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CLASSIFICATION OF INDEFINITE HYPER-KÄHLER SYMMETRIC SPACES*

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Abstract. We classify indefinite simply connected hyper-Kähler symmetric spaces. Any such space without flat factor has commutative holonomy group and signature $(4m, 4m)$. We establish a natural 1-1 correspondence between simply connected hyper-Kähler symmetric spaces of dimension $8m$ and orbits of the group $\mathrm{GL}(m, \mathbb{H})$ on the space $(S^4\mathbb{C}^n)^\tau$ of homogeneous quartic polynomials S in $n = 2m$ complex variables satisfying the reality condition $S = \tau S$, where τ is the real structure induced by the quaternionic structure of $\mathbb{C}^{2m} = \mathbb{H}^m$. We define and classify also complex hyper-Kähler symmetric spaces. Such spaces without flat factor exist in any (complex) dimension divisible by 4.

1. Introduction. We recall that a pseudo-Riemannian manifold (M, g) is called a symmetric space if any point $x \in M$ is an isolated fixed point of an involutive isometry s_x (called central symmetry with centre x). Since the product of two central symmetries s_x and s_y with sufficiently close centres is a shift along the geodesic (xy), the group generated by central symmetries acts transitively on M and one can identify M with the quotient $M = G/K$, where G is the connected component of the isometry group $\mathrm{Isom}(M, g)$ and K is the stabilizer of a point $o \in M$.

A symmetric space $(M = G/K, g)$ is called Kähler (respectively, hyper-Kähler) if its holonomy group $\mathrm{Hol}(M, g)$ is a subgroup of the pseudo-unitary group $\mathrm{U}(p, q)$ (respectively, of the pseudo-symplectic group $\mathrm{Sp}(p, q) \subset \mathrm{SU}(2p, 2q)$). Any hyper-Kähler symmetric space is in particular a homogeneous hypercomplex manifold. Homogeneous hypercomplex manifolds of compact Lie groups were constructed by Ph. Spindel, A. Sevrin, W. Troost, A. Van Proeyen [SSTVP] and by D. Joyce [J] and homogeneous hypercomplex structures on solvable Lie groups by M.L. Barberis and I. Dotti-Miatello [BD].

The classification of simply connected symmetric spaces reduces to the classification of involutive automorphisms σ of a Lie algebra $\mathfrak{g}$, such that the adjoint representation $\mathrm{ad}_{\mathfrak{k}}|\mathfrak{m}$ preserves a pseudo-Euclidean scalar product g, where

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m}, \quad \sigma|\mathfrak{k} = 1, \quad \sigma|\mathfrak{m} = -1,$$

is the eigenspace decomposition of the involution σ. Note that the eigenspace decomposition of an involutive automorphism is characterized by the conditions

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}.$$

Such a decomposition is called a symmetric decomposition.

In fact, for any pseudo-Riemannian symmetric space $M = G/K$ the conjugation with respect to the central symmetry s_o with centre $o = eK$ is an involutive automorphism of the Lie group G, which induces an involutive automorphism σ of its Lie algebra $\mathfrak{g}$. The pseudo-Riemannian metric of M induces a $\mathfrak{k}$-invariant scalar product on $\mathfrak{m} \cong T_oM$, where $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ is the symmetric decomposition defined by

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σ. Conversely, a symmetric decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ together with a $\mathfrak{k}$-invariant scalar product on $\mathfrak{m}$ determines a pseudo-Riemannian symmetric space $M = G/K$, where G is the simply connected Lie group with the Lie algebra $\mathfrak{g}$, K is the connected (and closed) subgroup of G generated by $\mathfrak{k}$, the pseudo-Riemannian metric on M is defined by g and the central symmetry is defined by the involutive automorphism σ associated to the symmetric decomposition.

Naturally identifying the space $\mathfrak{m}$ with the tangent space T_oM, the isotropy group is identified with $\mathrm{Ad}_K|\mathfrak{m}$ and the holonomy algebra is identified with $\mathrm{ad}_{\mathfrak{h}}$, where $\mathfrak{h} = [\mathfrak{m}, \mathfrak{m}]$. If one assumes that the holonomy algebra is irreducible then one can prove that the Lie algebra $\mathfrak{g}$ is semisimple. Hence the classification of pseudo-Riemannian symmetric spaces with irreducible holonomy reduces to the classification of involutive authomorphisms of semisimple Lie algebras. Such a classification was obtained by M. Berger [B1, B2] and A. Fedenko [F]. It includes the classification of Riemannian symmetric spaces (obtained earlier by E. Cartan), since according to de Rham's theorem any simply connected complete Riemannian manifold is a direct product of Riemannian manifolds with irreducible holonomy algebra and a Euclidean space.

A classification of pseudo-Riemannian symmetric spaces with non completely reducible holonomy is known only for signature $(1, n)$ (Cahen-Wallach [CW]) and for signature $(2, n)$ under the assumption that the holonomy group is solvable (Cahen-Parker [CP]). The classification problem for arbitrary signature looks very complicated and includes, for example, the classification of Lie algebras which admit a nondegenerate ad-invariant symmetric bilinear form. An inductive construction of solvable Lie algebras with such a form was given by V. Kac [K], see also [MR1], [Bo] and [MR2].

In this paper we give a classification of pseudo-Riemannian hyper-Kähler symmetric spaces. In particular, we prove that any simply connected hyper-Kähler symmetric space M has signature (4m,4m) and its holonomy group is commutative. The main result is the following, see Theorem 6.

Let (E, ω, j) be a complex symplectic vector space of dimension $4m$ with a quaternionic structure j such that $\omega(jx, jy) = \overline{\omega(x, y)}$ for all $x, y \in E$ and $E = E_+ \oplus E_-$ a j-invariant Lagrangian decomposition. Such a decomposition exists if and only if the Hermitian form $\gamma = \omega(\cdot, j\cdot)$ has real signature $(4m, 4m)$. We denote by τ the real structure in $S^{2r}E$ defined by $\tau(e_1 e_2 \dots e_{2r}) := j(e_1)j(e_2)\dots j(e_{2r})$, $e_i \in E$. Then any element $S \in (S^4E_+)^\tau$ defines a hyper-Kähler symmetric space M_S which is associated with the symmetric decomposition

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m},$$

where $\mathfrak{m} = (\mathbb{C}^2 \otimes E)^\rho$ is the fixed point set of the real structure ρ on $\mathbb{C}^2 \otimes E$ given by $\rho(h \otimes e) = j_H h \otimes je$, where j_H is the standard quaternionic structure on $\mathbb{C}^2 = \mathbb{H}$, $\mathfrak{h} = \mathrm{span}\{S_{e,e'} \,|\, e, e' \in E\}^\tau \subset sp(E)^\tau \cong \mathrm{sp}(m, m)$ with the natural action on $\mathfrak{m} \subset \mathbb{C}^2 \otimes E$ and the Lie bracket $\mathfrak{m} \wedge \mathfrak{m} \to \mathfrak{h}$ is given by

$$[h \otimes e,\, h' \otimes e'] = \omega_H(h, h')S_{e,e'},$$

where ω_H is the standard complex symplectic form on $\mathbb{C}^2$.

Moreover, we establish a natural 1-1 correspondence between simply connected hyper-Kähler symmetric spaces (up to isomorphism) and orbits of the group $GL(m, \mathbb{H})$ in $(S^4E_+)^\tau$.

We define also the notion of complex hyper-Kähler symmetric space as a complex manifold (M, g) of complex dimension $4n$ with holomorphic metric g such that for any point $x \in M$ there is a holomorphic central symmetry s_x with centre x and which has holonomy group $\mathrm{Hol}(M,g) \subset \mathrm{Sp}(n,\mathbb{C})$ ($\mathrm{Sp}(n,\mathbb{C}) \hookrightarrow \mathrm{Sp}(n,\mathbb{C}) \times \mathrm{Sp}(n,\mathbb{C}) \subset \mathrm{O}(4n,\mathbb{C})$ is diagonally embedded) and give a classification of such spaces. We establish a natural 1-1 correspondence between simply connected complex hyper-Kähler symmetric spaces and homogeneous polynomials of degree 4 in the vector space $\mathbb{C}^n$ considered up to linear transformations from $GL(n,\mathbb{C})$.

2. Symmetric Spaces.

2.1. Basic facts about pseudo-Riemannian symmetric spaces. A **pseudo-Riemannian symmetric space** is a pseudo-Riemannian manifold (M,g) such that any point is an isolated fixed point of an isometric involution. Such a pseudo-Riemannian manifold admits a transitive Lie group of isometries L and can be identified with L/L_o, where L_o is the stabilizer of a point o. More precisely, any simply connected pseudo-Riemannian symmetric space $M = G/K$ is associated with a symmetric decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m}\,, \quad [\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k}\,, \quad [\mathfrak{k},\mathfrak{m}] \subset \mathfrak{m}\,, \quad [\mathfrak{m},\mathfrak{m}] \subset \mathfrak{k} \tag{2.1}$$

of the Lie algebra $\mathfrak{g} = Lie\, G$ together with an Ad_K-invariant pseudo-Euclidean scalar product on $\mathfrak{m}$. We will assume that G acts almost effectively on M, i.e. $\mathfrak{k}$ does not contain any nontrivial ideal of $\mathfrak{g}$, that M and G are simply connected and that K is connected. Then, under the natural identification of the tangent space T_oM at the canonical base point $o = eK$ with $\mathfrak{m}$, the holonomy group $\mathrm{Hol} \subset \mathrm{Ad}_K|\mathfrak{m}$. We will denote by $\mathfrak{h}$ the holonomy Lie algebra. Since the isotropy representation is faithful it is identified with the subalgebra $\mathfrak{h} = [\mathfrak{m},\mathfrak{m}] := \mathrm{span}\{[x,y]|\, x,y \in \mathfrak{m}\} \subset \mathfrak{k}$. Recall that the curvature tensor R of a symmetric space M at o is $\mathfrak{h}$-invariant and determines the Lie bracket in the ideal $\mathfrak{h} + \mathfrak{m} \subset \mathfrak{g}$ as follows:

$$\mathfrak{h} = R(\mathfrak{m},\mathfrak{m}) := \mathrm{span}\{R(x,y)|x,y \in \mathfrak{m}\} \quad \text{and} \quad [x,y] = -R(x,y)\,, \quad x,y \in \mathfrak{m}\,.$$

The following result is well known:

PROPOSITION 1. *The full Lie algebra of Killing fields of a symmetric space has the form*

$$\mathrm{isom}(M) = \tilde{\mathfrak{h}} + \mathfrak{m}\,,$$

where the full isotropy subalgebra is given by

$$\tilde{\mathfrak{h}} = \mathrm{aut}(R) = \{A \in \mathrm{so}(\mathfrak{m})|A \cdot R = [A, R(\cdot,\cdot)] - R(A\cdot,\cdot) - R(\cdot,A\cdot) = 0\}\,. \tag{2.2}$$

2.2. Symmetric spaces of semisimple Lie groups. We will prove that in the case when $(M = G/K, g)$ is a pseudo-Riemannian symmetric space of a (connected) semisimple Lie group G then G is the maximal connected Lie group of isometries of M.

PROPOSITION 2. *Let $(M = G/K, g)$ be a pseudo-Riemannian symmetric space associated with a symmetric decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$. If G is semisimple and almost effective then*

(i) the restriction of the Cartan-Killing form B of $\mathfrak{g}$ to $\mathfrak{k}$ is nondegenerate and hence $\mathfrak{k}$ is a reductive subalgebra of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ is a B-orthogonal decomposition,

(ii) $\mathfrak{k} = [\mathfrak{m}, \mathfrak{m}]$ *and*

(iii) $\mathfrak{g} = \mathrm{isom}(M, g)$ *is the Lie algebra of the full isometry group of* M.

