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THE CLOSURE DIAGRAM FOR NILPOTENT ORBITS OF THE REAL FORM E IX OF E_8*

DRAGOMIR Ž. ĐOKOVIĆ†

Abstract. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be adjoint nilpotent orbits in a real semisimple Lie algebra. Write $\mathcal{O}_1 \geq \mathcal{O}_2$ if $\mathcal{O}_2$ is contained in the closure of $\mathcal{O}_1$. This defines a partial order on the set of such orbits, known as the closure ordering. We determine this order for the noncompact nonsplit real form of the simple complex Lie algebra E_8.

1. Introduction. The closure diagrams for adjoint nilpotent orbits in noncompact real forms of F_4 and G_2 were determined in [9], for E_6 in [10], and for the noncompact and nonsplit real forms of E_7 in [11]. In this paper we handle the noncompact and nonsplit real form of E_8.

By $\mathfrak{g}$ we denote a simple complex Lie algebra of type E_8, by $\mathfrak{g}_0$ the real form of $\mathfrak{g}$ of type E IX, and by G (respectively G_0) the adjoint group of $\mathfrak{g}$ (respectively $\mathfrak{g}_0$). As usual, let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be a Cartan decomposition of $\mathfrak{g}_0$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ its complexification, and θ the Cartan involution. Let σ be the complex conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$, and let $\mathfrak{h}$ be a σ-stable Cartan subalgebra of $\mathfrak{k}$. Since $\mathfrak{g}_0$ is of inner type, $\mathfrak{h}$ is also a Cartan subalgebra of $\mathfrak{g}$.

Denote by $\mathcal{N}$ the nilpotent variety of $\mathfrak{g}$ and set

$$\mathcal{N}_{\mathbf{R}} = \mathcal{N} \cap \mathfrak{g}_0, \quad \mathcal{N}_1 = \mathcal{N} \cap \mathfrak{p}.$$

Let K^0 be the connected subgroup of G with Lie algebra $\mathfrak{k}$. It is known that the orbit spaces $\mathcal{N}_{\mathbf{R}}/G_0$ and $\mathcal{N}_1/K^0$, equipped with the quotient topologies, are homeomorphic and that the Kostant-Sekiguchi bijection is a homeomorphism $\mathcal{N}_{\mathbf{R}}/G_0 \to \mathcal{N}_1/K^0$ (see [6, 1]). We can think of the closure diagram for adjoint nilpotent orbits in $\mathfrak{g}_0$ as describing the topology of $\mathcal{N}_{\mathbf{R}}/G_0$ (or, equivalently, $\mathcal{N}_1/K^0$).

Our main result is the closure diagram depicted in Figure 3. In order to construct this diagram and prove its correctness, it was necessary to perform extensive nontrivial computations. For this purpose, in addition to our own programs, we used heavily Maple [5] and, to a lesser extent, LiE [18].

2. Preliminaries. The closure diagram for adjoint nilpotent orbits in $\mathfrak{g}$ was determined by Mizuno [14] and verified later by Beynon and Spaltenstein [2]. We give this diagram in Figures 1 and 2 where each node represents a G-orbit in $\mathcal{N}$ and is labelled by the corresponding Bala-Carter symbol (see [6, 4]). This diagram is a modified form of the one given in [17, p. 249]. In our diagram, the orbits having the same dimension are positioned at the same level. Because of its length, the diagram is split into three pieces. The bottom and the top portions of the diagram are shown in Figure 1, while the middle part is shown separately in Figure 2. The dimensions of the orbits are indicated on the left of the diagrams.

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†Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario, N2L 3G1, Canada (djokovic@uwaterloo.ca). Supported in part by the NSERC Grant A-5285.

The line in the diagram [17, p. 249] joining the nodes $2A_4$ and $D_6(a_2)$ is redundant as the diagram contains the lines joining $2A_4$ to A_5+A_2 and the one joining A_5+A_2 to $D_6(a_2)$. (The Bala-Carter labels for the nodes $D_6(a_2)$, A_5+A_2, $2A_4$ in [17, p. 249] are $D_6(a_2)$, $E_7(a_5)$, $E_8(a_7)$, respectively.) Consequently, we have omitted that line from our diagram.

We remark that the closure diagram given in [4, p. 444] for this case has two errors: First, the line joining the nodes A_3 and $D_4(a_1)$ should be replaced by a line joining A_3 to A_3+A_1. Second, one should insert a new line joining the nodes $E_8(b_6)$ and E_6+A_1.

There are 70 adjoint nilpotent orbits in $\mathfrak{g}$ (including the trivial orbit). The nonzero ones are listed in Table 1. The k-th orbit, i.e., the one that appears as the k-th entry in Table 1, will be denoted by $\mathcal{O}^k$. The second column of this table contains the Bala-Carter symbol of $\mathcal{O}^k$, and the third one gives the weights of the weighted Dynkin diagram of $\mathcal{O}^k$. The complex dimension of $\mathcal{O}^k$ is recorded in the last column.

The nonzero G_0-orbits in $\mathcal{N}_{\mathbf{R}}$, or equivalently the nonzero K^0-orbits in $\mathcal{N}_1$, were classified in [7] (see also [6]). We shall keep the same numbering for these orbits as in these two references. The i-th nontrivial G_0-orbit in $\mathcal{N}_{\mathbf{R}}$ will be denoted by $\mathcal{O}_0^i$, and we denote by $\mathcal{O}_1^i$ the nontrivial K^0-orbit in $\mathcal{N}_1$ that corresponds to $\mathcal{O}_0^i$ under the Kostant-Sekiguchi bijection. In the fourth and fifth columns of Table 1 we list the superscripts i of the orbits $\mathcal{O}_0^i$ (or, equivalently, $\mathcal{O}_1^i$) which are contained in $\mathcal{O}^k$. This depends on the type of the real form $\mathfrak{g}_0$ of $\mathfrak{g}$ (for the sake of completeness we have included also the split real form E VIII). For instance, if $k=4$ then:

$$\begin{array}{lll} \text{E VIII:} & \mathcal{O}^4\cap\mathfrak{g}_0=\mathcal{O}_0^4\cup\mathcal{O}_0^5, & \mathcal{O}^4\cap\mathfrak{p}=\mathcal{O}_1^4\cup\mathcal{O}_1^5; \\ \text{E IX:} & \mathcal{O}^4\cap\mathfrak{g}_0=\mathcal{O}_0^6\cup\mathcal{O}_0^7\cup\mathcal{O}_0^8, & \mathcal{O}^4\cap\mathfrak{p}=\mathcal{O}_1^6\cup\mathcal{O}_1^7\cup\mathcal{O}_1^8. \end{array}$$

Recall that a triple (E,H,F) in $\mathfrak{g}$ is called a *standard triple* if $[H,E]=2E$, $[H,F]=-2F$, $[F,E]=H$ and E,H,F are nonzero. Such a triple is *normal* if also $H\in\mathfrak{k}$ and $E,F\in\mathfrak{p}$. We denote the root system of $(\mathfrak{g},\mathfrak{h})$ by R, and choose a system of positive roots $R^+\subset R$ and a base $B=\{\alpha_i : 1\le i\le 8\}\subset R^+$ of R (see Figure 4 in the Appendix). The simple roots $\alpha_i\in B$ are indexed as in [3].

Let us also introduce the subgroup $K=\{x\in G : \theta(x)=x\}$. Its identity component is the group K^0 defined above. In the case that concerns us here, namely E IX, we have $K=K^0$ (see [13]).

We extend the enumeration of simple roots α_i, $1\le i\le 8$, to the enumeration α_i, $1\le i\le 120$, of R^+. It is the same as the one used in [8]. We have reproduced it in the Appendix. A negative root $-\alpha_i$ will be also written as α_{-i}. The coroot of α_i is denoted by $H_i\in\mathfrak{h}$. Note that $H_{-i}=-H_i$. For $\alpha\in R$ we let $\mathfrak{g}^\alpha$ be the root space of α. A nonzero element $X_\alpha\in\mathfrak{g}^\alpha$ is called a *root vector* of α. We assume that a root vector X_i is fixed for each root α_i, $|i| \in \{1, \ldots, 120\}$.

