WILLMORE HYPERSURFACES IN A SPHERE*

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Abstract. Let $x : M \to S^{n+1}$ be an n-dimensional hypersurface in S^{n+1}, $x : M \to S^{n+1}$ is called a Willmore hypersurface if it is an extremal hypersurface to the following Willmore functional:

$$\int_M (S - nH^2)^{\frac{n}{2}} dv,$$

where $S = \sum_{i,j}(h_{ij})^2$ is the square of the length of the second fundamental form, H is the mean curvature of M. In this paper, through study of the Euler-Lagrange equation of the Willmore functional, we obtain an integral inequality of Simons' type for Willmore hypersurfaces in S^{n+1} and give a characterization of *Willmore tori* by use of our integral formula. We also classify all isoparametric Willmore hypersurfaces in S^{n+1}.

1. Introduction. Let M be an n-dimensional compact hypersurface of the $(n+1)$-dimensional unit sphere S^{n+1}. If h_{ij} denotes the components of the second fundamental form of M, S denotes the square of the length of the second fundamental form and H denotes the mean curvature, then we have

$$S = \sum_{i,j}(h_{ij})^2, \quad \mathbf{H} = \frac{1}{n}\sum_k h_{kk}e_{n+1}, \quad H = |\mathbf{H}|,$$

where e_{n+1} is an unit normal vector field of M in S^{n+1}.

We define the following non-negative function on M

$$\rho^2 = S - nH^2, \tag{1.1}$$

which vanishes exactly at the umbilic points of M.

Willmore functional is the following non-negative functional (see [4], [20] or [22])

$$\int_M \rho^n dv = \int_M (S - nH^2)^{\frac{n}{2}} dv.$$

It was shown in [4] and [20] that this functional is an invariant under Moebius (or conformal) transformations of S^{n+1}. We use the term Willmore hypersurfaces to call its critical points, because when $n = 2$, the functional essentially coincides with the well-known Willmore functional and its critical points are the Willmore surfaces.

In this paper, we first prove the following theorem

THEOREM 1. *Let M be an n-dimensional hypersurface in an $(n+1)$-dimensional unit sphere S^{n+1}. Then M is a Willmore hypersurface if and only if*

$$\begin{aligned} &-\rho^{n-2}(2HS - nH^3 - \sum_{i,j,k} h_{ij}h_{jk}h_{ki}) \\ &+(n-1)\Delta(\rho^{n-2}H) - \sum_{i,j}(\rho^{n-2})_{,ij}(nH\delta_{ij} - h_{ij}) = 0, \end{aligned} \tag{1.2}$$

*Received May 3, 2000; accepted for publication May 13, 2000.

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where Δ is the Laplacian, $(.)_{,ij}$ is the covariant derivative relative to the induced metric.

Remark 1.1. When $n = 2$, Theorem 1 was proved by R. Bryant [2] and J. Weiner [21], (1.2) reduces to the following well-known equation of Willmore surfaces (see [2,21])

$$\Delta H + H(S - 2H^2) = 0. \tag{1.3}$$

We note that Peter Li and S.-T. Yau [11] introduced a concept of conformal volume and obtained a partial solution of Willmore conjecture through estimating the eigenvalues of the Laplacian. We also note that Pinkall [14] constructed some compact non-minimal Willmore surfaces in S^3.

Remark 1.2. We should note that for $n \geq 2$, C. P. Wang [20] got the Euler-Lagrange equation of Willmore functional for any n-dimensional submanifold in an $(n+p)$-dimensional unit sphere S^{n+p} in terms of Moebius geometry.

In order to state our Theorem 3, we first give the following important example

EXAMPLE (C.F.[6]). The tori

$$W_{m,n-m} = S^m\left(\sqrt{\frac{n-m}{n}}\right) \times S^{n-m}\left(\sqrt{\frac{m}{n}}\right), \quad 1 \leq m \leq n-1 \tag{1.4}$$

are Willmore hypersurfaces. We call $W_{m,n-m}$, $1 \leq m \leq n-1$, *Willmore tori.* In fact, the principal curvatures $k_1, \cdots, k_n$ of $W_{m,n-m}$ are

$$k_1 = \cdots = k_m = \sqrt{\frac{m}{n-m}}, \quad k_{m+1} = \cdots = k_n = -\sqrt{\frac{n-m}{m}}. \tag{1.5}$$

We have from (1.5)

$$H = \frac{1}{n}\left(m\sqrt{\frac{m}{n-m}} - (n-m)\sqrt{\frac{n-m}{m}}\right), \quad S = \frac{m^2}{n-m} + \frac{(n-m)^2}{m},$$

$$\sum_{i,j,k} h_{ij}h_{jk}h_{ki} = \sum_i k_i^3 = m\left(\frac{m}{n-m}\right)^{\frac{3}{2}} - (n-m)\left(\frac{n-m}{m}\right)^{\frac{3}{2}},$$

thus we easily check that (1.2) holds, i.e., $W_{m,n-m}$ are Willmore hypersurfaces. In particular, We note that ρ^2 of $W_{m,n-m}$ for all $1 \leq m \leq n-1$ satisfy

$$\rho^2 = n. \tag{1.6}$$

We recall that well-known Clifford minimal tori are

$$C_{m,n-m} = S^m\left(\sqrt{\frac{m}{n}}\right) \times S^{n-m}\left(\sqrt{\frac{n-m}{n}}\right), \quad 1 \leq m \leq n-1. \tag{1.7}$$

It is remarkable that Willmore tori coincide with Clifford minimal tori if and only if $n = 2m$ for some m.

