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CONVEX POLYHEDRA IN LORENTZIAN SPACE-FORMS*

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Abstract. Aleksandrov [Ale51] characterized the metrics induced on convex polyhedra in E^3, H^3 and S^3. We give analogs for compact and complete polyhedra in Lorentzian space-forms.

There are three types of convex polyhedra in the de Sitter space S^3_1. One, which includes generalized hyperbolic polyhedra, was treated in [Sch98a]. For the second, we characterize the induced metrics, and show that each is obtained on a unique polyhedron satisfying a natural condition at infinity. For the last type — compact polyhedra bounding compact domains — we describe the induced metrics, and give an existence and uniqueness result for a smaller class of metrics.

The results on complete polyhedra are consequences of the study of the metrics induced on convex polyhedra in a natural extension of H^3 by S^3_1. We also characterize the metrics induced on compact, convex polyhedra in the Minkowski space E^3_1.

Those description are partly similar to those obtained in the Riemannian cases, but they also involve new elements of a metric and combinatorial nature.

Résumé. Aleksandrov [Ale51] a caractérisé les métriques induites sur les polyèdres convexes dans E^3, H^3 et S^3. On donne des résultats similaires pour les polyèdres compacts ou complets dans les formes d'espace lorentziennes.

On distingue trois types de polyèdres convexes dans l'espace de Sitter S^3_1. L'un d'eux, incluant les polyèdres hyperboliques généralisés, est étudié dans [Sch98a]. Pour le second, on caractérise les métriques induites, et on montre que chacune est obtenue sur un unique polyèdre satisfaisant une conditions naturelle à l'infini. Pour le troisième type — les polyèdres compacts bordant des domaines compacts — on donne une description des métriques induites, et un résultat d'existence et d'unicité pour une classe plus restreinte de métriques.

Les résultats concernant les polyèdres complets sont des conséquences de l'étude des métriques induites sur les polyèdres convexes dans une extension naturelle de H^3 par S^3_1. On caractérise les aussi les métriques induites sur les polyèdres convexes dans l'espace de Minkowski E^3_1.

Ces descriptions sont partiellement analogues à celles obtenues dans le cas riemannien, mais elles font aussi intervenir de nouveaux éléments de nature métrique et combinatoire.

1. Introduction and main results. The metrics induced on convex polyhedra of 3-dimensional Riemannian space-forms are completely described by the following well-known theorem:

THEOREM 1.1 (Aleksandrov [Ale51]). *Choose $K_0 \in \{-1, 0, 1\}$. A Riemannian metric g on S^2 with conical singularities is induced by a convex polyhedral embedding in the simply connected space M_{K_0} with constant curvature K_0 if and only if g has constant curvature K_0 except at a finite number of singular points $x_1, \cdots, x_N$, where the singular curvature is positive. The embedding is then unique modulo global isometries.*

This paper intends to give similar results for compact polyhedra in two Lorentzian space-forms, the de Sitter and the Minkowski spaces. We will also describe the induced metrics on complete polyhedra in the de Sitter space, along the lines of [Sch98a] for complete hyperbolic polyhedra.

Note that a basic reason to study the metrics induced on convex polyhedra in the de Sitter space is that there is a classical duality, in particular between convex polyhedra in H^3 and in S^3_1, which exchanges the edge lengths and the dihedral angles (this is recalled in section 2). The dihedral angles of a convex hyperbolic polyhedron are therefore related to the induced metric on its dual, which is a convex polyhedron

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in the de Sitter space (see [RH93]). Dihedral angles of convex polyhedra are interesting because of the Poincaré theorem, which says that the group generated by the reflections in the faces of a convex polyhedron is discrete if and only if its dihedral angles are $2\pi/k, k \in \mathbf{N} \setminus \{0\}$.

The description of the metrics induced on polyhedra is more complicated in Lorentzian space-forms than in Riemannian spaces. One reason is that all faces are not similar, since they can be space-like, light-like, or time-like (which means that the signature of the induced metric is $(1,1)$, or that their normal vector is space-like). This is the origin of many boring technical difficulties which will appear clearly below, but also of a rich combinatorics which in my opinion is of some interest.

The results concerning non-compact polyhedra appear naturally in the setting of [Sch98a], where it was shown that some important properties of hyperbolic convex polyhedra (e.g. rigidity, or some compactness results for sequences of polyhedra when the sequence of the induced metrics converge) extend to polyhedra in a space (called $\tilde{\mathrm{HS}}^3$, and defined below) with a complex distance, which contains both H^3 and the de Sitter space S^3_1. It is therefore possible to describe the metrics induced on the convex polyhedra in $\tilde{\mathrm{HS}}^3$, and to give some existence and uniqueness results for those metrics (see theorem 1.3 below). Non-compact polyhedra in S^3_1 or H^3 are then considered as the de Sitter or hyperbolic part of a "compact" polyhedron in $\tilde{\mathrm{HS}}^3$.

In all this paper, complete polyhedra have a finite number of faces, edges and vertices. Complete metrics also have a finite number of singular points. "Completeness" can be understood for instance as "geodesic completeness", since the metrics are, near the ends, isometric to domains in space-forms.

Consider a convex polyhedron in a Lorentzian 3-dimensional space-form, for instance in the Minkowski space E^3_1. Each face F has an induced metric modeled on a Riemannian, Lorentzian or degenerate 2-dimensional space-form (E^2, E^2_1 or $E^2_{1,0}$ for polyhedra in E^3_1), for which F is the interior of a convex polygon. The metrics on adjacent faces satisfy an obvious compatibility condition, namely that the restriction of each to the common edge is the same. We call such an object a **polyhedral metric**; each polyhedral immersion of a polyhedron P into a Lorentz space-form induces a polyhedral metric σ on P. Polyhedral metrics are considered here up to isometry, i.e. they do not include the decomposition into faces that appear in their definition.

We will actually consider induced metrics together with some additional combinatorial data on the way the polyhedron is embedded in the Lorentzian space-form. One way to understand why this is necessary is to remark that a space-like face (for instance) can "degenerate" by becoming thinner and thinner until it is reduced to an edge; the resulting edge should be considered as "space-like", although it bounds two time-like faces.

The same can happen with a space-like face becoming smaller and smaller until it is reduced to a vertex, and a time-like face can also "degenerate" to a space-like edge, bounding two space-like faces. This edge should be considered as a "degenerate time-like face".

We want to define the set of "space-like" points of a polyhedron in a way that is stable under this kind of deformations. So we call Σ the subset of P containing:

- the space-like faces of P;
- the space-like edges e of P which bound two time-like or light-like faces such that, in each neighborhood of e, some space-like geodesic intersects both;

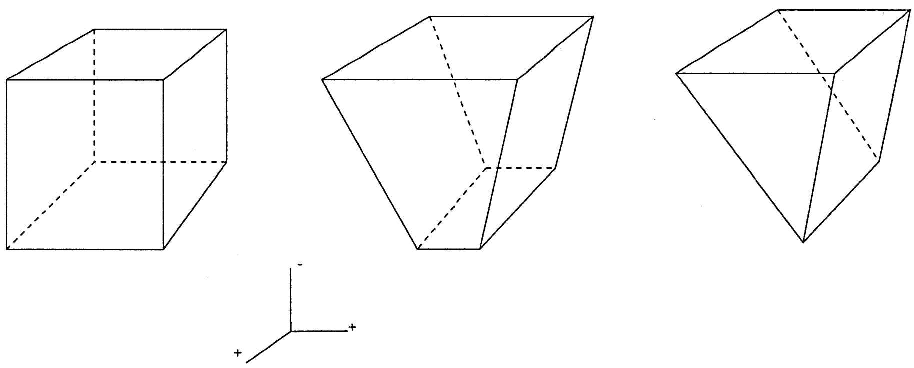

FIG. 1.1. *A space-like face becoming an edge*

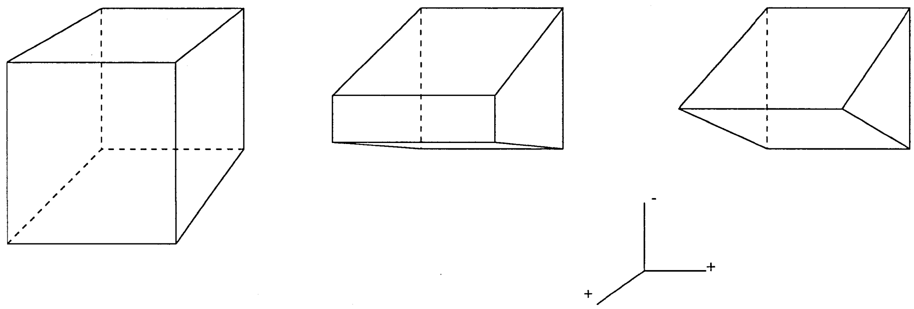

FIG. 1.2. *A time-like face becoming an edge*

- the vertices v of P such that all geodesics of P (for the induced metric σ) starting from v are time-like.

Again, one reason why we add those edges and vertices is because they look like "degenerate" space-like faces. But we remove from Σ:

- the space-like edges e of P bounding two space-like faces such that, in each neighborhood of e, some time-like geodesic intersects both;
- the vertices v of P such that all geodesics of P starting from v are space-like.

Those edges and vertices look like "degenerate" time-like faces.

We also call $\mathcal{T} := P \setminus \Sigma$. Σ should be considered as the set of points in space-like faces (which might be "degenerate") while $\mathcal{T}$ is in a sense the set of points in light-like or time-like faces. The couple (σ, Σ) is the **marked (polyhedral) metric** induced by the polyhedral immersion.

[Sch98a] contains the definition of a "convex" HS marked metric (see definition 3.1). Its main property is that the induced marked metric on a (strictly) convex polyhedron in $\tilde{HS}^3$ (or in any Lorentzian space-form) is convex, while, if a polyhedron is degenerate at a vertex v, then the induced marked metric is not convex at v. We can now state the main result in the Minkowski space.

THEOREM 1.2. *A marked metric (σ, Σ) on S^2 is induced on a convex polyhedron P in E^3_1 if and only if:*

1. *(σ, Σ) is flat (modeled on E^2, E^2_1 or $E^2_{1,0}$) except at M singular points;*
2. *(σ, Σ) is convex (as in definition 3.1) at the singular points;*
3. *Σ has two connected components Σ_+ and Σ_-, and each time-like geodesic in*

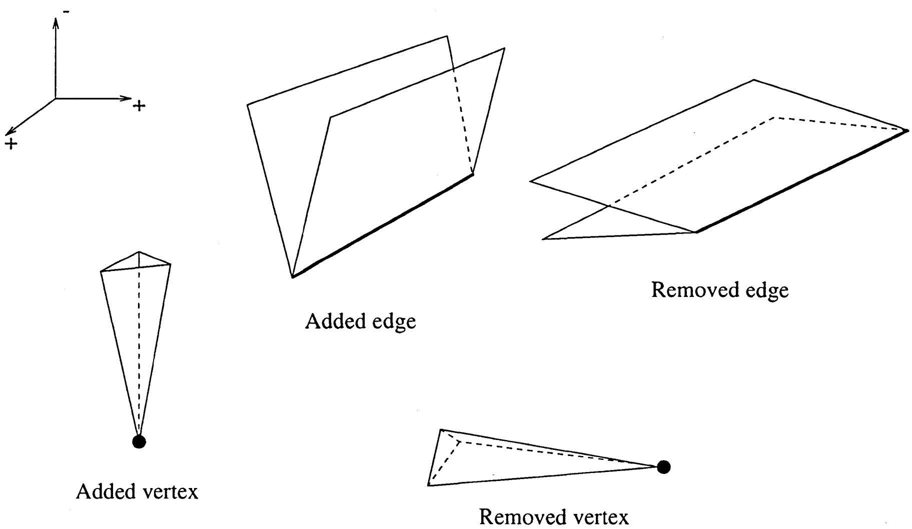

FIG. 1.3. *Edges and vertices added to/removed from* Σ

$\mathcal{T}$ connects one to the other.
P is then unique modulo global isometries of E^3_1.

This theorem should show clearly why the definition of Σ above is necessary. The topological condition (3) in theorem 1.2 would not be correct if Σ was made for instance only of the space-like faces, since Σ_+ and Σ_- can be made of a "special" edge or vertex corresponding to a "degenerate" space-like face.

We will not consider complete, non-compact polyhedra in the flat case. It might be possible to give some results, but I do not believe that uniqueness holds without some conditions at infinity (for instance on the limit cone, see [Pog80] or [PS85] for analogous situations for the Minkowski problem on complete surfaces).

There exist a projective model in S^3 of both S^3_1 (which sits between 2 spheres around the poles) and two copies of H^3 (in those two spheres) which are denoted by H^3_+ and H^3_-. This model is defined in section 3 (following [Sch98a]), we call it the "hyperbolic-de Sitter space" and denote it by $\tilde{\mathrm{HS}}^3$; it corresponds to a complex "distance" on S^3 minus two spheres. Two points inside one of the spheres have a real, positive distance, while two points outside both spheres have an imaginary distance if they correspond to de Sitter points joined by a space-like geodesic, and a real, negative distance if the corresponding points in S^3_1 are linked by a time-like geodesic. The distance is also defined between a "hyperbolic" and a "de Sitter" point.

If P is a polyhedron in $\tilde{\mathrm{HS}}^3$, it inherits an induced **marked HS metric** σ, namely, each 2-face F of P has a "complex distance" for which it is isometric to the interior of a convex polygon in one of three possible model spaces: $\tilde{\mathrm{HS}}^2$, $(S^2, -\mathrm{can})$ or the degenerate space $\tilde{\mathrm{HS}}^2_{1,0}$. The metrics on adjacent faces satisfy obvious compatibility conditions. The induced marked metric on P is the couple (σ, Σ), where Σ is included in the "de Sitter" part of $\tilde{\mathrm{HS}}^3$ and defined as above. We also call H the set of "hyperbolic" points of P (those which are in $\overline{H^3_+} \cup \overline{H^3_-}$), and $\mathcal{T} := P \setminus (H \cup \Sigma)$.

