ASIAN J. MATH.
Vol. 5, No. 3, pp. 433–440, September 2001

REGULARITY OF BUTTERWORTH REFINABLE FUNCTIONS*

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Abstract. Let Ψ_N be the refinable function with Butterworth filter $\cos^{2N}\frac{\xi}{2}(\cos^{2N}\frac{\xi}{2}+\sin^{2N}\frac{\xi}{2})^{-1}$ and let $s_p(\Psi_N)$ be the Fourier exponent of Ψ_N of order p $(0<p\le\infty)$. It is proved that

$$0 \le s_\infty(\Psi_N) - N\frac{\ln 3}{\ln 2} \le \frac{\ln(1+3^{-N})}{\ln 2} \qquad (N\ge 1)$$

and for $0<p<\infty$

$$-\frac{\ln(1+r_0^{Np})}{p\ln 2} \le s_p(\Psi_N) - N\frac{\ln 3}{\ln 2} \le \frac{\ln(1+3^{-N})}{\ln 2} \qquad (N\ge 1)$$

where $r_0\in(0,1)$ is independent of p and N.

1. Introduction and Result. In this paper we study the solutions of some *refinement equations* of the form

$$\phi(x) = \sum_{j\in\mathbb{Z}} c_j\phi(2x-j) \qquad (x\in\mathbb{R}) \tag{1.1}$$

where the coefficients c_j are supposed to satisfy the arithmetic condition $\sum_{j\in\mathbb{Z}} c_j = 2$ and the exponential decay condition $|c_j|\le Ce^{-\beta|j|}$ $(C,\beta>0$ constants). Solutions of a refinement equation are called *refinable functions.* The 2π-periodic function

$$m(\xi) = \frac{1}{2}\sum_{j\in\mathbb{Z}} c_j e^{-ij\xi}$$

is called the *filter* of the refinement equation (1.1). A continuous function ϕ is called a *cardinal interpolant* if $\phi(0)=1$ and $\phi(k)=0$ for all nonzero integer k. It is known that there is an important class of refinable functions which are cardinal interpolants and whose filters satisfy

$$m(\xi)+m(\xi+\pi)=1. \tag{1.2}$$

Such a filter $m(\xi)$ can be put into the factorized form

$$m(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^N R(\xi) \tag{1.3}$$

where N is a strictly positive integer and $R(\xi)$ is a 2π-periodic function whose Fourier coefficients decay exponentially. The minimal degree solution of (1.2) having the factorized form (1.3) is given by

$$m_N(\xi) = \cos^{2N}\frac{\xi}{2}\sum_{s=0}^{N-1}\binom{N-1+s}{s}\sin^{2s}\frac{\xi}{2}.$$

*Received April 8, 1999; accepted for publication June 19, 2000.

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The corresponding refinable functions, denoted by Φ_N, are the self-convolution of Daubechies' scaling functions, and they are cardinal interpolants (see [3, 4, 5]). We will study the solution of the equation (1.2) whose filter has a simpler factorized form (1.3) given by

$$\tilde{m}_N(\xi) = \cos^{2N}\frac{\xi}{2}\left(\cos^{2N}\frac{\xi}{2} + \sin^{2N}\frac{\xi}{2}\right)^{-1}. \tag{1.4}$$

These filters are well known in signal processing as the transfer functions of the "Butterworth filter" (see [8] for a detailed review). The corresponding refinable functions, denoted by Ψ_N, are said to be *Butterworth refinable functions*, which are also cardinal interpolants. Denote by $\hat{f}$ the Fourier transform of an integrable function or a tempered distribution f. In the form of Fourier transform, the equation (1.1) becomes $\widehat{\phi}(\xi) = m(\xi/2)\widehat{\phi}(\xi/2)$. Hence we get the useful formula

$$\widehat{\Psi}_N(\xi) = \left(\frac{\sin\xi/2}{\xi/2}\right)^{2N}\prod_{n=1}^{\infty}\left(\cos^{2N}2^{-n-1}\xi + \sin^{2N}2^{-n-1}\xi\right)^{-1}. \tag{1.5}$$