Remark 1.3. When $n = 2$, we can see from (1.3) that all minimal surfaces are Willmore surfaces. When $n \geq 3$, minimal hypersurfaces are not Willmore hypersurfaces in general, for example, Clifford minimal tori $C_{m,n-m} = S^m\left(\sqrt{\frac{m}{n}}\right) \times S^{n-m}\left(\sqrt{\frac{n-m}{n}}\right)$ are not Willmore hypersurfaces when $n \neq 2m$.

In the theory of minimal hypersurfaces, the following Simons' integral inequality is well-known

THEOREM 2. (Simons [17], Lawson [8], Chern-Do Carmo-Kobayashi [5]) Let M be an n-dimensional ($n \geq 2$) compact minimal hypersurface in $(n+1)$-dimensional unit sphere S^{n+1}. Then we have

$$\int_M S(n-S)dv \leq 0. \tag{1.8}$$

In particular, if

$$0 \leq S \leq n, \tag{1.9}$$

then either $S = 0$ and M is totally geodesic, or $S = n$ and M is one of the Clifford tori $C_{m,n-m}$, which are defined by (1.7).

In this paper we prove the following integral inequality of Simons' type for Willmore hypersurfaces.

THEOREM 3. Let M be an n-dimensional ($n \geq 2$) compact Willmore hypersurface in $(n+1)$-dimensional unit sphere S^{n+1}. Then we have

$$\int_M \rho^n(n-\rho^2)dv \leq 0. \tag{1.10}$$

In particular, if

$$0 \leq \rho^2 \leq n, \tag{1.11}$$

then either $\rho^2 = 0$ and M is totally umbilic, or $\rho^2 = n$ and M is one of the Willmore tori $W_{m,n-m}$, which are defined by (1.4).

2. Preliminaries. Let $x : M \to S^{n+1}$ be an n-dimensional hypersurface in an $(n+1)$-dimensional unit sphere S^{n+1}. Let $\{e_1, \cdots, e_n\}$ be a local orthonormal basis of M with respect to the induced metric, $\{\theta_1, \cdots, \theta_n\}$ their dual form. Let e_{n+1} be the local unit normal vector field. In this paper we make the following convention on the range of indices:

$$1 \leq i, j, k \leq n.$$

Then we have the structure equations

$$dx = \sum_i \theta_i e_i, \tag{2.1}$$

$$de_i = \sum_j \theta_{ij} e_j + \sum_j h_{ij}\theta_j e_{n+1} - \theta_i x, \tag{2.2}$$

$$de_{n+1} = -\sum_{i,j} h_{ij}\theta_j e_i. \tag{2.3}$$

The Gauss equations are

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk}), \tag{2.4}$$

$$R_{ik} = (n-1)\delta_{ik} + nHh_{ik} - \sum_j h_{ij}h_{jk}, \tag{2.5}$$

$$n(n-1)R = n(n-1) + n^2H^2 - S, \tag{2.6}$$

where R is the normalized scalar curvature of M and $S = \sum_{i,j} h_{ij}^2$ is the norm square of the second fundamental form and H is the mean curvature.

The Codazzi equations are

$$h_{ijk} = h_{ikj}, \tag{2.7}$$

where the covariant derivative of h_{ij} is defined by

$$\sum_k h_{ijk}\theta_k = dh_{ij} + \sum_k h_{kj}\theta_{ki} + \sum_k h_{ik}\theta_{kj}. \tag{2.8}$$

The second covariant derivative of h_{ij} is defined by

$$\sum_l h_{ijkl}\theta_l = dh_{ijk} + \sum_l h_{ljk}\theta_{li} + \sum_l h_{ilk}\theta_{lj} + \sum_l h_{ijl}\theta_{lk}. \tag{2.9}$$

By exterior differentiation of (2.8), we have the following Ricci identities

$$h_{ijkl} - h_{ijlk} = \sum_m h_{mj}R_{mikl} + \sum_m h_{im}R_{mjkl}. \tag{2.10}$$

We define the following non-negative function on M

$$\rho^2 = S - nH^2, \tag{2.11}$$

which vanishes exactly at the umbilical points of M.