Let P be a polyhedron with an HS marked metric (σ, Σ); there are two special kind of polygonal curves in P which play a special role. Recall that a geodesic in a Riemannian surface (for instance, a polyhedral metric like the one in theorem 1.1) is a

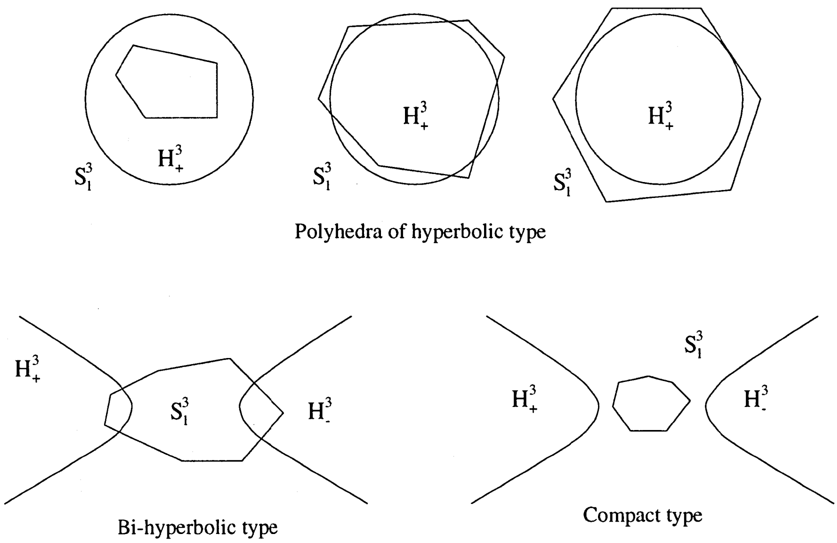

FIG. 1.4. *Types of polyhedra in* $\tilde{HS}^3$

curve such that each side is concave at each point. We say that a space-like polygonal curve g in P is **Σ-geodesic** if it remains in Σ and, at each vertex and on each side, either there is an element of $\mathcal{T}$ (face or edge), or the metric is concave. In the same way, g is **$\mathcal{T}$-geodesic** if it remains in $\mathcal{T}$ and, at each vertex and on each side, either there is an element of Σ, or the metric is concave. The precise definitions are given in section 2.

It is easy to see that there are three main types of convex polyhedra in $\tilde{HS}^3$, which are different in the way the induced metric behaves; more precisely, the topology of the time-like geodesics in $\mathcal{T}$ is different in each case. They are:

1. polyhedra which bound a convex domain intersecting $\overline{H^3_+}$, but not $\overline{H^3_-}$. We call them of **hyperbolic** type here. They contain for instance all hyperbolic polyhedra, as well as their duals. It is not difficult to see that, in those polyhedra, any time-like geodesic in $\mathcal{T}$ goes from Σ to H.
2. those bounding a convex domain intersecting both $\overline{H^3_+}$ and $\overline{H^3_-}$. They are of **bi-hyperbolic** type. Then $\Sigma = \emptyset$, and time-like geodesics of $\mathcal{T}$ go from one of the two connected components of H to the other.
3. those which bound a compact, convex domain in S^3_1, which are of **compact** type. Then $H = \emptyset$, and all time-like geodesics in $\mathcal{T}$ go from one of the two connected components of Σ to the other.

The class of polyhedra of hyperbolic type is invariant by the duality in $\tilde{HS}^3$ (defined in section 3). But this duality sends polyhedra of compact type to polyhedra of bi-hyperbolic type, and conversely. The results below on the induced metrics on each type of polyhedra can therefore be translated as results on the dual metrics on polyhedra of the other type.

Let c be a space-like polygonal curve in a Lorentzian surface; we will say that c is **simple** if c intersects any time-like geodesic at most once. We can now give the main theorem about the induced metrics on the convex polyhedra in $\tilde{HS}^3$:

THEOREM 1.3. *Let (σ, Σ) be a marked HS metric on S^2. Suppose that (σ, Σ)*

is induced on a convex polyhedron P in $\tilde{HS}^3$. Then (σ, Σ) satisfies the following properties:

A. *(σ, Σ) is convex (as in definition 3.1) at its singular points;*

B. *Closed Σ-geodesic curves of (σ, Σ) have length $L > 2\pi$, or $L = 2\pi$ if they bound a degenerate domain in $\mathcal{T}$;*

C. *Closed, simple $\mathcal{T}$-geodesic curves of (σ, Σ) have length $L < 2\pi$, or $L = 2\pi$ if they bound a degenerate domain in $\mathcal{T}$;*

D. *One of the following is true:*

1. *each time-like geodesic on $\mathcal{T}$ joins H to Σ (P is then of hyperbolic type);*
2. *$\Sigma = \emptyset$, H has two connected components H_+ and H_-, and each time-like geodesic in $\mathcal{T}$ joins H_+ to H_- (P is then of bi-hyperbolic type);*
3. *$H = \emptyset$, Σ has two connected components Σ_+ and Σ_-, and each time-like geodesic in $\mathcal{T}$ joins Σ_+ to Σ_-; and, moreover, Σ-geodesic segments in Σ_+ and Σ_- have length $L < \pi$ (P is then of compact type).*

Suppose now that (σ, Σ) satisfies properties (A), (B), (C), (D), and also:

E. *In case (D.3), Σ_+ and Σ_- are convex, with boundaries of length less than 2π.*

Then (σ, Σ) is induced on a unique convex polyhedron in $\tilde{HS}^3$.

Note that, in the first case, Σ or H can be empty; condition (D.1) then shows that $\mathcal{T} = \emptyset$, so that the polyhedron P is a finite volume hyperbolic polyhedron in the first case, and space-like and dual of a compact hyperbolic polyhedron in the second case. This corresponds to the polyhedra described by Aleksandrov [Ale51] and by Rivin-Hodgson [RH93] respectively.

I do not know whether all metrics satisfying conditions (A) to (D) are realized on convex polyhedra; condition (E) is necessary for technical reasons, namely because the proof of the connectedness of the space of metrics, which is necessary in the general proof, is difficult to carry on in full generality in case (D.3).

We will now translate this theorem in terms only of convex polyhedra in the de Sitter space. Those polyhedra might be compact or not, depending on whether their counterparts in $\tilde{\mathrm{HS}}^3$ are in the de Sitter part or intersect also the hyperbolic part. A similar line was followed in [Sch98a], but for complete polyhedra in H^3.

Already for complete, non compact polyhedra in H^3, a uniqueness assertion demands some conditions on the behavior at infinity on each end. We say that a polyhedral embedding ϕ in H^3 is **cylindrical** if, for each end y_i, there exists a 2-plane P in H^3 such that (the extension of) each face of ϕ at y_i is orthogonal to P. The following extension of theorem 1.1 holds (see [Sch98a]):

THEOREM 1.4. *Let g be a complete Riemannian metric on $S^2 \setminus \{y_1, \cdots, y_M\}$, $M \geq 2$. Suppose that the area of each end is infinite. Then g is induced by a convex, polyhedral embedding ϕ in H^3 if and only if g has constant curvature -1 except at a finite number of interior singular points $x_1, \cdots, x_N$, where it has positive singular curvature. There is a unique (up to global isometries) such ϕ which is cylindrical.*

The condition that $M \geq 2$ is not essential; for $M = 1$, however, "cylindrical" has to be understood in a slightly more general way, where the faces at the end can be either orthogonal to a given plane, or all going through a given point (which might be ideal). We leave the details to the reader, since this result follows from theorem 1.3 in all cases.

We now turn to complete polyhedra in the de Sitter space. We need an analog of the "cylindrical" condition defined for complete hyperbolic polyhedra above. We will say that a complete polyhedron in S^3_1 is **cylindrical** if, for each end e, there exists a point $x_e \in \tilde{\mathrm{HS}}^3$ such that (the extension of) each face at e goes through x_e. x_e might be on one of the "boundary" spheres separating S^3_1 from H^3_+ and H^3_-. In purely de Sitter term, this means that either all faces at e go through a point x_e (which might be an ideal point) or that all faces at e are orthogonal to a given space-like 2-plane x^*_e, which is the dual of $x_e \in H^3$.

We will also need a condition on the ends of a complete polyhedral metric. Let e be an end of such a metric, such that each face at e is modeled on S^2_1. Consider the holonomy h around e, it is an isometry of the universal cover $\tilde{S^2_1}$ of S^2_1, which might be of elliptic type (i.e. a translation in S^2_1, or a rotation in H^2), of hyperbolic type (a rotation in S^2_1) or of parabolic type. Define the **angle** $\theta(e)$ at e as the rotation angle of h if h is elliptic (then $\theta(e) \in \mathbf{R}$) or hyperbolic (and $\theta(e) \in i\mathbf{R}$), and $\theta(e) = 0$ if h is parabolic.

THEOREM 1.5. *Let (σ, Σ) be a marked HS metric on a surface S homeomorphic to S^2 minus N points ($N \geq 0$). Suppose that (σ, Σ) is induced on a convex complete polyhedron in S^3_1. Then (σ, Σ) satisfies the following properties:*

A. *(σ, Σ) is convex (as in definition 3.1) at its singular points;*

B. *Closed Σ-geodesic curves of (σ, Σ) have length $L > 2\pi$, or $L = 2\pi$ if they bound a degenerate domain in $\mathcal{T}$;*

C. *Closed, simple $\mathcal{T}$-geodesic curves of (σ, Σ) have length $L < 2\pi$;*

D. *The angle at each end is in $[0, 2\pi] \cup i\mathbf{R}_+$;*

E. *One of the following is true:*

1. *each time-like geodesic in $\mathcal{T}$ starts at Σ and goes to infinity;*
2. *$\Sigma = \emptyset$, $N = 2$, S has two ends, and each time-like geodesic joins one end to the other;*
3. *$N = 0$, Σ has two connected components Σ_+ and Σ_-, and each time-like geodesic in $\mathcal{T}$ joins Σ_+ to Σ_-; and, moreover, Σ-geodesic segments in Σ_+ and Σ_- have length $L < \pi$.*

Suppose now that (σ, Σ) satisfies properties (A), (B), (C), (D), (E) and also:

F. *The angle at each end is different from 2π, and, in case (E.3), Σ_+ and Σ_- are convex with boundaries of length less than 2π.*

Then (σ, Σ) is induced on a unique convex cylindrical polyhedron in S^3_1.

The condition that the angle at each end is different from 2π is not essential; when the angle is 2π, the polyhedron is not cylindrical, but satisfies another condition at the end, corresponding to the fact that the hyperbolic part of its extension in $\tilde{\mathrm{HS}}^3$ is flat, i.e. has no vertex.

The theorems above are illustrated in section 2 by several more explicit situations, concerning various kinds of polyhedra in E^3_1 and S^3_1. Section 3 contains elements from [Sch98a], as well as some "translations" in the context of convex polyhedra in S^3_1. The crucial results on degenerations of polyhedra are in section 4. Section 5 deals with convex caps in S^3_1, for which existence results are proved (or recalled from earlier works); connectedness lemmas are then deduced in section 6. The proof of theorem 1.3 is given in section 7, and section 8 contains the proof of theorem 1.5 from theorem 1.3, as well as some considerations about complete polyhedra. Section 9 contains some remarks concerning smooth analogues of those results, the relations between them, and a few other things. It shows, in particular, how the method of Volkov [Vol60] to prove results on convex caps can be considered as a polyhedral version of methods which are now classical for the elliptic Monge-Ampère PDEs associated to the smooth case.

The rigidity lemma of section 3 is also valid in the anti-de Sitter space H^3_1, and it is conceivable that the "compactness" results for $\tilde{\mathrm{HS}}^3$ (in section 4) also extends in some way to the space $\tilde{\mathrm{HS}}^3_1$ (see [Sch98a]) which is made of two copies of H^3_1. We might therefore hope that some results similar to those described in this paper hold in H^3_1.

Most of the statements above will be proved mainly using a "deformation method". The main technical results necessary, namely the rigidity of polyhedra and the description of degenerations when the induced metrics converge, were basically given in [Sch98a] (although the degenerations results there where not exactly as needed, and are stated and proved again here in section 4). This paper is therefore mainly concerned with questions of connectedness of spaces of metrics. The way to prove them is partly based on an alternative approach (following the ideas of Volkov [Vol60] and Milka [Mil86]) to the existence of convex caps with given induced metric, which then leads to connectedness of spaces of metrics through deformations of convex polyhedra.

I want to apologize for the changes in notations between [Sch98a] and this paper. I believe the notations used here are more convenient and coherent. Σ and $\mathcal{T}$ are used here instead of the combinatorial data given in [Sch98a] by the sets of edges A and B, and the sets of vertices S_A and S_B. The polygonal curves which were called A-admissible in [Sch98a] correspond to Σ-geodesics here (with the minor difference that Σ-geodesics can bound a degenerate domain in $\mathcal{T}$), and the B-admissible curves of [Sch98a] are now $\mathcal{T}$-geodesics.

2. Examples and applications. This section only gives translations and special cases of theorems 1.2, 1.3 and 1.5.

Consider first the compact, space-like polyhedra in S^3_1. There exist two kinds of such polyhedra. The first kind is made of those which are duals of hyperbolic convex polyhedra; their orthogonal projection on any totally geodesic space-like 2-plane is one-to-one, and they do not bound a compact domain in S^3_1. They are part of case (D.1) of theorem 1.3, and of case (E.1) of theorem 1.5. The metrics on those polyhedra have been characterized by Rivin and Hodgson in the following result, which can be considered as an extension of a classical result of Andreev [And70] on the dihedral angle of convex hyperbolic polyhedra with acute angles.

THEOREM 2.1 (Rivin, Hodgson [RH93], [Riv86]). *A Riemannian metric g on S^2 is induced by a convex polyhedral embedding of hyperbolic type in S^3_1 if and only if:*

- *g has constant curvature 1 except at a finite number of singular points $x_1, \cdots, x_N$;*
- *the singular curvature of g at $x_1, \cdots, x_N$ is negative;*
- *all closed geodesics of g have length $L > 2\pi$.*

The embedding is then unique modulo global isometries.

This theorem is obviously a special case of theorems 1.3 or 1.5. Theorem 1.3 also leads to the results of [Riv93] (see also [HIR92]) on the metrics induced on the duals of ideal hyperbolic polyhedra, and to results on the duals of polyhedra with some ideal and some non-ideal vertices.