The aim of this paper is to study the regularity of Ψ_N. The regularity of refinable functions is of central importance in the theory of wavelets. A usual approach is to study the Fourier exponents, which are also called Sobolev exponents in the literature. For a tempered distribution f with measurable Fourier transform, define its Fourier exponents $s_p(f)$ by

$$s_p(f) = \sup\left\{s:\ \int_{\mathbb{R}}|\hat{f}(\xi)|^p(1+|\xi|)^{ps}d\xi < \infty\right\} \qquad (0<p<\infty)$$
$$s_\infty(f) = \sup\left\{s:\ \hat{f}(\xi)(1+|\xi|)^s = O(1) \quad |\xi|\to\infty\right\}.$$

In [1] , Cohen and Daubechies studied the regularity of refinable functions Ψ_N and gave some numerical results on the Fourier exponents $s_p(\Psi_N)$ for $p = 1/2, 1, 2, 4$ and $N = 1, 2, \cdots, 19$. They noticed that for large value of N the Fourier exponent $s_p(\Psi_N)$ reveals a linear asymptotic behavior and the limit ratio $s_p(\Psi_N)/N$ indicates that the worst decay of $\widehat{\Psi}_N$ occurs at the points $2^{j+1}\pi/3$. In this paper, we confirm the above observation by proving

THEOREM 1. *Let Ψ_N be defined as above. Then*

$$0 \le s_\infty(\Psi_N) - \frac{N\ln 3}{\ln 2} \le \frac{\ln(1+3^{-N})}{\ln 2}$$

for all $N \ge 1$, and

$$-\frac{\ln(1+r_0^{Np})}{p\ln 2} \le s_p(\Psi_N) - \frac{N\ln 3}{\ln 2} \le \frac{\ln(1+3^{-N})}{\ln 2}$$

for all $N \ge 1$ and $0 < p < \infty$, where $r_0 \in (0,1)$ is a constant independent of p and N.

As a consequence of Theorem 1, we have

COROLLARY 1. *Let Ψ_N be defined as above. Then*

$$\lim_{N\to\infty}\frac{s_p(\Psi_N)}{N} = \frac{\ln 3}{\ln 2} \quad (0 < p \le \infty)$$

and

$$\lim_{N\to\infty} (s_p(\Psi_N) - s_q(\Psi_N)) = 0 \quad (0 < p, q \leq \infty).$$

2. Proof. To get the lower bound estimate of $s_p(\Psi_N)$, we introduce an auxiliary π-periodic even function defined by

$$h(\xi) = \max\{|\cos\xi/2|, |\sin\xi/2|\}. \tag{2.1}$$

It is clear that $h(\xi) = \cos\xi/2$ if $|\xi| \leq \pi/2$ and $h(\xi) = |\sin\xi/2|$ if $\pi/2 \leq |\xi| \leq \pi$. Furthermore, we have

LEMMA 1. *Let $h(\xi)$ be the function defined by* (2.1). *Then*

$$\begin{cases} h(\xi) \geq \dfrac{\sqrt{3}}{2}, & \xi \in [-\dfrac{\pi}{3}, \dfrac{\pi}{3}] + \pi\mathbb{Z}, \\ h(\xi)h(2\xi) \geq \dfrac{3}{4}, & \xi \in ([-\frac{5\pi}{12}, -\frac{\pi}{3}] \cup [\frac{\pi}{3}, \frac{5\pi}{12}]) + \pi\mathbb{Z}, \\ h(\xi)h(2\xi)h(4\xi) > (\dfrac{\sqrt{3}}{2})^3, & \xi \in ([-\dfrac{\pi}{2}, -\dfrac{5\pi}{12}] \cup [\dfrac{5\pi}{12}, \dfrac{\pi}{2}]) + \pi\mathbb{Z}. \end{cases} \tag{2.2}$$

Proof. For simplicity, we write $H_2(\xi) = h(\xi)h(2\xi)$ and $H_3(\xi) = h(\xi)h(2\xi)h(4\xi)$. Since h is an even function with period π, it suffices to prove (2.2) for $\xi \in [0, \pi/2]$. The first inequality of (2.2) follows from the facts that $h(\xi)$ decreases on $[0, \pi/2]$ and that $h(\pi/3) = \sqrt{3}/2$.

