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HARMONIC RADIAL COMBINATIONS AND DUAL MIXED VOLUMES*

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Abstract. For star bodies, p-harmonic radial combinations were introduced and studied in several papers. In this paper we study the relations of the dual quermassintegrals of p-harmonic radial combinations of star bodies to their dual quermassintegrals and obtain the upper bound for the dual quermassintegrals of p-harmonic radial combinations of star bodies.

For star bodies, p-harmonic radial combinations were introduced and studied in several papers. The aim of this article is to study them further, that is, we investigate the relations of the dual quermassintegrals of p-harmonic radial combinations of star bodies to their dual quermassintegrals and obtain the upper bound for the dual quermassintegrals of p-harmonic radial combinations of star bodies.

1. Preliminaries. By a *convex body* in $E^n, n \geq 2$, we mean a compact convex subset of E^n with nonempty interior. Let S^{n-1} denote the unit sphere centered at the origin in E^n, and write O_{n-1} for the $(n-1)$-dimensional volume of S^{n-1}. Let B be the closed unit ball in E^n, write ω_n for the n-dimensional volume of B. Note that

$$\omega_n = \pi^{n/2}/\Gamma(1+\frac{1}{n}), \qquad \text{and} \qquad O_{n-1} = n\omega_n.$$

For each direction $u \in S^{n-1}$, we define the support function $h(K,u)$ on S^{n-1} of the convex body K by

$$h(K,u) = \sup\{u \cdot x | x \in K\}$$

and the radial function $\rho(K,u)$ on S^{n-1} of the convex body K is

$$\rho(K,u) = \sup\{\lambda > 0 | \lambda u \in K\}.$$

If $\rho(K,u)$ is positive and continuous, call K a *star body* (about the origin), and write $\mathcal{S}$ for the set of star bodies (about the origin) of E^n. Sets A, B are called *homothetic* if $A = \lambda B + t$ with $t \in E^n$ and $\lambda > 0$ or one of them is a singleton (a one-point set).

The *polar body* of a convex body K, denoted by K^*, is another convex body defined by

$$K^* = \{y | x \cdot y \ \leq 1 \text{ for all } x \in K\}.$$

The polar body has the well known property that

$$h(K^*,u) = 1/\rho(K,u) \text{ and } \rho(K^*,u) = 1/h(K,u),$$

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Let K_j be a star body in E^n with $o \in K_j$, $1 \le j \le n$. Then we define the *dual mixed volumes* $\tilde{V}(K_1, \cdots, K_n)$ by

$$\tilde{V}(K_1, \cdots, K_n) = \frac{1}{n} \int_{S^{n-1}} \rho(K_1, u) \cdots \rho(K_n, u) du, \tag{1.1}$$

where du signifies the area element on S^{n-1}. Let

$$\tilde{V}_i(K_1, K_2) = \tilde{V}(\underbrace{K_1, \cdots, K_1}_{n-i}, \underbrace{K_2, \cdots, K_2}_{i}).$$

The *dual quermassintegrals* are the special dual mixed volumes defined by

$$\tilde{W}_i(K) = \tilde{V}_i(K, B).$$

Note that $\tilde{W}_0(K) = V(K)$ is the volume of K, while $\tilde{W}_n(K) = \omega_n$ does not depend on K.

2. Main Results. Fix a real $p \ge 1$. For $K, L \in \mathcal{S}$, and $\lambda, \mu \ge 0$ (not both zero), the *p-harmonic radial combination* $\lambda \cdot K \hat{+}_p \mu \cdot L \in \mathcal{S}$ is defined by

$$\rho(\lambda \cdot K \hat{+}_p \mu \cdot L, u)^{-p} = \lambda \rho(K, u)^{-p} + \mu \rho(L, u)^{-p}. \tag{2.1}$$

We obtain easily from the definition the following.

THEOREM 1 (Positive multisublinear). *Let $K, L \in \mathcal{S}$, a real $p \ge 1$, and $\lambda, \mu \ge 0$ (not both zero). Then, for any $M_2, \cdots, M_n \in \mathcal{S}$,*

$$\tilde{V}(\lambda \cdot K \hat{+}_p \mu \cdot L, M_2, \cdots, M_n) \le \lambda \tilde{V}(K, M_2, \cdots, M_n) + \mu \tilde{V}(L, M_2, \cdots, M_n)$$

where $\lambda + \mu = 1$.