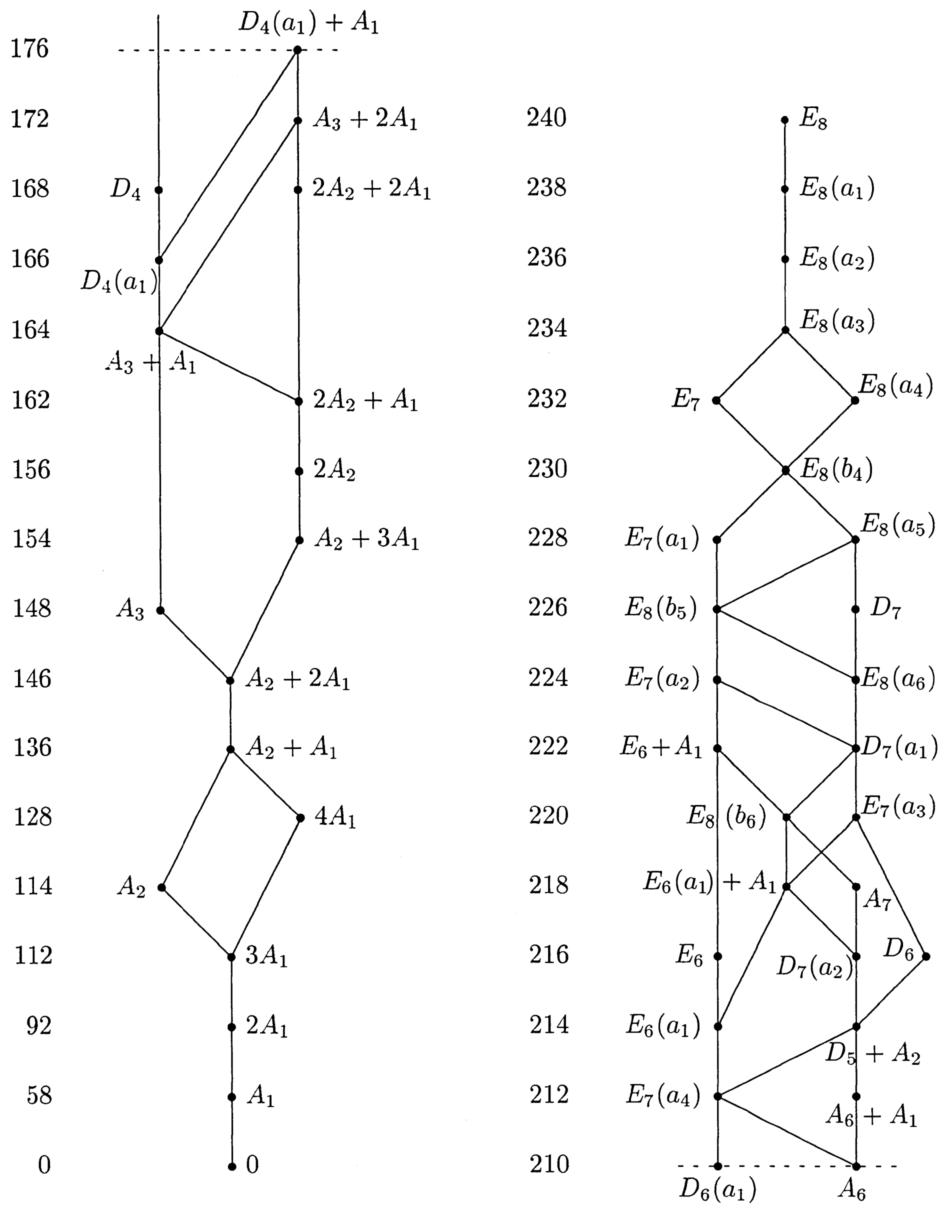

Figure 1: The bottom and top portions of the closure diagram of E_8

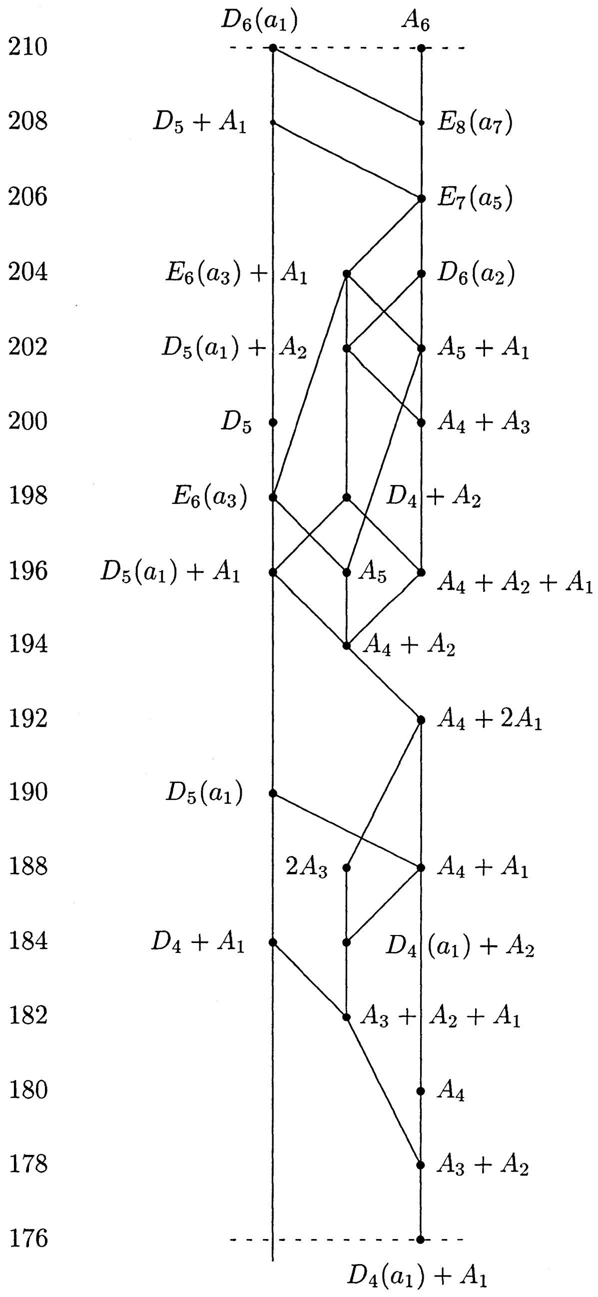

Figure 2: The middle part of the closure diagram of E_8

Table 1: Nonzero nilpotent adjoint orbits $\mathcal{O}^k$ in $\mathbf{E_8}$

k	Label	$\alpha_j(H)$	E VIII	E IX	dim
1	A_1	00000001	1	1	58
2	$2A_1$	10000000	2	2,3	92
3	$3A_1$	00000010	3	4,5	112
4	A_2	00000002	4,5	6,7,8	114
5	$4A_1$	01000000	6		128
6	A_2+A_1	10000001	7,8	9	136
7	A_2+2A_1	00000100	9,10	10,11	146
8	A_3	10000002	11	12,13	148
9	A_2+3A_1	00100000	12,13		154
10	$2A_2$	20000000	14,15,16	14	156
11	$2A_2+A_1$	10000010	17	15	162
12	A_3+A_1	00000101	18	16,17	164
13	$D_4(a_1)$	00000020	19,20	18,19,20	166
14	D_4	00000022	21	21,22	168
15	$2A_2+2A_1$	00001000	22		168
16	A_3+2A_1	00100001	23		172
17	$D_4(a_1)+A_1$	01000010	24,25		176
18	A_3+A_2	10000100	26,27,28	23	178
19	A_4	20000002	29,30	24,25	180
20	$A_3+A_2+A_1$	00010000	31,32		182
21	D_4+A_1	01000012	33		184
22	$D_4(a_1)+A_2$	02000000	34,35,36		184
23	A_4+A_1	10000101	37,38	26	188
24	$2A_3$	10001000	39		188
25	$D_5(a_1)$	10000102	40,41	27	190
26	A_4+2A_1	00010001	42,43,44		192
27	A_4+A_2	00000200	45,46	28	194
28	A_5	20000101	47	29	196
29	$D_5(a_1)+A_1$	00010002	48,49		196
30	$A_4+A_2+A_1$	00100100	50		196
31	D_4+A_2	02000002	51,52,53		198
32	$E_6(a_3)$	20000020	54,55	30,31	198
33	D_5	20000022	56	32,33	200
34	A_4+A_3	00010010	57		200
35	A_5+A_1	10010001	58		202
36	$D_5(a_1)+A_2$	00100101	59		202
37	$D_6(a_2)$	01100010	60,61		204
38	$E_6(a_3)+A_1$	10001010	62,63		204
39	$E_7(a_5)$	00010100	64,65		206
40	D_5+A_1	10001012	66		208
41	$E_8(a_7)$	00002000	67,68,69		208
42	A_6	20000200	70,71	34	210
43	$D_6(a_1)$	01100012	72,73		210

Table 1: (continued)

k	Label	$\alpha_j(H)$	E VIII	E IX	dim
44	A_6+A_1	10010100	74		212
45	$E_7(a_4)$	00010102	75,76		212
46	$E_6(a_1)$	20000202	77,78	35	214
47	D_5+A_2	00002002	79,80,81		214
48	E_6	20000222	82	36	216
49	D_6	21100012	83		216
50	$D_7(a_2)$	10010101	84,85		216
51	A_7	10010110	86		218
52	$E_6(a_1)+A_1$	10010102	87,88		218
53	$E_7(a_3)$	20010102	89,90		220
54	$E_8(b_2)$	00020002	91,92		220
55	$D_7(a_1)$	20002002	93,94,95		222
56	E_6+A_1	10010122	96		222
57	$E_7(a_2)$	01101022	97		224
58	$E_8(a_6)$	00020020	98,99		224
59	D_7	21101101	100		226
60	$E_8(b_5)$	00020022	101,102		226
61	$E_7(a_1)$	21101022	103		228
62	$E_8(a_5)$	20020020	104,105		228
63	$E_8(b_4)$	20020022	106,107		230
64	E_7	21101222	108		232
65	$E_8(a_4)$	20020202	109,110		232
66	$E_8(a_3)$	20020222	111,112		234
67	$E_8(a_2)$	22202022	113		236
68	$E_8(a_1)$	22202222	114		238
69	E_8	22222222	115		240

By adjoining the negative of the highest root: $\alpha_0 = -\alpha_{120} = \alpha_{-120}$ to B we obtain the so-called extended base $B_e = B \cup \{\alpha_0\}$. Let R_0 be the root system of $(\mathfrak{k}, \mathfrak{h})$ where we view R_0 as a subsystem of R. We set $R_0^+ = R_0 \cap R^+$ and denote by B_0 the unique base of R_0 contained in R_0^+. It turns out that $B_0 \subset B_e$. Explicitly we have:

$$\begin{array}{ll} \text{E VIII:} & B_0 = \{\alpha_0, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}, \\ \text{E IX:} & B_0 = \{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}. \end{array}$$

Given a K^0-orbit $\mathcal{O}_1^i \subset \mathcal{N}_1$, we can choose a normal triple (E, H, F) such that $E \in \mathcal{O}_1^i$, $H \in \mathfrak{h}$, and $\alpha(H) \geq 0$ for all $\alpha \in B_0$. The integers $\alpha(H)$ for $\alpha \in B_0$ determine uniquely H and, consequently, also the orbit $\mathcal{O}_1^i$.

As in [7], in the case E IX we set (see Figure 4)

$$\beta_i = \alpha_i, \quad (1 \leq i \leq 7); \quad \beta_8 = \alpha_0.$$

The technique developed in [9] to find the closure diagrams is especially convenient for real forms of inner type and will be employed in this paper. The closure diagram for the case E IX is given in Figure 3.