The second kind of convex, space-like polyhedra in S^3_1 is made of those which bound a compact domain in S^3_1. They correspond to cases (D.3) of theorem 1.3, and (E.3) of theorem 1.5, that is, they are of compact type in the terminology defined above. Their orthogonal projection on a totally geodesic space-like 2-plane is not surjective, and any point in the interior of the image has two inverse images. They are duals of bi-hyperbolic polyhedra. The metrics on those compact type polyhedra which have only space-like faces are described by the following result; here, as in the previous theorem, the "singular curvature" of a polyhedral metric at a singular point is 2π minus the sum of the angles of the faces at that point (this definition obviously does not depend on the decomposition into faces).

THEOREM 2.2. *A Riemannian metric g on S^2 is induced by a convex, polyhedral embedding ϕ of compact type in S^3_1 if and only if:*

1. *g has constant curvature 1 except at a finite number of singular points $x_1, \cdots, x_N$;*
2. *$S^2 = D_+ \cup D_-$, where D_+, D_- are (topological) disks intersecting on a curve $C = \partial D_+ = \partial D_-$ of length $L < 2\pi$ with vertices the singular points $x_1, \cdots, x_p$;*
3. *the singular curvature of g at $x_{p+1}, \cdots, x_N$ is negative;*
4. *D_+, D_- are convex for g, and strictly convex at each vertex of C;*
5. *geodesic segments of D_+, D_- have length less than π.*

ϕ is then unique modulo global isometries.

Note that condition (4) implies in particular that g has positive singular curvature at $x_1, \cdots, x_p$, so that the singular points of g which lie on C are determined by g. The condition on $L(C)$ is a direct consequence of condition (C) of theorem 1.3, because C is a $\mathcal{T}$-geodesic. D_+ and D_- have to be convex because of condition (A) of theorem 1.3, leading to the special form of the convexity condition (case (7) of definition 3.1) in this situation.

A similar existence result was proved by Il'khamov and Sokolov [IS90] in E^3_1; it is a direct consequence of theorem 1.2.

THEOREM 2.3 (Il'khamov, Sokolov [IS90]). *A Riemannian metric g on S^2 is induced on a convex, compact polyhedron P in E^3_1 if and only if:*

1. *g is flat except at a finite number of singular points $x_1, \cdots, x_N$;*
2. *$S^2 = D_+ \cup D_-$, where D_+, D_- are (topological) disks intersecting on a curve $C = \partial D_+ = \partial D_-$ with vertices the singular points $x_1, \cdots, x_p$;*
3. *the singular curvature of g at $x_{p+1}, \cdots, x_N$ is negative;*
4. *D_+, D_- are convex for g, and strictly convex at each vertex of C.*

P is then unique.

A simple example of application of theorem 2.2 is given by "convex caps" in S^3_1. A convex cap (see [Pog73]) is the image of a convex polyhedral embedding of a disk sending the boundary to a totally geodesic space P, such that its orthogonal projection on P is injective. They appear here if g is identical on D_+ and D_-, since then (by uniqueness) $\phi(S^2)$ has to be the union of two symmetrical convex caps in S^3_1. Therefore:

COROLLARY 2.4. *A Riemannian metric g on D^2 is induced on a convex cap in S^3_1 if and only if:*

- *g has constant curvature 1 except at a finite number of interior singular points $x_1, \cdots, x_N$;*
- *g has negative singular curvature at $x_1, \cdots, x_N$;*
- *∂D^2 is piecewise geodesic for g, strictly convex at each vertex, with length $L < 2\pi$;*
- *geodesic segments of D^2 have length less than π.*

g is then realized on a unique convex cap, modulo global isometries.

A similar result was proved by Milka [Mil96] in E^3_1 , using (refinements of) methods developed essentially by Volkov [Vol60] to prove a similar result in E^3. Of course, the length conditions do not appear in the Minkowski case. The existence part of corollary 2.4 will actually be proved (using the method of Volkov [Vol60] and Milka [Mil96]) before theorem 1.3, and it will be used to show the connectedness of a space of metrics, a necessary point for the proof of theorem 1.3 in case (D.3).

Theorem 2.2 is the special case of theorem 1.3, (D.3), when $\mathcal{T}$ is only a curve. Another special case is obtained when D_+, D_- are points, and the result is a "cigar-like" surface in S^3_1 all of whose faces are time-like:

COROLLARY 2.5. *Let g be a polyhedral metric on S^2, locally modeled on S^2_1, except at N points $x_1, \cdots, x_N$. Suppose that:*

- *the singular curvature of g at $x_3, \cdots, x_N$ is in $i(\mathbf{R}_- \setminus \{0\})$;*
- *each time-like geodesic in $S^2 \setminus \{x_1, x_2\}$ has one end on x_1 and one on x_2;*
- *all closed, simple, space-like geodesics of $S^2 \setminus \{x_1, x_2\}$ have length $L < 2\pi$.*

Then (S^2, g) has a unique polyhedral isometric embedding in S^3_1.

Again, a similar existence result (without the lengths conditions) was proved by Gajdalovich and Sokolov [GS86] in the Minkowski space E^3_1. Their result is a consequence of theorem 1.2.

Actually, [Sch98a] contains a description of all metrics induced on compact type polyhedra in S^3_1. Here is a more precise result describing those metrics, which is again a consequence of case (D.3) of theorem 1.3:

THEOREM 2.6. *Let P be a convex polyhedron of compact type in S^3_1. Then, in the induced metric g:*

1. *$S^2 = D_+ \cup \mathcal{T} \cup D_-$, with D_+, D_- disks and $\mathcal{T}$ and annulus, such that $D_+ \cap \mathcal{T}$ and $D_- \cap \mathcal{T}$ are graphs;*
2. *g is modeled on S^2 on D_+, D_- and on S^2_1 or $S^2_{1,0}$ on $\mathcal{T}$, away from the singular points;*
3. *time-like geodesics of $\mathcal{T}$ have one end on D_- and the other on D_+;*
4. *g is "convex" (in the sense of definition 3.1) at each singular point;*
5. *all closed, simple $\mathcal{T}$-geodesics of $\mathcal{T}$ have length $L < 2\pi$;*
6. *Σ-geodesic segments of D_+, D_- have length less than π.*

If a marked polyhedral metrics satisfies (1) to (6) and, moreover, D_+ and D_- are convex and their boundaries have length less than 2π, then it is induced on a unique convex polyhedron of compact type in S^3_1.

Condition (6) was missing in [Sch98a] (no existence statement was given there, only a description of the induced metrics). It is different from the other conditions; this is clear in theorem 2.6, where it is necessary not to define the boundary of the set of metrics which can be obtained, but to single out one connected component of the space of metrics satisfying the other hypothesis. This is discussed in section 5, where assertion 5.5 should clear up things.

I do not know whether all metrics satisfying (1)-(6) are actually induced on convex polyhedra, nor whether a uniqueness result holds, without the additional assumption at the end of the theorem: again, it should be possible to extend the realizability part of this theorem by defining a wider class of metrics (satisfying (1)-(6)) and proving that it is connected.

In this theorem, D_+, D_- might be made of several disks attached at points or by segments, and $\mathcal{T}$ might be made of disks joined together by segments. In this case, conditions (3) and (4) are empty on the segments.

There are two kinds of convex (not necessarily compact) polyhedra in S^3_1, which are not of compact type. The first, which we called above **hyperbolic type** polyhedra, are those which lie on one side of a totally geodesic, space-like 2-plane (for the compact ones, the definition is coherent with the one given right before theorem 2.1). They correspond to case (E.1) of theorem 1.5. The other kind, of **bi-hyperbolic type**, are those which cross every space-like 2-plane (they can not be compact). They correspond to case (E.2) of theorem 1.5.

We can then state the following result, which is a simple consequence of [Sch98a]. Actually, both this result and theorem 1.4 are consequences of the same theorem of [Sch98a].

THEOREM 2.7. *Let g be a geodesically complete polyhedral metric on $S^2 \setminus \{y_1, \cdots, y_M\}$. Suppose that the area of each end is infinite. Then g is induced by a convex polyhedral cylindrical embedding ϕ of hyperbolic type in S^3_1 if and only if:*

- *$S^2 \setminus \{y_1, \cdots, y_M\}$ can be decomposed into domains on which g is modeled on S^2, S^2_1 or the degenerate space $S^2_{1,0}$, each domain being convex with geodesic boundary;*
- *each y_i has a neighborhood which is isometric to a domain in a quotient of $\tilde{S}^2_1$ by a translation along a space-like vector of length $|L| < 2\pi$;*
- *g is "convex" at each singular point (in the sense of definition 3.1);*
- *each closed Σ-geodesic of g has length $L > 2\pi$;*
- *closed, simple $\mathcal{T}$-geodesics of g have length $L < 2\pi$.*

ϕ is then unique modulo global isometries.

Here again, "Σ-geodesics" (see definition 3.3) are generalizations of space-like geodesics in space-like faces of the metric. The case where some end has finite area is not too interesting; it is rather easy to see (in the projective model of section 3) that there can be only one face at that end, and that its induced metric is degenerate. This end is therefore dual to an ideal end of a hyperbolic polyhedron. Both theorems 1.4 and 2.7 are special cases of the main theorem of [Sch98a].

Finally, the metrics induced on convex polyhedra of bi-hyperbolic type are as follows:

THEOREM 2.8. *Let P be a non-degenerate bi-hyperbolic polyhedron in S^3_1. Then P is homeomorphic to $S^2 \setminus \{y_+, y_-\}$, and the induced polyhedral metric g is such that:*

- *$S^2 \setminus \{y_+, y_-\}$ is complete, modeled on S^2_1 except at a finite number of singular points $y_1, \cdots, y_N$;*
- *g is "convex" at each singular point, i.e. its singular curvature is in $i(\mathbf{R}_- \setminus \{0\})$;*
- *y_+ and y_- have a neighborhood which is isometric to a domain in a quotient of $\tilde{S}^2_1$ by a translation along a space-like geodesic, with angle $\theta(y_\pm) \in (0, 2\pi)$;*
- *closed, simple $\mathcal{T}$-geodesics of g has length $L < 2\pi$.*

Conversely, each such metric is induced on a unique cylindrical bi-hyperbolic polyhedron.

3. Some tools and previous results. We recall in this section various definitions and results of [Sch98a]. The basic point is that there is a natural model of H^3

and the de Sitter space S^3_1, which can be obtained by taking the "Hilbert metric" of a quadric in S^3 (or, more generally, in S^n or $\mathbf{RP}^n$). There is a natural notion of convex polyhedra in those spaces. We obtain here the same construction in a simpler (but probably less interesting for subsequent generalizations) way.

Consider the quadratic form:

$$q(x,y,z,t) = x^2 + y^2 + z^2 - t^2$$

in $\mathbf{R}^4$. Call $\langle , \rangle$ the associated bilinear form, which is just the scalar product of the Minkowski space. For $X \in \mathbf{R}^4$, define $r(X)$ to be the unique number in $\mathbf{R}_+ \cup i\mathbf{R}_+$ such that $r^2(X) = q(X)$. For X, Y in S^3 such that $q(X) \neq 0, q(Y) \neq 0$, let $d_H(x,y)$ be a complex number such that:

$$\cosh(d_H(X,Y)) = \frac{\langle X, Y\rangle}{r(X)r(Y)}$$

d_H is then well defined modulo $2\pi i\mathbf{Z}$ and modulo sign.

We then define the sign of $d_H(X,Y)$ as follows:

- if X and Y are in the same connected component of the set of directions where $q < 0$, then $d_H(X,Y) \geq 0$;
- if they are in different connected components of that set, then $d_H(X,Y) \in i\pi + \mathbf{R}_-$;
- if $q(X), q(Y) > 0$, then $d_H(X,Y) \in i[0,\pi]$ or $d_H(X,Y) \in \mathbf{R}_-$, depending on whether $\langle X, Y\rangle \leq q(X)q(Y)$ or $\langle X, Y\rangle \geq q(X)q(Y)$;
- if $q(X) < 0$ and $q(Y) > 0$, then $d_H(X,Y) \in i\pi/2 + \mathbf{R}$.

Note that the signs are obtained directly by the definition in term of "Hilbert metric", it is then not necessary to give them explicitly. It is then fairly straightforward to check (as in [Sch98a]) that S^3 is divided into 3 parts where $q \neq 0$:

- above the tropic of Cancer (the set of points $(x,y,z,t) \in S^3$ with $t > 0$ and $q(x,y,z,t) = 0$), there is a projective model of H^3, which we call H^3_+. The distance between two points is just the hyperbolic distance;
- the situation is identical below the tropic of Capricorn (the set of points $(x,y,z,t) \in S^3$ with $t < 0$ and $q(x,y,z,t) = 0$);
- between the tropics, we find a projective model of S^3_1, where 2 points which are joined by a time-like geodesic have a real, negative distance, and two points joined by a space-like geodesic have a distance in $i[0,\pi]$. Points joined by a light-like geodesic are at distance 0;
- if $X \in H^3_+$ and $Y \in H^3_-$, then $d_H(X,Y) = i\pi - r$, where r is the distance between X and the antipode of Y;
- the polar dual of a point is just the set of points at distance $i\pi/2$ (this gives a natural extension of the usual notion of polar dual for points in S^3_1) and the dual of a p-plane is the intersection of the hyperplanes duals to its points;
- if $X \in H^3_+$ and $Y \in S^3_1$, $d_H(X,Y) = i\pi/2 + r$, where r is the (oriented) distance between X and the polar dual of Y, or the opposite.

We call $\tilde{\mathrm{HS}}^3$ the "hyperbolic-de Sitter" space obtained, i.e. S^3 minus two spheres, with the complex distance d_H. HS^3 is the quotient of $\tilde{\mathrm{HS}}^3$ by the antipodal map, so it has only one copy of H^3 and a hemisphere of S^3_1. There is a natural notion of convex polyhedra in $\tilde{\mathrm{HS}}^3$: they are the convex polyhedra in S^3 with no vertex on the two limit spheres.

The main properties of S^3_1 are easy to see in this model. Through each tangent 2-plane P goes a totally geodesic 2-plane, which is either space-like (it is then isometric to S^2 with its canonical metric, and its extension P' in $\tilde{\mathrm{HS}}^3$ does not intersect ∂H^3_+ or ∂H^3_-), light-like (it is then isometric to a degenerate space $S^2_{1,0}$, and P' meets ∂H^3_+ and ∂H^3_- at one point each), or time-like (and it is isometric to S^2_1, and P' intersects H^3_+ and H^3_-).