Proof: For (i) see [O-V] Ch. 3 Proposition 3.6.

(ii) It is clear that $\bar{\mathfrak{g}} = [\mathfrak{m}, \mathfrak{m}] + \mathfrak{m}$ is an ideal of $\mathfrak{g}$. The B-orthogonal complement $\mathfrak{a} := \bar{\mathfrak{g}}^{\perp} \subset \mathfrak{k}$ is a complementary ideal of $\mathfrak{g}$. Since $[\mathfrak{a}, \mathfrak{m}] = 0$ the Lie algebra $\mathfrak{a}$ acts trivially on M. From the effectivity of $\mathfrak{g}$ we conclude that $\mathfrak{a} = 0$.

(iii) By Proposition 1, $\tilde{\mathfrak{g}} = \mathrm{isom}(M, g) = \tilde{\mathfrak{h}} + \mathfrak{m}$, where $\tilde{\mathfrak{h}} = \mathrm{aut}(R) = \{A \in \mathrm{so}(\mathfrak{m}) | A \cdot R = 0\}$. Now $\tilde{\mathfrak{h}}$ preserves $\mathfrak{m}$ and by the identity $A \cdot R = [A, R(\cdot, \cdot)] - R(A \cdot, \cdot) - R(\cdot, A \cdot)$ it also normalizes $\mathfrak{k}$. This shows that $\tilde{\mathfrak{h}}$ normalizes $\mathfrak{g}$ and hence $\mathfrak{g} \subset \tilde{\mathfrak{g}}$ is an ideal. Since $\mathfrak{g}$ is semisimple there exists a $\mathfrak{g}$-invariant complement $\mathfrak{b}$ in $\tilde{\mathfrak{g}}$. Note that $[\mathfrak{g}, \mathfrak{b}] \subset \mathfrak{g} \cap \mathfrak{b} = 0$. We can decompose any $X \in \mathfrak{b}$ as $X = Y + Z$, where $Y \in \tilde{\mathfrak{h}}$ and $Z \in \mathfrak{m}$. From $[\mathfrak{g}, \mathfrak{b}] = 0$ it follows that $[\mathfrak{g}, Y] = [\mathfrak{g}, Z] = 0$ and in particular $[\mathfrak{m}, Y] = 0$. This implies that $Y = 0$ and $X = Z \in \mathfrak{b} \cap \mathfrak{m} = 0$. This shows that $\mathfrak{b} = 0$ proving (iii). □

We recall that a pseudo-Riemannian Hermitian symmetric space is pseudo-Riemannian symmetric space $(M = G/K, g)$ together with an invariant (and hence parallel) g-orthogonal complex structure J.

PROPOSITION 3. *Let* $(M = G/K, g, J)$ *be a pseudo-Riemannian Hermitian symmetric space of a semisimple and almost effective Lie group* G. *Then the Ricci curvature of* M *is not zero.*

Proof: From Proposition 2 it follows that $\mathfrak{g} = \mathrm{isom}(M, g) = \tilde{\mathfrak{h}} + m$, where $\tilde{\mathfrak{h}} = \mathfrak{k} = [\mathfrak{m}, \mathfrak{m}]$. It is well known that the curvature tensor R of any pseudo-Kähler manifold (and in particular of any pseudo-Riemannian Hermitian symmetric space) is invariant under the operator J. This shows that $J \in \tilde{\mathfrak{h}} = \mathrm{aut}(R) = [\mathfrak{m}, \mathfrak{m}] = \mathfrak{h}$ (holonomy Lie algebra), which implies that the holonomy Lie algebra is not a subalgebra of $\mathrm{su}(\mathfrak{m}) \cong \mathrm{su}(p, q)$. Hence M is not Ricci-flat. In fact, we can write $J = \sum \mathrm{ad}[X_i, Y_i]$, for $X_i, Y_i \in \mathfrak{m}$. Then using the formulas $-2\mathrm{Ric}(X, JY) = \mathrm{tr} JR(X, Y)$ for the Ricci curvature of a pseudo-Kähler manifold and $R(X, Y) = -ad_{[X,Y]}|\mathfrak{m}$ for the curvature of a symmetric space we calculate:

$$-2 \sum \mathrm{Ric}(X_i, JY_i) = \sum \mathrm{tr} JR(X_i, Y_i) = -\sum \mathrm{tr} J \mathrm{ad}[X_i, Y_i] = -\mathrm{tr} J^2 \neq 0 .$$

□

3. Structure of Hyper-Kähler Symmetric Spaces.

3.1. Definitions. A (possibly indefinite) **hyper-Kähler manifold** is a pseudo-Riemannian manifold (M^{4n}, g) of signature $(4k, 4l)$ together with a **compatible** hypercomplex structure, i.e. three g-orthogonal parallel complex structures $(J_1, J_2, J_3 = J_1 J_2)$. This means that the holonomy group $\mathrm{Hol} \subset \mathrm{Sp}(k, l)$. Two hyper-Kähler manifolds (M, g, J_α) $(\alpha = 1, 2, 3)$ and (M', g', J'_α) are called **isomorphic** if there exists a triholomorphic isometry $\varphi : M \to M'$, i.e. $\varphi^* J'_\alpha = J_\alpha$ and $\varphi^* g' = g$.

A **hyper-Kähler symmetric space** is a pseudo-Riemannian symmetric space $(M = G/K, g)$ together with an invariant compatible hypercomplex structure. Consider now a simply connected hyper-Kähler symmetric space $(M = G/K, g, J_\alpha)$. Without restriction of generality we will assume that G acts almost effectively. M being hyper-Kähler is equivalent to $\mathrm{Ad}_K|\mathfrak{m} \subset \mathrm{Sp}(k, l)$, or, since K is connected, to $\mathrm{ad}_{\mathfrak{k}}|\mathfrak{m} \subset \mathrm{sp}(k, l)$. This condition means that $\mathfrak{k}$ commutes with the Lie algebra $Q = \mathrm{sp}(1) \subset \mathrm{so}(\mathfrak{m}) = \mathrm{so}(4k, 4l)$ spanned by three anticommuting complex structures J_1, J_2, J_3.

3.2. Existence of a transitive solvable group of isometries and solvability of the holonomy. In this subsection we prove that any simply connected hyper-Kähler symmetric space (M, g, J_α) admits a transitive solvable Lie group $G \subset \mathrm{Aut}(g, J_\alpha)$ of automorphisms and has solvable holonomy group.

PROPOSITION 4. *Let $(M = G/K, g, J_\alpha)$ be a simply connected hyper-Kähler symmetric space and $A = \mathrm{Aut}_0(g, Q) \supset \mathrm{Aut}_0(g, J_\alpha) \supset G$ the connected group of isometries which preserve the quaternionic structure $Q = \mathrm{span}\{J_\alpha\}$. Then*

(i) the stabilizer A_o of a point $o \in M$ contains a maximal semisimple subgroup of A,

(ii) the radical R of A acts transitively and triholomorphically on M and

(iii) the holonomy group of M is solvable.

Proof: We consider the quaternionic Kähler symmetric space $(M = A/A_o, g, Q)$. The Lie algebra $\mathfrak{a}_o$ of the stabilizer is given by

$$\mathfrak{a}_o = \mathrm{aut}(R, Q) = \{A \in \mathrm{so}(T_oM) | A \cdot R = 0, [A, Q] \subset Q\} .$$

Since the curvature tensor of a quaternionic Kähler manifold is invariant under the quaternionic structure Q we conclude that $Q \subset \mathfrak{a}_o$ and $\mathfrak{a}_o = Q \oplus Z_{\mathfrak{a}}(Q)$, where $Z_{\mathfrak{a}}(Q)$ denotes the centralizer of Q in $\mathfrak{a}$. Since $Q \cong \mathrm{sp}(1)$ is simple, we may choose a Levi-Malcev decomposition $\mathfrak{a} = \mathfrak{s} + \mathfrak{r}$ such that the Levi subalgebra $\mathfrak{s} \supset Q$. We put $\mathfrak{m}_r := [Q, \mathfrak{r}]$ and denote by $\mathfrak{m}_s$ a $Q \oplus Z_{\mathfrak{s}}(Q)$-invariant complement of Q in $[Q, \mathfrak{s}]$. The stabilizer has the decomposition $\mathfrak{a}_o = Q \oplus (Z_{\mathfrak{s}}(Q) + Z_{\mathfrak{r}}(Q))$.

LEMMA 1. *The complement $\mathfrak{m} = \mathfrak{m}_s + \mathfrak{m}_r$ to $\mathfrak{a}_o$ in $\mathfrak{a}$ is $\mathfrak{a}_o$-invariant and the decomposition*

$$\mathfrak{a} = \mathfrak{a}_o + \mathfrak{m}$$

is a symmetric decomposition.

Proof: It is clear that $\mathfrak{m}_r$ is $\mathfrak{a}_o$-invariant and $\mathfrak{m}_s$ is invariant under $Q \oplus Z_{\mathfrak{s}}(Q)$ by construction. It remains to check that $[Z_{\mathfrak{r}}(Q), \mathfrak{m}_s] \subset \mathfrak{m}$. Since $\mathfrak{m}_s = [Q, \mathfrak{m}_s]$, we have

$$[Z_{\mathfrak{r}}(Q), \mathfrak{m}_s] = [Z_{\mathfrak{r}}(Q), [Q, \mathfrak{m}_s]] = [Q, [Z_{\mathfrak{r}}(Q), \mathfrak{m}_s]] \subset [Q, \mathfrak{r}] = \mathfrak{m}_r \subset \mathfrak{m} .$$

This shows that $\mathfrak{a} = \mathfrak{a}_o + \mathfrak{m}$ is an $\mathfrak{a}_o$-invariant decomposition. We denote by $\mathfrak{a} = \mathfrak{a}_o + \mathfrak{p}$ a symmetric decomposition. Any other $\mathfrak{a}_o$-invariant decomposition is of the form $\mathfrak{a} = \mathfrak{a}_o + \mathfrak{p}_\varphi$, where $\varphi : \mathfrak{p} \to \mathfrak{a}_o$ is an $\mathfrak{a}_o$-equivariant map and $\mathfrak{p}_\varphi = \{X + \varphi(X) | X \in \mathfrak{p}\}$. If such non-zero equivariant map φ exists then $\mathfrak{p}$ and $\mathfrak{a}_o$ contain non-trivial isomorphic Q-submodules. Since $\mathfrak{p}$ is a sum of 4-dimensional irreducible Q-modules and $\mathfrak{a}_o$ is the sum of the 3-dimensional irreducible Q-module Q and the trivial complementary Q-module $Z_{\mathfrak{a}_o}(Q)$, we infer that there exists a unique $\mathfrak{a}_o$-invariant decomposition, which coincides with the symmetric decomposition $\mathfrak{a} = \mathfrak{a}_o + \mathfrak{p}$. □

To prove (i) we have to check that $\mathfrak{m}_s = 0$. We note that by the previous lemma $\mathfrak{s} = (Q \oplus Z_{\mathfrak{s}}(Q)) + \mathfrak{m}_s$ is a symmetric decomposition of the semisimple Lie algebra $\mathfrak{s}$. Since $[\mathfrak{m}_s, \mathfrak{m}_s] \subset Z_{\mathfrak{s}}(Q)$ it defines a hyper-Kähler symmetric space $N = L/L_o$, where L is the simply connected semisimple Lie group with Lie algebra $\mathfrak{l} = Z_{\mathfrak{s}}(Q) + \mathfrak{m}_s$ and L_o is the Lie subgroup generated by the subalgebra $Z_{\mathfrak{s}}(Q) \subset \mathfrak{l}$. Since N is in particular a Ricci-flat pseudo-Riemannian Hermitian symmetric space, from Proposition 3 we obtain that N is reduced to point. Therefore $\mathfrak{m}_s = 0$. This proves (i) and (ii). Finally, since the holonomy Lie algebra $\mathfrak{h}$ is identified with $\mathfrak{h} = [\mathfrak{m}, \mathfrak{m}] = [\mathfrak{m}_r, \mathfrak{m}_r] \subset \mathfrak{r}$ it is solvable as subalgebra of the solvable Lie algebra $\mathfrak{r}$. □