Table 2: Representatives of the nonzero nilpotent K-orbits $\mathcal{O}_1^i$ in $\mathfrak{p}$

k	i	$\beta_j(H^i)$	E^i	Type
1	1	0000001 1	X_{-8}	A_1
2	2	1000000 2	$(X_{-8}) + (X_{-74})$	$2A_1$
2	3	0000010 0	$(X_{118}) + (X_{-8})$	$2A_1$
3	4	0000001 3	$(X_{-15}) + (X_{-68}) + (X_{-101})$	$3A_1$
3	5	1000001 1	$(X_{119}) + (X_{-15}) + (X_{-68})$	$3A_1$
4	6	0000000 4	$X_{-8} + X_{-119}$	A_2
			$(X_{-8}) + (X_{-74}) + (X_{-104}) + (X_{-118})$	$(4A_1)''$
4	7	0000002 2	$(X_{119}) + (X_{-15}) + (X_{-68}) + (X_{-101})$	$(4A_1)''$
4	8	2000000 0	$X_{101} + X_{-8}$	A_2
			$(X_{101}) + (X_{119}) + (X_{-15}) + (X_{-68})$	$(4A_1)''$
6	9	1100000 1	$(X_{113} + X_{-42}) + (X_{-47})$	$A_2 + A_1$
7	10	1000010 2	$(X_{118} + X_{-68}) + (X_{-47}) + (X_{-81})$	$A_2 + 2A_1$
7	11	0001000 0	$(X_{110} + X_{-36}) + (X_{113}) + (X_{-50})$	$A_2 + 2A_1$
8	12	1000002 4	$X_{119} + X_{-74} + X_{-101}$	A_3
8	13	0000012 2	$X_{-68} + X_{118} + X_{-101}$	A_3
10	14	0000020 0	$(X_{96} + X_{-22}) + (X_{112} + X_{-62})$	$2A_2$
11	15	1000011 1	$(X_{107} + X_{-47}) + (X_{117} + X_{-81})$ $+(X_{-68})$	$2A_2 + A_1$
12	16	1000011 3	$(X_{-73} + X_{118} + X_{-98}) + (X_{-74})$	$A_3 + A_1$
12	17	1000003 1	$(X_{-47} + X_{104} + X_{-81}) + (X_{118})$	$A_3 + A_1$
13	18	0000020 4	$X_{118} + X_{-101} + X_{-68} + X_{-74}$	$D_4(a_1)$
			$(X_{-68} + X_{118} + X_{-101}) + (X_{-79})$ $+(X_{-102})$	$(A_3 + 2A_1)''$
13	19	0000004 0	$X_{-15} + X_{74} + X_{104} + X_{101}$	$D_4(a_1)$
			$(X_{-22} + X_{74} + X_{-62}) + (X_{104})$ $+(X_{118})$	$(A_3 + 2A_1)''$
			$(X_{74} + X_{-22}) + (X_{104} + X_{-67})$ $+(X_{118} + X_{-98})$	$3A_2$
13	20	2000002 2	$(X_{-47} + X_{104} + X_{-81}) + (X_{118})$ $+(X_{-74})$	$(A_3 + 2A_1)''$
14	21	0000004 8	$X_{-74} + X_{119} + X_{-104} + X_{-118}$	D_4
14	22	2000004 4	$X_{104} + X_{-101} + X_{118} + X_{-74}$	D_4
18	23	0110001 2	$(X_{-65} + X_{114} + X_{-87})$ $+(X_{112} + X_{-74})$	$A_3 + A_2$
19	24	4000000 4	$X_{101} + X_{-47} + X_{-112} + X_{119}$	A_4
			$(X_{-47} + X_{101} + X_{-81})+$ $(X_{-100} + X_{119} + X_{-110})$	$2A_3$
19	25	2000020 0	$X_{101} + X_{-47} + X_{96} + X_{-68}$	A_4
			$(X_{96} + X_{-68} + X_{112})+$ $(X_{-54} + X_{101} + X_{-77})$	$2A_3$
23	26	1010011 1	$(X_{104} + X_{-65} + X_{105} + X_{-74})$ $+(X_{-73})$	$A_4 + A_1$
25	27	0110003 4	$X_{114} + X_{-98} + X_{-79} + X_{112} + X_{113}$	$D_5(a_1)$
27	28	0002000 0	$(X_{102} + X_{-56} + X_{94} + X_{-67})$ $+(X_{107} + X_{-78})$	$A_4 + A_2$
28	29	1000031 3	$X_{-73} + X_{96} + X_{-74} + X_{112} + X_{-98}$	A_5

Table 2: (continued)

k	i	$\beta_j(H^i)$	E^i	Type
32	30	2000022 2	$(X_{-73}+X_{96}+X_{-74}+X_{112}+X_{-98})$ $+(X_{104})$	$(A_5+A_1)''$
32	31	0000040 4	$(X_{-79}+X_{96}+X_{-68}+X_{112}+X_{-102})$ $+(X_{-101})$	$(A_5+A_1)''$
			$X_{96}+X_{-74}+X_{112}+X_{114}+X_{-101}$ $+X_{113}$	$E_6(a_3)$
33	32	2000024 4	$X_{104}+X_{-101}+X_{-74}+X_{96}+X_{112}$	D_5
33	33	4000004 8	$X_{118}+X_{-101}+X_{104}+X_{-79}+X_{-102}$	D_5
42	34	0002020 0	$X_{102}+X_{-77}+X_{94}+X_{-68}+X_{96}$ $+X_{-79}$	A_6
46	35	4000040 4	$X_{-84}+X_{96}+X_{-73}+X_{101}+X_{-98}$ $+X_{112}+X_{-99}$	$(A_7)''$
			$X_{96}+X_{-73}+X_{101}+X_{105}+X_{-102}$ $+X_{104}$	$E_6(a_1)$
48	36	4000044 8	$X_{96}+X_{-101}+X_{-79}+X_{104}+X_{-102}$ $+X_{112}$	E_6

For any integer j we define:

$$\mathfrak{g}_H(0,j)=\{X\in\mathfrak{k}:[H,X]=jX\},$$
$$\mathfrak{g}_H(1,j)=\{X\in\mathfrak{p}:[H,X]=jX\},$$

and

$$\mathfrak{p}_i(H)=\sum_{j\geq i}\mathfrak{g}_H(1,j).$$

Let Q_H be the parabolic subgroup of K^0 with Lie algebra

$$\mathfrak{q}_H=\sum_{j\geq 0}\mathfrak{g}_H(0,j).$$

3. Statement of the main result. Recall that $\mathfrak{g}_0$ is assumed to be of type E IX. Hence $K=(\mathrm{E}_7\times\mathrm{SL}_2)/Z_2$, where E_7 is simply connected and Z_2 is the diagonal central subgroup of order 2. (By Z_k we denote a cyclic group of order k.) There are exactly 36 nontrivial K-orbits in $\mathcal{N}_1$: denoted by $\mathcal{O}_1^i$, $1\leq i\leq 36$. We choose a normal triple (E^i,H^i,F^i) with $E^i\in\mathcal{O}_1^i$, $H^i\in\mathfrak{h}$, and such that $\beta_j(H^i)\geq 0$ for $1\leq j\leq 8$.

Table 3: Root spaces in $\mathfrak{g}_{H^i}(1,2)$ and $\mathfrak{p}_3(H^i)$

i	Indices of roots
1	$-8;$
2	$-8, -15, -22, -29, -36, -42, -43, -50, -56, -62, -68, -74;$
3	$118, 119, -8, -15;$
4	$-15, -22, -29, -36, -42, -43, -47, -50, -54, -56, -60, -62, -65, -67, -68, -72, -73, -77, -78, -81, -83, -86, -87, -90, -94, -98, -101; -8$
5	$119, -15, -22, -29, -36, -42, -43, -50, -56, -62, -68; -8$
6	$-8, -15, -22, -29, -36, -42, -43, -47, -50, -54, -56, -60, -62, -65, -67, -68, -72, -73, -74, -77, -78, -79, -81, -83, -84, -86, -87, -88, -90, -91, -92, -94, -95, -96, -98, -99, -100, -101, -102, -103, -104, -105, -106, -107, -108, -109, -110, -111, -112, -113, -114, -115, -116, -117, -118, -119;$
7	$119, -15, -22, -29, -36, -42, -43, -47, -50, -54, -56, -60, -62, -65, -67, -68, -72, -73, -77, -78, -81, -83, -86, -87, -90, -94, -98, -101; -8$
8	$101, 104, 107, 109, 111, 113, 114, 115, 116, 117, 118, 119, -8, -15, -22, -29, -36, -42, -43, -50, -56, -62, -68, -74;$
9	$113, 115, 116, 117, 118, 119, -42, -47, -50, -56, -62, -68, -74; -8, -15, -22, -29, -36, -43$
10	$118, 119, -47, -54, -60, -65, -67, -68, -72, -74, -77, -81; -22, -29, -36, -42, -43, -50, -56, -62, -8, -15$
11	$110, 111, 112, 113, 114, 115, -36, -42, -43, -47, -50, -54; 116, 117, 118, 119, -8, -15, -22, -29$
12	$119, -74, -101; -47, -54, -60, -65, -67, -72, -73, -77, -78, -81, -83, -86, -87, -90, -94, -98, -15, -22, -29, -36, -42, -43, -50, -56, -62, -68, -8$
13	$118, -68, -73, -78, -83, -86, -87, -90, -94, -98, -101; -22, -29, -36, -42, -43, -47, -50, -54, -56, -60, -62, -65, -67, -72, -77, -81, 119, -15, -8$
14	$96, 100, 103, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, -22, -29, -36, -42, -43, -47, -50, -54, -56, -60, -62, -65, -67, -72, -77, -81; 118, 119, -8, -15$
15	$107, 109, 111, 113, 114, 115, 116, 117, -47, -54, -60, -65, -67, -68, -72, -77, -81; 118, -22, -29, -36, -42, -43, -50, -56, -62, 119, -15, -8$
16	$118, -73, -74, -78, -83, -86, -87, -90, -94, -98; 119, -47, -54, -60, -65, -67, -68, -72, -77, -81, -22, -29, -36, -42, -43, -50, -56, -62, -15, -8$
17	$104, 107, 109, 111, 113, 114, 115, 116, 117, 118, -47, -54, -60, -65, -67, -72, -73, -77, -78, -81, -83, -86, -87, -90, -94, -98; -15, -22, -29, -36, -42, -43, -50, -56, -62, -68, 119, -8$
18	$118, 119, -68, -73, -74, -78, -79, -83, -84, -86, -87, -88, -90, -91, -92, -94, -95, -98, -99, -101, -102, -104; -22, -29, -36, -42, -43, -47, -50, -54, -56, -60, -62, -65, -67, -72, -77, -81, -8, -15$