Proof. In the definition of the p-harmonic radial combination, if we use the fact that $f(x) = x^{-\frac{1}{p}} (p \ge 1)$ is convex, then

$$\rho(\lambda \cdot K \hat{+}_p \mu \cdot L, u) \le \lambda \rho(K, u) + \mu \rho(L, u), \qquad \lambda + \mu = 1.$$

So using the definition of the dual mixed volume, we easily obtain

$$\begin{aligned} \tilde{V}(\lambda \cdot K \hat{+}_p \mu \cdot L, M_2, \cdots, M_n) &= \frac{1}{n} \int_{S^{n-1}} \rho(\lambda \cdot K \hat{+}_p \mu \cdot L, u) \rho(M_2, u) \cdots \rho(M_n, u) du \\ &\le \frac{1}{n} \int_{S^{n-1}} \Big(\lambda \rho(K, u) + \mu \rho(L, u)\Big) \rho(M_2, u) \cdots \rho(M_n, u) du \\ &= \lambda \tilde{V}(K, M_2, \cdots, M_n) + \mu \tilde{V}(L, M_2, \cdots, M_n). \qquad \square \end{aligned}$$

Now we consider the dual quermassintegrals of the p-harmonic radial combinations.

In the following theorem we obtain the upper bound for the dual quermassintegrals of the p-harmonic radial combinations of star bodies.

THEOREM 2. *Let $K, L \in \mathcal{S}$, a real $p \ge 1$, and $\lambda, \mu \ge 0$ (not both zero). Then, for the p-harmonic radial combination $\lambda \cdot K \hat{+}_p \mu \cdot L$*

$$\tilde{W}_i(\lambda \cdot K \hat{+}_p \mu \cdot L) \le \left(\frac{1}{2\sqrt{\lambda \mu}}\right)^{\frac{n-i}{p}} \frac{1}{n} \left(\int_{S^{n-1}} \rho(K, u)^{\frac{(n-i)s}{2}} du\right)^{\frac{1}{s}} \left(\int_{S^{n-1}} \rho(L, u)^{\frac{(n-i)t}{2}} du\right)^{\frac{1}{t}}$$

for $i = 0, 1, \cdots, n$ and $s > 1$ and $\frac{1}{s} + \frac{1}{t} = 1$. Equality holds if and only if both K and L are balls such that $\lambda^{\frac{1}{p}} L = \mu^{\frac{1}{p}} K$.

Proof From the definition of the p-harmonic radial combination, we have

$$\begin{aligned}\frac{1}{\rho(\lambda \cdot K \hat{+}_p \mu \cdot L, u)^p} &= \lambda \frac{1}{\rho(K,u)^p} + \mu \frac{1}{\rho(L,u)^p} \\ &= \frac{\lambda \rho(L,u)^p + \mu \rho(K,u)^p}{\rho(K,u)^p \rho(L,u)^p}.\end{aligned}$$

It follows from the inequality between arithmetic and geometric means that

$$\begin{aligned}\rho(\lambda \cdot K \hat{+}_p \mu \cdot L, u)^p &= \frac{\rho(K,u)^p \rho(L,u)^p}{\lambda \rho(L,u)^p + \mu \rho(K,u)^p} \\ &\leq \frac{\rho(K,u)^p \rho(L,u)^p}{2\sqrt{\lambda \rho(K,u)^p \mu \rho(L,u)^p}} \\ &= \frac{1}{2\sqrt{\lambda\mu}} \Big(\rho(K,u)\rho(L,u)\Big)^{\frac{p}{2}}.\end{aligned}$$

It also follows that

$$\rho(\lambda \cdot K \hat{+}_p \mu \cdot L, u)^{n-i} \leq \left(\frac{1}{2\sqrt{\lambda\mu}}\right)^{\frac{n-i}{p}} \Big(\rho(K,u)\rho(L,u)\Big)^{\frac{n-i}{2}}$$

and, using Hölder's inequality for integrals,

$$\begin{aligned}\tilde{W}_i(\lambda \cdot K \hat{+}_p \mu \cdot L) &= \frac{1}{n} \int_{S^{n-1}} \rho(\lambda \cdot K \hat{+}_p \mu \cdot L, u)^{n-i} du \\ &\leq \left(\frac{1}{2\sqrt{\lambda\mu}}\right)^{\frac{n-i}{p}} \frac{1}{n} \int_{S^{n-1}} \Big(\rho(K,u)\rho(L,u)\Big)^{\frac{n-i}{2}} du \\ &\leq \left(\frac{1}{2\sqrt{\lambda\mu}}\right)^{\frac{n-i}{p}} \frac{1}{n} \Big(\int_{S^{n-1}} \rho(K,u)^{\frac{(n-i)s}{2}} du\Big)^{\frac{1}{s}} \\ &\quad \times \Big(\int_{S^{n-1}} \rho(L,u)^{\frac{(n-i)t}{2}} du\Big)^{\frac{1}{t}}\end{aligned} \tag{2.2}$$