3.3. Hyper-Kähler symmetric spaces and second prolongation of symplectic Lie algebras. Let $(M = G/K, g, J_\alpha)$ be a simply connected hyper-Kähler symmetric space associated with a symmetric decomposition (2.1). Without restriction of generality we will assume that G acts almost effectively and that $\mathfrak{k} = [\mathfrak{m}, \mathfrak{m}] = \mathfrak{h}$ (holonomy Lie algebra). The complexification $\mathfrak{m}^{\mathbb{C}}$ as $\mathfrak{h}^{\mathbb{C}}$-module can be written as $\mathfrak{m}^{\mathbb{C}} = H \otimes E$, such that $\mathfrak{h}^{\mathbb{C}} \subset \mathrm{Id} \otimes \mathrm{sp}(E) \cong \mathrm{sp}(E)$, where $H = \mathbb{C}^2$ and $E = \mathbb{C}^{2n}$ are complex symplectic vector spaces with symplectic form ω_H and ω_E, respectively, such that $g^{\mathbb{C}} = \omega_H \otimes \omega_E$ is the complex bilinear metric on $\mathfrak{m}^{\mathbb{C}}$ induced by g. Note that the symplectic forms are unique up to the transformation $\omega_H \mapsto \lambda\omega_H$, $\omega_E \mapsto \lambda^{-1}\omega_E$, $\lambda \in \mathbb{C}^*$. We have also quaternionic structures j_H and j_E on H and E, such that $\omega_H(j_H x, j_H y) = \overline{\omega_H(x,y)}$ for all $x, y \in H$ and $\omega_E(j_E x, j_E y) = \overline{\omega_E(x,y)}$ for all $x, y \in E$, where the bar denotes complex conjugation. This implies that $\gamma_H := \omega_H(\cdot, j_H \cdot)$ and $\gamma_E := \omega_E(\cdot, j_E \cdot)$ are Hermitian forms on H and E. For fixed ω_H and ω_E the quaternionic structures j_H and j_E are uniquely determined if we require that γ_H is positive definite and that $\rho = j_H \otimes j_E$ is the real structure on $\mathfrak{m}^{\mathbb{C}}$, i.e. the complex conjugation with respect to $\mathfrak{m}$. The metric $g^{\mathbb{C}}$ and the Hermitian form $g^{\mathbb{C}}(\cdot, \rho \cdot) = \gamma_H \otimes \gamma_E$ restrict to a real valued scalar product g of some signature $(4k, 4l)$ on $\mathfrak{m} = (H \otimes E)^\rho$, where $(2k, 2l)$ is the (real) signature of the Hermitian form $\gamma_E = \omega_E(\cdot, j_E \cdot)$. Note that for the holonomy algebra we have the inclusion

$$\mathfrak{h} = \mathrm{Id} \otimes (\mathfrak{h}^{\mathbb{C}})^{j_E} \hookrightarrow \mathrm{sp}(E)^{j_E} = \{A \in \mathrm{sp}(E) | [A, j_E] = 0\}$$

$$= \mathrm{aut}(E, \omega_E, j_E) \cong \mathrm{aut}(\mathfrak{m}, g, J_\alpha) \cong \mathrm{sp}(k, l)\,.$$

Using the symplectic forms we identify $H = H^*$ and $E = E^*$. Then the symplectic Lie algebras are identified with symmetric tensors as follows:

$$\mathrm{sp}(H) = S^2 H\,, \quad \mathrm{sp}(E) = S^2 E\,.$$

Since the curvature tensor R of any hyper-Kähler manifold M^{4n} at a point $p \in M$ can be identified with an element $R \in S^2\mathrm{sp}(k, l)$ it is invariant under the Lie algebra $\mathrm{sp}(1) = \mathrm{span}\{J_1, J_2, J_3\}$. Let $M = G/K$ be a hyper-Kähler symmetric space as above. By Proposition 1 we can extend the Lie algebra $\mathfrak{g} = \mathfrak{k} + \mathfrak{m} = \mathfrak{h} + \mathfrak{m}$ to a Lie algebra

$$\tilde{\mathfrak{g}} = \mathrm{sp}(1) + \mathfrak{h} + \mathfrak{m}$$

of Killing vector fields such that $[\mathrm{sp}(1), \mathfrak{h}] = 0$. In the $H \otimes E$-formalism the Lie algebra $\mathrm{sp}(1)$ is identified with $\mathrm{sp}(H)^{j_H} \otimes \mathrm{Id} \subset \mathrm{so}(\mathfrak{m})$.

LEMMA 2. *Denote by* $\tilde{\mathfrak{g}}^{\mathbb{C}} = \mathrm{sp}(1, \mathbb{C}) + \mathfrak{h}^{\mathbb{C}} + \mathfrak{m}^{\mathbb{C}}$ *the complexification of the Lie algebra* $\tilde{\mathfrak{g}}$. *Then the Lie bracket* $[\cdot, \cdot] : \wedge^2 \mathfrak{m}^{\mathbb{C}} \to \mathfrak{h}^{\mathbb{C}}$ *can be written as*

$$[h \otimes e, h' \otimes e'] = \omega_H(h, h') S_{e,e'}\,, \tag{3.1}$$

where $S \in (\mathfrak{h}^{\mathbb{C}})^{(2)} := \mathfrak{h}^{\mathbb{C}} \otimes S^2 E^* \cap E \otimes S^3 E^* = \mathfrak{h}^{\mathbb{C}} \otimes \mathfrak{h}^{\mathbb{C}} \cap S^4 E$. *Moreover* S *is* $\mathrm{sp}(1, \mathbb{C}) \oplus \mathfrak{h}^{\mathbb{C}}$*-invariant and satisfies the following reality condition:* $[S_{j_E e, e'} - S_{e, j_E e'}, j_E] = 0$.

Proof: The Lie bracket $[\cdot, \cdot] : \wedge^2 \mathfrak{m}^{\mathbb{C}} \to \mathfrak{h}^{\mathbb{C}}$ is an $\mathrm{sp}(1, \mathbb{C}) \oplus \mathfrak{h}^{\mathbb{C}}$-equivariant map, due to the Jacobi identity. We decompose the $\mathrm{sp}(H) \oplus \mathrm{sp}(E)$-module $\wedge^2 \mathfrak{m}^{\mathbb{C}}$:

$$\wedge^2 \mathfrak{m}^{\mathbb{C}} = \wedge^2 (H \otimes E) = \wedge^2 H \otimes S^2 E \oplus S^2 H \otimes \wedge^2 E = \omega_H \otimes S^2 E \oplus S^2 H \otimes \wedge^2 E\,.$$

Since $\mathfrak{h}^{\mathbb{C}} \subset S^2E$ the Lie bracket defines an $\mathrm{sp}(1,\mathbb{C})\oplus\mathfrak{h}^{\mathbb{C}}$-invariant element of the space $\omega_H\otimes S^2E\otimes S^2E\oplus S^2H\otimes\wedge^2E\otimes S^2E$. The second summand has no nontrivial $\mathrm{sp}(1,\mathbb{C})$-invariant elements. Hence the bracket is of the form (3.1), where $S \in S^2E^* \otimes \mathfrak{h}^{\mathbb{C}} \subset S^2E \otimes S^2E$. The Jacobi identity reads:

$$0 = [h\otimes e,[h'\otimes e',h''\otimes e'']] - [[h\otimes e,h'\otimes e'],h''\otimes e''] - [h'\otimes e',[h\otimes e,h''\otimes e'']]$$

$$= -\omega_H(h',h'')h\otimes S_{e',e''}e - \omega_H(h,h')h''\otimes S_{e,e'}e'' + \omega_H(h,h'')h'\otimes S_{e,e''}e' .$$

Since $\dim H = 2$ we may assume that $h, h' = h''$ is a symplectic basis, i.e. $\omega_H(h,h') = 1$, and the equation implies: $S_{e,e''}e' = S_{e,e'}e''$, i.e. $S \in (\mathfrak{h}^{\mathbb{C}})^{(2)}$. The Lie bracket of two real elements $h\otimes e + j_Hh\otimes j_Ee$ and $h\otimes e' + j_Hh\otimes j_Ee' \in \mathfrak{m} \subset \mathfrak{m}^{\mathbb{C}}$ is an element of $\mathfrak{h}$. This gives:

$$[h\otimes e + j_Hh\otimes j_Ee, h\otimes e' + j_Hh\otimes j_Ee'] = \omega_H(h,j_Hh)(S_{e,j_Ee'} - S_{j_Ee,e'}) \in \mathfrak{h} .$$

From the fact that the Hermitian form $\gamma_H = \omega_H(\cdot, j_H\cdot)$ is positive definite it follows that $\omega_H(h,j_Hh) \neq 0$. This establishes the reality condition since $\mathfrak{h} = \{A \in \mathfrak{h}^{\mathbb{C}} | [A,j_E] = 0\}$. □

In fact any tensor $S \in S^4E$ satisfying the conditions of the above lemma can be used to define a hyper-Kähler symmetric space as the following theorem shows. We can identify S^4E with the space $\mathbb{C}[E]^{(4)}$ of homogeneous quartic polynomials on $E \cong E^*$.

THEOREM 1. *Let $S \in S^4E$, $E = \mathbb{C}^{2n}$, be a quartic polynomial invariant under all endomorphisms $S_{e,e'} \in S^2E = \mathrm{sp}(E)$ and satisfying the reality condition*

$$[S_{j_Ee,e'} - S_{e,j_Ee'}, j_E] = 0 . \tag{3.2}$$

Then it defines a hyper-Kähler symmetric space, which is associated with the following complex symmetric decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} + H\otimes E , \quad \mathfrak{h}^{\mathbb{C}} = \mathrm{span}\{S_{e,e'} | e,e' \in E\} \subset \mathrm{sp}(E) . \tag{3.3}$$

The bracket $\wedge^2(H\otimes E) \to \mathfrak{h}^{\mathbb{C}}$ is given by (3.1). The real symmetric decomposition is defined as ρ-real form $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ of (3.3), where

$$\mathfrak{k} = \mathfrak{h} = \{A \in \mathfrak{h}^{\mathbb{C}} | [A,j_E] = 0\} = \mathrm{span}\{S_{j_Ee,e'} - S_{e,j_Ee'} | e,e' \in E\} , \quad \mathfrak{m} = (H\otimes E)^{\rho} .$$

The hyper-Kähler symmetric space M associated to this symmetric decomposition is the quotient $M = M_S = G/K$, where G is the simply connected Lie group with Lie algebra $\mathfrak{g}$ and $K \subset G$ is the connected (and closed) subgroup with Lie algebra $\mathfrak{k} = \mathfrak{h}$.

Moreover any simply connected hyper-Kähler symmetric space can be obtained by this construction. Two hyper-Kähler symmetric spaces M_S and $M_{S'}$ defined by quartics S and S' are isomorphic if and only if S and S' are in the same orbit of the group $\mathrm{Aut}(E,\omega_E,j_E) = \{A \in \mathrm{Sp}(E) | [A,j_E] = 0\} \cong \mathrm{Sp}(k,l)$.