Willmore functional is the following functional (see [4] or [20])

$$\int_M \rho^n dv = \int_M (S - nH^2)^{\frac{n}{2}} dv. \tag{2.12}$$

By Gauss equation (2.6), this is equivalent to

$$[n(n-1)]^{\frac{n}{2}} \int_M (H^2 - R + 1)^{\frac{n}{2}} dv. \tag{2.12$'$}$$

It was shown in [20] and [22] that this functional is an invariant under Moebius (or conformal) transformations of S^{n+1}. We use the term *Willmore hypersurfaces* to call its critical points. When $n = 2$, the functional essentially coincides with the well-known Willmore functional and its critical points are *Willmore surfaces.*

In order to prove Theorem 1, we need the following Reilly's result

LEMMA 2.1 (Theorem A of Reilly [15]). *Suppose that f is any smooth function of n variables. $x : M \to S^{n+1}$ a compact hypersurface. Consider a one-parameter family $x_t : M \to S^{n+1}$ with $x_0 = x$, $t \in (-\epsilon, \epsilon)$. Let $\xi = \frac{\partial x_t}{\partial t}|_{t=0}$, $\lambda =< \xi, e_{n+1} >$. Then*

$$\begin{aligned} &\frac{d}{dt}\bigg|_{t=0} \int_M f(S_1, ..., S_n) dv \\ &= \int_M \lambda\{-S_1 f(S_1, ..., S_n) + \sum_{r=1}^{n} (S_r S_1 - (r+1)S_{r+1}) D_r f(S_1, ..., S_n) \\ &\quad + \sum_{i,j,r=1}^{n} (D_r f(S_1, ..., S_n))_{,ij} T_{r-1}^{ij} + \sum_{r=1}^{n} D_r f(S_1, ..., S_n)(n - r + 1)S_{r-1}\} dv, \end{aligned} \tag{2.13}$$

where $D_r f(S_1, \cdots, S_n)$ denotes the partial derivative of f with respect to variable S_r, $(.)_{,ij}$ denotes the covariant derivative relative to the metric induced by x. $h = (h_{ij}) = (k_i\delta_{ij})$ and S_r the r-th elementary symmetric function of the the eigenvalues $k_1, \cdots, k_n$ of h, i.e.,

$$S_0 = 1, \quad S_1 = k_1 + \cdots + k_n, \quad \cdots, \quad S_n = k_1 \cdots k_n. \tag{2.14}$$

The Newton transformation T_r is defined inductively by

$$T_0^{ij} = \delta_{ij}, \quad T_{r+1}^{ij} = S_{r+1}\delta_{ij} - \sum_k T_r^{ik} h_{kj}, \quad r = 0, \cdots, n-1. \tag{2.15}$$

LEMMA 2.2 (see [16], c.f. [9,10]). *Let M be an n-dimensional ($n \geq 2$) hypersurface in S^{n+1}. Then we have*

$$\begin{aligned} \frac{1}{2}\Delta\rho^2 =& |\nabla h|^2 - n^2|\nabla H|^2 + \sum_{i,j,k}(h_{ij}h_{kki})_j \\ &+ nS - S^2 - n^2H^2 + nH\sum_{i,j,k} h_{ij}h_{jk}h_{ki} - \frac{1}{2}\Delta(nH^2), \end{aligned} \tag{2.16}$$

where $|\nabla h|^2 = \sum_{i,j,k} h_{ijk}^2$ and $|\nabla H|^2 = \sum_i H_i^2$.

Proof. By the definition of Δ and ρ^2, we have by use of (2.7) and (2.10)

$$\begin{aligned} &\frac{1}{2}\Delta\rho^2 \\ =&\frac{1}{2}\Delta(\sum_{i,j} h_{ij}^2) - \frac{1}{2}\Delta(nH^2) \\ =&\sum_{i,j,k} h_{ijk}^2 + \sum_{i,j,k} h_{ij}h_{kijk} - \frac{1}{2}\Delta(nH^2) \\ =&|\nabla h|^2 - n^2|\nabla H|^2 + \sum_{i,j,k}(h_{ij}h_{kki})_j + \sum h_{ij}(h_{lk}R_{lijk} + h_{il}R_{lj}) - \frac{1}{2}\Delta(nH^2). \end{aligned} \tag{2.17}$$

We can easily obtain (2.16) by putting (2.4) and (2.5) into (2.17)

3. Proof of Theorem 1. Choosing in Lemma 2.1

$$f(S_1, \cdots, S_n) = Q^n := [(n-1)S_1^2 - 2nS_2]^{\frac{n}{2}} = n^{\frac{n}{2}}(S - nH^2)^{\frac{n}{2}} = n^{\frac{n}{2}}\rho^n, \tag{3.1}$$

and noting that

$$D_1 f = n(n-1)Q^{n-2}S_1, \quad D_2 f = -n^2Q^{n-2}, \quad D_r f = 0, \ r \geq 3. \tag{3.2}$$

