^K_L \otimes T^I_K, \quad \text{and moreover} \quad S(Z^I_J) = (-1)^{|\hat{J}|(|\hat{I}|+|\hat{J}|)} T^{I'}_{J'}.$$

We shall, for convenience, formally set

$$Z^{\dagger}{}^{I'}_{J'} := S(Z^I_J) = (-1)^{|\hat{J}|(|\hat{I}|+|\hat{J}|)} T^{I'}_{J'},$$

where K' is the sequence K written in the reverse order. There is a close connection between E_R and H_R with the Hecke algebras of type A. The n-th Hecke algebra of type A, $\mathcal{H}_{n,q}$ is generated by elements $T_i, 1 \leq i \leq n-1$, subject to the relations

$$T_i T_j = T_j T_i, |i - j| \geq 2; \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, i = 1, ..., n-2;$$

$$T_i^2 = (q-1)T_i + q, \forall i.$$

To each element w of the symmetric group $\mathfrak{S}_n$ of permutations of the sets $\{1, 2, ..., n\}$, one can associate in a canonical way an element T_w of $\mathcal{H}_n = \mathcal{H}_{n,q}$, in particular, $T_1 = 1, T_{(i,i+1)} = T_i$. The set $\{T_w | w \in \mathfrak{S}_n\}$ form a k-basis for $\mathcal{H}_n$ [6].

R induces an action of the Hecke algebra $\mathcal{H}_n = \mathcal{H}_{q,n}$ on the tensor powers $V^{\otimes n}$ of V, $\rho_n(T_i) = R_i := \mathsf{id}_V^{\otimes i-1} \otimes R \otimes \mathsf{id}_V^{\otimes n-i-1}$. We shall therefore use the notation $R_w := \rho(T_w)$. On the other hand, E_R coacts on V by $\delta(x_i) = x_j \otimes z^j_i$. Since E_R is a bialgebra, it coacts on $V^{\otimes n}$ by means of the product. Explicitly, $\delta(Z_I) = X_J \otimes Z^J_I$, where $X_I := x_{i_1} \otimes \cdots \otimes x_{i_n}$.

Let us use the notation TEnd for the set of all endomorphisms of a vector superspaces. The following is a super version of the double centralizer theorem proved in [10, Theorem 2.1].

Proposition 1.1. *With the assumption that q is not a root of unity of degree greater then 1, we have the following isomorphism of algebras:*

(1.9) $$\mathsf{TEnd}_{\mathcal{H}_n}(V^{\otimes n}) \cong (E^n_R)^*$$

(1.10) $$\mathsf{TEnd}^{E_R}(V^{\otimes n}) \cong \rho_n(\mathcal{H}_n),$$

where, E^n_R is the n-th homogeneous component of E_R, which is a coalgebra. Since R is an even operator, (1.10) implies

$$\mathsf{End}^{E_R}(V^{\otimes n}) \cong \rho_n(\mathcal{H}_n). \tag{1.11}$$

Proof. It is sufficient to prove (1.9). (1.10) then follows by means of the density theorem [5, Vol. 2], for $\mathcal{H}_n$ is semisimple. It is to establish a non-degenerate bilinear form $\langle \mathsf{TEnd}_{\mathcal{H}_n}(V^{\otimes n}), E^n_R\rangle$, which is compatible with the product and coproduct.

Recall that E^n_R is a quotient of $(V^* \otimes V)^{\otimes n}$ by the relation in (1.2). Using the isomorphism θ_n, we identify this space with $(V^{\otimes n})^* \otimes V^{\otimes n}$. Then E^n_R is isomorphic to the quotient of the latter by the subspace

$$\sum_{i=1}^{n-1} \mathsf{Im}((R^{-1}{}_i)^* \otimes R_i).$$

The proof of [10, Theorem 2.1] now applies and gives us the desired result.□

The double centralizer theorem implies that a simple E_R-comodule is the image of the operator induced by a primitive idempotent of $\mathcal{H}_n$ and, conversely, each primitive idempotent of $\mathcal{H}_n$ induces an E_R comodule which is either zero or simple. On the other hand, irreducible representations of $\mathcal{H}_n$ are parameterized by partitions of n. Hence up to conjugation, primitive idempotents of $\mathcal{H}_n$ are parameterized by partitions of n, too. Note that by the semisimplicity of $\mathcal{H}_n$, $\rho_n(\mathcal{H}_n)$ is also a subalgebra of $\mathcal{H}_n$.

The primitive idempotents $x_n := \sum_w T_w/[n]_q!$ and $y_n := \sum_w (-q)^{-l(w)}T_w/[n]_{1/q}!$ induce the symmetrizer and anti-symmetrizer operators on $V^{\otimes n}$. Let $S_n := \mathsf{Im}\rho_n(x_n)$ and $\Lambda_n := \mathsf{Im}\rho_n(y_n)$. Then one can show that $S := \bigoplus_{n=0}^{\infty} S_n$ and $\Lambda := \bigoplus_{n=0}^{\infty} \Lambda_n$ are algebras [9]. They are called symmetric and exterior tensor algebras on the corresponding quantum superspace.

By definition, the Poincaré series $P_\Lambda(t)$ of Λ is $\sum_{n=0}^{\infty} t^n \mathsf{dim}_k(\Lambda_n)$. It is proved in [14] that this series is a rational function having negative roots and positive poles. Let r be the number of its roots and s be the number of its poles. Then, simple E_R-comodules are parameterized by partitions from $\Gamma^{rs}_n := \{\lambda \vdash n | \lambda_{r+1} \le s\}$ [loc.cit., Theorem 5.1]. Consequently, in the algebra $\rho_n(\mathcal{H}_n)$ primitive idempotents are parameterized by partitions from $\Gamma^{r,s}_n$, too.

The pair (r, s) is called the birank of R. Our main assumption on R is that

$$\mathsf{tr}(C) = -[s-r]_q := -\frac{q^{s-r}-1}{q-1}, \tag{1.12}$$

where C is the reflection operator introduced above. In fact, (1.12) holds for any Hecke symmetry. The proof will be given elsewhere.

Simple H_R-comodules are much more complicated. The problem of classifying all its simple comodules is still open.

The Hopf algebra H_R (resp. the bialgebra E_R) is called the (function algebra on) a quantum group (resp. quantum semigroup) of type $A_{r-1|s-1}$.

Let $R = R_{r|s}$ be the standard deformation defined above. $R_{r|s}$ has the birank (r, s). The associated Hopf algebra is called the (function algebra on the) standard quantum general linear supergroup $GL_q(r|s)$ [20]. In this case, the matrices C and D are diagonal, $\mathsf{tr}(C) = \mathsf{tr}(D) = -[s-r]_q$.

2. The Integral on H_R. Recall that by definition, a left integral on a Hopf algebra H over a field k is a non trivial lineal functional $\int : H \longrightarrow k$ with the invariance property:

$$\int(a) = \sum_{(a)} a_1 \int(a_2). \tag{2.1}$$

Since we are considering superalgebras, we shall also require that the integral is even, that means the value of an integral at an odd element of H_R is zero. The even part $H_{\bar{0}}$ of H is an ordinary Hopf algebra and if H possesses a non zero integral then its restriction on $H_{\bar{0}}$ is also non-zero and therefore by a theorem of Sullivan [23] is uniquely determined up to a scalar constant.