Proof: First of all we note that $\mathfrak{h}^{\mathbb{C}} = S_{E,E} := \mathrm{span}\{S_{e,e'} | e,e' \in E\}$ is a subalgebra of $\mathrm{sp}(E)$ because

$$[S_{e,e'}, S_{f,f'}] = (S_{e,e'}\cdot S)_{f,f'} - S_{S_{e,e'}f,f'} - S_{f,S_{e,e'}f'} = -S_{S_{e,e'}f,f'} - S_{f,S_{e,e'}f'} \in \mathfrak{h}^{\mathbb{C}} .$$

Since S is $\mathfrak{h}^{\mathbb{C}}$-invariant and completely symmetric we can check, as in Lemma 2, that the Jacobi identity is satisfied and that (3.3) defines a complex symmetric decomposition. We prove that $\mathfrak{h} := \mathrm{span}\{S_{j_E e,e'} - S_{e,j_E e'} | e, e' \in E\} \subset \{A \in \mathfrak{h}^{\mathbb{C}} | [A, j_E] = 0\}$ defines a real form of $\mathfrak{h}^{\mathbb{C}}$. Indeed for $e, e' \in E$ we have

$$S_{e,e'} = \frac{1}{2}(S_{e,e'} + S_{j_E e, j_E e'}) - \frac{\sqrt{-1}}{2}(\sqrt{-1}S_{e,e'} - \sqrt{-1}S_{j_E e, j_E e'})$$

$$= \frac{1}{2}(S_{j_E e'', e'} - S_{e'', j_E e'}) - \frac{\sqrt{-1}}{2}(S_{j_E e'', \sqrt{-1}e'} - S_{e'', j_E \sqrt{-1}e'}),$$

where $e'' = -j_E e$. Due to the reality condition the restriction of the Lie bracket $[\cdot,\cdot] : \wedge^2 \mathfrak{m}^{\mathbb{C}} \to \mathfrak{h}^{\mathbb{C}}$ to $\wedge^2 \mathfrak{m}$ has values in $\mathfrak{h}$ and $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ is a symmetric decomposition with $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{h}$. The metric $g^{\mathbb{C}} = \omega_H \otimes \omega_E$ defines a real valued scalar product g of some signature (p, q) on $\mathfrak{m} = (H \otimes E)^\rho$, which is invariant under the Lie algebra $\mathfrak{h}$. Since $[\mathfrak{h}, j_E] = 0$ the holonomy algebra $\mathfrak{h} \subset \mathrm{sp}(k, l)$, $p = 4k$, $q = 4l$. Hence this symmetric decomposition defines a hyper-Kähler symmetric space.

By Lemma 2 any hyper-Kähler symmetric space can be obtained by this construction. It is well known that a simply connected symmetric space M of signature (p, q) is determined by its abstract curvature tensor $R \in S^2(\wedge^2 V)$, $V = \mathbb{R}^{p,q}$, and two tensors R and R' define isometric symmetric spaces if and only if they belong to the same $\mathrm{O}(V)$ orbit. Similarly a simply connected hyper-Kähler symmetric space is determined up to isometry by its abstract curvature tensor $R \in S^2(\wedge^2 V)$, where $V = \mathbb{R}^{4k,4l}$ is the pseudo-Euclidean vector space with fixed hypercomplex structure $J_\alpha \in O(V)$. For a hyper-Kähler symmetric space the complexified curvature tensor has the form

$$R(h \otimes e, h' \otimes e') = -\omega_H(h, h') S_{e,e'},$$

where $S \in S^4 E$ is the quartic form of Lemma 2. Two such curvature tensors define isomorphic hyper-Kähler symmetric spaces if and only if they belong to the same orbit of $\mathrm{Aut}(\mathbb{R}^{4k,4l}, J_\alpha) = \mathrm{Sp}(k, l)$. The group $\mathrm{Sp}(k, l)$ acts on $V^{\mathbb{C}} = H \otimes E$ as $\mathrm{Id} \otimes \mathrm{Sp}(E)^{j_E} = \mathrm{Id} \otimes \mathrm{Aut}(E, \omega_E, j_E)$. Hence two curvature tensors $R = -\omega_H \otimes S$ and $R' = -\omega_H \otimes S'$ are in the same $\mathrm{Sp}(k, l)$-orbit if and only if S and S' are in the same $\mathrm{Sp}(k, l)$-orbit on $S^4 E$. □

4. Complex Hyper-Kähler Symmetric Spaces.

4.1. Complex hyper-Kähler manifolds. A **complex Riemannian manifold** is a complex manifold M equipped with a complex metric g, i.e. a holomorphic section $g \in \Gamma(S^2 T^* M)$ which defines a nondegenerate complex quadratic form. As in the real case any such manifold has a unique holomorphic torsionfree and metric connection (Levi-Civita connection). A **complex hyper-Kähler manifold** is a complex Riemannian manifold (M^{4n}, g) of complex dimension $4n$ together with a compatible hypercomplex structure, i.e. three g-orthogonal parallel complex linear endomorphisms $(J_1, J_2, J_3 = J_1 J_2)$ with $J_\alpha^2 = -1$. This means that the holonomy group $\mathrm{Hol} \subset \mathrm{Sp}(n, \mathbb{C}) = Z_{O(4n,\mathbb{C})}(\mathrm{Sp}(1, \mathbb{C}))$. The linear group $\mathrm{Sp}(n, \mathbb{C})$ is diagonally embedded into $\mathrm{Sp}(n, \mathbb{C}) \times \mathrm{Sp}(n, \mathbb{C}) \subset \mathrm{GL}(4n, \mathbb{C})$. Two complex hyper-Kähler manifolds (M, g, J_α) $(\alpha = 1, 2, 3)$ and (M', g', J'_α) are called **isomorphic** if there exists a holomorphic isometry $\varphi : M \to M'$ such that $\varphi^* J'_\alpha = J_\alpha$ and $\varphi^* g' = g$.

We will show that the complex hyper-Kähler structure can be described as a half-flat Grassmann structure of a certain type. A **Grassmann structure** on a complex Riemannian manifold (M, g) is a decomposition of the (holomorphic) tangent bundle $TM \cong H \otimes E$ into the tensor product of two holomorphic vector bundles H and E of rank $2m$ and $2n$ with holomorphic nondegenerate 2-forms ω_H and ω_E such that $g = \omega_H \otimes \omega_E$. The Grassmann structure will be called **parallel** if the Levi-Civita connection $\nabla = \nabla^{TM}$ can be decomposed as:

$$\nabla = \nabla^H \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^E ,$$

where ∇^H and ∇^E are (uniquely defined) symplectic connections in the bundles H and E. A parallel Grassmann structure will be called **half-flat** if ∇^H is flat. Note that a parallel Grassmann structure on a simply connected manifold is half-flat if and only if the holonomy group of the Levi-Civita connection is contained in $\mathrm{Id} \otimes \mathrm{Sp}(n, \mathbb{C}) \subset \mathrm{Sp}(m, \mathbb{C}) \otimes \mathrm{Sp}(n, \mathbb{C}) \subset \mathrm{O}(\mathbb{C}^{2m} \otimes \mathbb{C}^{2n})$.

PROPOSITION 5. *A complex hyper-Kähler structure (g, J_α) on a simply connected complex manifold M is equivalent to the following geometric data:*

(i) a half-flat Grassmann structure $(TM, g, \nabla) \cong (H, \omega_H, \nabla^H) \otimes (E, \omega_E, \nabla^E)$ and

(ii) an isomorphism of flat symplectic vector bundles $H \cong M \times \mathbb{C}^2$. Under this isomorphism $\omega_H = h_1^* \wedge h_2^*$, where (h_1, h_2) is the standard basis of $\mathbb{C}^2$ considered as parallel frame of the trivial bundle $H = M \times \mathbb{C}^2$.

More precisely,

$$J_1 = R_i \otimes \mathrm{Id}\,, \quad J_2 = R_j \otimes \mathrm{Id}\,, \quad \textit{and} \quad J_3 = R_k \otimes \mathrm{Id}\,, \tag{4.1}$$

where we have identified $\mathbb{C}^2 = \mathbb{C}h_1 \oplus \mathbb{C}h_2$ with $\mathbb{H} = \mathrm{span}_{\mathbb{R}}\{1, i, j, k\} = \mathrm{span}_{\mathbb{C}}\{1, j\} = \mathbb{C}1 \oplus \mathbb{C}j$ with the complex structure defined by left-multiplication by i and R_x denotes the right-multiplication by the quaternion $x \in \mathbb{H}$.

Proof: It is easy to check that the geometric data (i) and (ii) define a complex hyper-Kähler structure on M. Conversely let (g, J_α) be a complex hyper-Kähler structure on M. The endomorphism J_1 has eigenvalues $\pm i$ and the tangent space can be decomposed into a sum of eigenspaces

$$TM = E_+ \oplus E_- .$$

From the J_1-invariance of the metric g it follows that $g(E_\pm, E_\pm) = 0$ and we can identify $E_- = E^*$ with the dual space of $E = E_+$. Since J_2 anticommutes with J_1 it interchanges E and E^* and hence defines an isomorphism $E \xrightarrow{\sim} E^*$. Now $g(\cdot, J_2 \cdot)$ defines a symplectic form ω_E on E. Let $H = M \times \mathbb{C}^2 = M \times (\mathbb{C}h_1 \oplus \mathbb{C}h_2)$ be the trivial bundle with 2-form $\omega_H = h_1^* \wedge h_2^*$. Then we can identify

$$TM = E \oplus E^* = E \oplus E = h_1 \otimes E \oplus h_2 \otimes E = H \otimes E .$$

We check that under this identification we have $g = \omega_H \otimes \omega_E$. Note that both sides vanish on $h_1 \otimes E$ and $h_2 \otimes E$ and $\omega_H(h_1, h_2) = 1$. We calculate for $e, e' \in E = E_+ = h_1 \otimes E$:

$$g(e, J_2 e') = \omega_E(e, e') = \omega_H(h_1, h_2)\omega_E(e, e') = (\omega_H \otimes \omega_E)(h_1 \otimes e, h_2 \otimes e') .$$

Hence we have a Grassmann structure. The eigenspaces $E_\pm$ of the parallel endomorphism J_1 are invariant under parallel transport. Therefore the Levi-Civita connection

∇ induces a connection ∇^E in the bundle E. Since $\nabla g = 0$ and $\nabla J_2 = 0$ we have $\nabla^E \omega_E = 0$. We define a flat connection ∇^H on the trivial bundle $H = M \times \mathbb{C}^2$ by the condition $\nabla^H h_1 = \nabla^H h_2 = 0$. Then $\nabla = \nabla^H \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^E$. So the Grassmann structure is half-flat.

Finally, using the standard identification $\mathbb{C}^2 = \mathbb{H}$, one can easily check that the J_α are given by (4.1). □

4.2. Complexification of real hyper-Kähler manifolds. Let (M, g, J_α) be a (real) hyper-Kähler manifold. We will assume that it is real analytic. This is automatically true if the metric g is positive definite since it is Ricci-flat and a fortiori Einstein. Using analytic continuation we can extend (M, g, J_α) to a complex hyper-Kähler manifold $(M^{\mathbb{C}}, g^{\mathbb{C}}, J_\alpha^{\mathbb{C}})$ equipped with an antiholomorphic involution T. In complex local coordinates $z^j = x^j + iy^j$ which are extension of real analytic coordinates x^j, y^j the involution is given by the complex conjugation $z^j \to \bar{z}^j = x^j - iy^j$. We can reconstruct the (real) hyper-Kähler manifold as the fixed point set of T. We will call (M, g, J_α) a **real form** of $(M^{\mathbb{C}}, g^{\mathbb{C}}, J_\alpha^{\mathbb{C}})$ and $(M^{\mathbb{C}}, g^{\mathbb{C}}, J_\alpha^{\mathbb{C}})$ the complexification of (M, g, J_α).