Let $t = \cos^2\xi/2$. By a simple calculation, we obtain that

$$H_2(\xi)^2 = \cos^2\frac{\xi}{2}\sin^2\xi = 4t^2(1-t) \tag{2.3}$$

and that $t \in [(2 + \sqrt{2-\sqrt{3}})/4, 3/4]$ for any $\xi \in [\pi/3, 5\pi/12]$. Observe that

$$\frac{d}{dt}(t^2(1-t)) = 3t(2/3 - t).$$

This, together with (2.3), implies that $H_2(\xi)$ increases on the interval $\left[\dfrac{\pi}{3}, 2\arccos\sqrt{\dfrac{2}{3}}\right]$ and decreases on the interval $[2\arccos\sqrt{2/3}, 5\pi/12]$. Thus,

$$H_2(\xi) \geq \min\{H_2(\pi/3), H_2(5\pi/12)\} = H_2(\pi/3) = 3/4, \quad \forall\, \xi \in [\pi/3, 5\pi/12].$$

It is the second inequality of (2.2).

If $\xi \in [5\pi/12, \pi/2]$, we have $2\xi \in [5\pi/6, \pi]$ and $4\xi \in [5\pi/3, 2\pi] = [-\pi/3, 0] + 2\pi$. Therefore

$$H_3(\xi)^2 = \cos^2\frac{\xi}{2}\sin^2\xi\cos^2 2\xi = 4t^2(1-t)(8t^2 - 8t + 1)^2$$

where $t = \cos^2\xi/2 \in [1/2, (2 + \sqrt{2-\sqrt{3}})/4]$. Let

$$g_1(t) = t^2(1-t)(8t^2 - 8t + 1)^2, \qquad g_2(t) = 56t^3 - 88t^2 + 35t - 2.$$

Notice that

$$\frac{d}{dt}g_1(t) = t(-8t^2+8t-1)g_2(t), \qquad \frac{d}{dt}g_2(t) = 168t^2 - 176t + 35.$$

It follows that $\frac{d}{dt}g_2(t) < 0$ on $[1/2, (2+\sqrt{2-\sqrt{3}})/4]$. On the other hand, $g_2(1/2) = 1/2 > 0$ and

$$g_2((2+\sqrt{2-\sqrt{3}})/4) \le g_2(5/8) = -53/64 < 0.$$

Therefore there exists $t_0 \in [1/2, (2+\sqrt{2-\sqrt{3}})/4]$ such that $g_2(t) > 0$ on $[1/2, t_0]$ and $g_2(t) < 0$ on $[t_0, (2+\sqrt{2-\sqrt{3}})/4]$. Observe that $-8t^2+8t-1 = -\cos 2\xi > 0$. Thus $H_3(\xi)$ increases on $[5\pi/12, 2\arccos\sqrt{t_0}]$ and decreases on $[2\arccos\sqrt{t_0}, \pi/2]$. Hence

$$H_3(\xi) \ge \min\{H_3(5\pi/12), H_3(\pi/2)\} = H_3(5\pi/12) > (\sqrt{3}/2)^3.$$

Thus we have proved the third inequality of (2.2). □

For $N \ge 1$, let

$$R_N(\xi) = \left(\cos^{2N}\frac{\xi}{2} + \sin^{2N}\frac{\xi}{2}\right)^{-1}. \tag{2.4}$$

Clearly R_N is a π-periodic function and

$$\tilde{m}_N(\xi) = \cos^{2N}\frac{\xi}{2} R_N(\xi)$$

(see (1.4)). Note that $R_N(\xi) \le h(\xi)^{-2N}$. Therefore, by Lemma 1 and the strict monotonicity of $h(\xi), h(\xi)h(2\xi), h(\xi)h(2\xi)h(4\xi)$ on their respective intervals, we have