It easy to see that (2.1) is equivalent to

$$\sum_{(b)} \int (aS(b_1))b_2 = \sum_{(a)} a_1 \int (a_1 S(b)). \tag{2.2}$$

From the definition of H_R, an arbitrary element of H_R can be represented as a linear combination of monomials in z_i^j and t_k^l. On the other hand, using the relation on H_R and the axiom (iii) of R, we can represent a monomial like $t_j^i z_k^l$ as a linear combination of monomials of the form $z_p^q t_r^s$, i.e., we can interchange the order of z' and $t's$ in a tensor product. Namely, we have, according to (1.5),

$$(-1)^{\hat{l}(\hat{i}+\hat{j})} t_i^j z_k^l = (-1)^{\hat{r}((\hat{p}+\hat{q})} R_{ks}^{jp} z_p^q t_r^s P_{qi}^{rl}. \tag{2.3}$$

Thus, by using the rule (2.3), we can represent any element of H_R as a linear combination of monomials of the form

$$z_{i_1}^{j_1} \cdots z_{i_p}^{j_p} t_{k_1}^{l_1} \cdots t_{k_q}^{l_q}.$$

Therefore, it is sufficient to compute the integral at such monomials. Next, notice that the integral at those monomials with $p \neq q$ must vanish, by virtue of (2.2). Indeed, if the integral did not vanish at such monomials, (2.2) would give us a non-homogeneous relation between elements of H_R, which is a contradiction.

Let us denote, for $n = l(I)$,

$$I_n{}_{IK}^{JL} := \int \left(Z_I^L Z^{\dagger}{}_{K'}^{J'}\right) = (-1)^{|\hat{K}|(|\hat{K}|+|\hat{J}|)} \int \left(Z_I^L T_{K'}^{J'}\right)$$

where, as defined in the previous section, K' is K written in the reverse order. Thus, we can consider I_n as the matrix of an operator acting on $V^{\otimes n} \otimes V^{\otimes n}$.

We have the following conditions on I_n (cf. [11, Section 4]) :

(I1) I_n should be invariant with respect to the relations within z's and t's given in (1.2) and (1.5), respectively, that is, for all $i, j = 1, 2, \ldots, n-1$

$$(R_i \otimes R_j) I_n = I_n (R_j \otimes R_i).$$

(I2) when we contract I_n with respect to the relation (1.3), (1.7), we should get I_{n-1}, more precisely,

$$\delta_{j_n}^{i_n} I_n{}_{I_1 i_n K}^{J_1 j_n L} = I_{n-1}{}_{I_1 K_1}^{J_1 L_1} \delta_{kn}^{l_n}$$
$$C_{l_n}^{k_n} I_n{}_{IKk_n}^{JL_1 l_n} = I_{n-1}{}_{I i_p K_1}^{J_1 j_n L_1} C_{j_n}^{i_n}.$$

(I3) I_n should respect the rule (2.2), which reads

$$I_{n}{}^{JL}_{IM}Z^M_K = Z^L_N I_{n}{}^{JN}_{IK}.$$

These conditions are, in fact, sufficient for an integral on H_R. For, assume we have a collection of matrices I_n, satisfying the conditions (I1-I3) above, then we can extend it linearly on the whole H_R. The only ambiguity may occur is that, there may be more than one way of leading an element of H_R to a linear combination of monomials of the form $Z^J_I T^L_K$. However, the Yang-Baxter equation on R ensures that different ways of using rule (2.3) give us the same result, up to relations in z's and $t's$, respectively.

We thus reduced the problem to finding a family of matrices I_n satisfying conditions (I1-I3). Our next claim is that I_n can be found in the following way

$$I_n = \sum_{w\in\mathfrak{S}_n} q^{-l(w)}(P_n C^{\otimes n} R_{w^{-1}}) \otimes R_w \tag{2.4}$$

where $R_w = \rho(T_w)$ as in Section 1, C is the reflection operator introduced in Section 1 and P_n is a certain operator on $V^{\otimes n}$. More precisely, we have

LEMMA 2.1. *Assume that for each $n \geq 1$ the operator $P_n \in \rho_n(\mathcal{H}_n) = \mathsf{End}^{H_R}(V^{\otimes n}) \subset \mathsf{End}^k(V^{\otimes n})$ is in the center of $\rho_n(\mathcal{H}_n)$ and satisfies the condition*

$$P_{n-1} \otimes \mathsf{id}_V = P_n(L_n + \mathsf{tr}(C))$$

where L_n are the Murphy operators: $L_1 = 0$,

$$L_n = \sum_{i=1}^{n-1} q^{-i} R_{(n-i,n)}, \quad n \geq 2,$$

$(n-i, n)$ is the inversion that changes places of $n-i$ and n. Then the matrices I_n given in (2.4) satisfy the conditions (I1-I3).

Proof. The conditions (I1) and (I3) can be easily verified. In fact, (I1) is equivalent to the equations

$$\sum_{w\in\mathfrak{S}_n} q^{-l(w)}(R_i P_n C^{\otimes n} R_{w^{-1}}) \otimes R_w = \sum_{w\in\mathfrak{S}_n} q^{-l(w)} P_n C^{\otimes n} R_{w^{-1}} \otimes (R_w R_i),$$

for $i = 1, 2, \ldots, n-1$. By assumption, P_n commutes with all R_i. On the other hand, using the Yang-Baxter equation we can also show that $C^{\otimes n}$ commutes with all R_i. Therefore the equation above follows from

$$\sum_{w\in\mathfrak{S}_n} q^{-l(w)}(R_i R_{w^{-1}}) \otimes R_w = \sum_{w\in\mathfrak{S}_n} q^{-l(w)} R_{w^{-1}} \otimes (R_w R_i), \quad \text{in } \mathcal{H}_n \otimes \mathcal{H}_n, \tag{2.5}$$

which can be easily verified using the Hecke equation for R. The verification of (I3) is straightforward, it does not involve P_n and $C^{\otimes n}$ but rather a direct consequence of relations in (1.2).