The duality between points and 2-planes is also fairly easy to understand now; the dual of a hyperbolic point is a space-like 2-plane in the de Sitter space, while the dual of a de Sitter point is a hyperbolic plane and also the corresponding time-like plane in the de Sitter space. The dual of a light-like plane is its intersection with ∂H^3_+ or ∂H^3_- (depending on the orientation). The class of hyperbolic polyhedra is stable by duality in $\tilde{\mathrm{HS}}^3$, and contains both the hyperbolic polyhedra and their duals (as well as the duals of ideal polyhedra, etc). Polyhedra of compact type are duals of polyhedra of bi-hyperbolic type, and conversely.

As stated in the introduction, given a convex polyhedron P in $\tilde{\mathrm{HS}}^3$, there is also a natural notion of induced metric on P. That is, each 2-face of P has a "metric", for which it is isometric to the interior of a convex polygon in $\tilde{\mathrm{HS}}^2$, the degenerate space $\mathrm{HS}^2_{1,0}$ (see [Sch98a]) or the sphere S^2 (which actually appears here with a definite negative metric, which is minus the canonical metric). Those "metrics" on the 2-faces satisfy some natural compatibility conditions on the edges, and we call a **HS metric** such a structure on a polyhedron. Those metrics are considered here up to isometry, i.e. without the decomposition into faces which is needed for their definition. Marked metrics are as defined in section 1.

We also need a special class of metrics, for which we will prove existence and uniqueness results. Given a marked metric (σ, Σ), we call $S_{\mathcal{T}}$ the set of singular points p of σ in $\mathcal{T} \cap \overline{\Sigma}$ such that all geodesic segments starting from p are space-like. Then:

DEFINITION 3.1. *A marked metric (σ, Σ) is convex at a singular point s if one of the following conditions is satisfied:*

1. *$s \in H$, and σ has positive singular curvature at s;*
2. *s is in the interior of Σ, and σ has negative singular curvature at s;*
3. *s is in the interior of $\mathcal{T}$, and the sum of the angles at s of the incident faces is $2\pi + ir$, with $r > 0$ (i.e. the singular curvature is in $i(\mathbf{R}_- \setminus \{0\})$;*
4. *s is an isolated point of Σ;*
5. *$s \in S_{\mathcal{T}}$, $\Sigma \setminus \{s\}$ has two connected components in the neighborhood of s and, in each, the sum of the angles of the faces incident at s is in $[0, \pi)$; and $\mathcal{T} \setminus \{s\}$ has two connected components in the neighborhood of s and, in each, the sum of the angles is in $i\mathbf{R}_+$;*
6. *$s \in \overline{\Sigma} \cap \overline{\mathcal{T}} \setminus S_{\mathcal{T}}$, $\mathcal{T} \setminus \{s\}$ is connected in the neighborhood of s, and the angles θ_i at s of the faces in $\mathcal{T}$ and the angles θ'_j at s of the faces in Σ satisfy:*
$$\sum_i \theta_i = \pi - ir_1 \ , \ \sum_j \theta'_j = r_2$$
with $r_1 \in \mathbf{R}, r_2 \geq 0$, and either $r_1 > 0$ or $r_2 < \pi$;
7. *$s \in \overline{\Sigma} \cap \overline{\mathcal{T}} \setminus S_{\mathcal{T}}$, $\mathcal{T} \setminus \{s\}$ is not connected in the neighborhood of s and, for each connected component C of $\mathcal{T} \setminus \{s\}$ in the neighborhood of s, the sum α of the angles at s of the faces in C is in $\pi - i(\mathbf{R}_+ \setminus \{0\})$, or $\alpha = \pi$ and all faces in C are light-like; and the sum of all angles at s is not 2π.*

In the last case, we forbid the situation where, in the neighborhood of s, $\mathcal{T} \setminus \{s\}$

has two connected components, each made of light-like faces, and Σ only has edges meeting s.

We say that a convex polyhedron is degenerate at a vertex v if, in a neighborhood of v, P is in the union of two half-planes meeting along a geodesic containing v. The basic properties of definition 3.1 are proved in [Sch98a]:

LEMMA 3.2. *If P is a convex, non-degenerate polyhedron in $\tilde{HS}^3$, then the induced marked metric σ is convex at each vertex. If, on the other hand, P is convex but is degenerate at a vertex v, then σ is not convex at v.*

The important consequence is that the marked metrics induced on polyhedra which are on the boundary of the set of convex polyhedra are not convex.

A **polygonal curve** in a polyhedral metric is a continuous, piecewise geodesic curve.

DEFINITION 3.3. *If γ is a polyhedral curve for a HS metric g on S^2, we say that γ is* **Σ-geodesic** *if:*
- *γ remains in Σ;*
- *at each vertex s of γ, the connected components of $S^2 \setminus \gamma$ which are in Σ in the neighborhood of s are concave at s.*

This can be considered an analogue of the geodesics in a Riemannian polyhedral metric. More precisely, if g is Riemannian, the Σ-geodesic curves are the same as "usual" geodesics.

DEFINITION 3.4. *If γ is a polyhedral curve for a polyhedral metric g on S^2, we say that γ is* **$\mathcal{T}$-geodesic** *if:*
- *γ is space-like, and each of its segments is in the closure of $\mathcal{T}$;*
- *if s is a vertex of γ such that all faces and edges of σ at s on one side of γ are in $\mathcal{T}$, then the sum of their angles at s is in $\pi + i\mathbf{R}_+$.*

The second condition is actually a concavity condition, too; definitions 3.3 and 3.4 are therefore similar.

The following lemma is proved in [Sch98a], using an extension to $\tilde{\mathrm{HS}}^3$ of a remarkable construction of Pogorelov [Pog73], which brings rigidity problems from non-flat to flat space-forms (see also [LS00] for a related extension of this lemma, and [Sch00] for another use of it which is not related to the problems considered here).

LEMMA 3.5. *Convex, non-degenerate polyhedra in $\tilde{HS}^3$ are rigid, i.e. any infinitesimal polyhedral deformation inducing no variation of the induced metric is trivial.*

The construction in [Sch98a] actually extends to other spaces $\tilde{\mathrm{HS}}^n_k$, each containing a pair of pseudo-Riemannian space forms. It does not deal, however, with the Minkowski space. A rigidity result there could be obtained by composing two applications defined in [Sch98a] between flat and non-flat space-forms, but this would be very clumsy. The next lemma, on the other hand, is proved in a very simple manner, by using a clever trick discovered by Galeeva, Pelipenko and Sokolov [GPS82].

LEMMA 3.6. *Let P be a compact, convex non-degenerate polyhedron in E^3_1, and let V be an infinitesimal polyhedral deformation of P. If V induces no infinitesimal variation of the induced metric, then V is trivial, i.e. it is the restriction to P of an infinitesimal isometry of E^3_1.*

Proof. Consider P as an abstract polyhedron, with a polyhedral convex embedding $\phi : P \hookrightarrow E^3_1$. Choosing an orthogonal decomposition, we can write: $E^3_1 \simeq E^2 \ominus E^1$, and $\phi = (X, z)$, with $X : P \to E^2$, and $z : P \to E^1 \simeq \mathbf{R}$. $E^3 \simeq E^2 \oplus E^1$, so we find an embedding $\psi : P \hookrightarrow E^3$, and it is easy to check that it is convex.

Now $V = (\overset{\bullet}{X}, \overset{\bullet}{z})$ induces an infinitesimal variation of the induced metric, which is zero if and only if, for $m, m' \in P$ in the same face of $\phi(P)$:

$$\langle \phi(m') - \phi(m), V(m') - V(m) \rangle = 0$$

so if and only if:

$$\langle X(m') - X(m), \overset{\bullet}{X}(m') - \overset{\bullet}{X}(m) \rangle - (z(m') - z(m))(\overset{\bullet}{z}(m') - \overset{\bullet}{z}(m)) = 0$$

so if and only if $W := (\overset{\bullet}{X}, -\overset{\bullet}{z})$ is an infinitesimal isometric deformation of $\psi(P)$ in E^3.

Now it is known, basically since Cauchy [Cau13], that convex, compact polyhedra in E^3 are rigid. $(\overset{\bullet}{X}, -\overset{\bullet}{z})$ is therefore the restriction to $\psi(P)$ of an infinitesimal isometry of E^3, which we can write as $(\nu, -\eta)$ in $E^3 \simeq E^2 \oplus E^1$. Then (ν, η) is an infinitesimal isometry of $E^3_1 \simeq E^2 \ominus E^1$ inducing $(\overset{\bullet}{X}, \overset{\bullet}{z})$ on $\psi(P)$, so V is trivial. □

The same proof actually shows the rigidity of convex caps in E^3_1, from the rigidity of convex caps in E^3, because the deformations which have to be considered are those which, on a totally geodesic space-like 2-plane P_0, are tangent to P_0. P_0 of course has to be parallel to the factor E^2 in the decomposition of E^3_1 used.

4. Degenerations of polyhedra. This section contains the compactness results needed for the proofs of the main results of this paper. It describes how a sequence of polyhedra can behave when the sequence of induced metrics converges.

The simplest result concerns polyhedra in E^3_1:

LEMMA 4.1. *Let P be a (combinatorial) polyhedron homeomorphic to S^2, and $(\phi_n)_{n\in\mathbf{N}}$ a sequence of polyhedral convex embeddings of P into E^3_1. Suppose that the sequence of induced metrics $(\mu_n)_{n\in\mathbf{N}}$ converges to a limit μ_∞. Then there exists a sequence $(\rho_n)_{n\in\mathbf{N}}$ of isometries of E^3_1 such that $(\rho_n \circ \phi_n)_{n\in\mathbf{N}}$ has a converging subsequence.*

Proof. Restricting (ϕ_n) to a subsequence if necessary, we can choose two points $m_0, m_1 \in S^2$ and a sequence (ρ_n) of isometries such that, for all n, $(\rho_n \circ \phi_n)(m_0) = (0, 0, 0)$ and $(\rho_n \circ \phi_n)(m_1) = (0, 0, z_n)$ in the canonical coordinate system of E^3_1.

Then there exists $R > 0$ such that, for all n, the projection of $(\rho_n \circ \phi_n)(P)$ on the (x, y)-plane remains in the ball of radius R centered at 0. Otherwise, by convexity, there would exist a sequence of horizontal planes (P_n) such that the length of $P_n \cap (\rho_n \circ \phi_n)(P) \to \infty$; therefore, since P_n intersects only a bounded number of faces of $(\rho_n \circ \phi_n)(P)$, (μ_n) could not converge.

Now this implies that there exists some $Z(R) > 0$ such that, for all n, $z_n \in [-Z, Z]$; otherwise, for any vertical plane Π going through 0, the length of the time-like part of $\Pi \cap (\rho_n \circ \phi_n)(P)$ would go to infinity, which, again for the same reason, is impossible.

Therefore, $(\rho_n \circ \phi_n)(P)$ remains in a bounded region of E^3_1, so some subsequence of $(\rho_n \circ \phi_n)$ converges. □

Note that the proof above (and the statement of lemma 4.1) could be given in any flat pseudo-Riemannian space-form E^p_q without any important difference. This extension, however, is not necessary here.

[Sch98a] contains a description of degenerations of convex polyhedra of hyperbolic type in $\tilde{\mathrm{HS}}^{n+1}$ when the induced metrics converge; but it is both too general (because it deals with higher dimensions) and not specific enough for here (it is precise enough only for polyhedra of hyperbolic type, which were the main object of the study of [Sch98a]). Theorem 4.2 below is simpler than the corresponding result of [Sch98a]. It could be extended in higher dimension, but then we would need some additional definitions for marked metrics and concave or geodesic polyhedral hypersurfaces, so we have not done this here.

THEOREM 4.2. *Let P be a (combinatorial) polyhedron homeomorphic to S^2, and $(\phi_n)_{n\in\mathbf{N}}$ a sequence of polyhedral convex embeddings of P into $\tilde{HS}^3$. Suppose that the sequence of induced marked HS metrics $(\mu_n)_{n\in\mathbf{N}}$ converges to a limit μ_∞ which is a marked HS metric. Then:*

1. *either there exists a sequence $(\rho_n)_{n\in\mathbf{N}}$ of isometries of $\tilde{HS}^3$ such that $(\rho_n \circ \phi_n)_{n\in\mathbf{N}}$ has a subsequence converging to a polyhedral embedding ϕ_0 of P into $\tilde{HS}^3$;*
2. *or μ_∞ has a Σ-geodesic of length 2π;*
3. *or μ_∞ has a simple $\mathcal{T}$-geodesic of length 2π.*

Proof. In all the proof, we use the projective model of $\tilde{\mathrm{HS}}^3$, i.e. a map $\alpha_0 : \tilde{\mathrm{HS}}^3 \to S^3$ sending the geodesic segments to geodesic segments (see [Sch98a]). Convergence of sequences of polyhedra means convergence in S^3 of the images through the projective model. We denote by $\tilde{Q}^3$ the disjoint union of two spheres which are removed from S^3 to obtain $\tilde{\mathrm{HS}}^3$.

First we choose $(\rho_n)_{n\in\mathbf{N}}$ so that $(\alpha_0 \circ \rho_n \circ \phi_n(P))$ does not converge to a point x_0 in $\tilde{Q}^3$. This is possible because, if such a convergence occurs, applying a sequence of isometries of $\tilde{\mathrm{HS}}^3$ having x_0 as a repulsive fixed point destroys it.

Now $(\alpha_0 \circ \rho_n \circ \phi_n(P))$ is a sequence of convex polyhedra in S^3, so it has a subsequence $(\alpha_0 \circ \phi'_n)_{n\in\mathbf{N}}$ converging to a limit $\alpha_0 \circ \phi'_\infty$. If the limit sends no vertex of P on $\tilde{Q}^3$, then we are in the first case of theorem 4.2. We suppose now that this is not the case. So there exists a vertex $v \in P$ such that $\alpha_0 \circ f_n(v) \to s_0 \in \tilde{Q}^3$.