In general a complex hyper-Kähler manifold has no real form. A necessary condition is that the holonomy group of ∇^E is contained in $\mathrm{Sp}(k, l)$, $n = k + l$, and hence preserves a quaternionic structure. Then we can define a parallel antilinear endomorphism field $j_E : E \to E$ such that $j_E^2 = -1$ and $\omega_E(j_E x, j_E y) = \overline{\omega_E(x, y)}$ for all $x, y \in E$, where the bar denotes complex conjugation. We define a parallel antilinear endomorphism field $j_H : H \to H$ as the left-multiplication by the quaternion j on $H = M \times \mathbb{H}$. Then $\rho = j_H \otimes j_E$ defines a field of real structures in $TM = H \otimes E$. We denote by $\mathcal{D} \subset TM$ the real eigenspace distribution of ρ with eigenvalue 1. Here TM is considered as real tangent bundle of the real manifold M. If $M^\rho \subset M$ is a leaf of $\mathcal{D}$ of real dimension $4n$ then the data (g, J_α) induce on M^ρ a (real) hyper-Kähler structure.

4.3. Complex hyper-Kähler symmetric spaces. A complex Riemannian symmetric space is a complex Riemannian manifold (M, g) such that any point is an isolated fixed point of an isometric holomorphic involution. Like in the real case one can prove that it admits a transitive complex Lie group of holomorphic isometries and that any simply connected complex Riemannian symmetric M is associated to a complex symmetric decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m}, \quad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] = \mathfrak{k} \tag{4.2}$$

of a complex Lie algebra $\mathfrak{g}$ together with an $\mathrm{ad}_{\mathfrak{k}}$-invariant complex scalar product on $\mathfrak{m}$. More precisely $M = G/K$, where G is the simply connected complex Lie group with the Lie algebra $\mathfrak{g}$ and K is the (closed) connected subgroup associated with $\mathfrak{k}$. The holonomy group of such manifold is $H = \mathrm{Ad}_K|\mathfrak{m}$. Any pseudo-Riemannian symmetric space $M = G/K$ associated with a symmetric decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ has a canonical complexification $M^{\mathbb{C}} = G^{\mathbb{C}}/K^{\mathbb{C}}$ defined by the complexification $\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} + \mathfrak{m}^{\mathbb{C}}$ of the symmetric decomposition. Proposition 1 remains true for complex Riemannian symmetric spaces. Ignoring the reality condition we obtain the following complex version of Theorem 1.

THEOREM 2. *Let $S \in S^4E$, $E = \mathbb{C}^{2n}$, be a quartic polynomial invariant under all endomorphisms $S_{e,e'} \in S^2E = \mathrm{sp}(E)$. Then it defines a complex hyper-Kähler symmetric space, which is associated with the following complex symmetric decomposition*

$$\mathfrak{g} = \mathfrak{h} + H \otimes E, \quad \mathfrak{h} = S_{E,E} = \mathrm{span}\{S_{e,e'} | e, e' \in E\} \subset \mathrm{sp}(E). \tag{4.3}$$

The bracket $\wedge^2(H \otimes E) \to \mathfrak{h}$ *is given by (3.1). The complex hyper-Kähler symmetric space* M *associated to this symmetric decomposition is the quotient* $M = M_S = G/K$*, where* G *is the (complex) simply connected Lie group with Lie algebra* $\mathfrak{g}$ *and* $K \subset G$ *is the connected (and closed) subgroup with Lie algebra* $\mathfrak{k} = \mathfrak{h}$.

Moreover any simply connected complex hyper-Kähler symmetric space can be obtained by this construction. Two complex hyper-Kähler symmetric spaces M_S *and* $M_{S'}$ *defined by quartics* S *and* S' *are isomorphic if and only if* S *and* S' *are in the same orbit of* $\mathrm{Aut}(E, \omega_E) = \mathrm{Sp}(E) \cong \mathrm{Sp}(n, \mathbb{C})$.

COROLLARY 1. *There is a natural bijection between simply connected complex hyper-Kähler symmetric spaces of dimension* $4n$ *up to isomorphism and* $\mathrm{Sp}(n,\mathbb{C})$*-orbits on the space of quartic polynomials* $S \in S^4E$ *in the symplectic vector space* $E = \mathbb{C}^{2n}$ *such that*

$$S_{e,e'} \cdot S = 0 \quad \textit{for all} \quad e, e' \in E\,. \tag{4.4}$$

4.4. Classification of complex hyper-Kähler symmetric spaces. The following complex version of Proposition 4 (with similar proof) will be a crucial step in the classification of complex hyper-Kähler symmetric spaces.

PROPOSITION 6. *Let* $(M = G/K, g, J_\alpha)$ *be a simply connected complex hyper-Kähler symmetric space. Then the holonomy group of* M *is solvable and* M *admits a transitive solvable Lie group of automorphisms.*

Due to Corollary 1 the classification of simply connected complex hyper-Kähler symmetric spaces reduces to the determination of quartic polynomials S satisfying (4.4). Below we will determine all such polynomials. We will prove that the following example gives all such polynomials.

EXAMPLE 1: Let $E = E_+ \oplus E_-$ be a Lagrangian decomposition, i.e. $\omega(E_\pm, E_\pm) = 0$, of the symplectic vector space $E = \mathbb{C}^{2n}$. Then any polynomial $S \in S^4E_+ \subset S^4E$ satisfies the condition (4.4) and defines a simply connected complex hyper-Kähler symmetric space M_S with Abelian holonomy algebra $\mathfrak{h} = S_{E_+,E_+} \subset S^2E_+ \subset S^2E = \mathrm{sp}(E)$.

In fact, since E_+ is Lagrangian the endomorphisms from S^2E_+ form an Abelian subalgebra of $\mathrm{sp}(E)$, which acts trivially on E_+ and hence on S^4E_+.

THEOREM 3. *Let* $S \in S^4E$ *be a quartic polynomial satisfying (4.4). Then there exists a Lagrangian decomposition* $E = E_+ \oplus E_-$ *such that* $S \in S^4E_+$.

Proof: According to Theorem 2 the quartic S defines a hyper-Kähler symmetric space with holonomy Lie algebra $\mathfrak{h} = S_{E,E}$. Since, by Proposition 6, $\mathfrak{h}$ is solvable, Lie's theorem implies the existence of a one-dimensional $\mathfrak{h}$-invariant subspace $P = \mathbb{C}p \subset E$. There exists an ω-nondegenerate subspace $W \subset E$ such that the ω-orthogonal complement of P is $P^\perp = P \oplus W$. We choose a vector $q \in E$ such that $\omega(p, q) = 1$ and $\omega(W, q) = 0$ and put $Q := \mathbb{C}q$. Then we have

$$E = P \oplus W \oplus Q\,.$$

Since $\mathfrak{h}$ preserves P we have the following inclusion

$$\mathfrak{h} \subset PE + W^2 = P^2 + PW + PQ + W^2\,,$$

where we use the notation $XY = X \vee Y$ for the symmetric product of subspaces $X, Y \subset E$. Then the second prolongation $\mathfrak{h}^{(2)} = \{T \in S^4E | T_{e,e'} \in \mathfrak{h} \text{ for all } e, e' \in E\}$

has the following inclusion

$$\begin{aligned}\mathfrak{h}^{(2)} &\subset P^3E + P^2W^2 + PW^3 + W^4 \\ &= P^4 + P^3Q + P^3W + P^2W^2 + PW^3 + W^4 .\end{aligned} \tag{4.5}$$

Indeed $\mathfrak{h}^{(2)} \subset \mathfrak{h}^2 = P^4+P^3Q+P^3W+P^2Q^2+P^2WQ+P^2W^2+PQW^2+PW^3+W^4$. The projection $\mathfrak{h}^{(2)} \to P^2Q^2 + P^2WQ + PQW^2$ is zero because otherwise $S_{q,q} \in \mathfrak{h} \subset PE + W^2$ would have a nonzero projection to $Q^2 + WQ$ or $S_{w,q} \in \mathfrak{h}$ would have a nonzero projection to QW for appropriate choice of $w \in W$. By (4.5) we can write the quartic S as

$$S = p^3(\lambda p + \mu q + w_0) + p^2B + pC + D ,$$

where $\lambda, \mu \in \mathbb{C}$, $w_0 \in W$, $B \in S^2W$, $C \in S^3W$ and $D \in S^4W$. From now on we will identify S^dE with the space $\mathbb{C}[E^*]^{(d)}$ of homogeneous polynomials on E^* of degree d. Then the ω-contraction $T_x = \iota_{\omega x}T = T(\omega x, \dots)$ of a tensor $T \in S^dE$ with a vector $x \in E$ is identified with the following homogeneous polynomial of degree $d-1$:

$$T_x = \frac{1}{d}\partial_{\omega x}T ,$$

where $\partial_{\omega x}T$ is the derivative of the polynomial $T \in \mathbb{C}[E^*]^{(d)}$ in the direction of $\omega x = \omega(x, \cdot) \in E^*$. For example $p_q = \langle p, \omega q\rangle = \omega(q,p) = \partial_{\omega q}p = -\partial_{p^*}p = -1 = -q_p$.

From $S_{p,q} = -\frac{1}{4}\mu p^2$ and the condition $S_{p,q} \cdot S = 0$ we obtain $\mu = 0$, since $p^2 \cdot S = \mu p^4$. This implies $S_{p,\cdot} = 0$. Next we compute:

$$\begin{aligned}S_{q,q} &= \frac{1}{6}(6\lambda p^2 + 3pw_0 + B) \\ S_{q,w} &= -\frac{1}{12}(-3p^2\omega(w_0,w) + 2p\partial_{\omega w}B + \partial_{\omega w}C) \\ &= -\frac{1}{12}(-3p^2\omega(w_0,w) + 4pB_w + 3C_w) \\ S_{w,w'} &= \frac{1}{6}(p^2B_{w,w'} + 3pC_{w,w'} + 6D_{w,w'})\end{aligned}$$

for any $w, w' \in W$.

Now the condition (4.4) can be written as follows:

$$\begin{aligned}0 = 6S_{q,q} \cdot S &= (3pw_0 + B) \cdot S = (\frac{3}{2}(p \otimes w_0 + w_0 \otimes p) + B) \cdot S \\ &= \frac{3}{2}(2p^3B_{w_0} + 3p^2C_{w_0} + 4pD_{w_0}) + p^3Bw_0 + p^2B \cdot B + pB \cdot C + B \cdot D \\ &= -2p^3Bw_0 + \frac{9}{2}p^2C_{w_0} + p(6D_{w_0} + B \cdot C) + B \cdot D .\end{aligned}$$

Note that $Bw_0 = -B_{w_0}$ and $B \cdot B = [B,B] = 0$.

$$\begin{aligned}0 = -12S_{q,w} \cdot S &= (4pB_w + 3C_w) \cdot S \\ &= 2(p^4\omega(B_w,w_0) + 2p^3B^2w - 3p^2C_{Bw} - 4pD_{Bw}) \\ &\quad +3(p^3C_ww_0 + p^2C_w \cdot B + pC_w \cdot C + C_w \cdot D) \\ &= 2p^4\omega(B_w,w_0) + p^3(4B^2w + 3C_ww_0) + p^2(-6C_{Bw} + 3C_w \cdot B)\end{aligned}$$

$$+p(-8D_{Bw} + 3C_w \cdot C) + 3C_w \cdot D$$

$$\begin{aligned} 0 = 2S_{w,w'} \cdot S &= \frac{1}{2}(p \otimes C_{w,w'} + C_{w,w'} \otimes p) \cdot S + 2D_{w,w'} \cdot S \\ &= \frac{1}{2}(p^4\omega(C_{w,w'}, w_0) - 2p^3 BC_{w,w'} + 3p^2 C_{C_{w,w'}} + 4pD_{C_{w,w'}}) + \\ &\quad 2(p^3 D_{w,w'}w_0 + p^2 D_{w,w'} \cdot B + pD_{w,w'} \cdot C + D_{w,w'} \cdot D) \\ &= \frac{1}{2}p^4\omega(C_{w,w'}, w_0) + p^3(-BC_{w,w'} + 2D_{w,w'}w_0) + \\ &\quad p^2(\frac{3}{2}C_{C_{w,w'}} + 2D_{w,w'} \cdot B) + p(2D_{C_{w,w'}} + 2D_{w,w'} \cdot C) + 2D_{w,w'} \cdot D. \end{aligned}$$