The harder part is to verify (I2). Here we use the condition

$$P_{n-1} \otimes \mathsf{id}_V = P_n(L_n + \mathsf{tr}(C)).$$

It is known that each element T_w of $\mathcal{H}_n$ can be expressed in the form $T_w = T_k \cdots T_{n-1}T_{w_1}$ for some $w_1 \in \mathfrak{S}_{n-1}$, where $\mathfrak{S}_{n-1}$ is the subgroup of $\mathfrak{S}_n$, fixing n. We define a linear map $\mathcal{H}_n \longrightarrow \mathcal{H}_{n-1}$ setting

$$h_n(T_w) = \begin{bmatrix} T_k \cdots T_{n-2}T_{w_1} & \text{if } k \leq n-2; \\ \mathrm{tr}(C)T_{w_1} & \text{if } k = n-1. \end{bmatrix}$$

The two conditions in (I2) can then be rewritten as follows:

$$\sum_{w\in\mathfrak{S}_n} q^{-l(w)} h_n(P_nR_{w^{-1}}) \otimes R_w = \sum_{u\in\mathfrak{S}_{n-1}} q^{-l(u)} P_{n-1}R_{u^{-1}} \otimes R_u, \tag{2.6}$$

as elements of $\rho_{n-1}(H_{n-1}) \otimes \rho_n(\mathcal{H}_n)$, and

$$\sum_{w\in\mathfrak{S}_n} q^{-l(w)} (P_nR_{w^{-1}}) \otimes h_n(R_w) = \sum_{u\in\mathfrak{S}_{n-1}} q^{-l(u)} P_{n-1}R_{u^{-1}} \otimes R_u, \tag{2.7}$$

as elements of $\rho_n(\mathcal{H}_n) \otimes \rho_{n-1}(H_{n-1})$, (in the equations above, we have canceled the term $C^{\otimes n-1}$ on both sides).

The operator L_n comes into play because of the following identity in $\mathcal{H}_n \otimes \mathcal{H}_{n-1}$ (cf [11, Lemma 4.1.3])

$$\sum_{w\in\mathfrak{S}_n} q^{-l(w)} T_{w^{-1}} \otimes h_n(T_w) = \sum_{u\in\mathfrak{S}_{n-1}} q^{-l(w)} (L_n + \mathrm{tr}(C))T_{u^{-1}} \otimes T_u \tag{2.8}$$

here we identify $\mathcal{H}_{n-1}$ with the subalgebra of $\mathcal{H}_n$ generated by $T_u, u \in \mathfrak{S}_{n-1}$. In fact, we can replace $\mathrm{tr}(C)$ by any element of k. Replace T_w by R_w in (2.8) and plug in the identity $C^{i_n}_{j_n} R_w{}^{J_1 j_n}_{I_1 i_n} = h_n(R_w{}^J_I)$, we obtain (2.7).

On the other hand, (2.8) and (2.5) imply (2.6):

$$\begin{aligned}\sum_{w\in\mathfrak{S}_n} q^{-l(w)} h_n(P_nR_{w^{-1}}) \otimes R_w &= \sum_{w\in\mathfrak{S}_n} q^{-l(w)} h_n(R_{w^{-1}}) \otimes R_wP_n \\ &= \sum_{u\in\mathfrak{S}_{n-1}} q^{-l(u)} T_{u^{-1}} \otimes (L_n + \mathrm{tr}(C))T_uP_n \\ &= \sum_{u\in\mathfrak{S}_{n-1}} q^{-l(u)} P_{n-1}T_{u^{-1}} \otimes T_u.\end{aligned}$$

The lemma is therefore proved.□

Thus, we reduced the problem of finding an integral on H_R to constructing operators $P_n \in \rho_n(\mathcal{H}_n)$ satisfying certain conditions. The next step is to construct P_n.

3. The Construction of P_n. We want to construct operators $P_n \in \rho_n(\mathcal{H}_n)$ with the property

$$P_n(L_n + \mathrm{tr}(C)) = P_{n-1} \otimes \mathrm{id}_V.$$

The Murphy operators L_n were introduced by Dipper and James [7] following Murphy's construction, to describe a full set of mutually orthogonal primitive idempotents of the algebra $\mathcal{H}_n$:

$$E_{t_i(\lambda)} = \prod_{\substack{1\leq m\leq n \\ 1-m\leq k\leq m-1 \\ k\neq c_{t_i(\lambda)}(m)}} \frac{L_m - [k]_q}{[c_{t_i(\lambda)}(m)]_q - [k]_q}, \quad 1 \leq i \leq d_\lambda,\ \lambda \in \mathcal{P}_n,$$

where $\{t_i(\lambda), 1 \le i \le d_\lambda\}$ is the set of standard λ-tableaux, $c_{t_i(\lambda)}(m)$ is the content of m in the standard tableau $t_i(\lambda)$. The primitive idempotents $E_{t_i(\lambda)}$, $i = 1, 2, ..., d_\lambda$ belong to the same block that corresponds to λ, their sum $F_\lambda = \sum_{1\le i\le d_\lambda} E_{t_i(\lambda)}$ is the minimal central idempotent corresponding to λ.

It is known that L_m satisfies the equation $\prod_{k=-m-1}^{m+1}(L_m - [k]_q) = 0$. Therefore, for $1 \le m \le n$,

$$L_m E_{t_i(\lambda)} = E_{t_i(\lambda)} L_m = c_{t_i(\lambda)}(m) E_{t_i(\lambda)}. \tag{3.1}$$

Let $\Omega_n^{r,s}$ denote the set $\{\lambda | \lambda_{r+1} \le s, \lambda_r = s\}$. For $\lambda \in \Omega^{r,s}$, $[(s^r)] \subset [\lambda]$. We define

$$p_\lambda := \prod_{x\in[\lambda]\setminus[(s^r)]} \frac{q^{r-s}}{[c_\lambda(x) + r - s]_q},$$

and

$$P_n := \sum_{\substack{\lambda\in\Omega_n^{r,s}\\ 1\le i\le d_\lambda}} p_\lambda E_{t_i(\lambda)} = \sum_{\lambda\in\Omega_n^{r,s}} p_\lambda F_\lambda. \tag{3.2}$$

Recall that P_n are defined in the algebra $\rho_n(\mathcal{H}_n) \cong \mathsf{End}^{E_R}(V^{\otimes n})$, which is the factor algebra of $\mathcal{H}_n$ by the two-sided ideal generated by minimal central idempotents corresponding to partitions from $\mathcal{P}_n \setminus \Gamma_n^{r,s}$. Fixing an embedding $\rho_n(\mathcal{H}_n) \hookrightarrow \rho_{n+1}(\mathcal{H}_{n+1})$, $\rho_n(\mathcal{H}_n) \ni W \longmapsto W \otimes \mathsf{id}_V \in \rho_{n+1}(\mathcal{H}_{n+1})$, we identify $\rho_n(\mathcal{H}_n)$ with a subalgebra of $\rho_{n+1}(\mathcal{H}_{n+1})$.

LEMMA 3.1. *The operators P_n are central in $\rho_n(\mathcal{H}_n)$ and satisfy the equation (in $\rho_{n+1}(\mathcal{H}_{n+1})$)*

$$P_{n+1}(L_{n+1} - [s-r]_q) = P_n.$$

Proof. The operator P_n is obviously central, for it is a sum of central elements.

We check the equation above. First, notice that, if $\lambda \in \Omega_{n+1}^{r,s}$ and $t_i(\lambda)$ is a standard λ-tableau, then the node of $[\lambda]$, containing $n+1$, is removable, i.e., having removed it we still have a standard tableau. The tableau $t_i(\lambda)$ is called essential if this node is not the node (r, s), otherwise it is called non-essential. A tableau is essential iff the tableau, obtained from it by removing the node containing $n+1$ is again a γ-tableau with $\gamma \in \Omega_n^{r,s}$.