Let Q_0 be the tangent plane to $\tilde{Q}^3$ at s_0; by proposition 6.4 of [Sch98a], some neighborhood of s_0 in $\alpha_0 \circ \phi'_\infty(P)$ is contained in Q_0. The proof of this proposition is easy: otherwise, the distance between some vertex of P going to s_0 and some other vertex of P would go to infinity in the induced metric, and this contradicts the fact that (μ_n) converges.

Since Q_0 is a degenerate plane, $L(\partial(Q_0\cap\alpha_0\circ\phi'_\infty(\Pi))) = 2\pi$, and the corresponding curve in P also has length 2π for μ_∞. We have to show that this curve is either Σ-geodesic or $\mathcal{T}$-geodesic. There are now two cases to consider. In the first, the convex domain Ω_∞ bounded by $\alpha_0 \circ \phi'_\infty(P)$ is on the same side of Q_0 as the connected component of $\tilde{Q}^3$ containing s_0. In the second, it is on the other side.

The proof of theorem 6.3 of [Sch98a] applies almost as it is to the first case, we repeat it rapidly here. Let v be a vertex of $\partial(Q_0 \cap (\alpha_0 \circ \phi'_\infty(P)))$. Choose a small enough convex neighborhood C of v in $Q_0 \cap (\alpha_0 \circ \phi'_\infty(P))$, and consider the inverse images C_n of C in P by $\pi_0 \circ \alpha_0 \circ \phi'_n$, where π_0 is the orthogonal projection (in S^3) on Q_0. If μ_n is space-like on C_n, then (C_n) converges to a convex degenerate set for μ_∞, because the induced metric on Q_0 is degenerate. Therefore, if μ_∞ is space-like on the side corresponding to $Q_0 \cap \alpha_0 \circ \phi'_\infty(P)$, then $\partial(Q_0 \cap (\alpha_0 \circ \phi'_\infty(P)))$ is geodesic for μ_∞ on $Q_0 \cap (\alpha_0 \circ \phi'_\infty(P))$. We also have to prove that the other side is concave;

for this, we replace $Q_0 \cap (\alpha_0 \circ \phi'_\infty(P))$ by a slightly tilted space-like plane, to obtain a new convex polyhedron for which v is a vertex in the interior of Σ, with the same faces as $\phi'_\infty(P)$, except the faces corresponding to $Q_0 \cap \alpha_0 \circ \phi'_\infty(P)$, and with one additional space-like face F. Since this polyhedron is convex at v, its induced metric in $\tilde{\mathrm{HS}}^3$ is convex at v, i.e. the total angle at v is more than 2π. But since F is convex, its angle is less than π. Therefore, the total angle of all the other faces is more than π. This means that the complement of F is concave at v. Since this can be done with F as close as we want of $\alpha_0(Q_0 \cap \alpha_0 \circ \phi'_\infty(P))$ near v, is shows that the complement of $Q_0 \cap \alpha_0 \circ \phi'_\infty(P)$ is concave at P for μ_∞, as needed.

In the second case (when Ω_∞ and the connected component of $\tilde{Q}^3$ containing s_0 are on opposite sides of Q_0) the proof is similar. The proof that $\partial(Q_0 \cap \alpha_0 \circ \phi'_\infty(P))$ is geodesic at v for μ_∞ on $Q_0 \cap \alpha_0 \circ \phi'_\infty(P)$ is the same as above; for the other side, we replace $Q_0 \cap \alpha_0 \circ \phi'_\infty(P)$ by a time-like face F', so as to obtain a new polyhedron P' for which v is in the interior of $\mathcal{T}$. Then, if all faces on the complement of F' in the neighborhood of v in P' are time-like or degenerate, we apply lemma 3.2 (or more specifically proposition 8.7 of [Sch98a]), which shows that the total angle of P' at v is $2\pi + ir$, with $r > 0$. But F' is convex at v, so its angle at v is $\pi - ir'$, with $r' \geq 0$. The total angle of the other faces is therefore $\pi + ir$, $r \geq 0$, and this proves again that the side of $\partial(Q_0 \cap \alpha_0 \circ \phi'_\infty(P))$ opposite to $Q_0 \cap \alpha_0 \circ \phi'_\infty(P)$ is concave, as needed. □

An important point is that, for each type of polyhedra, only one kind of degeneration (along Σ-geodesics or $\mathcal{T}$-geodesics) always happens. It was proved in [Sch98a] for polyhedra of hyperbolic type, where a Σ-geodesic of length 2π always appears in the degenerate case (this is not to say that a $\mathcal{T}$-geodesic of length 2π can not also appear). In the other two cases — polyhedra of compact or of bi-hyperbolic type — it is easy to see that there is a $\mathcal{T}$-geodesic curve of length 2π in the limit, simply because, in the proof of theorem 4.2, $\alpha_0 \circ \phi'_\infty(P)$ and the connected component of $\tilde{Q}^3$ containing s_0 are always on opposite sides of Q_0.

5. Existence of convex caps. We prove in this section the existence of embeddings inducing certain polyhedral metrics on the disk D^2, and whose images are polyhedral "convex caps" in S^3_1. The corresponding result for convex caps in E^3_1 was proved by Milka; the method used here is similar, but new hypotheses are needed, namely that the length of the boundary curve of the cap is strictly less than 2π, and that geodesic segments in D^2 have length less than π.

In the next section, we will use the results of this section to prove the connectedness of some spaces of metrics, a key point in proving the existence and uniqueness of embeddings inducing given metrics on S^2 in S^3_1 or in E^3_1.

The methods used here were developed by Volkov [Vol60] for convex caps in E^3, and used also by Milka [Mil96] in E^3_1. We only refer to Milka's very carefully written paper for references to the tricky point of retriangulations, which appear here exactly in the same way as in [Mil96].

Recall that a polyhedral convex cap, in E^3_1 or S^3_1, is a polyhedral space-like embedding ϕ of the disk sending the boundary to a polygonal curve γ in a totally geodesic plane P_0, and such that the orthogonal projection to P_0 sends $\phi(D^2)$ to the inside of γ.

LEMMA 5.1 (Milka [Mil96]). *Let g be a flat, Riemannian polyhedral metric on D^2; suppose that g has negative curvature at the inner vertices, and that ∂D^2 is strictly convex at the boundary vertices, with $L(\partial D) < 2\pi$. Then g is induced on a (unique) polyhedral convex cap in E^3_1.*

We refer the reader to [Mil96] for the proof. But we will prove the de Sitter analogue (without the uniqueness now, it will come as a consequence of theorem 2.2). It is a reformulation of the existence part of corollary 2.4. The proof of lemma 5.2 below can be applied, with a few differences and simplifications, to prove the existence part of lemma 5.1.

LEMMA 5.2. *Let g be a spherical, polyhedral, Riemannian metric on D^2; suppose that g has negative curvature at the inner vertices, that ∂D^2 is strictly convex at the boundary vertices with $L(\partial D^2) < 2\pi$, and that geodesic segments of D^2 have length less than π. Then g is induced on a convex polyhedral cap in S^3_1.*

The last condition, concerning lengths of geodesic segments, might at first sight appear redundant. A closer look reveals that it is not, and that some metrics on D^2 satisfy the other conditions, but not this one. A crucial point, however, is that those metrics can not be obtained by deforming any "usual" metric on D^2 (say, the metric on a spherical disk of radius $R < \pi/2$) among metrics satisfying the other hypothesis of 5.2. In other words, the space of polyhedral, spherical metrics on D^2 with negative singular curvature, such that ∂D^2 is convex with $L(\partial D^2) < 2\pi$, is not connected; one connected component contains only metrics with no geodesic segment of length above π.

The proof of lemma 5.2 is parallel to the one in [Mil96], and the notations here are similar, of course with some added details concerning the lengths of "geodesic" curves. The approach used here, however, avoids some of the technicalities of [Mil96]. We call a **prism** a domain of S^3_1 bounded by a space-like convex polygon $\overline{P}$, by part of the cylinder made of the union of the (time-like) geodesics orthogonal to $\overline{P}$ and intersecting $\partial\overline{P}$, and by another space-like polygon P "above" $\overline{P}$, i.e. whose orthogonal projection on the plane containing $\overline{P}$ has image $\overline{P}$. For each vertex v of $\overline{P}$, the **height** of v is the length of the time-like geodesic joining v to the corresponding vertex of P.

A **prismatic cap** is a union of prisms glued along their "vertical" sides, so that the union of the "lower bases" $\overline{P}_i$ forms a convex surface $\overline{S}$ (which might have singular points), and so that the "upper bases" P_i meet the corresponding "lower bases" $\overline{P}_i$ on $\partial\overline{S}$. We call $\overline{S}$ the **lower surface**, and S the **upper surface**, which is the union of the polygons P_i. We say that the prismatic cap is **normal** if the singular curvature at each vertex of $\overline{S}$ is non-positive.

Given a spherical polyhedral metric g on D^2, call $\mathcal{C}$ the set of normal prismatic caps whose upper surface is isometric to g. If g has negative curvature at each vertex, then $\mathcal{C}$ is not empty, because it contains at least the "trivial" prismatic cap whose upper and lower surfaces both are (D^2, g), with any polygonal decomposition (e.g. a triangulation). Call $\mathcal{C}_0$ the connected component in $\mathcal{C}$ of this "trivial" prism.

We need to understand what is the link of a vertex v of S for some $C \in \mathcal{C}$. Let e be the vertical edge between v and the corresponding vertex $\overline{v}$ in $\overline{S}$. In a neighborhood of e, C is obtained by gluing prisms P_i along e. The link of each P_i at e is a segment (of length the dihedral angle of P_i at e), and the link of C at e is a closed curve of length the total angle θ_e around e (if C is normal, then $\theta_e \geq 2\pi$). The link of each point $p \in e \setminus \{v, \overline{v}\}$ is therefore obtained by taking the universal cover of $\tilde{\mathrm{HS}}^2$ minus two antipodal hyperbolic points (which correspond to e) and quotienting by a translation along a space-like vector of $\tilde{S}^2_1$ of length θ_e. We call $\tilde{\mathrm{HS}}^2_{\theta_e}$ this space. The link of S at v is a convex polyhedral curve in $\tilde{\mathrm{HS}}^2_{\theta_e}$.

For $C \in \mathcal{C}$, call **height** of C the sum of the heights of the vertices of the polyhedra

$\overline{P}_i$ (counted once each, i.e. without the multiplicities). The proof of the existence of a polyhedral cap proceeds by a maximization argument on the heights of prismatic caps.

PROPOSITION 5.3. *If each point of (D^2, g) is at distance strictly less than $\pi/2$ from the boundary, then there exists a normal prismatic cap of maximal height in $\mathcal{C}_0$.*

This maximal element is the required polyhedral cap:

PROPOSITION 5.4. *A normal prismatic cap of maximal height is a polyhedral cap, i.e. it is isometric to the convex domain in S^3_1 bounded by a convex cap and its orthogonal projection to the plane containing its boundary.*

The proof of lemma 5.2 clearly follows; the hypothesis of proposition 5.3 is satisfied because geodesic segments have length $L < \pi$, and, by proposition 5.4, the maximal prismatic cap is a polyhedral cap, whose upper surface is isometric to (D^2, g) as needed.

To prove proposition 5.3, we must show that prismatic caps can not "degenerate" when we raise the heights of the vertices. In other terms, we have to find upper bounds on the heights and on the diameter of the lower surface, depending only on the length of the boundary ∂D^2. The first point is that, for all prisms in $\mathcal{C}_0$, $\overline{S}$ contains no geodesic segment of length above π:

ASSERTION 5.5. *Let $(g_t)_{t\in[0,1]}$ be a 1-parameter family of spherical polyhedral metrics on D^2 with negative curvature at the interior singular points, such that ∂D^2 is convex with length less than 2π. Suppose that geodesic segments of g_0 have length less than π. Then geodesic segments of g_1 have length less than π.*

Proof. Since polyhedral, spherical metrics with negative singular curvature and convex boundary can be approximated by smooth metrics with curvature $K \leq 1$ and convex boundary, it is enough to prove the smooth analogue: if $(h_t)_{t\in[0,1]}$ is a 1-parameter family of metrics on D^2 with $K \leq 1$ and convex boundary of length $L(\partial D^2) < 2\pi$, if geodesic segments of h_0 have length less than π, then the same is true of h_1. The proof in the polyhedral case can actually be done along the same lines, with a little more care.

Let t_0 be the infimum of all $t \in [0, 1]$ such that h_t contains a geodesic segment of length at least π. The key point is that, in each h_t for $t \leq t_0$, geodesic segments of length less than π are minimizing. To see this, fix $t \leq t_0$, and let l_0 be the infimum of all l such that there exists a non-minimizing geodesic segment of length l. There are a priori 4 possibilities:

- some geodesic segment of length l_0 in $\overline{D}^2$ meets ∂D^2 at an interior points; this is impossible here since ∂D^2 is convex;
- some geodesic segments of length l_0 has conjugate endpoints, but then $l_0 \geq \pi$ since $K \leq 1$;
- there are two geodesic segments of length l_0 in D^2 with the same end-points, and with angle less than π at one end; this is impossible again, since it is easy to see, by moving a little one of the ends, that some geodesic segment of length less than l_0 should then be non-minimizing;
- there are two geodesics of length l_0 with the same end-points, meeting at angle π; those geodesics therefore add up to a closed geodesic γ_0. Since $K \leq 1$, some point in the disk D_0 bounded by γ_0 would have to be at distance at least $\pi/2$ from γ_0. This is proved in a classical way by considering the variation of the length of the curves made of points of D_0 at distance r from γ_0, and

showing that the derivatives of those lengths can not vanish for $r \leq \pi/2$. But then D_0 should contain a geodesic segment of length strictly larger than π, which contradicts the definition of t_0.

Now consider h_{t_0}; it contains a maximal geodesic segment of length π, with end-points x_0, y_0 on ∂D^2. As we have seen above, this geodesic segment is minimizing. Therefore, each connected component of $\partial D^2 \setminus \{x_0, y_0\}$, which is a path joining x_0 to y_0, has length at least π, so $L(\partial D^2) \geq 2\pi$, a contradiction. □

This means that heights can not become too large, because of the:

ASSERTION 5.6. *For each $d < \pi/2$, there exists $h > 0$ such that, if γ_0 is a space-like geodesic segment in S^2_1, and γ a polygonal space-like line in S^2_1, sharing an end-point with γ_0; if the orthogonal projection of γ to γ_0 is one-to-one, and if $L(\gamma_0) \leq d$, then the distance between γ_0 and each point of γ is at most h.*

"Distance" here means the length of the (time-like) curve orthogonal to γ_0 joining γ_0 to each point of γ. The proof is a simple exercise in the geometry of S^2_1, so we leave it to the reader.