This gives the following system of equations:

$$\begin{aligned} &(1)\ Bw_0 = 0 \\ &(2)\ C_{w_0} = 0 \\ &(3)\ 6D_{w_0} + B \cdot C = 0 \\ &(4)\ B \cdot D = 0 \\ &(5)\ \omega(B_w, w_0) = 0 \\ &(6)\ 4B^2 w + 3C_w w_0 = 0 \\ &(7)\ -2C_{Bw} + C_w \cdot B = 0 \\ &(8)\ -8D_{Bw} + 3C_w \cdot C = 0 \\ &(9)\ C_w \cdot D = 0 \\ &(10) \omega(C_w w', w_0) = 0 \\ &(11) -BC_w w' + 2D_{w,w'}w_0 = 0 \\ &(12) \frac{3}{2}C_{C_w w'} + 2D_{w,w'} \cdot B = 0 \\ &(13) D_{C_w w'} + D_{w,w'} \cdot C = 0 \\ &(14) D_{w,w'} \cdot D = 0. \end{aligned}$$

Note that (5) and (10) follow from (1) and (2) and that using (2) equation (6) says that the endomorphism B has zero square:

(6′) $B^2 = 0\,.$

Eliminating D_{w_0} in equations (3) and (11) we obtain:

$$\begin{aligned} (15)\quad 0 = (B \cdot C)_w w' + 3BC_w w' &= BC_w w' - C_{Bw}w' - C_w Bw' + 3BC_w w' \\ &= 4BC_w w' - C_{Bw}w' - C_w Bw'\,. \end{aligned}$$

We can rewrite (7) as:

(7′) $-2C_{Bw}w' + C_w Bw' - BC_w w' = 0\,.$

Eliminating $C_{Bw}w'$ in (7′) and (15) we obtain:

(16) $-3BC_w w' + C_w Bw' = 0\,.$

Since the first summand is symmetric in w and w' we get

(17) $C_w Bw' = C_{w'} Bw = C_{Bw} w'\,.$

Now using (17) we can rewrite (15) as:

(15′) $2BC_w w' - C_w Bw' = 0\,.$

The equations (15′) and (16) show that $BC_w w' = C_w B w' = C_{Bw} w' = 0$ and hence also $B \cdot C = 0$. This implies $D_{w_0} = 0$, by (3). Now we can rewrite (1-14) as:

$$Bw_0 = C_{w_0} = D_{w_0} = 0 \tag{4.6}$$

$$B \cdot C = B \cdot D = 0 \tag{4.7}$$

$$B^2 = 0 \tag{4.8}$$

$$C_{Bw} = C_w B = BC_w = 0 \tag{4.9}$$

$$-8D_{Bw} + 3C_w \cdot C = 0 \tag{4.10}$$

$$C_w \cdot D = 0 \tag{4.11}$$

$$\frac{3}{2} C_{C_w w'} + 2D_{w,w'} \cdot B = 0 \tag{4.12}$$

$$D_{C_w w'} + D_{w,w'} \cdot C = 0 \tag{4.13}$$

$$D_{w,w'} \cdot D = 0. \tag{4.14}$$

Now to proceed further we decompose $K := \ker B = W_0 \oplus W'$, where $W_0 = \ker \omega|K$ and W' is a (nondegenerate) complement. Let us denote by W_1 a complement to K in W such that $\omega(W', W_1) = 0$. Then $W_0 + W_1$ is the ω-orthogonal complement to the B-invariant nondegenerate subspace W'. This shows that $BW_1 \subset (W_0 + W_1) \cap K = W_0$. Moreover since $W_1 \cap K = 0$ the map $B : W_1 \to W_0$ is injective and hence $\dim W_1 \leq \dim W_0$. On the other hand $\dim W_1 \geq \dim W_0$, since W_0 is an isotropic subspace of the symplectic vector space $W_0 + W_1$. This shows that $B : W_1 \to W_0$ is an isomorphism.

LEMMA 3. *$C \in S^3 K$ and $D \in S^4 K$.*

Proof: Since $W_0 = BW$ the equation (4.9) shows that $C_{W_0} = 0$, which proves the first statement. From (4.10) and the identity

$$(C_x \cdot C)_y = [C_x, C_y] - C_{C_x y} \tag{4.15}$$

we obtain

$$D_{Bx,y} + D_{By,x} = \frac{3}{8}((C_x \cdot C)_y + (C_y \cdot C)_x) = -\frac{3}{4} C_{C_x y}. \tag{4.16}$$

The equation $B \cdot D = 0$ (4.7) reads:

$$0 = (B \cdot D)_{x,y} = [B, D_{x,y}] - D_{Bx,y} - D_{x,By}.$$

Using this (4.12) yields:

$$D_{Bx,y} + D_{By,x} = [B, D_{x,y}] = -D_{x,y} \cdot B = \frac{3}{4} C_{C_x y}. \tag{4.17}$$

Now from (4.16) and (4.17) we obtain that

$$0 = C_{C_x y} z = C_z C_x y$$

for all $x, y, z \in W$. This implies $[C_x, C_y] = 0$ for all $x, y \in W$ and hence

$$C_x \cdot C = 0 \tag{4.18}$$

for all $x \in W$, by (4.15). Finally this shows that $D_{W_0} = 0$ by (4.10). This proves the second statement. □

LEMMA 4. $D_{x,y} C_z = C_z D_{x,y} = 0$ *for all* $x, y, z \in W$.
Proof: Using (4.13) we compute:

$$D_{x,y} C_z w = D_{C_z w, x} y = -(D_{z,w} \cdot C)_x y = -([D_{z,w}, C_x] y - C_{D_{z,w} x} y) . \tag{4.19}$$

From (4.11) we get:

$$0 = (C_x \cdot D)_{z,w} y = [C_x, D_{z,w}] y - D_{C_x z, w} y - D_{z, C_x w} y$$

$$= C_x D_{z,w} y - D_{z,w} C_x y - D_{y,w} C_x z - D_{z,y} C_x w ,$$

and hence:

$$[D_{z,w}, C_x] y = -D_{y,w} C_x z - D_{z,y} C_x w ,$$

and

$$C_{D_{z,w} x} y = C_y D_{z,w} x = D_{z,w} C_x y + D_{x,w} C_y z + D_{z,x} C_y w .$$

Now we eliminate the CD-terms from (4.19) arriving at:

$$D_{x,y} C_z w = (D_{y,w} C_x z + D_{z,y} C_x w + D_{z,w} C_x y + D_{x,w} C_y z + D_{z,x} C_y w) . \tag{4.20}$$

Considering all the permutations of (x, y, z, w) we get 6 homogeneous linear equations for the 6 terms of equation (4.20) with the matrix:

$$\begin{pmatrix} -1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -1 \end{pmatrix}$$

This is the matrix of the endomorphism $-2\,\mathrm{Id} + e \otimes e$ in the arithmetic space $\mathbb{R}^6$, where $e = e_1 + \ldots + e_6$; (e_i) the standard basis. It has eigenvalues $(4, -2, -2, -2, -2, -2)$. This shows that the matrix is nondegenerate and proves the lemma. □

For a symmetric tensor $T \in S^d W$ we denote by

$$\Sigma_T := \mathrm{span}\{T_{x_1, x_2, \ldots, x_{d-2}} x_{d-1} | x_1, x_2, \ldots x_{d-1} \in W\} \subset W$$

the **support** of T.

LEMMA 5. *The supports of the tensors $B \in S^2W$, $C \in S^3W$ and $D \in S^4W$ admit the following inclusions*

$$\Sigma_B + \Sigma_C \subset \ker B \cap \ker C \cap \ker D\,, \quad \Sigma_D \subset \ker B \cap \ker C\,.$$

Moreover $\Sigma_B + \Sigma_C$ is isotropic and $\omega(\Sigma_D, \Sigma_B + \Sigma_C) = 0$.

Proof: The first statement follows from $B^2 = BC_x = BD_{x,y} = C_xB = C_xC_y = C_xD_{y,z} = D_{x,y}B = D_{x,y}C_z = 0$ for all $x, y, z \in W$. The second statement follows from the first and the definition of support, e.g. if $z = C_xy \in \Sigma_C$ and $w \in \Sigma_B + \Sigma_C + \Sigma_D \subset \ker C$ we compute:

$$\omega(z, w) = \omega(C_xy, w) = -\omega(y, C_xw) = 0\,.$$

□

LEMMA 6. *The Lie algebra $D_{W,W} \subset S^2W \cong \mathrm{sp}(W)$ is solvable.*

Proof: This follows from Proposition 6, since $D \in S^4W$ satisfies (4.4) and hence defines a complex hyper-Kähler symmetric space with holonomy Lie algebra $D_{W,W}$. It also follows from the solvability of $S_{E,E}$ as we show now. In terms of the decomposition $E = P + W + Q$ an endomorphism

$$S_{x,y} = (\lambda p^4 + p^3w_0 + p^2B + pC + D)_{x,y} = (p^2B + pC + D)_{x,y} = B(x, y)p^2 - pC_xy + D_{x,y}\,,$$

where $x, y \in W$, is represented by

$$\begin{pmatrix} 0 & -(C_xy)^t & B(x,y) \\ 0 & D_{x,y} & -C_xy \\ 0 & 0 & 0 \end{pmatrix}.$$

Since the Lie algebra $S_{E,E}$ is solvable this implies that the Lie algebra $D_{W,W}$, which corresponds to the induced representation of $S_{E,E}$ on $P^\perp/P \cong W$, is also solvable. □

LEMMA 7.

$$\omega(w_0, \Sigma_B + \Sigma_C + \Sigma_D) = 0\,.$$

Proof: Note that $w_0 \in \ker B \cap \ker C \cap \ker D$, due to equations (1-3) and (4.7). This implies the lemma. In fact, if e.g. $y = Bx \in \Sigma_B$ then

$$\omega(w_0, y) = \omega(w_0, Bx) = -\omega(Bw_0, x) = 0\,,$$

which shows that $\omega(w_0, \Sigma_B) = 0$ □

Now to finish the proof of Theorem 3 we will use induction on the dimension $\dim E = 2n$. If $n = 1$ the (solvable) holonomy algebra $\mathfrak{h}$ is a proper subalgebra of $S^2E \cong \mathrm{sl}(2, \mathbb{C})$. Without loss of generality we may assume that either
a) $\mathfrak{h} = \mathbb{C}p^2$ or
b) $\mathfrak{h} = \mathbb{C}pq$ or
c) $\mathfrak{h} = \mathbb{C}p^2 + \mathbb{C}pq$,
where (p, q) is a symplectic basis of E. In the all three cases the Lie algebra $S_{E,E} = \mathfrak{h} \subset \mathbb{C}p^2 + \mathbb{C}pq$ and hence

$$S = \lambda p^4 + \mu p^3q\,.$$

In the cases b) and c) we have that $pq \in \mathfrak{h}$ and since

$$pq \cdot S = \frac{1}{2}(p \otimes q + q \otimes p) \cdot S = -2\lambda p^4 - \mu p^3 q = 0$$

it follows that $S = 0$. In the case a) from $S_{E,E} = \mathfrak{h} = \mathbb{C}p^2$ we have that $S = \lambda p^4$. This tensor is invariant under $\mathfrak{h} = \mathbb{C}p^2$ and belongs to the fourth symmetric power of the Lagrangian subspace $\mathbb{C}p \subset E$. This establishes the first step of the induction. Now by induction using equation (4.14) and Lemma 6 we may assume that Σ_D is isotropic. Now Lemma 5 and Lemma 7 show that $\mathbb{C}w_0 + \Sigma_B + \Sigma_C + \Sigma_D$ is isotropic and hence is contained in some Lagrangian subspace $E_+ \subset E$. This implies that $S \in S^4E_+$. □

Now we give a necessary and sufficient condition for a symmetric manifold $M = M_S$, $S \in S^4E_+$, to have no flat de Rham factor.