Observe, that if $t_i(\lambda)$ is non-essential, then $c_{t_i(\lambda)}(n+1) = s - r$, hence

$$E_{t_i(\lambda)}(L_{n+1} - [s-r]_q) = 0,$$

by virtue of Equation (3.1). We reshuffle the terms of P_{n+1} in groups as follows

$$P_{n+1} = \sum_{\substack{\gamma\in\Omega_n^{r,s}\\ 1\le i\le d_\gamma}} \sum_{\substack{\lambda\in\Omega_{n+1}^{r,s}\\ t(\lambda)\supset t_i(\gamma)}} p_\lambda E_{t(\lambda)} + \sum_{\substack{t(\lambda) \text{ is}\\ \text{non-eessential}}} p_\lambda E_{t(\lambda)}.$$

That is, for each $t_i(\gamma)$, $\gamma \in \Omega_n^{r,s}$, we pick up into a group those standard tableaux $t(\lambda)$, $\lambda \in \Omega_{n+1}^{r,s}$ that contain $t_i(\gamma)$ as a subtableau. The above observation implies that the last sum in the right-hand side of the above equation is killed by $L_{n+1}-[s-r]_q$. Thus, it is sufficient to prove, for a fixed $t_i(\gamma)$, $\gamma\in\Omega_n^{r,s}$,

$$\sum_{\substack{\lambda\in\Omega_{n+1}^{r,s}\\ t(\lambda)\supset t_i(\gamma)}} p_\lambda E_{t(\lambda)}(L_{n+1}-[s-r]_q)=p_\gamma E_{t_i(\gamma)}.$$

We have $(L_{n+1}-[s-r]_q)E_{t(\lambda)}=[c_{t(\lambda)}(n+1)]_q-[s-r]_qE_{t(\lambda)}$ and thus $p_\lambda([c_\lambda(x)]_q-[s-r]_q)=p_\gamma$ whenever $[\gamma]$ is obtained from $[\lambda]$ by removing the node x. Since, for any two standard tableaux $t(\gamma)$ and $t(\lambda)$ with $\gamma\subset\lambda$ as above, the number $n+1$ should lie in the node x, for which $[\lambda]\setminus[x]=[\gamma]$, we deduce that the equation to be proved is equivalent to

$$\sum_{\substack{\lambda\in\Omega_{n+1}^{r,s}\\ t(\lambda)\supset t_i(\gamma)}} E_{t(\lambda)}=E_{t_i(\gamma)}. \tag{3.3}$$

Since $\prod_{k=-n-1}^{n+1}(L_{n+1}-[k]_q)=0$,

$$\sum_{m=-n-1}^{n+1}\prod_{\substack{k=-n-1,\\ k\neq m}}^{n+1}\frac{L_{n+1}-[k]_q}{[m]_q-[k]_q}=1.$$

Therefore

$$E_{t_i(\gamma)}=\sum_{m=-n-1}^{n+1}E_{t_i(\gamma)}\prod_{\substack{k=-n-1,\\ k\neq m}}^{n+1}\frac{L_{n+1}-[k]_q}{[m]_q-[k]_q}.$$

Remember that we are working in the algebra $\rho_{n+1}(\mathcal{H}_{n+1})$, in which $E_\lambda\neq 0$ if and only if $\lambda\in\Omega_{n+1}^{r,s}$. Each term on the right-hand side of the above equation is either zero or a primitive idempotent of the form $E_{t(\lambda)}$ with $t(\lambda)$ containing $t_i(\gamma)$ as a subtableau. Since the left-hand side of (3.3) contains all primitive idempotents in $\rho_{n+1}(\mathcal{H}_{n+1})$ that correspond to standard tableaux containing $t_i(\gamma)$ as a subtableau, the equation (3.3) follows. Lemma 3.1 is therefore proved.□

As a corollary of Lemmas 2.1 and 3.1, we have

THEOREM 3.2. *The Hopf algebra H_R associated to a Hecke symmetry R, which satisfies the condition (1.12), possesses an integral, which is uniquely determined up to a scalar multiple. Let (r,s) be the birank of R. Then an integral can be given as follows: if $l(I)=l(K)<rs$, $\int(Z_I^J Z^{\dagger}{}_K^L)=0$ and if $l(I)=l(K)=n\geq rs$,*

$$\int(Z_I^J Z^{\dagger}{}_{K'}^{L'})=\sum q^{-l(w)}(P_nC^{\otimes n}R_{w^{-1}})_I^L R_w{}_K^J, \tag{3.4}$$

$(K'=(k_n,k_{n-1},\ldots,k_1))$.

REMARK. Since P_n, C, R are even operators, the integral in (3.4) vanishes unless $|\hat{I}|=|\hat{L}|, |\hat{J}|=|\hat{K}|$.

There is a symmetric bilinear form on the Hecke algebra $\mathcal{H}_n$, given by $(T_u,T_v)=q^{l(u)}\delta_{v^{-1}}^u$. With respect to this bilinear form, $\{R_w, w\in\mathfrak{S}_n\}$ and $\{q^{-l(w)}R_{w^{-1},w\in\mathfrak{S}_n}\}$ are dual bases [6].

Let $\{E_\lambda^{ij},\lambda\vdash n, 1\leq i,j\leq d_\lambda\}$ be a basis of $\mathcal{H}_n$ with the following properties

1. $\{E^{ij}_\lambda, \leq i,j \leq d_\lambda\}$ is a basis of the block in $\mathcal{H}_n$, corresponding to λ;
2. $E^{ij}_\lambda E^{kl}_\mu = \delta^\mu_\lambda \delta^j_k E^{il}_\lambda$.

and let $k_\lambda = (E^{ii}_\lambda, E^{ii}_\lambda)$, $(\cdot,\cdot)$ be the mentioned above bilinear form. Then, $\{E^{ij}_\lambda, \leq i,j \leq d_\lambda\}$ and $\{E^{ji}_\lambda / k_\lambda, \leq i,j \leq d_\lambda\}$ are dual bases with respect to the above form. Hence, using standard argument we can easily show

$$\sum_{w\in\mathfrak{S}_n} q^{-l(w)} R_{w^{-1}} \otimes R_w = \sum_{\substack{\lambda\vdash n\\ 1\leq i,j\leq d_\lambda}} \frac{1}{k_\lambda} E^{ij}_\lambda \otimes E^{ji}_\lambda. \tag{3.5}$$

The number k_λ can be computed explicitly: $k_\lambda = q^{n(\lambda)}\prod_{x\in[\lambda]}[h(x)]_q^{-1}$, where $n(\lambda) = \sum_i \lambda_i(i-1)$, $h(x)$ is the hook length of x in the diagram $[\lambda]$: $h_\lambda(x) = \lambda_i + \lambda'_j - i - j + 1$, where (i,j) is the coordinate of x in the diagram $[\lambda]$ (cf. [12]).