LEMMA 2. *Let R_N be defined as above and let $q = (4/3)^N$. Then for any $0 < \delta \le \frac{\pi}{24}$, there exists $0 < r = r(\delta) < 1$ such that*

$$\begin{cases} R_N(\xi) \le q, & \xi \in [-\frac{\pi}{3}, \frac{\pi}{3}] + \pi\mathbb{Z} \\ R_N(\xi)R_N(2\xi) \le q^2, & \xi \in ([-\frac{5\pi}{12}, -\frac{\pi}{3}) \cup (\frac{\pi}{3}, \frac{5\pi}{12}]) + \pi\mathbb{Z} \\ R_N(\xi)R_N(2\xi)R_N(4\xi) \le q^3, & \xi \in ([-\frac{\pi}{2}, -\frac{5\pi}{12}) \cup (\frac{5\pi}{12}, \frac{\pi}{2}]) + \pi\mathbb{Z} \end{cases}$$

and

$$\begin{cases} R_N(\xi) \le r^N q, & \xi \in [-\frac{\pi}{3}+\delta, \frac{\pi}{3}-\delta] + \pi\mathbb{Z} \\ R_N(\xi)R_N(2\xi) \le r^N q^2, & \xi \in ([-\frac{5\pi}{12}, -\frac{\pi}{3}-\delta] \cup [\frac{\pi}{3}+\delta, \frac{5\pi}{12}]) + \pi\mathbb{Z} \\ R_N(\xi)R_N(2\xi)R_N(4\xi) \le r^{2N} q^3, & \xi \in ([-\frac{\pi}{2}, -\frac{5\pi}{12}) \cup (\frac{5\pi}{12}, \frac{\pi}{2}]) + \pi\mathbb{Z}. \end{cases}$$

In particular, r can be chosen as

$$\max\left\{\frac{3}{4}\left(h\left(\frac{\pi}{3}-\delta\right)\right)^{-2}, \left(\frac{3}{4}\right)^2\left(H_2\left(\frac{\pi}{3}+\delta\right)\right)^{-2}, \left(\frac{3}{4}\right)^{3/2}\left(H_3\left(\frac{5\pi}{12}\right)\right)^{-2}\right\}.$$

Define

$$I_k(\xi) = \{j:\ 1 \le j \le k, 2^j\xi \in \cup_{m\in\mathbb{Z}}[-\pi/4, \pi/4] + m\pi\}$$

and let $i_k(\xi)$ be the cardinality of the set $I_k(\xi)$.

LEMMA 3. *Let R_N be defined as above and let $q = (4/3)^N$. Then there exists a positive constant C_N such that for any $k \geq 1$*

$$\prod_{j=1}^{k} R_N(2^j\xi) \leq C_N r_0^{N i_k(\xi)} q^k \tag{2.5}$$

where $r_0 = r(\pi/24)$ is defined in Lemma 2.

Proof. The idea of proof was used in [7]. It is clear that the assertion in Lemma 3 holds for $k = 1, 2, 3$ if C_N is chosen large enough. We assume that (2.5) holds for all $k < l$ with $l \geq 3$. For $k = l$, we distinguish five cases.

(i) If $2\xi \in [-\pi/4, \pi/4] + \pi\mathbb{Z}$, then $i_k(\xi) = i_{k-1}(2\xi) + 1$. Write

$$\prod_{j=1}^{k} R_N(2^j\xi) = R_N(2\xi) \prod_{j=1}^{k-1} R_N(2^j(2\xi)).$$

Thus (2.5) holds by using Lemma 2 and the induction hypothesis.

(ii) If 2ξ or $-2\xi \in (\pi/4, \pi/3] + \pi\mathbb{Z}$, then $i_k(\xi) = i_{k-1}(2\xi)$. Again the induction hypothesis together with Lemma 2 implies (2.5).

(iii) If 2ξ or $-2\xi \in (\pi/3, 3\pi/8) + \pi\mathbb{Z}$, then $i_k(\xi) = i_{k-2}(4\xi)$. It suffices to write

$$\prod_{j=1}^{k} R_N(2^j\xi) = R_N(2\xi)R_N(4\xi) \prod_{j=1}^{k-2} R_N(2^j(4\xi))$$

and then to apply Lemma 2 and the induction hypothesis.