PROPOSITION 7. *The complex hyper-Kähler symmetric space M_S, $S \in S^4E_+$, has no flat de Rham factor if and only if the support $\Sigma_S = E_+$.*

Proof: If $M = M_S = G/K$ has a flat factor M_0, such that $M = M_1 \times M_0$, then this induces a decomposition $E = E^1 \oplus E^0$ and $S \in S^4E_1$; hence $\Sigma_S \subset E_1 \cap E_+ \neq E_+$. Conversely let $S \in S^4E_+$, assume that $E^1_+ = \Sigma_S \subset E_+$ is a proper subspace and choose a complementary subspace E^0_+. We denote by E^1_- and E^0_- the annihilator of ωE^0_+ and ωE^1_+ respectively. Let us denote $E^1 = E^1_+ \oplus E^1_-$, $E^0 = E^0_+ \oplus E^0_-$, $\mathfrak{m}^1 = H \otimes E^1$ and $\mathfrak{m}^0 = H \otimes E^0$. Then $E^0, E^1 \subset E$ are ω-nondegenerate complementary subspaces and $\mathfrak{m}^0, \mathfrak{m}^1 \subset \mathfrak{m} = T_oM$ are g-nondegenerate complementary subspaces. Since $S \in S^4E^1_+$ the Lie algebra $\mathfrak{g} = \mathfrak{h} + \mathfrak{m} = (\mathfrak{h} + \mathfrak{m}^1) \oplus \mathfrak{m}^0$ has the Abelian direct summand $\mathfrak{m}^0$, see (3.1), which gives rise to a flat factor $M^0 \subset M_S = M^1 \times M^0$.

THEOREM 4. *Any simply connected complex hyper-Kähler symmetric space without flat de Rham factor is isomorphic to a complex hyper-Kähler symmetric space of the form M_S, where $S \in S^4E_+$ and $E_+ \subset E$ is a Lagrangian subspace of the complex symplectic vector space $E = \mathbb{C}^{2n}$. Moreover there is a natural 1-1 correspondence between simply connected complex hyper-Kähler symmetric spaces without flat factor up to isomorphism and orbits $\mathcal{O}$ of the group $\mathrm{Aut}(E,\omega,E_+)|E_+ = \{A \in Sp(E) | AE_+ = E_+\}|E_+ \cong \mathrm{GL}(E_+) \cong \mathrm{GL}(n,\mathbb{C})$ on the space S^4E_+ such that $\Sigma_S = E_+$ for all $S \in \mathcal{O}$.*

Proof: This is a corollary of Theorem 2, Theorem 3 and Proposition 7. □

Let $M = G/K$ be a simply connected complex hyper-Kähler symmetric space without flat factor. By Theorem 3 and Proposition 7 it is associated to quartic polynomial $S \in S^4E_+$ with support $\Sigma_S = E_+$. Now we describe the Lie algebra $\mathrm{aut}(M_S)$ of the full group of automorphisms, i.e. isometries which preserve the hypercomplex structure, of M_S.

THEOREM 5. *Let $M_S = G/K$ be as above. Then the full automorphism algebra is given by*

$$\mathrm{aut}(M_S) = \mathrm{aut}(S) + \mathfrak{g}\,,$$

where $A \in \mathrm{aut}(S) = \{B \in \mathrm{gl}(E_+) | B \cdot S = 0\}$ acts on $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ as follows. It preserves the decomposition and acts on $\mathfrak{h} = S_{E,E}$ by

$$[A, S_{x,y}] = S_{Ax,y} + S_{x,Ay}$$

for all $x, y \in E$ and on $\mathfrak{m} = H \otimes E$ by

$$[A, h \otimes e] = h \otimes Ae\,,$$

where $\mathrm{gl}(E_+)$ *is canonically embedded into* $\mathrm{sp}(E)$.

Proof: By (the complex version of) Proposition 1 it is sufficient to determine the centralizer $\mathfrak{c}$ of $\mathrm{sp}(1,\mathbb{C})$ in the full isotropy algebra $\tilde{\mathfrak{h}} = \mathrm{aut}(R) \supset \mathrm{sp}(1,\mathbb{C}) \oplus \mathfrak{h}$. Equation (2.2) shows that

$$\mathfrak{c} = \{\mathrm{Id} \otimes A |\, A \in \mathrm{sp}(E)\,,\; A \cdot S = [A, S(\cdot,\cdot)] - S(A\cdot,\cdot) - S(\cdot, A\cdot) = 0\}\,.$$

From $A \cdot S = 0$ we obtain that the commutator $[A, S_{x,y}] = S_{Ax,y} + S_{x,Ay}$ for all $x, y \in E$ and $A\Sigma_S = AE_+ \subset E_+$. This implies $\mathfrak{c} = \mathrm{aut}(S)$. □

5. Classification of Hyper-Kähler Symmetric Spaces. Using the description of complex hyper-Kähler symmetric spaces given in Theorem 4 we will now classify (real) hyper-Kähler symmetric spaces. Recall that a simply connected pseudo-Riemannian manifold is called **indecomposable** if it is not a Riemannian product of two pseudo-Riemannian manifolds. Any simply connected pseudo-Riemannian manifold can be decomposed into the Riemannian product of indecomposable pseudo-Riemannian manifolds. By Wu's theorem [W] a simply connected pseudo-Riemannian manifold is indecomposable if and only if its holonomy group is **weakly irreducible**, i.e. has no invariant proper nondegenerate subspaces. Therefore it is sufficient to classify (real) hyper-Kähler symmetric spaces with *indecomposable* holonomy.

Let $(M = G/K, g, J_\alpha)$ be a hyper-Kähler symmetric space associated to a symmetric decomposition (2.1). The complexified tangent space of M is identified with $\mathfrak{m}^{\mathbb{C}} = H \otimes E$, the tensor product of to complex symplectic vector spaces with quaternionic structure j_H and j_E such that $\rho = j_H \otimes j_E$ is the complex conjugation of $\mathfrak{m}^{\mathbb{C}}$ with respect to $\mathfrak{m}$. By Theorem 1 it is defined by a quartic polynomial $S \in S^4E$ satisfying the conditions of the theorem. Moreover the holonomy algebra $\mathfrak{h}$ acts trivially on H and is identified with the real form of the complex Lie algebra $S_{E,E} \subset \mathrm{sp}(E)$ given by $\mathfrak{h} = \mathrm{span}\{S_{je,e'} - S_{e,je'} | e, e' \in E\} = \{A \in S_{E,E} | [A, j_E] = 0\} \subset \mathrm{sp}(E)^{j_E}$.

The quartic polynomial S defines also a complex hyper-Kähler symmetric space $M^{\mathbb{C}} = G^{\mathbb{C}}/K^{\mathbb{C}}$, which is the complexification of $M = G/K$. By Theorem 3, $S \in S^4L$ for some Lagrangian subspace $L \subset E$. Recall that the symplectic form $\omega = \omega_E$ together with the quaternionic structure $j = j_E$ define a Hermitian metric $\gamma = \gamma_E = \omega_E(\cdot, j_E\cdot)$ of (real) signature $(4k, 4l)$, $n = k + l$, which coincides with the signature of the pseudo-Riemannian metric g (we normalize $\gamma_H = \omega_H(\cdot, j_H\cdot)$ to be positive definite). We may decompose γ-orthogonally $L = L^0 \oplus L^+ \oplus L^-$, such that γ vanishes on L^0 is positive definite on L^+ and negative definite on L^-.

LEMMA 8.

(i) $jL^0 = L^0$ *and*

(ii) $L^+ + L^- + jL^+ + jL^- \subset E$ *is an* ω*-nondegenerate and* γ*-nondegenerate* $\mathfrak{h}$*-invariant subspace (with trivial action of* $\mathfrak{h}$*).*

Proof: We show first that $L + jL^0$ is ω-isotropic and hence $L + jL^0 = L$ since L is Lagrangian. Indeed $L \supset L^0$ is ω-isotropic and also jL^0 because ω is j-invariant. So it suffices to remark that $\omega(L, jL^0) = 0$:

$$\omega(L, jL^0) = \gamma(L, L^0) = 0\,.$$

This implies that $jL^0 \subset L$. Since $\gamma(L, jL^0) = -\omega(L, L^0) = 0$, we conclude that $jL^0 \subset \ker\gamma|L = L^0$. This proves (i).

To prove (ii) it is sufficient to check that the subspace $L^+ + L^- + jL^+ + jL^- \subset E$ is nondegenerate with respect to γ, since it is j-invariant. First we remark that γ is positive definite on L^+ and jL^+ and negative definite on L^- and jL^-, due to the j-invariance of γ: $\gamma(jx, jx) = \gamma(x, x)$, $x \in E$. So to prove (ii) it is sufficient to check that $jL^+ \oplus jL^-$ is γ-orthogonal to the γ-nondegenerate vector space $L^+ + L^-$:

$$\gamma(L^+ + L^-, jL^+ + jL^-) = \omega(L^+ + L^-, L^+ + L^-) = 0\,.$$

□

By Theorem 1 the quartic polynomial S must satisfy the reality condition $[S_{je,e'} - S_{e,je'}, j] = 0$. Now we describe all such polynomials.

The quaternionic structure j on E is **compatible** with ω, i.e. $\omega(jx, jy) = \overline{\omega(x, y)}$ for all $x, y \in E$ and it induces a real structure (i.e. an antilinear involution) $\tau := j \otimes j \otimes \cdots \otimes j$ on all even powers $S^{2r}E \subset E \otimes E \otimes \cdots \otimes E$. For $S \in S^{2r}E$ and $x_1, \cdots, x_{2r} \in E$ we have

$$(\tau S)(x_1, \cdots, x_{2r}) = \overline{S(jx_1, \cdots, jx_{2r})}\,.$$

Note that the fixed point set $\mathrm{sp}(E)^\tau = \{A \in \mathrm{sp}(E) | [A, j] = 0\} \cong \mathrm{sp}(k, l)$.