The formula (3.4) can therefore be rewritten as follows:

$$\int (Z^J_I Z^{\dagger}{}^{L'}_{K'}) = \sum_{\substack{\lambda\vdash n\\ 1\leq i,j\leq d_\lambda}} \frac{1}{k_\lambda}(C^{\otimes n}P_n E^{ij}_\lambda)^L_I E^{ji}_\lambda{}^J_K = \sum_{\substack{\lambda\in\Omega^{r,s}_n\\ 1\leq i,j\leq d_\lambda}} \frac{p_\lambda}{k_\lambda}(C^{\otimes n} E^{ij}_\lambda)^L_I E^{ji}_\lambda{}^J_K. \tag{3.6}$$

4. Characters of H_R and Quantum Analogue of Super HCIZ Integral Formula. In this section, we would like to give a quantum analogue of the super HCIZ integral formula. Recall that the super HCIZ integral formula (0.2) computes the integral on the compact supergroup $U(r|s)$ at the function on the variable U running in $U(r|s)$: $\mathsf{str}(MUNU^{-1})^n$, where M, N are fixed hermitian supermatrices. First, notice the following equality for supermatrices

$$\left[\mathsf{str}(MUN^{-1}U^{-1})\right]^n = \mathsf{str}((MUNU^{-1})^{\otimes n}) = (-1)^{|\hat{I}|} M^I_J U^J_K N^{\dagger}{}^{K'}_{L'} U^{\dagger}{}^{L'}_{I'},$$

where we adopt the notion in Section 1 of Z^I_J and $Z^{\dagger}{}^I_J$. In fact, the first equation above is obvious, the second equation follows from the following recurrent relation

$$(-1)^{|\hat{I}|+\hat{i}} M^{Ii}_{Jj} U^{Jj}_{Kk} N^{\dagger}{}^{kK'}_{lL'} U^{\dagger}{}^{lL'}_{iI'} = (-1)^{|\hat{I}|} M^I_J U^J_K N^{\dagger}{}^{K'}_{L'} U^{\dagger}{}^{L'}_{I'} \cdot (-1)^{\hat{i}} M^i_j U^j_k N^{\dagger}{}^k_l U^{\dagger}{}^l_i.$$

By definition, an A-point of H_R is an algebra homomorphism $\phi : H_R \longrightarrow A$, where A is a superalgebra over k. Set $M = \phi(Z)$. Then the entries of M commute by the same rule as the entries of Z. M is called a quantum supermatrix with values from A. We set $M^I_J := \phi(Z^I_J)$ and $M^{\dagger}{}^{I'}_{J'} := \phi(S(Z^I_J)) = \phi(Z^{\dagger}{}^{I'}_{J'})$. Our quantum analogue of HCIZ formula will compute the integral at the element $D^P_I M^I_J Z^J_K N^{\dagger}{}^{K'}_{L'} Z^{\dagger}{}^{L'}_{P'}$, where D^P_I is the entries of the tensor power of the matrix D (which consists of scalars from k), M and N are points of H_R with entries anti-commuting with the elements of H_R. In other words, let M and N be quantum supermatrices with values from an algebra A. $A \otimes H_R$ has a natural structure of a superalgebra as the product of two superalgebras in the category of vector superspaces. The integral on H_R induces a map $A \otimes H_R \longrightarrow A$. We want to compute the value of this map at the element $D^P_I M^I_J Z^J_K N^{\dagger}{}^{K'}_{L'} Z^{\dagger}{}^{L'}_{P'}$ of the algebra $A\otimes H_R$ in terms of the values of certain irreducible characters of H_R at the matrices M and N.

We first need the notion of coribbon coquasitriangular Hopf superalgebras. The notion of coquasitriangular Hopf superalgebras does not differ from the notion of coquasitriangular Hopf algebras (see e.g. [17]), except that a sign may appear whenever we permute two adjacent elements in a tensor product. Explicitly, we define a coquasitriangular structure on a Hopf superalgebra H as an even linear map $r : H\otimes H \longrightarrow k$, subject to the following conditions (see, e.g. [13]):

$$(-1)^{\hat{b}_1\hat{a}_2}r(a_1,b_1)a_2b_2 = (-1)^{\hat{b}_1\hat{a}}b_1a_1r(a_2,b_2),$$
$$r(ab,c) = (-1)^{\hat{b}\hat{c}_1}r(a,c_1)r(b,c_2), \quad r(a,bc) = r(a_1,c)r(a_2,b),$$
$$(-1)^{\hat{b}_1\hat{a}_2}r(a_1,b_1)r^{-1}(a_2,b_2) = (-1)^{\hat{b}_1\hat{a}_2}r^{-1}(a_1,b_1)r(a_2,b_2) = \varepsilon(ab).$$

The following properties of r are consequences of the definition:

$$r(S(a),b) = r^{-1}(a,b) \quad r(a,1) = r(1,a) = \varepsilon(a).$$

The coquasitriangular structure $r : H\otimes H \longrightarrow k$ induces a braiding in the category of H-comodules making this category a braided category. Explicitly, the braiding is given by

$$\tau_{M,N} : M\otimes N \longrightarrow N\otimes M, \quad m\otimes n \longmapsto (-1)^{\hat{m}\hat{n}_0}n_0\otimes m_0r(m_1,n_1), \quad m\in M, n\in N.$$

Define a linear map $u : H \longrightarrow k$, $u(a) := (-1)^{\hat{a}_1\hat{a}_2}r(a_2,S(a_1))$. Then the square of the antipode can be computed by means of u

$$S^2(a) = u^{-1}(a_1)a_2u(a_3), \tag{4.1}$$

where u^{-1}, its convolution inverse is given by $u^{-1}(a) = (-1)^{\hat{a}_1\hat{a}_2}r(S^2(a_2),a_1)$ (cf. [13, Section 2]).

A ribbon form on a coquasitriangular Hopf superalgebras is an even linear map $t : H \longrightarrow k$, subject to the following conditions:

$$t(a_1)a_2 = a_1t(a_2), \quad t(S(a)) = t(a), \quad t(a_1)t^{-1}(a_2) = t^{-1}(a_1)t(a_2) = \varepsilon(a),$$
$$t(ab) = (-1)^{\hat{a}\hat{b}_2}t(a_1)t(b_1)r(b_2,a_3)r(a_3,b_3).$$

The ribbon form satisfies:

$$t(a_1)t(a_2) = u^{-1}(a_1)u^{-1}(S(a_2)). \tag{4.2}$$

The proof is to expand the right-hand side using the formula for the square of the antipode given above.

Notice that, since r and t are even linear map, $r(a,b) = 0$ unless $\hat{a}+\hat{b} = \bar{0}$ and $t(a) = 0$ unless $\hat{a} = \bar{0}$.

The ribbon form t induces a twist in the category of finite dimensional H-comodules, making this category a ribbon category [22, 17]. The twist is given by

$$\theta_M : M \longrightarrow M, \quad m \longmapsto m_0t(m_1).$$

Let H be a coribbon Hopf superalgebra, and M an H-comodule of finite dimension. Fix a basis $x_1, x_2, \ldots, x_d$ and let $a^i_j, 1 \le i,j \le d$ be the corresponding multiplicative matrix, that is, the coaction of H on M is given by $\delta(x_i) = x_k\otimes a^k_i$. Let T_M be the matrix of the twist morphism θ_M with respect to this basis. Then we have ${T_M}^i_j = t(a^i_j)$. Similarly, let ${D_M}^i_j = (-1)^{\hat{i}\hat{j}}u(S(a^i_j)) = (-1)^{\hat{i}\hat{j}}r(a^i_k, S(a^k_j))$. Let f be an endomorphism of M and F be its matrix. We define

$$\Phi(f) := \mathsf{tr}(D_MT_MFA)$$

to be the character of the morphism f, where A is the matrix (a^i_j) (cf. [12]). The character of the comodule M is $\Phi(M) := \Phi(\mathsf{id}_M) = \mathsf{tr}(D_M T_M A)$.