Proof of proposition 5.3. By assertion 5.5, each prism in $\mathcal{C}_0$ has a lower surface $\overline{S}$ with no geodesic segment of length at least π. So each point of those $\overline{S}$ is at distance less than $\pi/2$ from $\partial\overline{S}$. By compactness, there exists some $d < \pi/2$ such that, for each $c \in \mathcal{C}_0$, each point of $\overline{S}$ is at distance at most d from $\partial\overline{S}$. Assertion 5.6 then shows that the heights of all vertices of the prisms $c \in \mathcal{C}_0$ is bounded by some $h > 0$. Moreover, the diameters of the "lower bases" are bounded by π. Therefore, a maximal element exists. □

Proof of proposition 5.4. Let P_0 be a normal prismatic cap whose height is maximal. Let $\overline{V}_0$ be the set of vertices of $\overline{S}$ which are on the boundary, or at which the singular curvature of $\overline{S}$ is 0, and let $\overline{V}_1$ be the set of the other vertices of $\overline{S}$. Call V_0, V_1 the sets of vertices corresponding to $\overline{V}_0, \overline{V}_1$ on the upper surface. We suppose (by contradiction) that $V_1 \neq \emptyset$.

The first point is to show that it is possible to find a triangulation of S which is compatible with the metric of the prismatic cap, that is, such that any geodesic segment of S, which joins a vertex of V_0 and a vertex of V_1, and only cuts edges where the sum of the angles of the prisms is π, is an edge of the triangulation. It is proved in [Mil86] that such a "retriangulation" is possible in the E^3_1 setting, but it applies just in the same way in S^3_1, so we refer the reader to it for details.

Consider now the modification of P_0 obtained by leaving invariant the heights of vertices in V_0, and raising the height of vertices in V_1 by some constant ϵ. For ϵ small enough, the retriangulation alluded to means that the result is again a convex prismatic cap , which we call P_1, with upper surface S' and lower surface $\overline{S}'$. We will show that, again for ϵ small enough, it is also normal, contradicting the definition of P_0.

Let $\overline{v}$ be a vertex of $\overline{S}$ in V_1. By definition of V_1, the curvature of $\overline{S}$ at $\overline{v}$ is negative, so, if ϵ is small enough, it remains so in the new prismatic cap.

Consider a vertex $\overline{v} \in \overline{V}_0$ of $\overline{S}$, and let v be the corresponding vertex on S. If $\overline{v}$ is a boundary vertex, there is nothing to prove; if $\overline{v}$ is an inner vertex, we must show that the singular curvature of $\overline{S}$ at $\overline{v}$ remains non-positive.

Let L be the link of S at v. As explained above, it is a convex polygon in $\tilde{\mathrm{HS}}^2_{2\pi} = \tilde{\mathrm{HS}}^2$. The link L' of S' at v is a convex polygon in $\tilde{\mathrm{HS}}^2_\theta$, and we want to prove that $\theta \geq 2\pi$. L' has the same edge length as L (because they correspond to the inside angles of the faces of S and S', which are isometric). Since P_1 is obtained from P_0 by raising some vertices around v, L' is obtained from L by moving some vertices towards the concave side of L, keeping the edge lengths fixed, but going from a polygon in $\tilde{\mathrm{HS}}^2 = \tilde{\mathrm{HS}}^2_{2\pi}$ to a polygon in $\tilde{\mathrm{HS}}^2_\theta$.

Now note that the duality defined above in $\tilde{\mathrm{HS}}^2$ extends to $\tilde{\mathrm{HS}}^2_\theta$, at least for convex, space-like polygons in the de Sitter part of $\tilde{\mathrm{HS}}^2_\theta$; this can be seen by defining the dual of such a polygon as the hyperbolic polygon whose vertices are the dual points of the edges, each defined as the only point in $\tilde{\mathrm{HS}}^2_\theta$ which is at distance $i\pi/2$ on a geodesic segment orthogonal to the edge (with the right orientation).

Let L^* be the dual polygon of L, and L'^* the dual of L'. L^* is a convex polygon in the hyperbolic part of $\tilde{\mathrm{HS}}^2$, and L'^* is a convex polygon in the hyperbolic part of $\tilde{\mathrm{HS}}^2_\theta$, around one of the singular points of $\tilde{\mathrm{HS}}^2_\theta$. Since L and L' have the same edge lengths, L^* and L'^* have the same angles θ_i. L'^* is obtained from L^* by moving some of its edges towards the convex side. If $\theta < 2\pi$, this implies that the area A' of the convex domain bounded by L'^* is less than the area A of the convex domain bounded by L^*.

Applying the Gauss-Bonnet theorem to L^* shows that:

$$\sum_i \theta_i = 2\pi + A$$

while the same theorem applied to L'^* leads to:

$$\sum_i \theta_i = \theta + A'$$

But if $\theta \leq 2\pi$, then $A' < A$, and we find a contradiction.

But the height of P_0 was assumed to be maximal, so that V_1 had to be empty, so that P_0 was already a convex cap. □

As stated above, the same proof works for convex caps in E^3_1; it corresponds there to a slightly simplified version of the proof given by Milka [Mil86]. It is also possible to use the same kind of ideas to give the existence part of theorem 2.1. In this case, the order relation should not be with respect to the sum of the height, but to the set of the heights at all vertices of the "lower surface" (which in this case is a sphere). The maximal element is no longer unique, but still corresponds to a polyhedral isometric embedding in S^3_1.

6. Spaces of metrics. We will define in this section some natural spaces of metrics (those appearing in the main theorems of the introduction) and prove that they are connected.

We need further combinatorial data about the way the upper and lower boundaries of the time-like part $\mathcal{T}$ of a metric might share some vertices and/or edges, and about how the singular points in the interior of $\mathcal{T}$ are located with respect to those shared edges.

Let $q_-, q_+ \in \mathbf{N}^*$ be the number of vertices of those two curves. The vertices of the lower curve can be associated to elements of $\mathbf{Z}/q_-\mathbf{Z}$, and those of the upper curve to elements of $\mathbf{Z}/q_+\mathbf{Z}$. We need to describe first which vertices of the upper curve coincide with which vertices of the lower curve.

Call $\overline{\mathcal{Q}}(q_-, q_+)$ the set of subsets $\overline{Q} \subset \mathbf{Z}/q_-\mathbf{Z} \times \mathbf{Z}/q_+\mathbf{Z}$ such that:

- for each $r \in \mathbf{Z}/q_-\mathbf{Z}$ (resp. $r \in \mathbf{Z}/q_+\mathbf{Z}$), there exists at most one pair $(s,t) \in \overline{Q}$ such that $s = r$ (resp. such that $t = r$);

- if $(s_1, t_1), (s_2, t_2), (s_3, t_3) \in \overline{Q}$ with $s_1 \leq s_2 \leq s_3$, then $t_1 \leq t_2 \leq t_3$ (i.e. t_1, t_2 and t_3 are ordered in the cyclic order of $\mathbf{Z}/q_+\mathbf{Z}$).

Each such $\overline{Q} \in \overline{\mathcal{Q}}(q_-, q_+)$ has a cyclic ordering, defined by: $(s_1, t_1) \leq (s_2, t_2) \leq (s_3, t_3)$ if $s_1 \leq s_2 \leq s_3$. For $\overline{Q} \in \overline{\mathcal{Q}}(q_-, q_+)$, define $F_{\overline{Q},r}$ as the set of maps from $\overline{Q}$ to $\mathbf{Z}/r\mathbf{Z}$ respecting the cyclic ordering on both sides.

Let:

$$\mathcal{Q}(q_-, q_+, r) = \{(\overline{Q}, f) \mid \overline{Q} \in \overline{\mathcal{Q}}(q_-, q_+), f \in F_{\overline{Q},r}\}/\simeq$$

where $(\overline{Q}, f) \simeq (\overline{Q}', f')$ if there exists $p_{-,0} \in \mathbf{Z}/q_-\mathbf{Z}, p_{+,0} \in \mathbf{Z}/q_+\mathbf{Z}, s_0 \in \mathbf{Z}/r\mathbf{Z}$ such that:

$$(p_-, p_+) \in \overline{Q} \Leftrightarrow (p_- + p_{-,0}, p_+ + p_{+,0}) \in \overline{Q}'$$
$$\forall (p_-, p_+) \in \overline{Q}, f'(p_- + p_{-,0}, p_+ + p_{+,0}) = f(p_-, p_+) + s_0 \ .$$

If C_-, C_+ are two simple closed space-like polygonal curves in a Lorentzian surface, bounding a compact connected domain Ω containing r marked points, with q_- and q_+ vertices respectively, we say that $Q = (\overline{Q}, f) \in \mathcal{Q}(q_-, q_+, r)$ describes the intersection of C_- and C_+ if:

- there is an order-preserving bijection between the vertices of C_- (resp. C_+) and $\mathbf{Z}/q_-\mathbf{Z}$ (resp. $\mathbf{Z}/q_+\mathbf{Z}$) such that $\overline{Q}$ corresponds to the pairs (v_-, v_+) where v_- and v_+ are vertices of C_- and C_+ respectively, which coincide;
- if (s, t) and (s', t') are successive elements of $\overline{Q}$, then $f(s', t') - f(s, t)$ is equal to the number of marked points of Ω which are in the subdomain of Ω bounded by the points $v, v' \in C_- \cap C_+$ corresponding respectively to (s, t) and to (s', t').

Here are the necessary definitions of the spaces of metrics. First, for theorem 1.2:

DEFINITION 6.1. *For $p_+, p_-, q_+, q_-, r \in \mathbf{N}$, and $Q \in \mathcal{Q}(q_-, q_+, r)$, $E(p_+, p_-, q_+, q_-, r, Q)$ is the set of metrics on S^2 satisfying the hypothesis of theorem 1.2, with p_+ singular points in the interior of Σ_+, p_- in the interior of Σ_-, q_+ on $\partial\Sigma_+$, q_- on $\partial\Sigma_-$, r points in the interior of $\mathcal{T}$, and such that Q describes the intersection between the lower and upper boundaries of $\mathcal{T}$. $E_C(p_+, p_-, q_+, q_-, r, Q)$ is the set of those metrics for which Σ_+ and Σ_- have convex boundary.*

For polyhedra of compact type in $\tilde{\mathrm{HS}}^3$:

DEFINITION 6.2. *For $p_+, p_-, q_+, q_-, r \in \mathbf{N}$ and $Q \in \mathcal{Q}(q_-, q_+, r)$, $C(p_+, p_-, q_+, q_-, r, Q)$ is the set of metrics on S^2 satisfying (A), (B), (C), (D.3) and (E) of theorem 1.5, with p_+ singular points in the interior of Σ_+, p_- in the interior of Σ_-, q_+ on $\partial\Sigma_+$, q_- on $\partial\Sigma_-$, r points in the interior of $\mathcal{T}$, and such that Q describes the intersection between the lower and the upper boundaries of $\mathcal{T}$.*

Nothing is necessary for polyhedra of hyperbolic type, since they were already studied in [Sch98a], where the corresponding part of theorem 1.3 was already proved. Finally, the corresponding definition is simpler for polyhedra of bi-hyperbolic type:

DEFINITION 6.3. *For $p_+, p_-, r \in \mathbf{N}$, $B(p_+, p_-, r)$ is the set of metrics on S^2 satisfying (A), (B), (C), (D.2) and (E) of theorem 1.3, with p_+ singular points in the interior of H_+, p_- in the interior of H_-, and r points in the interior of $\mathcal{T}$.*

The main result of this section is:

LEMMA 6.4. *Let $p_+, p_-, q_+, q_-, r \in \mathbf{N}$ and $Q \in \mathcal{Q}(q_-, q_+, r)$. If $p_+ + p_- + r \geq 3$, $B(p_+, p_-, r)$ is connected. If $q_+, q_- \geq 2$, then $E(p_+, p_-, q_+, q_-, r, Q)$ and $C(p_+, p_-, q_+, q_-, Q, r)$ are connected.*

The proof of lemma 6.4 in the setting of theorem 1.2 (that is, the connectedness of E) is done by first proving that any metric on the upper of lower space-like part of σ can be deformed to a metric which has convex boundary, without changing the lengths of the boundary edges (proposition 6.7). Lemma 5.1 is then used to deform metrics on the "upper" and "lower" parts of σ, and a "surgery" on the time-like part of σ finishes the proof.

The space of metrics corresponding to this time-like part is described as follows. Remember that a space-like curve in a Lorentzian surfaces is simple if it intersects any maximal time-like curve exactly once.

DEFINITION 6.5. *Let $q_+, q_- \geq 2$, $r \in \mathbf{N}$, and $Q \in \mathcal{Q}(q_-, q_+, r)$. Let*

$$L^+ = (l_1^+, l_2^+, \cdots, l_{q+}^+) \in (\mathbf{R}_+ \setminus \{0\})^{q+},$$
$$L^- = (l_1^-, l_2^-, \cdots, l_{q_-}^-) \in (\mathbf{R}_+ \setminus \{0\})^{q-}.$$

$T^E(q_+, q_-, r, Q; L^+, L^-)$ is the space of polyhedral metrics g on $S^1 \times [0,1]$ such that:

- *$S^1 \times [0,1]$ has a decomposition into pieces on which g is isometric to the inside of a convex polygon in $\mathbf{R}^2_1$ or $\mathbf{R}^2_{1,0}$ such that the metrics on each side of an edge have the same restriction;*
- *at each interior vertex, the sum of the angles of the adjacent faces (for any triangulation) is in $2\pi + i(\mathbf{R}_+ \setminus \{0\})$;*
- *$S^1 \times \{1\}$ and $S^1 \times \{0\}$ are simple space-like curves, piecewise geodesic with q_+ and q_- segments respectively, with lengths iL^+ and iL^- respectively;*
- *Q describes the intersection between $S^1 \times \{0\}$ and $S^1 \times \{1\}$.*

$T^E(q_+, q_-, r, Q)$ is the union of the $T^E(q_+, q_-, r; L^+, L^-)$ for all possible values of L^+, L^-.