Putting (3.1) and (3.2) into (2.13), we obtain by use of (2.14)

$$\begin{aligned} & n^{\frac{n}{2}}\frac{d}{dt}\Big|_{t=0}\int_M (S - nH^2)^{\frac{n}{2}} dv = \frac{d}{dt}\Big|_{t=0}\int_M ((n-1)S_1^2 - 2nS_2)^{\frac{n}{2}} dv \\ =& \int_M \lambda[-S_1Q^n + (S_1^2 - 2S_2)n(n-1)Q^{n-2}S_1 - n^2Q^{n-2}(S_2S_1 - 3S_3) \\ & + n(n-1)\Delta(Q^{n-2}S_1) - n^2\sum_{i,j}(Q^{n-2})_{,ij}(S_1\delta_{ij} - h_{ij})]dv. \end{aligned} \tag{3.3}$$

Thus the Euler-Lagrange equation of Willmore functional (2.12) is

$$\begin{aligned}(3.4)\qquad &-S_1Q^n + (S_1^2 - 2S_2)n(n-1)Q^{n-2}S_1 - n^2Q^{n-2}(S_2S_1 - 3S_3)\\ &+n(n-1)\Delta(Q^{n-2}S_1) - n^2\sum_{i,j}(Q^{n-2})_{,ij}(S_1\delta_{ij} - h_{ij}) = 0.\end{aligned}$$

From $Q = n^{\frac{1}{2}}\rho$, we know that (3.4) is equivalent to

$$\begin{aligned}(3.4)'\qquad &-nS_1\rho^n + (S_1^2 - 2S_2)n(n-1)\rho^{n-2}S_1 - n^2\rho^{n-2}(S_2S_1 - 3S_3)\\ &+n(n-1)\Delta(\rho^{n-2}S_1) - n^2\sum_{i,j}(\rho^{n-2})_{,ij}(S_1\delta_{ij} - h_{ij}) = 0.\end{aligned}$$

Noting that

$$S_1 = nH,\ S_2 = \frac{1}{2}(S_1^2 - S),\ S_3 = \frac{1}{3}(\sum_i k_i^3 - S_1S + S_2S_1),$$

we know that (3.4)′ is equivalent to (1.2). This completes the proof of Theorem 1.

4. Lemmas and Proof of Theorem 3. We first prove the following lemma (c.f. [7])

LEMMA 4.1. *Let M be an n-dimensional ($n \geq 2$) hypersurface in S^{n+1}, then we have*

$$(4.1)\qquad |\nabla h|^2 \geq \frac{3n^2}{n+2}|\nabla H|^2,$$

where $|\nabla h|^2 = \sum_{i,j,k} h_{ijk}^2$, $|\nabla H|^2 = \sum_i H_i^2$, $H_i = \nabla_i H$.

Proof. We decompose the tensor ∇h:

$$(4.2)\qquad h_{ijk} = E_{ijk} + F_{ijk},$$

where

$$E_{ijk} = \frac{n}{n+2}(H_i\delta_{jk} + H_j\delta_{ik} + H_k\delta_{ij}).$$

Then we can easily compute that

$$|E|^2 = \sum_{i,j,k} E_{ijk}^2 = \frac{3n^2}{n+2}|\nabla H|^2,\ < E_{ijk}, F_{ijk} >=< E_{ijk}, h_{ijk} - E_{ijk} >= 0,$$

i.e., E and F are orthogonal components of ∇h. Then

$$|\nabla h|^2 \geq |E|^2 = \frac{3n^2}{n+2}|\nabla H|^2,$$

which proves the Lemma 4.1.

Define trace-free tensor

$$(4.3)\qquad \tilde{h}_{ij} = h_{ij} - H\delta_{ij}.$$

We have by a direct calculation

$$(4.4)\qquad \sum_k \tilde{h}_{kk} = 0,\quad \sum_{i,j}\tilde{h}_{ij}^2 = \rho^2 = S - nH^2,$$

$$\sum_{i,j,k} h_{ij}h_{jk}h_{ki} = \sum_{i,j,k} \tilde{h}_{ij}\tilde{h}_{jk}\tilde{h}_{ki} + 3H\rho^2 + nH^3. \tag{4.5}$$

From (4.4), (4.5) and Theorem 1, we have

LEMMA 4.2. *Let M be an n-dimensional hypersurface in the $(n+1)$-dimensional unit sphere S^{n+1}. Then M is a Willmore hypersurface if and only if*

$$(n-1)\Delta(\rho^{n-2}H)+\rho^{n-2}(H\rho^2+\sum_{i,j,k}\tilde{h}_{ij}\tilde{h}_{jk}\tilde{h}_{ki})-\sum_{i,j}(\rho^{n-2})_{,ij}(nH\delta_{ij}-h_{ij})=0, \tag{4.6}$$

where Δ is the Laplacian, $(.)_{,ij}$ is the covariant derivative relative to the induced metric.