Table 3: (continued)

i	Indices of roots
19	$74, 79, 84, 88, 91, 92, 95, 96, 99, 100, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, -15, -22, -29, -36, -42, -43, -47, -50, -54, -56, -60, -62, -65, -67, -68, -72, -73, -77, -78, -81, -83, -86, -87, -90, -94, -98, -101; 119, -8$
20	$104, 107, 109, 111, 113, 114, 115, 116, 117, 118, -47, -54, -60, -65, -67, -72, -73, -74, -77, -78, -81, -83, -86, -87, -90, -94, -98; 119, -15, -22, -29, -36, -42, -43, -50, -56, -62, -68, -8$
21	$119, -74, -79, -84, -88, -91, -92, -95, -96, -99, -100, -102, -103, -104, -105, -106, -107, -108, -109, -110, -111, -112, -113, -114, -115, -116, -117, -118; -15, -22, -29, -36, -42, -43, -47, -50, -54, -56, -60, -62, -65, -67, -68, -72, -73, -77, -78, -81, -83, -86, -87, -90, -94, -98, -101, -8$
22	$104, 107, 109, 111, 113, 114, 115, 116, 117, 118, -74, -101; -47, -54, -60, -65, -67, -72, -73, -77, -78, -81, -83, -86, -87, -90, -94, -98, 119, -15, -22, -29, -36, -42, -43, -50, -56, -62, -68, -8$
23	$112, 113, 114, -65, -72, -74, -77, -78, -79, -83, -87; 115, 116, 117, 118, -50, -54, -56, -60, -62, -67, -68, -73, 119, -42, -43, -47, -15, -22, -29, -36, -8$
24	$101, 104, 107, 109, 111, 113, 114, 115, 116, 117, 118, 119, -47, -54, -60, -65, -67, -72, -73, -77, -78, -79, -81, -83, -84, -86, -87, -88, -90, -91, -92, -94, -95, -96, -98, -99, -100, -102, -103, -105, -106, -108, -110, -112; -8, -15, -22, -29, -36, -42, -43, -50, -56, -62, -68, -74$
25	$96, 100, 101, 103, 104, 105, 106, 108, 110, 112, -47, -54, -60, -65, -67, -68, -72, -74, -77, -81; 107, 109, 111, 113, 114, 115, 116, 117, -22, -29, -36, -42, -43, -50, -56, -62, 118, 119, -8, -15$
26	$104, 105, 108, 110, 112, -65, -72, -73, -74, -77, -81; 107, 109, 111, 113, -47, -54, -60, -67, -68, 114, 115, 116, 117, -43, -50, -56, -62, 118, -22, -29, -36, -42, 119, -15, -8$
27	$112, 113, 114, -74, -79, -98, -101; 115, 116, 117, 118, -81, -86, -90, -94, -65, -72, -77, -78, -83, -87, -50, -54, -56, -60, -62, -67, -68, -73, 119, -42, -43, -47, -15, -22, -29, -36, -8$
28	$94, 98, 99, 101, 102, 103, 104, 105, 106, 107, 108, 109, -56, -60, -62, -65, -67, -68, -72, -73, -74, -78, -79, -84; 110, 111, 112, 113, 114, 115, -36, -42, -43, -47, -50, -54, 116, 117, 118, 119, -8, -15, -22, -29$
29	$96, 100, 103, 105, 106, 108, 110, 112, -73, -74, -78, -83, -86, -87, -90, -94, -98; 107, 109, 111, 113, 114, 115, 116, 117, -68, -47, -54, -60, -65, -67, -72, -77, -81, 118, -22, -29, -36, -42, -43, -50, -56, -62, 119, -15, -8$
30	$96, 100, 103, 104, 105, 106, 108, 110, 112, -73, -74, -78, -83, -86, -87, -90, -94, -98; 107, 109, 111, 113, 114, 115, 116, 117, -47, -54, -60, -65, -67, -68, -72, -77, -81, 118, -22, -29, -36, -42, -43, -50, -56, -62, 119, -15, -8$

Table 3: (continued)

i	Indices of roots
31	$96, 100, 103, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, -68, -73, -74, -78, -79, -83, -84, -86, -87, -88, -90, -91, -92, -94, -95, -98, -99, -101, -102, -104; 118, 119, -22, -29, -36, -42, -43, -47, -50, -54, -56, -60, -62, -65, -67, -72, -77, -81, -8, -15$
32	$96, 100, 103, 104, 105, 106, 108, 110, 112, -74, -101; 107, 109, 111, 113, 114, 115, 116, 117, -73, -78, -83, -86, -87, -90, -94, -98, 118, -47, -54, -60, -65, -67, -68, -72, -77, -81, -22, -29, -36, -42, -43, -50, -56, -62, 119, -15, -8$
33	$104, 107, 109, 111, 113, 114, 115, 116, 117, 118, -79, -84, -88, -91, -92, -95, -96, -99, -100, -101, -102, -103, -105, -106, -108, -110, -112; 119, -47, -54, -60, -65, -67, -72, -73, -74, -77, -78, -81, -83, -86, -87, -90, -94, -98, -15, -22, -29, -36, -42, -43, -50, -56, -62, -68, -8$
34	$94, 96, 98, 99, 100, 101, 102, 104, -68, -73, -74, -77, -78, -79, -81, -84; 103, 105, 106, 107, 108, 109, -56, -60, -62, -65, -67, -72, 110, 111, 112, 113, 114, 115, -36, -42, -43, -47, -50, -54, 116, 117, -22, -29, 118, 119, -8, -15$
35	$96, 100, 101, 103, 104, 105, 106, 108, 110, 112, -73, -78, -79, -83, -84, -86, -87, -88, -90, -91, -92, -94, -95, -98, -99, -102; 107, 109, 111, 113, 114, 115, 116, 117, -47, -54, -60, -65, -67, -68, -72, -74, -77, -81, 118, 119, -22, -29, -36, -42, -43, -50, -56, -62, -8, -15$
36	$96, 100, 103, 104, 105, 106, 108, 110, 112, -79, -84, -88, -91, -92, -95, -99, -101, -102; 107, 109, 111, 113, 114, 115, 116, 117, -73, -74, -78, -83, -86, -87, -90, -94, -98, 118, -47, -54, -60, -65, -67, -68, -72, -77, -81, 119, -22, -29, -36, -42, -43, -50, -56, -62, -15, -8$

These 36 K-orbits are listed in Table 2. For each $i \in \{1, \dots, 36\}$ we record in the first column the integer k for which $\mathcal{O}_1^i \subset \mathcal{O}^k$. In the third column we list the integers $\beta_j(H^i)$, $1 \le j \le 8$. They determine uniquely the element $H^i \in \mathfrak{h}$. Since $\mathfrak{k}$ is of type $E_7 + A_1$ and $\{\beta_i : 1 \le i \le 7\}$ is a base for this E_7, we separate the last integer $\beta_8(H^i)$ from the first seven. In the fourth column we give a representative $E^i \in \mathcal{O}_1^i$. In each case, $E^i \in \mathfrak{g}_{H^i}(1,2)$ and E^i is a sum of root vectors. The last column gives the type of the representative E^i (to be defined below). In some cases we give several representatives of different types.

A subalgebra of $\mathfrak{g}$ is called *regular* if it is normalized by a Cartan subalgebra of $\mathfrak{g}$. We shall say that a regular subalgebra of $\mathfrak{g}$ is *standard* if it is normalized by $\mathfrak{h}$. Of course, every regular subalgebra is G-conjugate to a standard one. Most of the time, two isomorphic regular semisimple subalgebras are G-conjugate but there are 5 exceptions:

$$4A_1,\ A_3 + 2A_1,\ 2A_3,\ A_5 + A_1,\ A_7.$$

In each of the exceptional cases there are two G-conjugacy classes, say $(4A_1)'$ and $(4A_1)''$, and similarly in other cases. As mentioned in [8], the subalgebras $(X)'$ are Levi subalgebras while $(X)''$ are not.

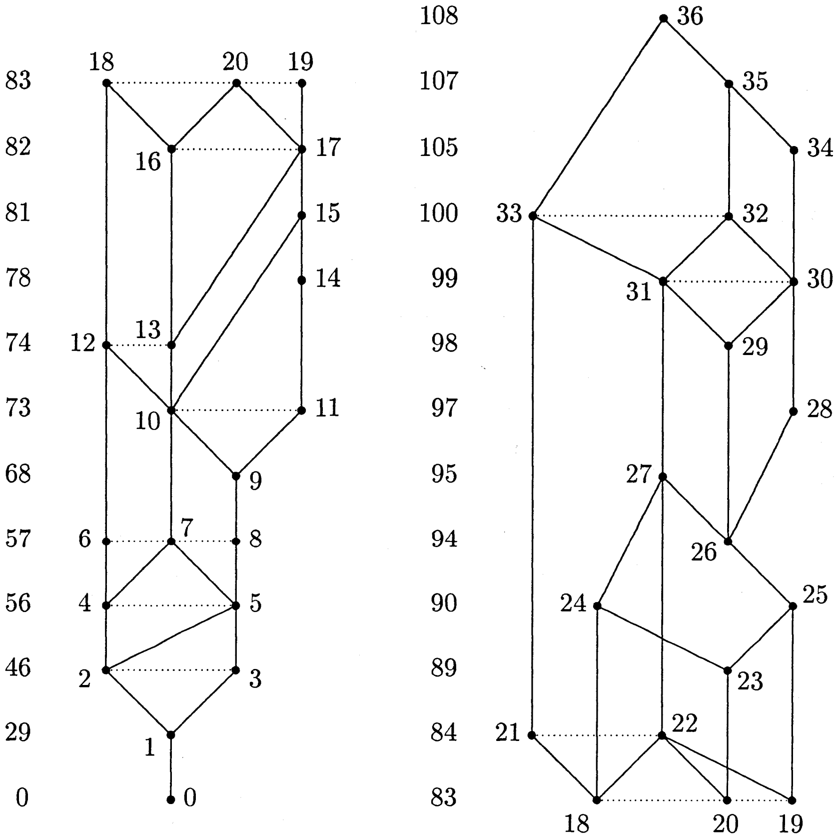

Figure 3: The closure diagram of adjoint nilpotent orbits of E IX

In most cases, the representative $E^i \in \mathcal{O}_1^i$ is the sum of root vectors for simple roots of a standard regular semisimple subalgebra and the type of E^i is, by definition, the type of that subalgebra (up to G-conjugacy). If this is not the case, then the type of E^i is the Bala-Carter symbol of the orbit $\mathcal{O}^k$ containing $\mathcal{O}_1^i$.