The third condition means that the successive lengths of the geodesic segments in $S^1 \times \{1\}$ are $l_1^+, \cdots, l_{q+}^+$ in this order, and similarly for L^-.

We also need spaces of metrics corresponding to the upper and lower parts of σ:

DEFINITION 6.6. *Let $p, q \in \mathbf{N}$ with $q \geq 2$. $B^E(p,q)$ is the space of polyhedral metrics h on the disk D^2 which are flat Riemannian (modeled on $\mathbf{R}^2$) except at p interior points, where the singular curvature is negative, with q vertices on the boundary; $C^E(p,q)$ is the set of elements of $B^E(p,q)$ for which the boundary is convex at all boundary vertices. Let $L = (l_1, \cdots, l_q) \in (\mathbf{R}_+ \setminus \{0\})^q$; $B^E(p,q;L)$ is the set of elements of $B^E(p,q)$ with successive boundary edge lengths $l_1, \cdots, l_q$, and $C^E(p,q;L) = B^E(p,q;L) \cap C^E(p,q)$.*

The connectedness of E follows from the following propositions, which will be proved below:

PROPOSITION 6.7. *Let $p \in \mathbf{N}, q \geq 2, L \in (\mathbf{R}_+ \setminus \{0\})^q$. For any $g_0 \in B^E(p,q;L)$, there exists a continuous path $(g_t)_{t \in [0,1]}$ in $B^E(p,q;L)$ such that $g_1 \in C^E(p,q;L)$, and, moreover, that all boundary vertices which are convex for g_0 remain convex for all $g_t, t \in [0,1]$.*

PROPOSITION 6.8. *If $q_+, q_- \geq 2$ and $g_0 \in T^E(q_+, q_-, r+1, Q; L^+, L^-)$, there exists a continuous path $(g_t)_{t \in [0,1)}$ in $T^E(q_+, q_-, r+1, Q; L^+, L^-)$ such that $\lim_{t \to 1} g_t \in T^E(q_+, q_-, r, Q'; L^+, L^-)$ for some $Q' \in \mathcal{Q}(q_-, q_+, r)$.*

PROPOSITION 6.9. *If $q_+, q_- \geq 2$ and $Q \in \mathcal{Q}(q_-, q_+, 0)$, then $T^E(q_+, q_-, 0, Q)$ is (empty or) connected.*

PROPOSITION 6.10. *Let $q \geq 2$, and $g_0, g_1 \in C^E(p,q)$. Denote by $L^0 = (l^0_1, l^0_2, \cdots, l^0_q)$ the successive lengths of the segments of ∂D^2 for g_0, and by $L^1 = (l^1_1, l^1_2, \cdots, l^1_q)$ the lengths of the segments of ∂D^2 for g_1. Choose a continuous path $(L^t)_{t\in[0,1]}$ connecting L^0 to L^1 in $C^E(p,q)$. There exists a continuous path $(g_t)_{t\in[0,1]}$ connecting g_0 to g_1 in $C^E(p,q)$, such that L^t is the q-uple of lengths of ∂D^2 for g_t.*

The connectedness of E follows from those propositions, which will be proved later in this section. First, proposition 6.7 can be used to deform any metric $\sigma \in E(p_+, p_-, q_+, q_-, r, Q)$ to a $\sigma' \in E^C(p_+, p_-, q_+, q_-, r, Q)$ in $E(p_+, p_-, q_+, q_-, r, Q)$. The metric remains convex (in the sense of definition 3.1) at the boundary vertices through this deformation, because the convexity there is achieved if either the Σ side or the $\mathcal{T}$ side is convex; since proposition 6.7 moves the space-like part of the metric without making a convex vertex concave, all boundary vertices remain convex in the sense of 3.1. So it is enough to prove that $E^C(p_+, p_-, q_+, q_-, r, Q)$ is connected.

Then, for $q_+, q_- \geq 2$, $T^E(q_+, q_-, Q, 0)$ is connected by proposition 6.9. A crucial point is that the convexity condition of the metric at the boundary vertices is empty, because the Σ part of the metric is convex there, and that is enough according to definition 3.1. Proposition 6.8 and 6.9 can then be used to show inductively on r that $T^E(q_+, q_-, r, Q)$ is connected for $q_+, q_- \geq 2$ and $r \in \mathbf{N}$; namely, it follows from the connectedness of $T^E(q_+, q_-, r, Q)$ that some neighborhood of $T^E(q_+, q_-, r, Q)$ in $T^E(q_+, q_-, r+1, Q)$ is connected, and then proposition 6.8 shows that $T^E(q_+, q_-, r+1, Q)$ is connected.

Given $\sigma_0, \sigma_1 \in E^C(p_+, p_-, q_+, q_-, r, Q)$, let $g_0, g_1 \in T^E(q_+, q_-, r, Q)$ correspond to the time-like/degenerate part of σ_0, σ_1 respectively, h^+_0, h^+_1 to the upper space-like parts, and h^-_0, h^-_1 to the lower space-like parts. g_0 can be connected to g_1 by a continuous path $(g_t)_{t\in[0,1]}$ in $T^E(q_+, q_-, r, Q)$; let L^+_t and L^-_t be the q_+-uple and q_--uple of lengths of the geodesic segments on the upper and lower boundaries of g_t. By proposition 6.10, there exist continuous paths $(h^+_t)_{t\in[0,1]}$ and $(h^-_t)_{t\in[0,1]}$ joining h^+_0 to h^+_1 and h^-_0 to h^-_1 respectively in $C^E(p_+, q_+)$ and in $C^E(p_-, q_-)$, with boundaries made of segments of lengths (L^+_t) and (L^-_t) respectively. For each $t \in [0,1]$, g_t, h^+_t and h^-_t can be glued along their boundaries to obtain $\sigma_t \in E^C(p_+, p_-, q_+, q_-, r, Q)$. $E^C(p_+, p_-, q_+, q_-, r, Q)$ is therefore connected.

The connectedness of $C(p_+, p_-, q_+, q_-, r, Q)$ is proved along the same lines, but more care is needed to take into account the length conditions. This is actually why we give the proof only in the case where Σ_+ and Σ_- are convex with boundary lengths less than 2π, whence condition (E) of theorem 1.3. The relevant definitions are:

DEFINITION 6.11. *Let $q_+, q_- \geq 2$, $r \in \mathbf{N}$, and $Q \in \mathcal{Q}(q_-, q_+, r)$. Let $L^+ = (l^+_1, l^+_2, \cdots, l^+_{q_+}) \in (\mathbf{R}_+ \setminus \{0\})^{q_+}, L^- = (l^-_1, l^-_2, \cdots, l^-_{q_-}) \in (\mathbf{R}_+ \setminus \{0\})^{q_-}$ with $\sum_i l^+_i < 2\pi, \sum_i l^-_i < 2\pi$. $T^S(q_+, q_-, r, Q; L^+, L^-)$ is the space of polyhedral metrics g on $S^1 \times [0,1]$ such that:*

- *$S^1 \times [0,1]$ has a decomposition into domains isometric to the inside of convex polygons in S^2_1 or in $S^2_{1,0}$, such that both sides of an edge agree;*
- *the sum of the angles at each vertex is in $2\pi + i(\mathbf{R}_+ \setminus \{0\})$;*
- *$S^1 \times \{1\}$ and $S^1 \times \{0\}$ are piecewise geodesic with q_+ and q_- segments respectively, with lengths iL^+ and iL^- respectively;*
- *Q describes the intersection between $S^1 \times \{0\}$ and $S^1 \times \{1\}$;*

- *closed, simple $\mathcal{T}$-geodesic of g have length less than 2π.*

$T^S(q_+, q_-, r, Q)$ is the union of the $T^S(q_+, q_-, r; L^+, L^-)$ for all possible values of L^+, L^-.

DEFINITION 6.12. *Let $p, q \in \mathbf{N}$ with $q \geq 2$. $C^S(p,q)$ is the space of polyhedral metrics h on D^2 such that:*
- *g is spherical (modeled on $(S^2, -\mathrm{can})$) except at p interior points, where the singular curvature is negative;*
- *∂D^2 is convex and piecewise geodesic with q segments, and has length less than 2π;*
- *Σ-geodesic segments of (D^2, h) have length less than π.*

The analogs of propositions 6.8, 6.9 and 6.10 are:

PROPOSITION 6.13. *If $q_+, q_- \geq 2$, $Q \in \mathcal{Q}(q_-, q_+, r)$, $r \in \mathbf{N}$, and $g_0 \in T^S(q_+, q_-, r{+}1, Q; L^+, L^-)$, there exists a continuous path $(g_t)_{t\in[0,1)}$ in $T^S(q_+, q_-, r{+}1, Q; L^+, L^-)$ such that $\lim_{t\to 1} g_t \in T^S(q_+, q_-, r, Q'; L^+, L^-)$ for some $Q' \in \mathcal{Q}(q_-, q_+, r)$.*

PROPOSITION 6.14. *If $q_+, q_- \geq 2$ and $Q \in \mathcal{Q}(q_-, q_+, 0)$, then $T^S(q_+, q_-, 0, Q)$ is connected.*

PROPOSITION 6.15. *Let $q \geq 2$, and $g_0, g_1 \in C^S(p,q)$. Denote by $L^0 = (l_1^0, l_2^0, \cdots, l_q^0)$ the successive lengths of the segments of ∂D^2 for g_0, and by $L^1 = (l_1^1, l_2^1, \cdots, l_q^1)$ the lengths of the segments of ∂D^2 for g_1. Choose a continuous path $(L^t)_{t\in[0,1]}$ connecting L^0 to L^1 and such that, for each t, $\sum_i l_i^t < 2\pi$. There exists a continuous path $(g_t)_{t\in[0,1]}$ connecting g_0 to g_1 in $C^S(p,q)$, such that L^t is the q-uple of lengths of ∂D^2 for g_t.*

The connectedness of C follows, as for the connectedness of E^C above.

A slightly different approach is needed to show that $H(p_+, p_-, r)$ is connected. The main points of the proof are:

PROPOSITION 6.16. *For each $\sigma_0 \in H(p_+ + 1, p_-, r)$ $(p_+ \geq 1)$ there exists a continuous path $(\sigma_t)_{t\in[0,1)}$ in $H(p_+ + 1, p_-, r)$ such that $\lim_{t\to 1} \sigma_t \in H(p_+, p_-, r)$.*

PROPOSITION 6.17. *For each $\sigma_0 \in H(p_+, p_-, r+1)$ $(r \geq 1)$, there exists a continuous path $(\sigma_t)_{t\in[0,1)}$ in $H(p_+, p_-, r+1)$ such that $\lim_{t\to 1} \sigma_t \in H(p_+, p_-, r)$.*

PROPOSITION 6.18. *$H(1,1,1)$, $H(1,0,2)$ and $H(2,1,0)$ are connected.*

The connectedness of H follows by a simple inductive argument.

We now turn to the proofs of the propositions stated above.

Proof of proposition 6.7. Let $g \in B^E(p, q; L)$; call $(\theta_i)_{i\in \mathbf{N}_q}$ the (exterior) angles of the boundary at the boundary vertices $(v_i)_{i\in\mathbf{N}_q}$, and $(K_j)_{j\in\mathbf{N}_p}$ the singular curvature at the interior vertices, $(w_j)_{j\in\mathbf{N}_p}$. Then $\forall j, K_j < 0$, and $\theta_i \in [0, \pi)$ when D^2 is convex at v_i, while $\theta_i \in (-\pi, 0]$ when D^2 is concave at v_i.

The proof rests on the following three elementary remarks:
1. by the Gauss-Bonnet theorem, $\sum_i \theta_i = 2\pi - \sum_j K_j > 2\pi$;
2. if $Q = (u_1, u_2, u_3, u_4)$ is a non-degenerate convex 4-gon in E^2, there exists a deformation of Q leaving its edge lengths fixed, decreasing the first and third angles, and increasing the other two angles;

3. if $Q = (u_1, u_2, u_3, u_4)$ is a non-degenerate 4-gon in E^2 which is concave at u_1 and convex at the other 3 vertices, then it admits a deformation leaving the edge lengths fixed, decreasing the angle at u_1 and increasing the other 3 angles.

Now call Θ the sum of the $|\theta_i|$ over all boundary vertices where $\theta_i \leq 0$; we will show that we can decrease Θ down to 0 (so that the resulting surface is convex) without changing the boundary edge lengths, increasing the curvature at the interior vertices, or making a convex boundary vertex concave.

To do this, consider the following situations:

1. there exists a non-degenerate convex 4-gon $Q = (u_1, u_2, u_3, u_4)$ in D^2 with no singular point in its interior, with u_1 and u_3 boundary vertices, one of which is concave, and u_2 and u_4 either convex boundary vertices of interior vertices; then, by remark (2) above, we can deform the metric, decreasing the (total) angle at u_1 and u_3 and increasing the total angle at u_2 and u_4, until Q is degenerate or one of the convex boundary points is almost concave. Θ decreases in this deformation.
2. there exists a non-degenerate 4-gon $Q = (u_1, u_2, u_3, u_4)$ in D^2 with no singular point in its interior, which is concave at the concave boundary point u_1, with u_2, u_3 and u_4 convex boundary vertices or interior vertices; then we can deform the metric along remark (3) above, increasing the angle at u_1 and decreasing it at u_2, u_3 and u_4, again until one of the convex boundary vertex is almost concave. Again, this decreases Θ.
3. there exists a non-degenerate convex 4-gon $Q = (u_1, u_2, u_3, u_4)$ in D^2 with no singular point in its interior, with u_1 a convex boundary vertex which is adjacent to a concave boundary vertex u_0, u_3 a boundary vertex, and u_2 and u_4 interior or convex boundary vertices; then, following remark (2) above again, we can deform the metric to increase the convexity at u_1, and, repeating the operation for the other boundary vertex adjacent to u_0, we can get into case (1), using u_0 as the concave boundary vertex in case (1).