PROPOSITION 8. *Let (E, ω, j) be a complex symplectic vector space with a quaternionic structure j such that $\omega(jx, jy) = \overline{\omega(x, y)}$ for all $x, y \in E$. Then a quartic polynomial $S \in S^4E$ satisfies the reality condition $[S_{je,e'} - S_{e,je'}, j] = 0$ if and only if $S \in (S^4E)^\tau = \mathrm{span}\{T + \tau T | T \in S^4E\}$.*

Proof: The reality condition for $S \in S^4E$ can be written as

$$[S_{jx,jy} + S_{x,y}, j]z = 0$$

for all $x, y, z \in E$. Contracting this vector equation with $jw \in E$ by means of ω and using the compatibility between j and ω we obtain the equivalent condition

$$\begin{aligned} 0 &= -\omega(jw, [S_{jx,jy} + S_{x,y}, j]z) \\ &= S(jx, jy, jz, jw) - \overline{S(jx, jy, z, w)} + S(x, y, jz, jw) - \overline{S(x, y, z, w)}\,. \end{aligned} \tag{5.1}$$

Now putting $x = y = z = w = u$ we obtain:

$$0 = S(ju, ju, ju, ju) - \overline{S(ju, ju, u, u)} + S(u, u, ju, ju) - \overline{S(u, u, u, u)}$$

and putting $x = iu$ and $y = z = w = u$ we obtain:

$$0 = -iS(ju, ju, ju, ju) - i\overline{S(ju, ju, u, u)} + iS(u, u, ju, ju) + i\overline{S(u, u, u, u)}\,.$$

Comparing these two equations we get $S(ju, ju, juj, u) = \overline{S(u, u, u, u)}$, i.e. $S = \tau S$. This shows that the reality condition implies that $S \in (S^4E)^\tau$. Conversely the condition $S = \tau S$ can be written as

$$S(jx, jy, jz, jw) = \overline{S(x, y, z, w)} \quad \text{for all} \quad x, y, z, w \in E\,.$$

Changing $z \to jz$ and $w \to jw$ in this equation we obtain

$$S(jx, jy, z, w) = \overline{S(x, y, jz, jw)} \quad \text{for all} \quad x, y, z, w \in E\,.$$

These two equations imply (5.1) and hence the reality condition. □

Now we are ready to classify simply connected hyper-Kähler symmetric spaces. We will show that the following construction gives all such symmetric spaces.

Let (E,ω,j) be a complex symplectic vector space of dimension $2n$ with a quaternionic structure j such that $\omega(jx,jy)=\overline{\omega(x,y)}$ for all $x,y\in E$ and $E=E_+\oplus E_-$ a j-invariant Lagrangian decomposition. Such a decomposition exists if and only if the Hermitian form $\gamma=\omega(\cdot,j\cdot)$ has real signature $(4m,4m)$, where $\dim_{\mathbb{C}} E=2n=4m$. Then any polynomial $S\in(S^4E_+)^\tau=S^4E_+\cap(S^4E)^\tau$ satisfies the condition (4.4) and the reality condition, by Proposition 8. Hence by Theorem 1 it defines a (real) simply connected hyper-Kähler symmetric space M_S with Abelian holonomy algebra $\mathfrak{h}=(S_{E_+,E_+})^\tau=S_{E_+,E_+}\cap(S^2E)^\tau=\operatorname{span}\{S_{je,e'}-S_{e,je'}|e,e'\in E\}\subset\operatorname{sp}(E)^\tau\cong\operatorname{sp}(m,m)$.

THEOREM 6. *Any simply connected hyper-Kähler symmetric space without flat de Rham factor is isomorphic to a hyper-Kähler symmetric space of the form M_S, where $S=T+\tau T$, $T\in S^4E_+$ and $E_+\subset E$ is a j-invariant Lagrangian subspace of the complex symplectic vector space E with compatible quaternionic structure j. A hyper-Kähler symmetric space of the form M_S has no flat factor if and only if it complexification has no flat factor, which happens if and only if the support $\Sigma_S=E_+$. Moreover there is a natural 1-1 correspondence between simply connected hyper-Kähler symmetric spaces without flat factor up to isomorphism and orbits $\mathcal{O}$ of the group* $\operatorname{Aut}(E,\omega,j,E_+)|E_+=\{A\in Sp(E)|[A,j]=0,AE_+=E_+\}|E_+\cong\operatorname{GL}(m,\mathbb{H})$ *on the space $(S^4E_+)^\tau$ such that $\Sigma_S=E_+$ for all $S\in\mathcal{O}$.*

Proof: Let M be a simply connected hyper-Kähler symmetric space. We first assume that it is indecomposable. Then the holonomy algebra $\mathfrak{h}$ is weakly irreducible. By Theorem 1, $M=M_S$ for some quartic polynomial $S\in S^4E$ satisfying (4.4) and the reality condition (3.2). By Proposition 8 the reality condition means that $S\in(S^4E)^\tau$. On the other hand, by Theorem 3 $S\in S^4L$ for some Lagrangian subspace L of E. Now the weak irreducibility of $\mathfrak{h}$ and Lemma 8 imply that $L=L^0$ is j-invariant. This proves that $S\in(S^4E_+)^\tau$, where $E_+=L=L^0$ is a j-invariant Lagrangian subspace of E. This shows that M is obtained from the above construction. Any simply connected hyper-Kähler symmetric space M without flat factor is the Riemannian product of indecomposable ones, say $M=M_1\times M_2\times\cdots\times M_r$, and we may assume that $M_i=M_{S_i}$, $S_i\in S^4E_i$. Therefore M is associated to the quartic polynomial $S=S_1\oplus S_2\oplus\cdots\oplus S_r\in S^4E$, $E=E_1\oplus E_2\oplus\cdots\oplus E_r$. Moreover S satisfies (4.4) and (3.2) if the S_i satisfy (4.4) and (3.2). This shows that any simply connected hyper-Kähler symmetric space is obtained from the above construction.

It is clear that the complexification $M_S^{\mathbb{C}}$ has a flat factor if M_S has a flat factor. Conversely let us assume that $M_S^{\mathbb{C}}$ has a flat factor, hence $\Sigma_S\subset E_+$ is a proper subspace. Since $j\Sigma_S=jS_{E,E}E=S_{E,E}jE=S_{E,E}E=\Sigma_S$ there exists a j-invariant complementary subspace E'_+ in E_+. Denote by E'_- the annihilator of Σ_S in E_- then $E'=E'_+\oplus E'_-$ is an ω-nondegenerate and j-invariant subspace of E on which the holonomy $\mathfrak{h}^{\mathbb{C}}=S_{E,E}\subset S^2\Sigma_S$ acts trivially. Then the corresponding real subspace $(H\otimes E')^\rho\subset\mathfrak{m}=(H\otimes E)^\rho$ is a g-nondegenerate subspace on which the holonomy $\mathfrak{h}$ acts trivially. By Wu's theorem [W] it defines a flat de Rham factor.

Now the last statement follows from the corresponding statement in Theorem 1. □

COROLLARY 2. *Any hyper-Kähler symmetric space without flat factor has signature $(4m,4m)$. In particular its dimension is divisible by 8.*

COROLLARY 3. *Let $M=M_S$ be a complex hyper-Kähler symmetric space without flat factor associated with a quartic $S\in S^4E_+$, where $E_+\subset E$ is a Lagrangian subspace. It admits a real form if and only if there exists a quaternionic structure j*

on E compatible with ω preserving E_+ such that $\tau S = S$, where τ is the real structure on S^4E induced by j. In particular $\dim_{\mathbb{C}} M$ has to be divisible by 8.

Let $M = G/K$ be a simply connected hyper-Kähler symmetric space without flat factor. By Theorem 6 it is associated to a quartic polynomial $S \in (S^4E_+)^\tau$ with support $\Sigma_S = E_+$. Now we describe the Lie algebra $\mathrm{aut}(M_S)$ of the full group of automorphisms, i.e. isometries which preserve the hypercomplex structure of M_S.

THEOREM 7. *Let $M_S = G/K$ be as above. Then the full automorphism algebra is given by*

$$\mathrm{aut}(M_S) = \mathrm{aut}(S) + \mathfrak{g}\,,$$

where $\mathrm{aut}(S) = \{A \in \mathrm{gl}(E_+) | [A, j] = 0\,,\ A \cdot S = 0\}$ *acts on*

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m}\,, \mathfrak{h} = \{A \in S_{E,E} | [A, j] = 0\} = \mathrm{span}\{S_{jx,y} - S_{x,jy} | x, y \in E\}\,, \mathfrak{m} = (H \otimes E)^\rho$$

as in Theorem 5.

Proof: The proof is similar to that of Theorem 5. □

6. Low Dimensional Hyper-Kähler Symmetric Spaces.

6.1 Complex hyper-Kähler symmetric spaces of dimension ≤ 8.

Dimension 4

Assume that M is a simply connected complex hyper-Kähler symmetric space of dimension 4. Applying Theorem 4 we conclude that $M = M_S$ for some $S \in S^4E_+$, where $E_+ \subset E$ is a one-dimensional subspace $E_+ = \mathbb{C}e$. This proves:

THEOREM 8. *There exists up to isomorphism only one non-flat simply connected complex hyper-Kähler symmetric space of dimension 4: $M = M_S$ associated with the quartic $S = e^4$.*

Dimension 8

Any eight-dimensional simply connected complex hyper-Kähler symmetric space is associated with a quartic $S \in S^4E_+$, where $E_+ \subset E$ is a Lagrangian subspace of $E = \mathbb{C}^4$. We denote by (e, e') a basis of E_+.

THEOREM 9. *Eight-dimensional simply connected complex hyper-Kähler symmetric space are in natural 1-1 correspondence with the orbits of the group $\mathrm{CO}(3, \mathbb{C}) = \mathbb{C}^* \cdot \mathrm{SO}(3, \mathbb{C})$ on the space $S^2_0\mathbb{C}^3$ of traceless symmetric matrices. The complex hyper-Kähler symmetric space associated with a traceless symmetric matrix A is the manifold $M_{S(A)}$, where $S(A) \in S^4\mathbb{C}^2$ is the quartic polynomial which corresponds to A under the $\mathrm{SO}(3, \mathbb{C})$-equivariant isomorphism $S^2_0\mathbb{C}^3 \cong S^2_0 \wedge^2 \mathbb{C}^3 \cong S^2_0 S^2\mathbb{C}^2 = S^4\mathbb{C}^2$.*

The classification of $\mathrm{SO}(3, \mathbb{C})$-orbits on $S^2_0\mathbb{C}^3$ was given by Petrov [P] in his classification of Weyl tensors of Lorentzian 4-manifolds.

Proof: By Theorem 4 the classification of eight-dimensional simply connected complex hyper-Kähler symmetric spaces reduces to the description of orbits of the group $\mathrm{GL}(E_+) = \mathrm{GL}(2, \mathbb{C})$ on $S^4\mathbb{C}^2 \subset S^2S^2\mathbb{C}^2$. Fixing a volume form σ on $\mathbb{C}^2$ we can identify $S^2\mathbb{C}^2$ with $\mathrm{sp}(1, \mathbb{C}) \cong \mathrm{so}(3, \mathbb{C})$. Then the Killing form B is an $\mathrm{SL}(2, \mathbb{C})$-invariant and we have the $\mathrm{GL}(2, \mathbb{C})$-invariant decomposition: $S^2S^2\mathbb{C}^2 = S^2_0S^2\mathbb{C}^2 \oplus \mathbb{C}B$. The action of $\mathrm{SL}(2, \mathbb{C})$ on $S^2\mathbb{C}^2$ is effectively equivalent to the adjoint action of $\mathrm{SO}(3, \mathbb{C})$. The problem thus reduces essentially to the determination of the orbits of $\mathrm{SO}(3, \mathbb{C})$ on $S^2_0\mathbb{C}^3$. □

5.1. Hyper-Kähler symmetric spaces of dimension ≤ 8. By Corollary 3 the minimal dimension of non-flat hyper-Kähler symmetric spaces is 8.

THEOREM 10. *Eight-dimensional simply connected hyper-Kähler symmetric space are in natural 1-1 correspondence with the orbits of the group* $\mathbb{R}^+ \cdot \mathrm{SO}(3)$ *on the space* $S_0^2\mathbb{R}^3$ *of traceless symmetric matrices. The hyper-Kähler symmetric space associated with a traceless symmetric matrix* A *is the manifold* $M_{S(A)}$, *where* $S(A) \in (S^4\mathbb{C}^2)^\tau$ *is the quartic polynomial which corresponds to* A *under the* SO(3)*-equivariant isomorphism* $S_0^2\mathbb{R}^3 \cong (S^4\mathbb{C}^2)^\tau$.

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