Let M^* be the dual vector superspace to M and $\xi^1, \xi^2, \dots, \xi^d$ be the dual basis on M^*. The coaction of H on M^* is given by $\delta(\xi^i) = (-1)^{\hat{k}(\hat{k}+\hat{\imath})}\xi^k \otimes S(a^i_k)$. Hence, the character of the dual comodule M^* is

$$\Phi(M^*) = \mathsf{tr}(C_M T_M S(A)),$$

where $C_M{}^i_j := (-1)^{\hat{\imath}\hat{\jmath}} u(a^i_j) = r(a^k_j, S(a^i_k))$.

The equality in (4.2) can be rewritten as follows:

$$D_{M\otimes N} T_{M\otimes N} = (D_M T_M) \otimes (D_N T_N), \tag{4.3}$$

where the matrices $D_{M\otimes N}$ and $T_{M\otimes N}$ are defined similarly for $M \otimes N$, with respect to certain bases of M and N. Notice that the definition of the ribbon form t implies that T_M commutes with D_M. We have the following properties of Φ:

$$\Phi(f \oplus g) = \Phi(f) + \Phi(g), \quad \Phi(f \otimes g) = \Phi(f) \cdot \Phi(g), \quad \Phi(f \circ h) = \Phi(h \circ f), \tag{4.4}$$

where f, h and g are endomorphisms of the comodules M and N, respectively. Indeed, the first equation is obvious, the second one follows from (4.3), the last one follows from the naturality of θ. Since $M\otimes N \cong N\otimes M$ by means of the braiding, which is a natural transformation, i.e. commute with all morphisms, we have $\Phi(f) \cdot \Phi(g) = \Phi(g) \cdot \Phi(f)$.

H_R is a coribbon Hopf superalgebra (cf. [15]). The coquasitriangular structure is given by

$$r(z^i_j, z^k_l) = (-1)^{\hat{\jmath}\hat{k}} R^{ki}_{jl}, r(t^i_j, z^k_l) = (-1)^{\hat{\imath}\hat{k}} {R^{-1}}^{ik}_{lj},$$
$$r(z^i_j, t^k_l) = (-1)^{\hat{\jmath}\hat{l}} P^{ki}_{jl}, r(t^i_j, t^k_l) = (-1)^{\hat{\imath}\hat{l}} R^{ki}_{jl},$$

and the ribbon form is given by $t(z^i_j) = q^{(r-s+1)/2}\delta^i_j$, where (r, s) is the birank of the Hecke symmetry R. The form u satisfies $u(z^i_j) = (-1)^{\hat{\imath}\hat{\jmath}} P^{il}_{jl}$, and $u(z^i_j) = (-1)^{\hat{\imath}\hat{\jmath}} P^{li}_{lj}$.

The double centralizer theorem 1.1 implies that simple comodules of E_R are parameterized by partitions from the set $\Gamma^{rs} = \{\lambda | \lambda_{r+1} \le s\}$. For each partition $\lambda \in \Gamma^{r,s}_n$, denote M_λ the corresponding simple E_R-comodule, M_λ is isomorphic to $\mathsf{Im}\rho(E_\lambda)$ for a primitive idempotent E_λ. Since the map $E_R \longrightarrow H_R$ is injective (cf. [11, Theorem 2.3.5]), M_λ is a simple H_R-comodule, too. Therefore $\Phi(M_\lambda) = \Phi(E_\lambda)$, hence

$$S_\lambda := \Phi(M_\lambda) = \mathsf{tr}(D_{V^{\otimes n}} T_{V^{\otimes n}} E_\lambda Z^{\otimes n}).$$

Remember that in the definition of $Z^{\otimes n}$, the signs are also inserted.

The equality in (4.3) for t and u implies

$$D_{V^{\otimes n}} T_{V^{\otimes n}} = D_V^{\otimes n} T_V^{\otimes n}.$$

Since $T_V = t(Z) = q^{r-2+1}\mathsf{id}$, we have

$$S_\lambda = \Phi(M_\lambda) = q^{n(r-s+1)/2} D^I_J E_\lambda{}^J_K Z^K_I. \tag{4.5}$$

Analogously, we have

$$S_{-\lambda} := \Phi(M^*_\lambda) = q^{n(r-s+1)/2} C^I_J E_\lambda{}^J_K {Z^\dagger}^{K'}_{I'}, \tag{4.6}$$

where $C^i_j := (-1)^{\hat{i}\hat{j}} u(z^i_j) = P^{li}_{lj}$.

We are now ready to formulate a quantum analogue of the HCIZ integral formula for quantum supergroups of type $A_{r|s}$.

THEOREM 4.1. *Let M and N be A-points of H_R. Assume that entries of M and N anti-commute with the entries of Z and T. Then*

$$\int (D^P_I M^I_J Z^J_K N^{\dagger K'}_{L'} Z^{\dagger L'}_{P'}) = q^{-n(r-s+1)} \sum_{\lambda\in\Gamma^{r,s}_n} \frac{d_\lambda p_\lambda}{k_\lambda} S_\lambda(M) S_{-\lambda}(N).$$

Proof. Choosing a basis $\{E^{ij}_\lambda, \lambda \vdash n, 1 \le i,j \le d_\lambda$, such that $E_\lambda = E^{ii}_\lambda$ for some i, we have

$$\begin{aligned}\int (D^P_I M^I_J Z^J_K N^{\dagger K'}_{L'} Z^{\dagger L'}_{P'}) &= \int ((-1)^{(|\hat{J}|+|\hat{K}|)(|\hat{K}|+|\hat{L}|)} D^P_I M^I_J N^{\dagger K'}_{L'} Z^J_K Z^{\dagger L'}_{P'}) \\ \text{(by (3.6))} &= \sum_{\substack{1\le i,j\le d_\lambda \\ \lambda\in\Omega^{r,s}_n}} \frac{p_\lambda}{k_\lambda} (-1)^{(|\hat{J}|+|\hat{K}|)(|\hat{K}|+|\hat{L}|)} D^P_I M^I_J N^{\dagger K'}_{L'} C^L_Q E^{ij}_\lambda{}^Q_K E^{ji}_\lambda{}^J_P \\ &= \sum_{\substack{1\le i,j\le d_\lambda \\ \lambda\in\Omega^{r,s}_n}} \frac{p_\lambda}{k_\lambda} D^P_I M^I_J E^{ji}_\lambda{}^J_P N^{\dagger K'}_{L'} C^L_Q E^{ij}_\lambda{}^Q_K \\ &= \sum_{\substack{1\le i,j\le d_\lambda \\ \lambda\in\Omega^{r,s}_n}} q^{-n(r-s+1)} \frac{p_\lambda}{k_\lambda} \Phi(E^{ji}_\lambda)(M) \cdot \Phi(E^{ij*}_\lambda)(N) \\ &= \sum_{\lambda\in\Omega^{r,s}_n} q^{-n(r-s+1)} d_\lambda \frac{p_\lambda}{k_\lambda} S_\lambda(M) S_{-\lambda}(N).\end{aligned}$$