The following lemma is a key step of the proof of Theorem 3

LEMMA 4.3. *Let M be the n-dimensional hypersurface in the $(n+1)$-dimensional unit sphere S^{n+1}, then we have*

$$\begin{aligned}
&\frac{1}{2}\Delta(\rho^n)\\
=&\frac{1}{2}n(n-2)\rho^{n-2}|\nabla\rho|^2+\frac{n}{2}\rho^{n-2}\{(|\nabla h|^2-\frac{3n^2}{n+2}|\nabla H|^2)+(\frac{3n^2}{n+2}-n)|\nabla H|^2\\
&-n(n-1)|\nabla H|^2+\sum_{i,j,k}(h_{ij}h_{kki})_j+\rho^2(n+nH^2-\rho^2)\\
&+nH\sum_{i,j,k}\tilde{h}_{ij}\tilde{h}_{jk}\tilde{h}_{ki}-\frac{1}{2}\Delta(nH^2)\}\\
\geq&\frac{n}{2}\rho^{n-2}\{-n(n-1)|\nabla H|^2+\sum_{i,j,k}(h_{ij}h_{kki})_j+\rho^2(n+nH^2-\rho^2)\\
&+nH\sum_{i,j,k}\tilde{h}_{ij}\tilde{h}_{jk}\tilde{h}_{ki}-\frac{1}{2}\Delta(nH^2)\}
\end{aligned} \tag{4.7}$$

Proof. First it is easy to check the following identity

$$\frac{1}{2}\Delta(\rho^n)=\frac{1}{2}n(n-2)\rho^{n-2}|\nabla\rho|^2+\frac{n}{4}\rho^{n-2}\Delta(\rho^2). \tag{4.8}$$

By use of (4.4) and (4.5), (2.16) can be written as

$$\begin{aligned}
\frac{1}{2}\Delta\rho^2=&|\nabla h|^2-n^2|\nabla H|^2+\sum_{i,j,k}(h_{ij}h_{kki})_j\\
&+\rho^2(n+nH^2-\rho^2)+nH\sum_{i,j,k}\tilde{h}_{ij}\tilde{h}_{jk}\tilde{h}_{ki}-\frac{1}{2}\Delta(nH^2).
\end{aligned} \tag{4.9}$$

Putting (4.9) into (4.8) and noting $\frac{3n^2}{n+2}-n=\frac{2n(n-1)}{n+2}>0$, we obtain (4.7) by use of Lemma 4.1.

LEMMA 4.4. *Let M be an n-dimensional compact Willmore hypersurface in the*

$(n+1)$-dimensional unit sphere S^{n+1}, then we have

(4.10)

$$
\begin{aligned}
&-n(n-1)\int_M \rho^{n-2}|\nabla H|^2\\
=&-n(n-1)\int_M \rho^{n-2}(HH_i)_i+n\int_M H(\rho^{n-2})_{,ij}(nH\delta_{ij}-h_{ij})-n(n-1)\int_M H^2\Delta(\rho^{n-2})\\
&-2n(n-1)\int_M H\nabla(\rho^{n-2})\cdot\nabla H-n\int_M H\rho^{n-2}(H\rho^2+\sum_{i,j,k}\tilde{h}_{ij}\tilde{h}_{jk}\tilde{h}_{ki}).
\end{aligned}
$$

Proof. We first note the following identity

$$
\begin{aligned}
&-n(n-1)\rho^{n-2}|\nabla H|^2\\
(4.11)\quad =&-n(n-1)(\rho^{n-2}(HH_i)_i-\rho^{n-2}H\Delta H)\\
=&-n(n-1)[\rho^{n-2}(HH_i)_i-H\Delta(\rho^{n-2}H)+H^2\Delta(\rho^{n-2})+2H\nabla(\rho^{n-2})\cdot\nabla H].
\end{aligned}
$$

Integrating (4.11) over M, we have (4.10) by use of (4.6).

LEMMA 4.5. *Let M be an n-dimensional compact hypersurface in the $(n+1)$-dimensional unit sphere S^{n+1}, then we have*

$$
(4.12)\quad \int_M \rho^{n-2}\sum_{i,j,k}(h_{ij}h_{kki})_j=n\int_M H\sum_{i,j}h_{ij}(\rho^{n-2})_{,ij}+n^2\int_M H\nabla(\rho^{n-2})\cdot\nabla H.
$$