(iv) If 2ξ or $-2\xi \in [3\pi/8, 5\pi/12] + \pi\mathbb{Z}$, then $i_k(\xi) \leq i_{k-2}(4\xi) + 1$. By using the induction hypothesis and Lemma 2, we have

$$\begin{aligned}\prod_{j=1}^{k} R_N(2^j\xi) &\leq r^N q^2 \left[C_N r^{N\ i_{k-2}(4\xi)} q^{k-2}\right] \\ &\leq C_N r^{N i_k(\xi)} q^k.\end{aligned}$$

(v) If 2ξ or $-2\xi \in (5\pi/12, \pi/2] + \pi\mathbb{Z}$, then $i_k(\xi) \leq i_{k-3}(8\xi) + 2$. Hence

$$\begin{aligned}\prod_{j=1}^{k} R_N(2^j\xi) &= R_N(2\xi)R_N(4\xi)R_N(8\xi) \prod_{j=1}^{k-3} R_N(2^j(8\xi)) \\ &\leq r^{2N} q^3 \left[C_N r^{N i_{k-3}(8\xi)} q^{k-3}\right] \\ &\leq C_N r^{N i_k(\xi)} q^k.\end{aligned}$$

□

Let $k \geq 2$. For $(\epsilon_1, \cdots, \epsilon_k) \in \{0,1\}^k$, let

$$Q(\epsilon_1, \cdots, \epsilon_k) = \{i : \ \epsilon_i = \epsilon_{i+1}\}$$

and $q(\epsilon_1, \cdots, \epsilon_k)$ be the cardinality of the set $Q(\epsilon_1, \cdots, \epsilon_k)$. For $0 \leq q \leq k-1$, let

$$G_{q,k} = \{(\epsilon_1, \cdots, \epsilon_k) \in \{0,1\}^k : \ q(\epsilon_1, \cdots, \epsilon_k) = q\}.$$

Then, for any $(\epsilon_1, \cdots, \epsilon_k) \in G_{q,k}$ there exist unique integers $1 \le i_1 < i_2 < \ldots < i_q \le k-1$ such that $\epsilon_{i_s} = \epsilon_{i_s+1}$ for all $1 \le s \le q$. On the other hand, given any $\epsilon_1 \in \{0,1\}$ and integers $1 \le i_1 < i_2 < \ldots < i_q \le k-1$, we may find one and only one $(\epsilon_1, \ldots, \epsilon_k) \in G_{q,k}$ such that $\epsilon_{i_s} = \epsilon_{i_s+1}$ for any $1 \le s \le q$. Therefore, the cardinality of $G_{q,k}$ is $2\binom{k-1}{q}$ for any $0 \le q \le k-1$.

LEMMA 4. *Let $k \ge 2, \xi \in [0,\pi)$ and let $i_k(\xi)$ and $q(\epsilon_1, \ldots, \epsilon_k)$ be defined as above. Write $\xi/\pi = \sum_{i=1}^k \epsilon_i 2^{-i} + \eta$ with $0 \le \eta < 2^{-k}$ and $\epsilon_i \in \{0,1\}$ for $1 \le i \le k$. Then $i_k(\xi) \ge q(\epsilon_1, \cdots, \epsilon_k) - 1$.*

Proof. For any $i \in Q(\epsilon_1, \cdots, \epsilon_k)$ and $i \ge 2$, we have $\epsilon_i = \epsilon_{i+1}$ and

$$2^{i-1}\xi = \frac{3}{4}\epsilon_i\pi + \eta'\pi + m\pi$$

with $0 \le \eta' < \frac{1}{4}$ and $m \in \mathbb{Z}$. Therefore $2^{i-1}\xi \in [0, \pi/4] + \pi\mathbb{Z}$ if $\epsilon_i = 0$ and $2^{i-1}\xi \in [-\pi/4, 0] + \pi\mathbb{Z}$ if $\epsilon_i = 1$. This implies that $i-1 \in I_k(\xi)$. Thus $i_k(\xi) \ge q(\epsilon_1, \cdots, \epsilon_k) - 1$. □