In Table 3 we list, for each i, the indices k of the roots $\alpha = \alpha_k$ for which $\mathfrak{g}^\alpha \subset \mathfrak{p}_2(H^i)$. We list first those indices for which $\mathfrak{g}^\alpha \subset \mathfrak{g}_{H^i}(1,2)$ and separate them by a semi-colon from the indices for which $\mathfrak{g}^\alpha \subset \mathfrak{p}_3(H^i)$.

THEOREM 3.1. *Let* $\mathfrak{g}_0$ *be of type* E IX. *Then the closure ordering of the nilpotent* K*-orbits in* $\mathfrak{p}$ *is as given in Figure* 3.

The horizontal dotted lines indicate that the K-orbits joined by these lines are contained in the same G-orbit. The numbers on the left hand side of the diagram are the complex dimensions of the orbits on that level.

Table 4: Elements $E \in \mathfrak{p}_2(H^i) \cap \mathcal{O}_1^j$

i	j	Type	E
2,3	1	A_1	X_{-8}
4,5	2	$2A_1$	$(X_{-15}) + (X_{-68})$
5	3	$2A_1$	$(X_{119}) + (X_{-15})$
6,7	4	$3A_1$	$(X_{-15}) + (X_{-68}) + (X_{-101})$
7,8	5	$3A_1$	$(X_{119}) + (X_{-15}) + (X_{-68})$
12	6	A_2	$X_{-74} + X_{-101}$
10	7	$(4A_1)''$	$(X_{119}) + (X_{-47}) + (X_{-68}) + (X_{-81})$
9	8	A_2	$X_{113} + X_{-42}$
10	9	$A_2 + A_1$	$(X_{118} + X_{-68}) + (X_{-47})$
11	9	$A_2 + A_1$	$(X_{113} + X_{-42}) + (X_{-47})$
12	10	$A_2 + 2A_1$	$(X_{119} + X_{-74}) + (X_{-47}) + (X_{-81})$
13,15	10	$A_2 + 2A_1$	$(X_{118} + X_{-68}) + (X_{-47}) + (X_{-81})$
14	11	$A_2 + 2A_1$	$(X_{114} + X_{-47}) + (X_{113}) + (X_{-50})$
18	12	A_3	$X_{119} + X_{-74} + X_{-101}$
16,17	13	A_3	$X_{-86} + X_{118} + X_{-87}$
15	14	$2A_2$	$(X_{113} + X_{-67}) + (X_{114} + X_{-65})$
17	15	$2A_2 + A_1$	$(X_{113} + X_{-67}) + (X_{114} + X_{-65}) + (X_{-68})$
18,20	16	$A_3 + A_1$	$(X_{-86} + X_{118} + X_{-87}) + (X_{-74})$
19,20	17	$A_3 + A_1$	$(X_{-77} + X_{114} + X_{-78}) + (X_{113})$
21	18	$(A_3 + 2A_1)''$	$(X_{-88} + X_{119} + X_{-95}) + (X_{-86}) + (X_{-87})$
22	18	$D_4(a_1)$	$X_{118} + X_{-101} + X_{-68} + X_{-74}$
24	18	$(A_3 + 2A_1)''$	$(X_{-73} + X_{118} + X_{-98}) + (X_{-84}) + (X_{-99})$
22	19	$(A_3 + 2A_1)''$	$(X_{113} + X_{-101} + X_{114}) + (X_{-36}) + (X_{-50})$
25	19	$(A_3 + 2A_1)''$	$(X_{-54} + X_{104} + X_{-77}) + (X_{96}) + (X_{112})$
22,23	20	$(A_3 + 2A_1)''$	$(X_{-77} + X_{114} + X_{-78}) + (X_{113}) + (X_{-74})$
33	21	D_4	$X_{-79} + X_{104} + X_{-101} + X_{-102}$
27	22	D_4	$X_{113} + X_{-101} + X_{114} + X_{-74}$
24	23	$A_3 + A_2$	$(X_{-65} + X_{114} + X_{-87}) + (X_{113} + X_{-79})$
25	23	$A_3 + A_2$	$(X_{96} + X_{-68} + X_{112}) + (X_{101} + X_{-54})$
27	24	A_4	$X_{113} + X_{-79} + X_{-98} + X_{114}$
26	25	A_4	$X_{104} + X_{-65} + X_{105} + X_{-74}$
27	26	$A_4 + A_1$	$(X_{114} + X_{-98} + X_{112} + X_{-74}) + (X_{-73})$
28	26	$A_4 + A_1$	$(X_{104} + X_{-65} + X_{105} + X_{-74}) + (X_{-73})$
29	26	$A_4 + A_1$	$(X_{112} + X_{-74} + X_{96} + X_{-83}) + (X_{-54})$
31	27	$D_5(a_1)$	$X_{114} + X_{-98} + X_{-79} + X_{112} + X_{113}$
30	28	$A_4 + A_2$	$(X_{108} + X_{-74} + X_{103} + X_{-73}) + (X_{104} + X_{-65})$
30,31	29	A_5	$X_{-73} + X_{96} + X_{-74} + X_{112} + X_{-98}$
32	30	$(A_5 + A_1)''$	$(X_{-73} + X_{96} + X_{-74} + X_{112} + X_{-98}) + (X_{104})$
34	30	$(A_5 + A_1)''$	$(X_{96} + X_{-68} + X_{94} + X_{-77} + X_{102}) + (X_{-54})$
32	31	$E_6(a_3)$	$X_{96} + X_{-74} + X_{112} + X_{114} + X_{-101} + X_{113}$
33	31	$(A_5 + A_1)''$	$(X_{-79} + X_{107} + X_{-101} + X_{117} + X_{-102}) + (X_{-68})$
35	32	D_5	$X_{112} + X_{-102} + X_{-73} + X_{96} + X_{101}$
36	33	D_5	$X_{118} + X_{-101} + X_{104} + X_{-79} + X_{-102}$
35	34	A_6	$X_{112} + X_{-98} + X_{101} + X_{-73} + X_{96} + X_{-84}$
36	35	$(A_7)''$	$X_{-73} + X_{96} + X_{-84} + X_{104} + X_{-99} + X_{112} + X_{-98}$

4. Proof of the main result. Let i, j be a pair of nodes in the diagram of Figure 3, with i above j, which are joined by a solid line. We prove that $\mathcal{O}_1^i > \mathcal{O}_1^j$ by showing that $\mathcal{O}_1^j$ meets $\mathfrak{p}_2(H^i)$ (see [9, Theorem 3.1]). In Table 4 we list all such pairs i, j (with $j \neq 0$) and for each of them provide an element $E \in \mathfrak{p}_2(H^i) \cap \mathcal{O}_1^j$. We also indicate the type of E.

The fact that $E \in \mathfrak{p}_2(H^i)$ can be checked by using Table 3. In most cases the verification of the claim that $E \in \mathcal{O}_1^j$ is straightforward: This element belongs to $\mathfrak{g}_{H^j}(1,2)$ and, being of the right type, is in fact a generic element for the action of the centralizer $Z_K(H^j)$. This argument is not applicable when (i, j) is one of the following pairs:

$$\begin{matrix} (33,21), & (25,23), & (27,26), & (29,26), \\ (34,30), & (35,32), & (35,34). & \end{matrix}$$

Since $\mathcal{O}^{18} \cap \mathfrak{p} = \mathcal{O}_1^{23}$ (see Table 1), we conclude that in the case $(25, 23)$ we must have $E \in \mathcal{O}_1^{23}$. A similar argument is applicable to the pairs $(27, 26)$, $(29, 26)$, and $(35, 34)$. The remaining three pairs $(33, 21)$, $(34, 30)$, and $(35, 32)$ require a more elaborate argument.

Let us consider in detail the pair $(33, 21)$. In this case

$$E = X_{-79} + X_{104} + X_{-101} + X_{-102}$$

is a standard principal nilpotent element of type D_4. Hence

$$E \in \mathcal{O}^{14} \cap \mathfrak{p} = \mathcal{O}_1^{21} \cup \mathcal{O}_1^{22}$$

and we have to show that in fact $E \in \mathcal{O}_1^{21}$. We do this by finding a normal triple (E, H, F) inside the standard regular simple subalgebra of type D_4 having $\{\alpha_{-79}, \alpha_{104}, \alpha_{-101}, \alpha_{-102}\}$ as a base for its root system. The element H is given by

$$\begin{aligned} H &= 6H_{-79} + 10H_{104} + 6H_{-101} + 6H_{-102} \\ &= -2(2H_1 + 5H_2 + 6H_3 + 10H_4 + 9H_5 + 8H_6 + 5H_7 + 4H_8). \end{aligned}$$

We do not need to compute F but we remark that

$$F \in \langle X_{79}, X_{-104}, X_{101}, X_{102} \rangle.$$

Next we compute

$$\alpha_m(H) = 4, 0, 0, 0, 0, -4, 4, -6 \quad (1 \leq m \leq 8)$$

and deduce that

$$\beta_m(H) = 4, 0, 0, 0, 0, -4, 4, -6 \quad (1 \leq m \leq 8).$$

Finally by applying a suitable element w of the Weyl group of $(\mathfrak{k}, \mathfrak{h})$ to H we obtain the element $H' = w(H)$ such that

$$\beta_m(H') = 0, 0, 0, 0, 0, 0, 4, 8 \quad (1 \leq m \leq 8).$$

By looking up Table 2, we conclude that indeed $E \in \mathcal{O}_1^{21}$.

The argument is similar in the other two cases. We only state that for the pair $(34, 30)$ we have

$$\alpha_m(H) = 2, 2, -4, 2, 0, 0, 4, -8 \quad (1 \leq m \leq 8),$$

and for $(35, 32)$

$$\alpha_m(H) = -2, -2, 4, 0, 0, 6, -6, 8 \quad (1 \leq m \leq 8).$$

By inspection of Figure 3, we see that in order to complete the proof of the theorem it suffices to show that $\mathcal{O}_1^i \not> \mathcal{O}_1^j$ when (i, j) is one of the following pairs:

$$\begin{array}{lll} (6,3), & (14,4), & (34,6), \\ (21,11), & (15,13), & (19,16), \\ (24,19), & (35,21), & (33,28). \end{array} \tag{4.1}$$

This assertion is valid for the pair $(15, 13)$ because $\mathcal{O}_1^{15} \subset \mathcal{O}^{11}$, $\mathcal{O}_1^{13} \subset \mathcal{O}^8$, and $\mathcal{O}^{11} \not> \mathcal{O}^8$ (see Table 1 and Figure 1). Another proof of this fact will be given below.