Proof. We have the following calculation

$$
\begin{aligned}
\int_M \rho^{n-2}\sum_{i,j,k}(h_{ij}h_{kki})_j&=\int_M\sum_{i,j,k}(\rho^{n-2}h_{ij}h_{kki})_j-\int_M\sum_{i,j,k}(\rho^{n-2})_{,j}h_{ij}h_{kki}\\
&=-\int_M\sum_{i,j,k}(\rho^{n-2})_{,j}h_{ij}h_{kki}\\
&=-\int_M\sum_{i,j,k}((\rho^{n-2})_{,j}h_{ij}h_{kk})_i+\int_M\sum_{i,j,k}(\rho^{n-2})_{,ji}h_{ij}h_{kk}\\
&\quad+n^2\int_M\sum_j H(\rho^{n-2})_{,j}H_j\\
&=n\int_M H\sum_{i,j}h_{ij}(\rho^{n-2})_{,ij}+n^2\int_M H\nabla(\rho^{n-2})\cdot\nabla H.
\end{aligned}
$$

Proof of Theorem 3. Integrating (4.7) over M, we have

$$
\begin{aligned}
&0\ge\frac{n}{2}\{-n(n-1)\int_M\rho^{n-2}|\nabla H|^2+\int_M\rho^{n-2}\sum_{i,j,k}(h_{ij}h_{kki})_j+\int_M\rho^n(n+nH^2-\rho^2)\\
(4.13)\quad &+n\int_M H\rho^{n-2}\sum_{i,j,k}\tilde{h}_{ij}\tilde{h}_{jk}\tilde{h}_{ki}-\frac{1}{2}\int_M\rho^{n-2}\Delta(nH^2)\}.
\end{aligned}
$$

Putting (4.10) and (4.12) into (4.13), we get

(4.14)

$$\begin{aligned}
0 \geq& \frac{n}{2}\{[-n(n-1)\int_M \rho^{n-2}(HH_i)_i + n\int_M H\sum_{i,j}(\rho^{n-2})_{,ij}(nH\delta_{ij}-h_{ij}) \\
&-n(n-1)\int_M H^2\Delta(\rho^{n-2}) - 2n(n-1)\int_M H\nabla(\rho^{n-2})\cdot\nabla H - n\int_M H\rho^{n-2}(H\rho^2 \\
&+\sum_{i,j,k}\tilde{h}_{ij}\tilde{h}_{jk}\tilde{h}_{ki})] + [n\int_M H\sum_{i,j}h_{ij}(\rho^{n-2})_{,ij} + n^2\int_M H\nabla(\rho^{n-2})\cdot\nabla H] \\
&+\int_M \rho^n(n+nH^2-\rho^2) + n\int_M H\rho^{n-2}\sum_{i,j,k}\tilde{h}_{ij}\tilde{h}_{jk}\tilde{h}_{ki} - \frac{1}{2}\int_M \rho^{n-2}\Delta(nH^2)\} \\
=& \frac{n}{2}\{-n(n-1)\int_M \rho^{n-2}(HH_i)_i + n^2\int_M H^2\Delta(\rho^{n-2}) - n(n-1)\int_M H^2\Delta(\rho^{n-2}) \\
&-2n(n-1)\int_M H\nabla(\rho^{n-2})\cdot\nabla H + n^2\int_M H\nabla(\rho^{n-2})\cdot\nabla H + \int_M \rho^n(n-\rho^2) \\
&-\frac{1}{2}\int_M \rho^{n-2}\Delta(nH^2)\} \\
=& \frac{n}{2}\int_M \rho^n(n-\rho^2).
\end{aligned}$$

Thus we reach the following integral inequality of Simons' type

$$\int_M \rho^n(n-\rho^2) \leq 0. \tag{4.15}$$

Therefore we have proved the integral inequality (1.10) in Theorem 3.

If (1.11) holds, then we conclude from (4.15) that either $\rho^2 \equiv 0$, or $\rho^2 \equiv n$. In the first case, we know that $S \equiv nH^2$, i.e. M is totally umbilic; in the latter case, i.e., $\rho^2 \equiv n$, we have from (4.7)

$$\int_M \rho^{n-2}(\frac{3n^2}{n+2} - n)|\nabla H|^2 = 0,$$

we have $H = constant$, thus we have again from (4.7)

$$\nabla h = 0.$$

It easily follows that M is an isoparametric hypersurface with two distinct constant principal curvatures, M is one of the Willmore tori (see Theorem 5.1), that is, $M = W_{m,n-m}$ for some m with $1 \leq m \leq n-1$. We complete the proof of Theroem 3.

5. Isoparametric Willmore hypersurfaces. In this section, we give the classification of isoparametric hypersurfaces in S^{n+1}. We need the following result

LEMMA 5.1 (see [1,3,12,18,19]). *Let M be an n-dimensional compact isoparametric hypersurface (i.e. hypersurface with constant principal curvatures) in S^{n+1}. Let $k_1 > k_2 > \cdots > k_g$ be the distinct principal curvatures with multiplicities $m_1, \cdots, m_g$ (so that $n = m_1 + m_2 + \cdots + m_g$). Then*

(a) g is either $1, 2, 3, 4$, or 6.

(b) If $g = 1$, M is totally umbilic.