where $s > 1$ and $\frac{1}{s} + \frac{1}{t} = 1$. From the equality conditions of the arithmetic-geometric mean inequality and Hölder's inequality for integrals in last two inequality of (2.2), equality holds if and only if both K and L are balls such that $\lambda^{\frac{1}{p}} L = \mu^{\frac{1}{p}} K$. □

We obtain the relations of the dual quermassintegrals of p-harmonic radial combinations of star bodies to their dual quermassintegrals.

COROLLARY 1. *Let $K, L \in \mathcal{S}$, a real $p \geq 1$, and $\lambda, \mu \geq 0$ (not both zero). Then, for the p-harmonic radial combination $\lambda \cdot K \hat{+}_p \mu \cdot L$,*

$$\tilde{W}_i^2(\lambda \cdot K \hat{+}_p \mu \cdot L) \leq \left(\frac{1}{4\lambda\mu}\right)^{\frac{n-i}{p}} \tilde{W}_i(K)\tilde{W}_i(L).$$

Equality holds if and only if K is homothetic to L such that $\lambda^{\frac{1}{p}} L = \mu^{\frac{1}{p}} K$.

Proof. In Theorem 2 take $s = t = 2$. Then

$$\tilde{W}_i(\lambda \cdot K \hat{+}_p \mu \cdot L) \leq \left(\frac{1}{2\sqrt{\lambda\mu}}\right)^{\frac{n-i}{p}} \frac{1}{n} \left(\int_{S^{n-1}} \rho(K,u)^{n-i} du\right)^{\frac{1}{2}} \left(\int_{S^{n-1}} \rho(L,u)^{n-i} du\right)^{\frac{1}{2}}$$

which implies that

$$\begin{aligned}\tilde{W}_i^2(\lambda \cdot K \hat{+}_p \mu \cdot L) &\leq \left(\frac{1}{4\lambda\mu}\right)^{\frac{n-i}{p}} \frac{1}{n^2} \left(\int_{S^{n-1}} \rho(K,u)^{n-i} du\right) \left(\int_{S^{n-1}} \rho(L,u)^{n-i} du\right) \\ &= \left(\frac{1}{4\lambda\mu}\right)^{\frac{n-i}{p}} \tilde{W}_i(K)\tilde{W}_i(L).\end{aligned}$$

The equality condition follows from the equality cases of the arithmetic-geometric mean inequality and Hölder's inequality for $s = t = 2$. □

From Theorem 2 we also obtain the following.

COROLLARY 2. *Let $K, L \in \mathcal{S}$, a real $p \geq 1$, and $\lambda, \mu \geq 0$ (not both zero). Then*

$$V(\lambda \cdot K \hat{+}_p \mu \cdot L) \leq \left(\frac{1}{2\sqrt{\lambda\mu}}\right)^{\frac{n}{p}} \frac{1}{n} \int_{S^{n-1}} \Big(\rho(K,u)\rho(L,u)\Big)^{\frac{n}{2}} du.$$

Equality holds if and only if K is homothetic to L such that $\lambda^{\frac{1}{p}} L = \mu^{\frac{1}{p}} K$.

Proof. Take $i = 0$ in (2.2). The equality condition follows from the equality case of the arithmetic-geometric mean inequality. □

We also obtain the upper bound for the dual quermassintegrals of any star body and its polar body. K is homothetic to K^* if and only if K is a ball. So we obtain the following.

COROLLARY 3. *Let $K \in \mathcal{S}$, a real $p \geq 1$, and $\lambda, \mu \geq 0$ (not both zero) and K^* the polar body of K. Then*

$$\tilde{W}_i(\lambda \cdot K \hat{+}_p \mu \cdot K^*) \leq \left(\frac{1}{2\sqrt{\lambda\mu}}\right)^{\frac{n-i}{p}} \omega_n$$

for $i = 0, 1, \cdots, n$, with equality if and only if K is a ball.