In the third equation above the term $(-1)^{(|\hat{J}|+|\hat{K}|)(|\hat{K}|+|\hat{L}|)}$ disappears according to the remark following Theorem 3.2, in the last equation, we have, by means of (4.4), $\Phi(E^{ij}_\lambda) = \Phi(E^{ij}_\lambda E^{jj}_\lambda) = \Phi(E^{jj}_\lambda E^{ij}_\lambda) = 0$ if $i \ne j$. □

EXAMPLE. Let us consider the case of standard quantum general linear super group $GL_q(r|s)$, determined in terms of the symmetry $R_{r|s}$ given Section 1. In this case, any diagonal matrix with commuting entries is a point of E_R. Thus, assume that M and N are diagonal matrix with entries commuting each other and with the entries of Z and T, $A = \mathsf{diag}(a_1, a_2, ..., a_{r+s})$, $B = \mathsf{diag}(b_1, b_2, ..., b_{r+s})$. Then, we have

$$S_{(n)}(M) = \sum_{k=0}^{n} h_{n-k}(qa_1, ..., q^r, a_r) e_k(-q^r a_{r+1}, ..., -q^{r-s+1} a_{r+s}),$$

h_k and e_k are the k-th complete and elementary symmetric functions in r and s variables, respectively [19, Chapter I]. Hence

$$S_\lambda(M) = s_\lambda(qa_1, q^2 a_2, ..., q^r a_r / - q^r a_{r+1}, ..., -q^{r-s+1} a_{r+s}),$$

s_λ are the Hook-Schur functions in $r+s$ variables (cf. [loc.cit., Ex. I.3.23]). Therefore, if $\lambda \in \Gamma^{r,s}_n$, thus, $\lambda = (r^s) + \mu \cup \nu'$, $\mu \in \mathcal{P}^r, \nu \in \mathcal{P}^s$, we have [loc.cit.]

$$S_\lambda(A) = (-1)^{|\nu|} \prod_{i=1,j=1}^{r,s} (q^i a_i - q^{r-j+1} a_{r+j}) s_\mu(qa_1, \ldots, q^r a_r) \times s_\nu(q^r a_{r+1}, \ldots, q^{r+1-s} a_{r+s}),$$

where s_μ (resp. s_ν) are the Schur functions in r variables (resp. s variables). Analogously, we have

$$S_{-\lambda}(B) = (-1)^{|\nu|} \prod_{i=1,j=1}^{r,s} (q^{r-s-i+1}b_i - q^{j-s}b_{r+j}) s_\mu(q^{r-s}b_1, \dots, q^{1-s}b_r) \times s_\nu(q^{1-s}b_{r+1}, \dots, b_{r+s}).$$

The quantum super HCIZ is then given by, ($n \geq rs$),

$$\int_{GL_q(r|s)} (D^I_J A_J B_K Z^J_K Z^{\dagger}{}^{K'}_{I'}) = \sum_{\substack{\mu\in\mathcal{P}^r,\nu\in\mathcal{P}^s,\\ |\mu|+|\nu|=n-rs\\ \lambda=(s^r)+\mu\cup\nu'}} \frac{d_\lambda p_\lambda}{k_\lambda} \prod_{\substack{i=1\\j=1}}^{r,s} (q^i a_i - q^{r-j+1}a_{r+j}) \times (q^{r-s-i+1}b_i - q^{j-s}b_{r+j}) s_\mu(\{q^i a_i\}) s_\nu(\{q^{r+1-i}a_{r+i}\}) \times s_\mu(\{q^{r-s+1-i}b_i\}) s_\nu(\{q^{i-s}b_{r+i}\}).$$

For $q = 1$, $\dfrac{d_\lambda p_\lambda}{k_\lambda} = \dfrac{(-1)^{|\nu|}n!}{|\mu|!|\nu|!}\dfrac{d_\mu d_\nu}{r_\mu r_\nu}$ and the above formula reduces to (0.2).

5. The Orthogonal Relations. We are now interested in the orthogonal relations. Let M_λ, M_μ be two simple comodules corresponding to partitions λ and μ of n. Let $M_\lambda = \mathsf{Im}E_\lambda$, $M_\mu = \mathsf{Im}E_\mu$. Then, choosing a basis E^{ij}_λ of $\mathcal{H}_n$, such that $E_\lambda = E^{ii}_\lambda$ for some i, and using (3.6), we have

$$\begin{aligned}(\Phi(M_\lambda), \Phi(M_\mu)) := \int(\Phi(M_\lambda)\Phi(M^*_\mu)) &= q^{n(r-s+1)}\int(D^I_J E_\lambda{}^J_K Z^K_I C^M_N E_\mu{}^N_P Z^{\dagger}{}^{P'}_{M'})\\ &= \sum_\nu \frac{p_\nu}{k_\nu} D^I_J E_\lambda{}^J_K C^M_N E_\mu{}^N_P (C^{\otimes n}E^{ij}_\nu)^P_I E^{ji}_\nu{}^K_M\\ \text{(by (1.8) or (4.2))} &= \delta^\mu_\lambda \delta^\lambda_{\Gamma^{r,s}} \frac{p_\lambda}{k_\lambda}\mathsf{tr}(C^{\otimes n}E_\lambda) = 0.\end{aligned}$$

Here $\delta^\lambda_{\Gamma^{r,s}}$ indicates, whether λ belongs to $\Gamma^{r,s}$, it is zero if $\lambda \notin \Gamma^{r,s}$ and 1 otherwise. On the other hand, for λ form $\Gamma^{r,s}$, $\mathsf{tr}(C^{\otimes n}E_\lambda) = 0$, for it is the quantum rank of M_λ (cf. [15]).

We have seen that the scalar product above cannot be used to define the orthogonal relations. So we compute instead the integral $\int(Z_\lambda S(Z_\mu))$, where Z_λ is a coefficient matrix of the simple comodule M_λ, i.e., Z_λ is the multiplicative matrix corresponding to certain basis of M_λ.

Let us fix a primitive idempotent E_λ, $\lambda \vdash n$, and set $M_\lambda = \mathsf{Im}\rho(E_\lambda) \subset V^{\otimes n}$. Fix a basis $e_1, e_2, \dots, e_{m_\lambda}$ of M_λ, $m_\lambda := \mathsf{dim}M_\lambda$. Let Q^a_b, $1 \leq a, b \leq m_\lambda$, be such that $Q^a_b e_c = \delta^a_c e_b$. Since E_λ is a projection on M_λ, we can consider it as a linear map from M_λ into $V^{\otimes n}$ and from $V^{\otimes n}$ onto M_λ. Hence we define $P^a_b := E_\lambda Q^a_b E_\lambda$ to be endomorphism of $V^{\otimes n}$. We have $P^a_b P^c_d = \delta^a_d P^c_b$.