Proof of Theorem 1. The upper bound estimate of $s_p(\Psi_N)$ will be proved by a modification of the method used in [2]. (The method is also used in [7]). By (1.5) and $R_N(2\pi/3) = 2^{2N}(1+3^N)^{-1}$, we have $\widehat{\Psi}_N(2\pi/3) \ne 0$ and

$$\widehat{\Psi}_N(2^{k+1}\pi/3) = (1+3^N)^{-k}\widehat{\Psi}_N(2\pi/3) \quad \forall\, k \ge 1.$$

This implies that $s_\infty(\Psi_N) \le \ln(1+3^N)/\ln 2$.

By the continuity of $\widehat{\Psi}_N$ and R_N, for any $\epsilon > 0$ there exists $0 < \delta < 1$ such that for all $\xi \in [-\delta, \delta]$ we have

$$|R_N(2\pi/3 + \xi)| = |R_N(-2\pi/3 - \xi)| \ge (1-\epsilon)2^{2N}(1+3^N)^{-1}$$

and

$$|\widehat{\Psi}_N(2\pi/3 + \xi)| \ge (1-\epsilon)|\widehat{\Psi}_N(2\pi/3)| > 0.$$

This together with (1.5) implies that for all $\xi \in [-\delta, \delta]$ and $k \ge 1$,

$$\begin{aligned}\widehat{\Psi}_N(2^{k+1}\pi/3 + \xi) &= \prod_{j=1}^k \tilde{m}_N(2^{k-j+1}\pi/3 + 2^{-j}\xi)\widehat{\Psi}_N(2\pi/3 + 2^{-k}\xi) \\ &\ge C(1+3^N)^{-k}(1-\epsilon)^k\end{aligned}$$

where C is a positive constant independent of k. Therefore for $0 < p < \infty$ and $k \ge 1$, we have

$$\int_{2^{k-1}\pi/3+1}^{2^{k+1}\pi/3+1} |\widehat{\Psi}_N(\xi)|^p d\xi \ge C_1 \int_{-\delta}^{\delta} |\widehat{\Psi}_N(2^{k+1}\pi/3 + \xi)|^p d\xi \ge C_2\delta(1+3^N)^{-kp}(1-\epsilon)^{kp}$$

where C_1 and C_2 are positive constants independent of k. This gives the desired upper bound estimate of $s_p(\Psi_N)$ for $0 < p < \infty$.

For $k \ge 1$ and $2^{k-1}\pi \le |\xi| \le 2^k\pi$, it follows from (1.5) and Lemma 3 that

$$|\widehat{\Psi}_N(\xi)| \le C_1 \prod_{j=1}^k |\tilde{m}_N(2^{-j}\xi)| \le C_2|\xi|^{-2N} \prod_{j=1}^k |R_N(2^j(2^{-k}\xi))| \le C_3 3^{-Nk}$$

where C_1, C_2 and C_3 are positive constants independent of k. This leads to the desired lower bound estimate of $s_\infty(\Psi_N)$.

Let $r_0 = r(\pi/24)$. Then for any $k \geq 1$ and $0 < p < \infty$, there exist positive constants C_i $(1 \leq i \leq 4)$ independent of k such that

$$\int_{2^{k-1}\pi \leq |\xi| \leq 2^k\pi} |\widehat{\Psi}_N(\xi)|^p d\xi = 2\int_{2^{k-1}\pi}^{2^k\pi} |\widehat{\Psi}_N(\xi)|^p d\xi$$

$$\leq C_1 3^{-kNp} \int_{2^{k-1}\pi}^{2^k\pi} r_0^{pNi_k(2^{-k}\xi)} d\xi$$

$$\leq C_2 3^{-kNp} \sum_{(\epsilon_1,\cdots,\epsilon_k)\in\{0,1\}^k} \int_{\sum_{j=1}^k 2^{k-j}\epsilon_j\pi}^{\sum_{j=1}^k 2^{k-j}\epsilon_j\pi+\pi} r_0^{Npq(\epsilon_1,\cdots,\epsilon_k)} d\xi$$