Let $V = \mathfrak{g}$ be the adjoint $\mathfrak{g}$-module (of dimension 248). It can be equipped with the Z_2-grading $V = V_0 \oplus V_1$, where $V_0 = \mathfrak{k}$ and $V_1 = \mathfrak{p}$. Thus $\dim V_0 = 136$, and $\dim V_1 = 112$. We introduce the integers

$$d_i(j, k) = \dim V_i \cap \ker \rho(E^k)^j$$

where $i = 0, 1$; $j \geq 1$; $1 \leq k \leq 36$, and $\rho = \mathrm{ad}$ is the adjoint representation of $\mathfrak{g}$ on V. They are easy to compute and are displayed in Table 5.

By applying [9, Theorem 4.1] and using Table 5, we see that $\mathcal{O}_1^i \not> \mathcal{O}_1^j$ when (i, j) is one of the following pairs.

$$\begin{array}{lllll} (6,3), & (11,4), & (6,5), & (23,6), & (12,11), \\ (19,12), & (23,12), & (15,13), & (19,16), & (18,17), \\ (23,19), & (21,20), & (31,21), & (34,21), & (24,22), \\ (28,22), & (28,24), & (34,32), & (34,33). & \end{array} \tag{4.2}$$

In particular this means that the pairs $(6, 3)$, $(15, 13)$, and $(19, 16)$ from (4.1) have been taken care of.

The remaining six pairs in the list (4.1) are handled by using the theory of prehomogeneous vector spaces (PV) [15, 16].

Table 5: The integers $d_i(j,k)$ for the adjoint module of E_8 equipped with a Z_2-gradation

k	$d_0(j,k);\ j=1,2,\ldots$	$d_1(j,k);\ j=1,2,\ldots$
1	107 136	83 111 112
2	90 134 136	66 100 112
3	90 126 136	66 108 112
4	80 134 135 136	56 83 111 112
5	80 118 135 136	56 99 111 112
6	79 134 135 135 136	55 56 111 112
7	79 108 135 136	55 82 111 111 112
8	79 102 135 135 136	55 88 111 112
9	68 102 129 135 136	44 87 105 112
10	63 100 125 135 136	39 80 101 110 112
11	63 96 125 133 136	39 84 101 112
12	62 90 107 134 135 136	38 55 83 100 111 111 112
13	62 82 107 126 135 136	38 63 83 108 111 111 112
14	58 84 122 126 136	34 72 98 108 112
15	55 84 115 126 135 136	31 71 91 108 111 112
16	54 82 107 126 134 136	30 62 83 100 110 111 112
17	54 82 107 118 134 136	30 62 83 108 110 111 112
18	53 82 107 125 134 136	29 54 83 92 110 110 112
19	53 82 107 109 134 136	29 54 83 108 110 110 112
20	53 75 107 118 134 135 136	29 61 83 99 110 111 112
21	52 80 80 107 107 134 134 135 135 136	28 28 56 56 83 83 110 110 111 111 112
22	52 64 80 91 107 118 134 135 135 136	28 44 56 72 83 99 110 110 111 111 112
23	47 70 102 117 129 134 136	23 59 78 98 105 111 112
24	46 68 90 101 123 134 135 135 136	22 44 66 88 99 100 111 112
25	46 60 90 101 123 126 135 135 136	22 52 66 88 99 108 111 112
26	42 60 89 101 120 126 134 135 136	18 51 65 88 96 108 110 112
27	41 58 80 91 107 118 129 134 135 136	17 43 56 72 83 98 105 110 111 111 112
28	39 54 87 96 117 122 133 133 136	15 48 63 84 93 104 109 112
29	38 50 73 84 106 116 124 126 134 136	14 37 49 72 82 97 100 108 110 111 112
30	37 50 73 84 106 109 124 126 134 135 136	13 36 49 71 82 97 100 108 110 111 112
31	37 50 73 83 106 116 124 126 134 136	13 36 49 72 82 90 100 108 110 110 112
32	36 48 63 74 89 92 107 109 124 126 134 135 135 136	12 27 39 54 65 80 83 98 100 108 110 110 111 111 112
33	36 48 63 74 89 100 107 117 124 134 134 135 135 136	12 27 39 54 65 72 83 90 100 100 110 110 111 111 112
34	31 38 63 69 90 96 114 115 125 126 133 133 136	7 32 39 60 66 84 90 100 101 108 109 112
35	29 34 55 59 80 84 98 101 115 118 125 126 133 134 135 135 136	5 26 31 52 56 70 74 88 91 98 101 108 109 110 111 112

Table 5: (continued)

k	$d_0(j,k);\ j=1,2,\ldots$	$d_1(j,k);\ j=1,2,\ldots$
36	28 32 46 49 63 66 80 83 97 100 107 109 116 118 125 126 133 134 134 135 135 136	4 18 22 36 39 53 56 70 73 80 83 90 92 99 101 108 109 109 110 110 111 111 112

In order to determine the closure of an orbit $\mathcal{O}_1^k$ we shall employ the following recursive procedure. The centralizer $Z = Z_K(H^k)$ is a connected reductive subgroup of K which can be easily determined from the integers $\beta_i(H^k)$ given in Table 2. Furthermore Z is a Levi factor of the parabolic subgroup Q_{H^k} of K. The centralizer of E^k in Z is reductive, and consequently the PV $(Z, \mathfrak{g}_{H^k}(1,2))$ is regular [15]. Hence the singular set S of this PV is a union of irreducible conical hypersurfaces S_i defined by equations $f_i = 0$, where the f_i's are the basic relative invariants of this PV. One knows how to compute the number m of the basic relative invariants [16, Proposition 4]. In particular, m cannot exceed the length of $\mathfrak{g}_{H^k}(1,2)$ as a Z-module. In all cases that we encounter below m is actually equal to the length of this module. The pair $(Q_{H^k}, \mathfrak{p}_2(H^k))$ is also a PV and its singular set is the union of the hypersurfaces $\hat{S}_i = S_i + \mathfrak{p}_3(H^k)$. In most cases each of the hypersurfaces $\hat{S}_i$ contains a dense open Q_{H^k}-orbit and we are able to identify the K-orbit that contains this Q_{H^k}-orbit. Then the closure of $\mathcal{O}_1^k$ is the union of $\mathcal{O}_1^k$ and the closures of K-orbits (of smaller dimension) which contain a dense open subset of one of the hypersurfaces $\hat{S}_i$.

We start with the pair (14,4). The PV $(Z_K(H^{14}), \mathfrak{g}_{H^{14}}(1,2))$ is irreducible and its singular set S is an irreducible hypersurface. The singular set of $(Q_{H^{14}}, \mathfrak{p}_2(H^{14}))$ is the irreducible hypersurface $\hat{S} = S + \mathfrak{p}_3(H^{14})$. The element

$$E = (X_{114} + X_{-47}) + (X_{113}) + (X_{-50})$$

belongs to $\mathfrak{p}_2(H^{14}) \cap \mathcal{O}_1^{11}$ (see Table 4). A computation shows that $Q_{H^{14}} \cdot E$ has dimension 35. As $\mathfrak{p}_2(H^{14})$ has dimension 36, this orbit is dense in $\hat{S}$. Consequently

$$\overline{\mathcal{O}_1^{14}} = \mathcal{O}_1^{14} \cup \overline{\mathcal{O}_1^{11}}.$$

By (4.2) we know that $\mathcal{O}_1^{11} \not> \mathcal{O}_1^4$, and so $\mathcal{O}_1^{14} \not> \mathcal{O}_1^4$.

Next we consider the pair (24,19). The argument is similar to the one used above. The singular set S of $(Z_K(H^{24}), \mathfrak{g}_{H^{24}}(1,2))$ is the union of two irreducible hypersurfaces S_1 and S_2. The singular set of $(Q_{H^{24}}, \mathfrak{p}_2(H^{24}))$ is also the union of two irreducible hypersurface $\hat{S}_i = S_i + \mathfrak{p}_3(H^{24})$. Let E_1 and E_2 be the elements from Table 4 in the rows with $i = 24$ and $j = 23, 18$. Hence E_1 belongs to $\mathfrak{p}_2(H^{24}) \cap \mathcal{O}_1^{23}$ and E_2 to $\mathfrak{p}_2(H^{24}) \cap \mathcal{O}_1^{18}$. A computation shows that the orbits $Q_{H^{14}} \cdot E_i$ have dimension 55. As $\mathfrak{p}_2(H^{24})$ has dimension 56, these orbits are dense in $\hat{S}_i$. Consequently

$$\overline{\mathcal{O}_1^{24}} = \mathcal{O}_1^{24} \cup \overline{\mathcal{O}_1^{23}} \cup \overline{\mathcal{O}_1^{18}}.$$

By (4.2) we know that $\mathcal{O}_1^{23} \not> \mathcal{O}_1^{19}$, and so $\mathcal{O}_1^{24} \not> \mathcal{O}_1^{19}$.

Let us now consider the pair (21,11). The argument is similar to the one used above but we have to use it several times. We shall indicate only the main steps.