(c) If $g = 2$, $M = S^m(r_1) \times S^{n-m}(r_2)$, $r_1^2 + r_2^2 = 1$.

(d) If $g=3$, $m_1=m_2=m_3=2^k$, $(k=0,1,2,3)$.

(e) If $g=4$, $m_1=m_3$ *and* $m_2=m_4$. *Moreover,* $(m_1,m_2)=(2,2)$ *or* $(4,5)$, *or* m_1+m_2+1 *is a multiple of* $2^{\phi(m_1-1)}$. *Here* $\phi(l)$ *is the number of integers* s *with* $1\le s\le l$ *and* $s\equiv 0,1,2,4 \mod 8$.

(f) If $g=6$, $m_1=m_2=\cdots=m_6=1$ *or* 2.

(g) There exists an angle θ, $0<\theta<\frac{\pi}{g}$, *such that*

$$k_\alpha=\cot(\theta+\frac{\alpha-1}{g}\pi),\quad \alpha=1,\cdots,g. \tag{5.1}$$

In the isoparametric case, $\rho^2=$ constant, $H=$ constant, we get from Theorem 1

LEMMA 5.2. *Let* M *be an* n*-dimensional isoparametric Willmore hypersurface in* S^{n+1}. *Then*

$$2HS-nH^3-\sum_i k_i^3=0, \tag{5.2}$$

where $h_{ij}=k_i\delta_{ij}$.

THEOREM 5.1. *Let* M *be an* n*-dimensional compact isoparametric Willmore hypersurface in* S^{n+1}. *Then*

(1) *If* $g=1$, M *is a totally umbilic hypersphere, satisfying* $\rho^2=0$.

(2) *If* $g=2$, M *is one of the Willmore tori* $W_{m,n-m}$, *which are defined by* (1.4), *satisfying* $\rho^2=n$.

(3) *If* $g=3$, $k_1=\sqrt{3}, k_2=0, k_3=-\sqrt{3}$. $n=3,6,12$ *or* 24. M *are Cartan minimal hypersurfaces, satisfying* $\rho^2=2n$.

(4) *If* $g=4$,

$$k_1=\lambda,\quad k_2=\frac{\lambda-1}{\lambda+1},\quad k_3=-\frac{1}{\lambda},\quad k_4=-\frac{\lambda+1}{\lambda-1}, \tag{5.3}$$

where $A=(\lambda-\frac{1}{\lambda})^2$ *is the positive solution of the following algebraic equation*

$$m_1(m_1+2m_2)^2x^2+4m_1m_2(m_2-m_1)x-16m_2(m_2+2m_1)^2=0, \tag{5.4}$$

and $(m_1,m_2)=(2,2)$ *or* $(4,5)$, *or* m_1+m_2+1 *is a multiple of* $2^{\phi(m_1-1)}$. *Here* $\phi(l)$ *is the number of integers* s *with* $1\le s\le l$ *and* $s\equiv 0,1,2,4 \mod 8$.

Among these isoparametric Willmore hypersurfaces, the only isoparametric minimal hypersurfaces are case with $m_1=m_2=m_3=m_4=2$, *principal curvatures are*

$$k_1=1+\sqrt{2},\quad k_2=\sqrt{2}-1,\quad k_3=1-\sqrt{2},\quad k_2=-(1+\sqrt{2}).$$

(5) *If* $g=6$, *then*

$$k_1=2+\sqrt{3},\quad k_2=1,\quad k_3=2-\sqrt{3},\quad k_4=-(2-\sqrt{3}),\quad k_5=-1,\quad k_6=-(2+\sqrt{3}).$$

In this case, $n=6$ *or* 12. *These Willmore hypersurfaces are minimal and satisfying* $\rho^2=5n$.

Proof. (1) Case $g=1$ is trivial.

(2) If $g=2$, let distinct principal curvatures are k_1 (multiplicity m) and k_2 (multiplicity $n-m$). Then by (*c*) of Lemma 5.1 and (5.2), we have

$$1+k_1k_2=0, \tag{5.5}$$

(5.6)
$$\frac{2(mk_1+(n-m)k_2)}{n}(mk_1^2+(n-m)k_2^2)-\frac{1}{n^2}(mk_1+(n-m)k_2)^3-(mk_1^3+(n-m)k_2^3)=0.$$

Putting (5.5) into (5.6), we have

$$(n-m)k_1^6+(2n-3m)k_1^4+(n-3m)k_1^2-m=0, \tag{5.7}$$

that is,

$$((n-m)k_1^2-m)(k_1^2+1)^2=0. \tag{5.7$'$}$$

Thus

$$k_1^2=\frac{m}{n-m},\quad M=W_{m,n-m}=S^m\left(\sqrt{\frac{n-m}{n}}\right)\times S^{n-m}\left(\sqrt{\frac{m}{n}}\right),\quad 1\le m\le n-1.$$