Proof. If we take $L = K^*$ in (2.2) and use $\rho(K^*, u) = 1/h(K, u)$, then we obtain

$$\begin{aligned}\tilde{W}_i(\lambda \cdot K \hat{+}_p \mu \cdot K^*) &\leq \left(\frac{1}{2\sqrt{\lambda\mu}}\right)^{\frac{n-i}{p}} \frac{1}{n} \int_{S^{n-1}} \Big(\rho(K,u)\rho(K^*,u)\Big)^{\frac{n-i}{2}} du \\ &= \left(\frac{1}{2\sqrt{\lambda\mu}}\right)^{\frac{n-i}{p}} \frac{1}{n} \int_{S^{n-1}} \left(\frac{rho(K,u)}{h(K,u)}\right)^{\frac{n-i}{2}} du \\ &\leq \left(\frac{1}{2\sqrt{\lambda\mu}}\right)^{\frac{n-i}{p}} \frac{1}{n} \int_{S^{n-1}} du \\ &= \left(\frac{1}{2\sqrt{\lambda\mu}}\right)^{\frac{n-i}{p}} \omega_n.\end{aligned}$$

The second inequality follows since $h(K, u) \geq \rho(K, u)$. □

COROLLARY 4. *Let $K \in \mathcal{S}$, a real $p \geq 1$, and $\lambda, \mu \geq 0$ (not both zero). Then*

$$V(\lambda \cdot K \hat{+}_p \mu \cdot K^*) \leq \left(\frac{1}{2\sqrt{\lambda\mu}}\right)^{\frac{n}{p}} \omega_n,$$

with equality if and only if K is a ball.

Proof. Take $i = 0$ in Corollary 3. □

We obtain the relations of the dual quermassintegrals of p-harmonic radial combinations of star bodies to their dual quermassintegrals.

THEOREM 3. *Let $K, L \in \mathcal{S}$ and $\lambda, \mu \geq 0$ (not both zero). Then, for the p-harmonic radial combination $\lambda \cdot K \hat{+}_p \mu \cdot L$,*

$$\tilde{W}_i(\lambda \cdot K \hat{+}_p \mu \cdot L) \leq \lambda \tilde{W}_i(K) + \mu \tilde{W}_i(L), \qquad \lambda + \mu = 1$$

for $p > 0$ and $0 < i < n$, with equality if and only if $K = L$.

Proof. From the definition of the p-harmonic radial combination,

$$\rho(\lambda \cdot K \hat{+}_p \mu \cdot L, u)^{-p} = \lambda\rho(K, u)^{-p} + \mu\rho(L, u)^{-p}.$$

It follows that

$$\rho(\lambda \cdot K \hat{+}_p \mu \cdot L, u)^{n-i} \leq \lambda\rho(K, u)^{n-i} + \mu\rho(L, u)^{n-i}, \quad \lambda + \mu = 1.$$

The inequality above follows from the Hölder inequality.

Therefore

$$\begin{aligned}
\tilde{W}_i(\lambda \cdot K \hat{+}_p \mu \cdot L) &= \frac{1}{n} \int_{S^{n-1}} \rho(\lambda \cdot K \hat{+}_p \mu \cdot L, u)^{n-i} du \\
&\leq \frac{1}{n} \int_{S^{n-1}} \left(\lambda\rho(K, u)^{n-i} + \mu\rho(L, u)^{n-i}\right) du \\
&= \frac{\lambda}{n} \int_{S^{n-1}} \rho(K, u)^{n-i} du + \frac{\mu}{n} \int_{S^{n-1}} \rho(L, u)^{n-i} du \\
&= \lambda \tilde{W}_i(K) + \mu \tilde{W}_i(L). \quad \square
\end{aligned}$$

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REFERENCES

[1] R. GARDNER, *Geometric Tomography*, Cambridge Univ. Press, Cambridge, 1995.

[2] E. LUTWAK, *Centroid bodies and dual mixed volumes*, Proc. London Math. Soc., 60 (1990), pp. 365–391.

[3] E. LUTWAK AND G. ZHANG, *Blaschke Santaló inequalities*, J. Differential Geom., 45 (1997), pp. 1–16.

[4] R. SCHNEIDER, *Convex Bodies: The Brunn-Minkowski Theory*, Cambridge Univ. Press, Cambridge, 1993.

[5] G. ZHANG, *Centered bodies and dual mixed volumes*, Trans. Amer. Soc., 345 (1994), pp. 777–801.