The singular set of $(Q_{H^{21}}, \mathfrak{p}_2(H^{21}))$ is the union of two irreducible hypersurfaces. Let E_1 be the element from Table 4 in the row with $i = 21$, $j = 18$ and let $E_2 = X_{-8} + X_{-74} + X_{-104} + X_{-118}$. The orbits $Q_{H^{21}} \cdot E_i$ are dense in the two hypersurfaces. As E_2 has type $(4A_1)''$, it belongs (see Table 2) to one of the orbits $\mathcal{O}_1^6$, $\mathcal{O}_1^7$, $\mathcal{O}_1^8$. We have also computed the dimension of the orbit $G \cdot E_2$. Since this dimension is 114, we obtain another proof (see Table 1) that E_2 belongs to one of the three K-orbits mentioned above. (By embedding E_2 in a suitable normal triple one can show that in fact it belongs to $\mathcal{O}_1^6$.) As $E_1 \in \mathcal{O}_1^{18}$ and $\overline{\mathcal{O}_1^{18}} \supset \mathcal{O}_1^6 \cup \mathcal{O}_1^7 \cup \mathcal{O}_1^8$, we conclude that

$$\overline{\mathcal{O}_1^{21}} = \mathcal{O}_1^{21} \cup \overline{\mathcal{O}_1^{18}}. \tag{4.3}$$

The singular set of $(Q_{H^{18}}, \mathfrak{p}_2(H^{18}))$ is also the union of two irreducible hypersurfaces. Let E_3 (resp. E_4) be the element from Table 4 in the row with $i = 18$, $j = 16$ (resp. $i = 18$, $j = 12$). Then a computation shows that the orbits $Q_{H^{18}} \cdot E_3$ and $Q_{H^{18}} \cdot E_4$ have co-dimension 1 in $\mathfrak{p}_2(H^{18})$, and so they are dense in the two hypersurfaces. Consequently

$$\overline{\mathcal{O}_1^{18}} = \mathcal{O}_1^{18} \cup \overline{\mathcal{O}_1^{16}} \cup \overline{\mathcal{O}_1^{12}}. \tag{4.4}$$

We now claim that

$$\overline{\mathcal{O}_1^{16}} = \mathcal{O}_1^{16} \cup \overline{\mathcal{O}_1^{13}}. \tag{4.5}$$

The argument in this case is more complicated and we shall provide more details. As a $Z_K(H^{16})$-module, $\mathfrak{g}_{H^{16}}(1,2)$ is the direct sum of two 1-dimensional modules $\langle X_{118}\rangle$ and $\langle X_{-74}\rangle$, and the 8-dimensional simple module

$$\langle X_{-98}, X_{-94}, X_{-90}, X_{-87}, X_{-86}, X_{-83}, X_{-78}, X_{-73}\rangle.$$

This PV has 3 basic relative invariants:

$$f_1(X) = a, \quad f_2(X) = b, \quad f_3(X) = cj - di + eh - fg,$$

where

$$\begin{aligned} X = aX_{118} + bX_{-74} + cX_{-98} + dX_{-94} + eX_{-90} + fX_{-87} + gX_{-86} \\ +hX_{-83} + iX_{-78} + jX_{-73}. \end{aligned}$$

If we set $\hat{f}_i(Z) = f_i(X)$ for $Z = X + Y$ with $X \in \mathfrak{g}_{H^{16}}(1,2)$ and $Y \in \mathfrak{p}_3(H^{16})$, then we obtain the basic relative invariants of $(Q_{H^{16}}, \mathfrak{p}_2(H^{16}))$. The elements E_i' defined by

$$\begin{aligned} E_1' &= (X_{-74} + X_{119}) + (X_{-73}) + (X_{-98}), \\ E_2' &= X_{-86} + X_{118} + X_{-87}, \\ E_3' &= (X_{118} + X_{-98}) + (X_{-47}) + (X_{-74}), \end{aligned}$$

belong to the hypersurfaces $\hat{S}_i$ defined by $\hat{f}_i(Z) = 0$ and their $Q_{H^{16}}$-orbits are dense in these hypersurfaces. This follows from the fact that each of these three orbits has co-dimension 1 in $\mathfrak{p}_2(H^{16})$. From Table 4 we see that $E_2' \in \mathcal{O}_1^{13}$. Both E_1' and E_3' have type $A_2 + 2A_1$ and so they belong either to $\mathcal{O}_1^{10}$ or $\mathcal{O}_1^{11}$. It is important to show that they both belong to $\mathcal{O}_1^{10}$. This can be shown by embedding, say, E_1' into a principal normal triple (E_1', H, F_1') of $A_2 + 2A_1$. An example of such a computation was given in the beginning of this proof. We omit most of the details but we point out that $H \in \mathfrak{h}$ has the following labels:

$$\alpha_m(H) = 2, 0, 0, 0, 0, -1, 2, -4 \quad (1 \leq m \leq 8).$$

In the case of E_3', the corresponding labels are

$$\alpha_m(H) = 2, 1, -2, 0, 0, 2, -1, -3 \quad (1 \leq m \leq 8).$$

As $\mathcal{O}_1^{13} > \mathcal{O}_1^{10}$, our claim is proved.

The singular set of $(Q_{H^{13}}, \mathfrak{p}_2(H^{13}))$ is the union of two irreducible hypersurfaces. Let E_4' be the element from Table 4 in the row with $i = 13$, $j = 10$ and let $E_5' = X_{-68} + X_{-101} + X_{119} + X_{-15}$. A computation shows that the orbits $Q_{H^{13}} \cdot E_4'$ and $Q_{H^{13}} \cdot E_5'$ have co-dimension 1 in $\mathfrak{p}_2(H^{13})$, and so are dense in these two hypersurfaces. The element E_4' belongs to $\mathcal{O}_1^{10}$ (see Table 4). The element E_5' has type $(4A_1)''$ and is contained in $\mathfrak{g}_{H^7}(1,2)$. Consequently it belongs to $\mathcal{O}_1^7$. Hence

$$\overline{\mathcal{O}_1^{13}} = \mathcal{O}_1^{13} \cup \overline{\mathcal{O}_1^{10}}. \tag{4.6}$$

Since $\mathcal{O}_1^{12} \not> \mathcal{O}_1^{11}$ by (4.2), it follows from (4.3-6) that $\mathcal{O}_1^{21} \not> \mathcal{O}_1^{11}$.

We move now to the pair (33,28). The singular set of $(Q_{H^{33}}, \mathfrak{p}_2(H^{33}))$ is the union of three irreducible hypersurfaces. Let E_1 and E_2 be the elements from Table 4 in the rows with $i = 33$ and $j = 21$ and $j = 31$, respectively. Let

$$E_3 = (X_{-78} + X_{118} + X_{-94}) + (X_{-79} + X_{104} + X_{-102}).$$

A computation shows that each of the orbits $Q_{H^{21}} \cdot E_i$ has co-dimension 1 in $\mathfrak{p}_2(H^{33})$ and so is dense in one of these hypersurfaces. Clearly, $E_1 \in \mathcal{O}_1^{21}$ and $E_2 \in \mathcal{O}_1^{31}$. Since E_3 has type $2A_3$ and belongs to $\mathfrak{g}_{H^{24}}(1,2)$, it follows that it belongs to $\mathcal{O}_1^{24}$. Hence

$$\overline{\mathcal{O}_1^{33}} = \mathcal{O}_1^{33} \cup \overline{\mathcal{O}_1^{21}} \cup \overline{\mathcal{O}_1^{31}}. \tag{4.7}$$

The singular set of $(Q_{H^{31}}, \mathfrak{p}_2(H^{31}))$ is the union of two irreducible hypersurfaces. Let E_4 (resp. E_5) be the element from Table 4 in the row with $i = 31$, $j = 27$ (resp. $i = 31$, $j = 29$). Then a computation shows that the orbits $Q_{H^{18}} \cdot E_4$ and $Q_{H^{18}} \cdot E_5$ have co-dimension 1 in $\mathfrak{p}_2(H^{31})$, and so they are dense in these two hypersurfaces. Consequently

$$\overline{\mathcal{O}_1^{31}} = \mathcal{O}_1^{31} \cup \overline{\mathcal{O}_1^{27}} \cup \overline{\mathcal{O}_1^{29}}. \tag{4.8}$$

The $Z_K(H^{29})$-module $\mathfrak{g}_{H^{29}}(1,2)$ is a direct sum of the 1-dimensional module $\langle X_{-74} \rangle$ and two nonisomorphic simple 8-dimensional modules:

$$V_8 = \langle X_{-73}, X_{-78}, X_{-83}, X_{-86}, X_{-87}, X_{-90}, X_{-94}, X_{-98} \rangle,$$
$$W_8 = \langle X_{112}, X_{110}, X_{108}, X_{106}, X_{105}, X_{103}, X_{100}, X_{96} \rangle.$$

The singular set of $(Q_{H^{29}}, \mathfrak{p}_2(H^{29}))$ is the union of three irreducible hypersurfaces $\hat{S}_i$ $(i = 1, 2, 3)$. Let E'_1 be the element from Table 4 in the row with $i = 29$ and $j = 10$. Further, let

$$E'_2 = (X_{-73} + X_{96} + X_{-74} + X_{113}) + (X_{-98}),$$
$$E'_3 = X_{-73} + X_{96} + X_{112} + X_{113} + X_{-11} + X_{-98}.$$

A computation shows that the orbits $Q_{H^{29}} \cdot E'_i$ have co-dimension 1 in $\mathfrak{p}_2(H^{29})$, and so are dense in the three hypersurfaces. Note that the elements E'_1 and E'_2 are both of type $A_4 + A_1$ and so belong to $\mathcal{O}_1^{26}$ (see Table 1). However the closures of their $Q_{H^{29}}$-orbits are two different hypersurface, say $\hat{S}_1$ and $\hat{S}_2$. This follows from the observation that the V_8-component of E'_1, namely X_{-98}, is not generic in the module V_8 while that of E'_2, namely $X_{-73} + X_{-98}$, is. (The situation is just the opposite for the W_8-components.) It is not obvious to which K-orbit the element E'_3 belongs. But a computation shows that the orbit $G \cdot E'_3$ has dimension 166, and so (see Table 1) E'_3 belongs to one of the orbits $\mathcal{O}_1^{18}$, $\mathcal{O}_1^{19}$, $\mathcal{O}_1^{20}$. Consequently, the closure of the orbit $Q_{H^{29}} \cdot E'_3$ is the third hypersurface $\hat{S}_3$. As the orbits $\mathcal{O}_1^i$ for $i = 18, 19, 20$, and 26 have dimensions less then that of $\mathcal{O}_1^{28}$, we conclude that $\mathcal{O}_1^{29} \not> \mathcal{O}_1^{28}$. Now (4.7) and (4.8) imply that $\mathcal{O}_1^{33} \not> \mathcal{O}_1^{28}$.

In connection with this case, we make an interesting observation. Consider the G-orbits $\mathcal{O}^{28}$ and $\mathcal{O}^{27}$ with Bala-Carter labels A_5 and $A_4 + A_2$, respectively. By Table 1 we have

$$\mathfrak{p} \cap \mathcal{O}^{28} = \mathcal{O}_1^{29}, \quad \mathfrak{p} \cap \mathcal{O}^{27} = \mathcal{O}_1^{28}.$$

Our observation is that

$$\mathcal{O}_1^{29} \not> \mathcal{O}_1^{28}$$

although (see Figure 2)

$$\mathcal{O}^{28} > \mathcal{O}^{27}.$$

The same phenomenon was observed in our previous paper [11].