(3) If $g=3$, by (d) of Lemma 5.1, $m_1=m_2=m_3:=m$, $n=3m$.
From (g) of Lemma 5.1, we have

$$k_1=\cot\theta,\quad k_2=\cot(\theta+\frac{\pi}{3})=\frac{k_1-\sqrt{3}}{1+\sqrt{3}k_1},\quad k_3=\cot(\theta+\frac{2}{3}\pi)=\frac{k_1+\sqrt{3}}{1-\sqrt{3}k_1}. \tag{5.8}$$

Putting (5.8) into (5.2) and noting $n=3m$, we obtain

$$k_1(k_1^2-3)(k_1^2+1)^3=0. \tag{5.9}$$

Thus we have $k_1=\sqrt{3}, k_2=0, k_3=-\sqrt{3}$.
(4) If $g=4$, in this case

$$m_1=m_3,\qquad m_2=m_4.$$

By (g) of Lemma 5.1,

$$\begin{aligned}k_1&=\cot\theta:=\lambda,\quad k_2=\cot(\theta+\frac{\pi}{4})=\frac{\lambda-1}{\lambda+1},\\ k_3&=\cot(\theta+\frac{2\pi}{4})=-\frac{1}{\lambda},\quad k_4=\cot(\theta+\frac{3\pi}{4})=-\frac{\lambda+1}{\lambda-1}.\end{aligned} \tag{5.10}$$

Write

$$A=\lambda-\frac{1}{\lambda},\qquad B=k_2-\frac{1}{k_2}. \tag{5.11}$$

Noting $nH=m_1A+m_2B$ and

$$S=m_1A^2+m_2B^2+2(m_1+m_2),\quad \sum_i k_i^3=m_1A^3+m_2B^3+3(m_1A+m_2B), \tag{5.12}$$

we have by putting (5.12) into (5.2)

$$\begin{aligned}&\frac{2}{n}(m_1A+m_2B)(m_1A^2+m_2B^2+2m_1+2m_2)-\frac{1}{n^2}(m_1A+m_2B)^3\\ &\quad-(m_1A^3+m_2B^3+3m_1A+3m_2B)=0.\end{aligned} \tag{5.13}$$

Putting $n = 2(m_1 + m_2)$ into (5.13) and noting $AB = -4$, we get that

$$\begin{aligned} m_1(m_1+2m_2)^2A^6 &+ 4m_1(m_1^2+3m_1m_2+5m_2^2)A^4 \\ &- 16m_2(m_2^2+3m_1m_2+5m_1^2) - 64m_2(m_2+2m_1)^2 = 0, \end{aligned} \tag{5.14}$$

which can be written as

$$(A^2+4)[m_1(m_1+2m_2)^2A^4 + 4m_1m_2(m_2-m_1)A^2 - 16m_2(m_2+2m_1)^2] = 0, \tag{5.15}$$

that is equivalent to

$$m_1(m_1+2m_2)^2A^4 + 4m_1m_2(m_2-m_1)A^2 - 16m_2(m_2+2m_1)^2 = 0. \tag{5.16}$$

(5) If $g = 6$, in this case, by (f) of Lemma 5.1 we have

$$m_1 = m_2 = m_3 = m_4 = m_5 = m_6 = 1, \quad \text{or} \quad 2.$$

By (g) of Lemma 5.1,

$$\begin{aligned} k_1 &= \cot\theta, \quad k_2 = \frac{\sqrt{3}k_1 - 1}{k_1 + \sqrt{3}}, \quad k_3 = \frac{k_1 - \sqrt{3}}{1 + \sqrt{3}k_1}, \\ k_4 &= -\frac{1}{k_1}, \quad k_5 = \frac{k_1 + \sqrt{3}}{1 - \sqrt{3}k_1}, k_6 = \frac{1 + \sqrt{3}k_1}{\sqrt{3} - k_1}. \end{aligned} \tag{5.17}$$

Putting (5.17) into (5.2), we can get by a direct calculation

$$(k_1^2-1)(k_1^4-14k_1^2+1)[(k_1^2-1)^2(k_1^4-14k_1^2+1)^2 + 4k_1^2(3k_1^4-10k_1^2+3)^2] = 0. \tag{5.18}$$

Thus we obtain

$$k_1 = 2+\sqrt{3}, \quad k_2 = 1, \quad k_3 = 2-\sqrt{3}, \quad k_4 = -(2-\sqrt{3}), \quad k_5 = -1, \quad k_6 = -(2+\sqrt{3}).$$

We complete the proof of Theorem 5.1.

ACKNOWLEDGEMENTS. The author has done this research works during his stay in department of mathematics of Harvard university as a visitor in academic year of 1999-2000. He would like to express his thanks to Prof. S.-T. Yau for his encouragements and help. He also would like to express his thanks to Prof. C. L. Terng for her help and useful comments.

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