Let C_λ be the restriction of $C^{\otimes n}$ on M_λ, which is an invariant space of $C^{\otimes n}$ and $C_\lambda{}^a_b$ be the matrix element of C_λ with respect to the above basis. Then we have

$$C_\lambda{}^a_b = \mathsf{tr}(C^{\otimes n}P^a_b).$$

Consider the basis $X_I = x_{i_1}\otimes x_{i_2}\otimes\cdots\otimes x_{i_n}$ of $V^{\otimes n}$, $1 \leq i_j \leq d$. The corresponding multiplicative matrix is Z^I_J. Then the multiplicative matrix for M_λ, corresponding to the basis $e_1, e_2, \dots e_{m_\lambda}$ is given by $Z_{\lambda}{}^{a}_{b} = P^{a}_{b}{}^{I}_{J} Z^{J}_{I}$. Now, choosing a basis E^{ij}_{λ} of $\mathcal{H}_n$ such that $E_\lambda = E^{ii}_{\lambda}$ for some i, we have

$$\begin{aligned}\int (Z_{\lambda}{}^{a}_{b} S(Z_{\lambda}{}^{c}_{d})) &= \int (P^{a}_{b}{}^{I}_{J} Z^{J}_{I})(P^{c}_{d}{}^{K}_{L} Z^{\dagger}{}^{L'}_{K'}) \\ &= \sum_{\substack{\mu \in \Gamma^{r,s}_{n} \\ 1 \le i,j \le d_\mu}} k_\mu^{-1} p_\mu \mathrm{tr}(C^{\otimes n} E^{ij}_{\mu} P^{a}_{b} E^{ji}_{\mu} P^{c}_{d}) \\ &= k_\lambda^{-1} p_\lambda \delta^a_d \mathrm{tr}(C^{\otimes n} P^c_b) \\ &= k_\lambda^{-1} p_\lambda \delta^a_d C_{\lambda}{}^{c}_{b}.\end{aligned}$$

Thus, we have proved

PROPOSITION 5.1. *Let M_λ be the simple comodule of H_R, corresponding to partition λ and $e_1, e_2, \cdots, e_{m_\lambda}$ be its basis. Let Z_λ be the corresponding multiplicative matrix. Let C_λ be the restriction of the operator $C^{\otimes n}$ on M_λ and $C_{\lambda}{}^{a}_{b}$ be its matrix elements with respect to the basis above. Then we have the following orthogonal-type relations:*

$$\int (Z_{\lambda}{}^{a}_{b} S(Z_\lambda)^{c}_{d}) = k_\lambda^{-1} p_\lambda \delta^a_d C_{\lambda}{}^{c}_{b}. \tag{5.1}$$

REMARK. The fact that the left-hand side and the right-hand side of (5.1) are proportional follows directly from (2.2) and (4.1). However, this direct proof can not give us the coefficient $k_\lambda^{-1} p_\lambda$.

REFERENCES

[1] J. ALFARO, R. MEDINA, AND L.F. URRUTIA, *The Itzykson-Zuber integral for* U$(m|n)$, J. Math. Phys., 36(6) (1995), pp. 3085–3093.

[2] J. ALFARO, R. MEDINA, AND L.F. URRUTIA, *The orthogonality relations for the supergroup* U$(m|n)$, J. Phys. A, 28(16) (1995), pp. 4581–4588.

[3] F.A. BEREZIN, *Introduction to superanalysis*, D. Reidel Publishing Co., Dordrecht, 1987.

[4] C. C. ITZYKSON AND J.B. ZUBER, *The planar approximation. II*, J. Math. Phys, 21 (1980), pp. 411–421.

[5] P.M. COHN, *Algebra*, John Wiley and Sons, 1982.

[6] R. DIPPER AND G. JAMES, *Representations of Hecke Algebras of General Linear Groups*, Proc. London Math. Soc., 52(3) (1986), pp. 20–52.

[7] R. DIPPER AND G. JAMES, *Block and Idempotents of Hecke Algebras of General Linear Groups*, Proc. London Math. Soc., 54(3) (1987), pp. 57–82.

[8] Y. DOI, *Homological coalgebra*, J. Math. Soc. Japan, 33(1) (1981), pp. 31–50.

[9] D.I. GUREVICH, *Algebraic Aspects of the Quantum Yang-Baxter Equation*, Leningrad Math. Journal, 2(4) (1991), pp. 801–828.

[10] P. H. HAI, *Koszul Property and Poincaré Series of Matrix Bialgebra of Type A_n*, Journal of Algebra, 192(2) (1997), pp. 734–748.

[11] P. H. HAI, *On Matrix Quantum Groups of Type A_n*, Int. Journal of Math., 11(9) (2000), pp. 1115–1146.

[12] P. H. HAI, *Characters of representations of Quantum Groups of Type A_n*, Preprint ICTP, available at xxx.lanl.gov, 1998.

[13] P. H. HAI, *Central bialgebras in braided categories and coquasitriangular structures*, Journal of Pure and Applied Algebra, 140 (1999), pp. 229–250.

[14] P. H. HAI, *Poincaré Series of Quantum Spaces Associated to Hecke Operators*, Acta Math. Vietnam, 24(2) (1999), pp. 236–246.

[15] P. H. HAI, *Hecke Symmetries*, J. of Pure and Appl. Algebra, 152 (2000), pp. 109–121.

[16] H. CHANDRA, *A formula for semisimple lie groups*, Amer. J. Math, 79 (1957), pp. 733–760.

[17] CH. KASSEL, *Quantum Groups*, volume 155 of Graduate Texts in Mathematics, Springer-Verlag, 1995, 531p.

[18] B.I. LIN, *Semiperfect coalgebras*, Journal of Algebra, 49(2) (1977), pp. 357–373.
[19] I.G. MACDONALD, *Symmetric functions and the Hall polynomials*, Oxford University Press, New York, 1979 (Second edition 1995).
[20] YU.I. MANIN, *Multiparametric Quantum Deformation of the General Linear Supergroups.* Comm. Math. Phys., 123 (1989), pp. 163–175.
[21] HOCHSCHILD G. P., *Structure of Lie Groups*, Holden-Day, San Francisco, 1965.
[22] N. RESHETIKHIN AND V. TURAEV, *Ribbon Graph and Their Invariant Derived from Quantum Groups*, Comm. Math. Phys., 127 (1990), pp. 1–26.
[23] J.B. SULLIVAN, *The Uniqueness of Integral for Hopf Algebras and Some Existence Theorems of Integrals for Commutative Hopf Algebras*, Journal of Algebra, 19 (1971), pp. 426–440.
[24] M. SWEEDLER, *Hopf Algebras*, Benjamin, New York, 1969.
[25] S.L. WORONOWICZ, *Compact matrix pseudogroups*, Commun. Math. Phys, 111 (1987), pp. 613–665.
[26] SOIBELMAN YA. AND VASKMAN L, *Function algebra on quantum group* $SU(2)$, Adv. in Sov. Math., 22(1988), pp. 1–14.