We next consider the pair (34,6) from the list (4.1). The proof in this case is quite different from the ones above. (A similar proof was used in our paper [10].) The idea is to construct a closed subset of $\mathfrak{p}$ which contains $\mathcal{O}_1^{34}$ but not $\mathcal{O}_1^6$. We begin by observing that, as a K-module, $\mathfrak{p}$ is isomorphic to $V \otimes W$ where V is the 56-dimensional simple module for E_7 and W is the 2-dimensional simple module for SL_2. Furthermore, $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$ where $\mathfrak{p}^+$ resp. $\mathfrak{p}^-$ is the subspace spanned by the positive resp. negative root spaces in $\mathfrak{p}$. Both $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are isomorphic to V as E_7-modules. We shall identify V with $\mathfrak{p}^+$. An explicit E_7-isomorphism $\varphi : \mathfrak{p}^+ \to \mathfrak{p}^-$ is given by $\varphi(X) = [X_{-120}, X]$.

Let $\pi : \mathfrak{p} \to \mathfrak{p}^+$ be the projector with kernel $\mathfrak{p}^-$. Note that π commutes with the action of E_7. Any $Z \in \mathfrak{p}$ can be written uniquely as $Z = X + \varphi(Y)$ with $X, Y \in \mathfrak{p}^+$. The orbit $\mathrm{SL}_2 \cdot Z$ consists of all vectors

$$(aX + bY) + \varphi(cX + dY)$$

with $ad - bc = 1$. Hence

$$\pi(\mathrm{SL}_2 \cdot Z) = \langle X, Y \rangle \setminus \{0\}.$$

The pair $(\mathrm{E}_7 \times \mathrm{GL}_1, \mathfrak{p}^+)$, where GL_1 is the maximal torus of the SL_2 factor of K leaving $\mathfrak{p}^+$ and $\mathfrak{p}^-$ invariant, is a regular PV (see [15, p. 147]). This PV was studied by S.J. Haris in [12]. The singular set is the hypersurface $S = \mathfrak{p}^+ \setminus \mathcal{O}_1^6$ and consists of 4 orbits. Let $E \in \mathcal{O}_1^{34}$ be the representative from Table 2 and write $E = X + \varphi(Y)$ where

$$X = X_{102} + X_{94} + X_{96}, \quad \varphi(Y) = X_{-77} + X_{-68} + X_{-79}.$$

We have

$$\pi(\mathrm{SL}_2 \cdot E) \subset \langle X, Y \rangle \subset \langle X_{102}, X_{94}, X_{96}, X_{100}, X_{93}, X_{95} \rangle \subset S.$$

The last inclusion follows from the fact that the $(\mathrm{E}_7 \times \mathrm{GL}_1)$-orbit of the element

$$X_{102} + X_{94} + X_{96} + X_{100} + X_{93} + X_{95}$$

has dimension 55 and the observation that this orbit contains all linear combinations of the root vectors X_{102}, X_{94}, X_{96}, X_{100}, X_{93}, X_{95} with all coefficients nonzero.

Hence $\pi(\mathcal{O}_1^{34}) \subset S$ and so $\mathcal{O}_1^{34} \subset S + \mathfrak{p}^-$. Thus $\overline{\mathcal{O}_1^{34}} \subset S + \mathfrak{p}^-$ and consequently $\mathcal{O}_1^6 \not\subset \overline{\mathcal{O}_1^{34}}$, i.e., $\mathcal{O}_1^{34} \not> \mathcal{O}_1^6$.

It remains to consider the pair (35,21) from the list (4.2). The singular set of $(Q_{H^{35}}, \mathfrak{p}_2(H^{35}))$ is the union of three irreducible hypersurfaces. Let E_1 and E_2 be the elements from Table 4 in the rows with $i = 35$ and $j = 34$ and $j = 32$, respectively. Let

$$E_3 = (X_{-84} + X_{96} + X_{-73} + X_{-98}) + (X_{-99}).$$

A computation shows that each of the orbits $Q_{H^{35}} \cdot E_i$ has co-dimension 1 in $\mathfrak{p}_2(H^{35})$ and so is dense in one of the hypersurfaces. Clearly, $E_1 \in \mathcal{O}_1^{34}$ and $E_2 \in \mathcal{O}_1^{32}$. Since E_3 has type $(A_5 + A_1)''$, it follows that it belongs to either $\mathcal{O}_1^{30}$ or $\mathcal{O}_1^{31}$. (One can show that it belongs to $\mathcal{O}_1^{31}$, but we do not need this fact.) Hence

$$\overline{\mathcal{O}_1^{35}} = \mathcal{O}_1^{35} \cup \overline{\mathcal{O}_1^{34}} \cup \overline{\mathcal{O}_1^{32}}.$$

Since $\mathcal{O}_1^{34} \not> \mathcal{O}_1^{21}$ by (4.2), it remains to show that $\mathcal{O}_1^{32} \not> \mathcal{O}_1^{21}$. The $Z_K(H^{32})$-module $\mathfrak{g}_{H^{32}}(1,2)$ is a direct sum of three 1-dimensional modules $\langle X_{104} \rangle$, $\langle X_{-74} \rangle$, $\langle X_{-101} \rangle$, and the 8-dimensional simple module

$$V_8 = \langle X_{96}, X_{100}, X_{103}, X_{105}, X_{106}, X_{108}, X_{110}, X_{112} \rangle.$$

We shall write an arbitrary element $Z \in \mathfrak{p}_2(H^{32})$ as $Z = X + Y$ where $X \in \mathfrak{g}_{H^{32}}(1,2)$, $Y \in \mathfrak{p}_3(H^{32})$, and

$$\begin{aligned} X = aX_{96} + bX_{100} + cX_{103} + dX_{105} + d'X_{106} + c'X_{108} \\ + b'X_{110} + a'X_{112} + uX_{104} + vX_{-74} + wX_{-101}. \end{aligned}$$

The basic relative invariants of $(Q_{H^{32}}, \mathfrak{p}_2(H^{32}))$ are given by:

$$\hat{f}_1(Z) = u, \quad \hat{f}_2(Z) = v, \quad \hat{f}_3(Z) = w,$$
$$\hat{f}_4(Z) = aa' - bb' + cc' - dd'.$$

The singular set $\hat{S}$ of this PV is the union of the three hyperplanes $\hat{S}_i : \hat{f}_i(Z) = 0$, $i = 1, 2, 3$, and the irreducible quadric $\hat{S}_4 : \hat{f}_4(Z) = 0$. Let E_1, E_3, and E_4 be the elements from Table 4 listed in the rows with $i = 32$ and $j = 31, 30$, and 27, respectively. It is easy to check that $E_i \in \hat{S}_j$ is true if and only if $j = i$. A computation shows that the $Q_{H^{32}}$-orbit through E_i has co-dimension 1 in $\mathfrak{p}_2(H^{32})$ and so this orbit is dense in $\hat{S}_i$ $(i = 1, 3, 4)$. The situation with the hyperplane $\hat{S}_2$ is quite different. The group $Q_{H^{32}}$ has no open dense orbit in this hyperplane. A computation shows that the maximum dimension of an $Q_{H^{32}}$-orbit in $\hat{S}_2$ is 45 while $\hat{S}_2$ has dimension 47. The maximum dimension of an orbit $K \cdot Z$ for $Z \in \hat{S}_2$ is 83 and so all these orbits are contained in the closure of $\mathcal{O}_1^{31}$ (see Figure 3). Consequently,

$$\overline{\mathcal{O}_1^{32}} = \mathcal{O}_1^{32} \cup \overline{\mathcal{O}_1^{30}} \cup \overline{\mathcal{O}_1^{31}}.$$

Now (4.2) implies that $\mathcal{O}_1^{32} \not> \mathcal{O}_1^{21}$.

5. Appendix.

The simple roots α_i, $1 \leq i \leq 8$, are numbered as in Figure 4.

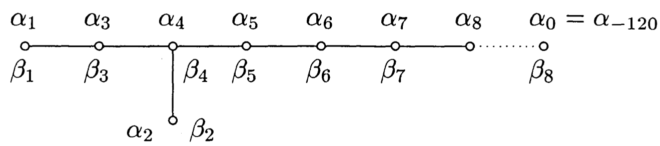

Figure 4: The extended base of E_8 and a base of $\mathfrak{k}$

In Table 6 we give our enumeration of the positive roots of E_8. This enumeration is the same as in our paper [8].

Table 6: Positive roots of $\mathbf{E_8}$

i	α_i	i	α_i	i	α_i	i	α_i
1	10000000	31	10111100	61	01122210	91	12232221
2	01000000	32	01121000	62	01122111	92	11233221
3	00100000	33	01111100	63	11232100	93	12343210
4	00010000	34	01011110	64	11222110	94	12243211
5	00001000	35	00111110	65	11221111	95	12233221
6	00000100	36	00011111	66	11122210	96	11233321
7	00000010	37	11121000	67	11122111	97	22343210
8	00000001	38	11111100	68	01122211	98	12343211
9	10100000	39	10111110	69	12232100	99	12243221

Table 6: (continued)

i	α_i	i	α_i	i	α_i	i	α_i
10	01010000	40	01121100	70	11232110	100	12233321
11	00110000	41	01111110	71	11222210	101	22343211
12	00011000	42	01011111	72	11222111	102	12343221
13	00001100	43	00111111	73	11122211	103	12243321
14	00000110	44	11221000	74	01122221	104	22343221
15	00000011	45	11121100	75	12232110	105	12343321
16	10110000	46	11111110	76	11232210	106	12244321
17	01110000	47	10111111	77	11232111	107	22343321
18	01011000	48	01122100	78	11222211	108	12344321
19	00111000	49	01121110	79	11122221	109	22344321
20	00011100	50	01111111	80	12232210	110	12354321
21	00001110	51	11221100	81	12232111	111	22354321
22	00000111	52	11122100	82	11233210	112	13354321
23	11110000	53	11121110	83	11232211	113	23354321
24	10111000	54	11111111	84	11222221	114	22454321
25	01111000	55	01122110	85	12233210	115	23454321
26	01011100	56	01121111	86	12232211	116	23464321
27	00111100	57	11222100	87	11233211	117	23465321
28	00011110	58	11221110	88	11232221	118	23465421
29	00001111	59	11122110	89	12243210	119	23465431
30	11111000	60	11121111	90	12233211	120	23465432

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