$$\leq C_3 3^{-kNp} \sum_{q=0}^{k-1} r_0^{Npq} \sum_{q(\epsilon_1,\cdots,\epsilon_k)=q} 1$$

$$\leq C_4 3^{-kNp}(1 + r_0^{Np})^k$$

where we have used (5) and Lemma 3 in the first inequality, Lemma 4 in the second one, the fact that the cardinality of $G_{q,k}$ is $2\binom{k-1}{q}$ in the last one. Hence we obtain the desired lower bound estimate of $s_p(\Psi_N)$ for $0 < p < \infty$. □

3. Remarks. From the above proof, we see that r_0 in the theorem can be chosen to be 0.9787028. When N is large, $s_p(\Psi_N)$ is well approximated by $N \ln 3/\ln 2$. Let us compare the numerical results obtained in [1] for $p = 1/2, 1, 4$ and the approximation given by $N\frac{\ln 3}{\ln 2}$ (see Table 1). We point out that the differences between the last two columns are small and that when $N \geq 20$ we can use $N \log_2 3$ to get rather precise approximation for $s_p(\Psi_N)$.

N	$p = \frac{1}{2}$	$p = 1$	$p = 4$	$\frac{1}{2}N \ln 3/\ln 2$
2	0.677350	1.256211	1.604344	1.584963
3	1.561362	2.044109	2.365870	2.377444
4	2.370365	2.843768	3.148599	3.169925
5	3.183890	3.648646	3.940563	3.962406
6	3.999055	4.456118	4.735925	4.754888
7	4.815040	5.264533	5.532265	5.547369
8	5.630616	6.072947	6.328326	6.339850
9	6.446191	6.881125	7.123827	7.132331
10	7.260947	7.688598	7.918627	7.924813
11	8.075292	8.495600	8.712863	8.717294
12	8.888817	9.301894	9.506534	9.509775
13	9.701520	10.107480	10.299921	10.302256
14	10.513813	10.912358	11.093166	11.094738
15	11.325284	11.716526	11.885986	11.887218
16	12.135933	12.519984	12.678805	12.679700
17	12.946170	13.322968	13.471625	13.472181
18	13.755996	14.125241	14.264159	14.264662
19	14.564999	14.927039	15.056836	15.057144

Table 1: Fourier exponent $s_{2p}(\tilde{\Psi}_N) = \frac{1}{2}s_p(\Psi_N)$ and its approximation $\frac{1}{2}N \log_2 3$

Let $\tilde{\Psi}_N$ be the refinable function with corresponding filter $\left(\frac{1+e^{-i\xi}}{2}\right)^N \left(\cos^{2N}\xi/2+\sin^{2N}\xi/2\right)^{-1/2}$. Then $\widehat{\Psi}_N(\xi) = |\widehat{\tilde{\Psi}_N}(\xi)|^2$ and $s_p(\Psi_N) = 2s_{p/2}(\tilde{\Psi}_N)$. In fact, the original numerical results in [1] is about the Fourier exponents $s_p(\tilde{\Psi}_N)$ with $p = 1, 2, 4, 8$ and $N = 1, 2, \cdots, 19$.

For the Daubechies scaling functions Φ_N, there are many papers devoted to the estimates of $s_p(\Phi_N)$ (see [1, 6, 7, 9] and references therein). In [7], Lau and Sun proved that

$$-\frac{C}{N} \leq s_p(\Phi_N) - 2N + \frac{\ln P_N(3/4)}{\ln 2} \leq 0$$

for $0 < p < \infty$ and

$$s_\infty(\Phi_N) = 2N - \frac{\ln P_N(3/4)}{\ln 2}$$

where C is a positive constant independent of N and

$$P_N(t) = \sum_{s=0}^{N-1} \binom{N+k-1}{k} t^s.$$

By the idea we used in the proof of the theorem, the term $-\frac{C}{N}$ in the above lower estimate can be improved to be $-Cr_0^N$ for some $0 < r_0 < 1$.

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