Now, because of remark (1) above, it is not difficult to check that, unless the boundary is convex, the metric is always in one of cases (1), (2) or (3), so that applying a finite number of times one of those 3 arguments leads to $\Theta = 0$, and so to a metric with convex boundary, as needed. □

Proof of proposition 6.8. The first point is that, if $g_0 \in T^E(q_+, q_-, r{+}1, Q; L^+, L^-)$, then there exists a geodesic space-like segment γ_0 joining two singular points, one of which is interior. This is because g_0 has an interior singular point m; if no space-like geodesic segment joined m to another singular point m' (interior or on the boundary) then no geodesic segment starting at m could reach the boundary, so the universal cover of $(S^1 \times [0,1], g_0)$ should contain the union of all space-like geodesics through m. This is clearly impossible, for instance because the area of g_0 could then not be bounded.

By definition 3.1, the sum of the angles of g_0 at m is $2\pi + 2i\theta_m$, for some $\theta_m \in \mathbf{R}_+ \setminus \{0\}$. Let γ_0^+, γ_0^- be the maximal (space-like) geodesic segments starting from m, making angles $\pm i\theta_m$ with γ_0 at m. For $\theta \geq 0$ small enough, let $\gamma_\theta^+, \gamma_\theta^-$ be the maximal (space-like) geodesic segments starting at m', making an angle $\pm i\theta$ with γ_0 at m'. $\gamma_\theta^+, \gamma_\theta^-$ are well defined and intersect γ_0^+, γ_0^- for $\theta \in [0, \theta_0]$, where θ_0 is such that one of the following alternatives is true:

1. m' is an interior singular point of g_0, and the sum of the angles at m' is

$2\pi + 2i\theta_0$;

2. m' is on the boundary, and $\gamma^+_{\theta_0}$ or $\gamma^-_{\theta_0}$ follows the boundary in a neighborhood of m';
3. $\gamma^+_{\theta_0}$ or $\gamma^-_{\theta_0}$ goes through a singular point of g_0.

For $\theta \in [0, \theta_0)$, define a new metric g_θ by removing from g_0 the two triangles bounded by $(\gamma_0, \gamma^+_0, \gamma^+_\theta)$ and by $(\gamma_0, \gamma^-_0, \gamma^-_\theta)$ respectively, and by gluing the corresponding edges. m is not a singular point of g_θ (for $\theta > 0$), because the sum of the angles at of g_θ at m is 2π. But another singular point appears at the intersection of γ^+_0 and γ^+_θ (which is identified with the intersection of γ^-_0 and γ^-_θ in the metric γ_θ). So, for $\theta \in [0, \theta_0)$, $g_\theta \in T^E(q_+, q_-, r+1, Q; L^+, L^-)$.

In case (1) above, g_{θ_0} has one interior singular point less than g_0, so the proposition is proved. In cases (2) and (3), the same procedure can be used inductively for $g_1 := g_{\theta_0}$, leading to a new metric $g_2 := g_{\theta_1}$, and so on. Since the area decreases at each step, the minimal (for the area) metric obtained in this way from g_0 has to be one for which case (1) applies. □

The proof of proposition 6.9 is elementary: each metric $\sigma \in T^E(q_+, q_-, 0, Q)$ is isometric to the domain bounded by two polygonal space-like simple curves (with q_+ and q_- vertices respectively) in the quotient of E^2_1 or $E^2_{1,0}$ by a translation. We leave it to the reader to check, by deforming the boundary curves, that two such metrics can be connected by a continuous path in $T^E(q_+, q_-, 0, Q)$.

Proposition 6.10 is a simple consequence of lemma 5.1. Both g_0 and g_1 can be realized on convex caps C_0, C_1 in E^3_1, with boundaries Γ_0, Γ_1 in $E^2 \subset E^3_1$. Given $(L^t)_{t\in[0,1]}$, it is easy to deform Γ_0 into Γ_1 through polygonal convex curves $(\Gamma_t)_{t\in[0,1]}$ in $E^2 \subset E^3_1$. It is also easy, then, to find convex caps $(C_t)_{t\in[0,1]}$ in E^3_1 such that $\partial C_t = \Gamma_t$. The induced metrics $(g_t)_{t\in[0,1]}$ on (C_t) provide the path we need.

The proof of proposition 6.13 is similar to the proof of proposition 6.8, but more details are needed because of the length conditions. In the first step, we choose γ_0 of length less than π. The existence of such a space-like geodesic segment is assured only because simple $\mathcal{T}$-geodesic curves of g_0 have length $L < 2\pi$: this means that g_0 can not contain the union of all space-like geodesic segments of length π starting from m.

The construction of step 2 is as for proposition 6.8, but a crucial point is that g_θ remains in $T^S(q_+, q_-, r+1, Q)$ because (since g_θ is obtained from g_0 by removing material) the length of the longest $\mathcal{T}$-geodesic curve can not increase. In fact, for any metric $g \in T^S(q_+, q_-, Q, r+1)$, the longest simple space-like curve is $\mathcal{T}$-geodesic (it is possible to deform any space-like curve into a $\mathcal{T}$-geodesic, by replacing parts of it by geodesic segments, therefore increasing the length); any $\mathcal{T}$-geodesic c of g_θ can be "lifted" to a longer space-like curve c' of g (by adding a geodesic segment in the part of g which has been removed) and c is therefore shorter than a closed $\mathcal{T}$-geodesic of g.

Proposition 6.14 is proved as proposition 6.9; and proposition 6.15 as proposition 6.10, using the condition that the sum of the elements of each L^t is less than 2π to realize L^t as the successive lengths of the segments of a convex polygonal curve in a space-like 2-plane in S^3_1.

To prove proposition 6.16, we use the same ideas as in [Pog73]: we choose a geodesic segment γ_0 joining two singular points $m_0, , m_1$ in the upper hyperbolic part of σ_0, and call $K_0 = 2\theta_0$ and $K_1 = 2\theta_1$ the singular curvatures of σ_0 at m_0 and m_1

respectively. Call T_θ the hyperbolic triangle with vertices v_0, v_1, v_2 such that:

- the distance between v_0 and v_1 is the length of γ_0, i.e. the distance between m_0 and m_1 along γ_0;
- the angle at v_0 is θ_0;
- the angle at v_1 is θ.

Now define σ_θ (for $\theta \in [0, \theta_1]$) by cutting σ_0 open along γ_0, and gluing in two copies of T_θ. Each copy of the segment (v_0, v_1) is glued to one side of γ_0, and both copies of (v_0, v_2) and of (v_1, v_2) are glued together. For $\theta \in [0, \theta_1)$, σ_θ has two singular points, one at $v_1 \simeq m_1$, the other at v_2 — the singular point at $v_0 \simeq m_0$ disappears since the total angle after gluing is exactly 2π. For $\theta = \theta_1$, the singular point at v_1 also disappears, so $\sigma_{\theta_1} \in H(p_+, p_-, r)$ as needed.

Proposition 6.17 is proved just like proposition 6.8 (or proposition 6.13); again, the existence of a space-like geodesic segment of length less than π between two singular points comes from the condition that $\mathcal{T}$-geodesic curves have length less than 2π.

Finally, proposition 6.18 is just about special triangles in $\tilde{\mathrm{HS}}^3$, because a triangle in $\tilde{\mathrm{HS}}^3$ is determined by its edge lengths (see [Sch98a]) so that a HS metric with 3 singular points is made of two copies of a triangle glued along their boundaries. We leave it to the reader to check that the three spaces of triangles under consideration are connected.

7. Main proofs in $\tilde{\mathbf{HS}}^3$ and E^3_1. The results of the previous sections are brought together here to give the proof of theorems 1.3 and 1.2. Since the important lemmas have been given above, the essential point that remains to be proved is that the number of inverse images of each metric, which is constant by a general argument given below, is in fact one. We prove this in a simpler and more general way than in [Sch98a].

Note that there are only three independent statements to prove; all the others results stated in the introduction about compact polyhedra are either consequences or proved elsewhere. Those three results are theorem 1.2, for compact polyhedra in the Minkowski space, case (D.3) of theorem 1.3, for compact type polyhedra in the de Sitter space, and case (D.2) of theorem 1.3 for polyhedra of bi-hyperbolic type in $\tilde{\mathrm{HS}}^3$. Case (D.1) of theorem 1.3 is a consequence of [Sch98a]. Theorem 2.2 and corollaries 2.4 and 2.5 are consequences of theorem 2.6, which is itself a translation of case (D.2) of theorem 1.3, for compact type polyhedra. The next section contains the proof of theorem 1.5 from theorem 1.3. The results of the introduction on complete non-compact polyhedra follow.

The proofs of all three results are based on a "deformation" method: we define a natural operator Φ from a space of polyhedra to a space of metrics with the same dimension. Then we prove that Φ is locally injective (polyhedra are rigid) and proper (sequences of polyhedra can not degenerate if the induced metrics converge). Φ is therefore a covering of the space of metrics by the space of polyhedra. Then we prove inductively on the number of vertices/singular points that it is actually a homeomorphism. To avoid some technical complications concerning the space of metrics, we first consider only (as in [Sch98a]) polyhedra having at most one triangular face which is degenerate (and metrics with the corresponding property).

Consider first the setting of theorem 1.2.

DEFINITION 7.1. *Let $p_+, p_-, q_+, q_-, r \in \mathbf{N}$ and $Q \in \mathcal{Q}(q_-, q_+, r)$ with $q_+, q_- \geq 2$. A convex, non degenerate polyhedron P in E^3_1 is in $\mathcal{E}'(p_+, p_-, q_+, q_-, r, Q)$ if it* *has* p_+ *vertices in the interior of* D_+, p_- *in the interior of* D_-, q_+ *on* ∂D_+, q_- *in* ∂D_- *and* r *in the interior of* $\mathcal{T}$, *and if* Q *describes the intersection of* ∂D_- *and* ∂D_+. $\mathcal{E}(p_+,p_-,q_+,q_-,r,Q)$ *is the quotient of* $\mathcal{E}'(p_+,p_-,q_+,q_-,r,Q)$ *by the action of* $\mathrm{Isom}(E^3_1)$, $\mathcal{E}^0(p_+,p_-,q_+,q_-,r,Q)$ *the subset of polyhedra with at most one triangular degenerate face, and* $\mathcal{E}^1(p_+,p_-,q_+,q_-,r,Q) := \mathcal{E}(p_+,p_-,q_+,q_-,r,Q) \setminus \mathcal{E}^0(p_+,p_-,q_+,q_-,r,Q)$.

"At most one degenerate triangular face" means here that the polyhedra can have at most one degenerate face, and that this face has to be triangular. An analogous definition holds in the space of metrics: E^0 is the space of metrics in E which have at most one degenerate triangular face, and $E^1 = E \setminus E^0$.

We now fix p_+,p_-,q_+,q_-,r,Q and stop writing them except when necessary; for instance, "$\mathcal{E}$" means "$\mathcal{E}(p_+,p_-,q_+,q_-,r,Q)$".

E^0 has a natural manifold structure. To define it around a point $\sigma_0 \in E^0$, choose a triangulation τ of σ_0 such that, if σ_0 has a degenerate triangular face, then it is a triangle of τ. Then define a map ϕ from a neighborhood U of σ_0 in E^0 to $\mathbf{R}^A$, where A is the number of edges of τ, sending $\sigma \in U$ to the A-uple of the squares of the (complex) lengths of its edges. Then (as proved in [Sch98a], proposition 9.2) ϕ sends U to a neighborhood of $\phi(\sigma_0)$ in $\mathbf{R}^A$. Moreover, the maps coming from different triangulations of σ_0 are compatible, so E^0 is a manifold. We call $\mathcal{T}_{E^0}$ the associated topology on E^0.

$\mathcal{E}^0$ also has a natural (and simple) manifold structure: it is locally a product of $3N$ copies of E^3_1 (where N is the number of vertices) quotiented by an action of a 6-dimensional group. A rather simple counting argument then shows the:

LEMMA 7.2. *Both* $\mathcal{E}^0$ *and* E^0 *have dimension* $3N-6$, *where* N *is the total number of vertices in of polyhedra in* $\mathcal{E}$, *and of singular points of metrics in* E.

There is a natural morphism $\Phi_E : \mathcal{E} \to E$, sending a convex polyhedron to the induced metric. Moreover, Φ_E restricts to $\Phi^0_E : \mathcal{E}^0 \to E^0$. With the topologies defined above, it is easy to check that Φ^0_E is C^1 - of course it is important here that the manifold structure on E^0 was defined using the squares of the lengths of the edges. Now, a direct consequence of lemma 3.6 is that:

LEMMA 7.3. Φ^0_E *is locally injective, i.e. its differential is an isomorphism.*

We can also reformulate lemma 4.1 as:

LEMMA 7.4. Φ^0_E *is proper.*

Φ^0_E is therefore a covering of E^0 by $\mathcal{E}^0$. Moreover, lemma 6.4 shows that E is connected; E^0 is therefore also connected, because any path in E can be deformed to avoid metrics with more than one triangular degenerate face (see [Sch98a] for details, this is mainly because the condition to have at more than one triangular degenerate face is of codimension at least 2). We will then prove, inductively on the total number of vertices/singular points, the:

LEMMA 7.5. Φ^0_E *is a homeomorphism from* $\mathcal{E}^0$ *to* E^0.

To prove this lemma, we need to show that, when we add a vertex to the polyhedra and a singular point to the metrics (for instance by adding 1 to p_+,p_-,r or one of the other parameters) each point of E^0 still has only one inverse image. We give the proof for the case where p_+ is increased by one unit, the other cases are similar. Instead of writing all the parameters each time, we denote by E^0 and $\mathcal{E}^0$ the spaces corresponding to the initial value of p_+, and by $\overline{E}^0$ and $\overline{\mathcal{E}}^0$ the spaces for the value of p_+ increased by 1.

Since the number of inverse images is constant on $\overline{E}^0$, we only need to prove the result for a special element of $\overline{E}^0$. Choose $P_0 \in \mathcal{E}^0$ such that D_+ has at least one triangular face T, and let x_0 be an interior point of T. Call e_1, e_2, e_3 the edges of T, and v_1, v_2, v_3 its vertices. Let $\sigma_0 = \Phi^0_E(P_0)$.

Note that (again with the manifold structure defined on E^0 and on $\overline{E}^0$ above) E^0 has a natural mapping into $\partial\overline{E}^0$, since a metric with N singular points can be obtained as a limit of a sequence of metrics with $N+1$ singular points. So we can consider σ_0 as in $\partial\overline{E}^0$.

ASSERTION 7.6. *There exists a neighborhood U of σ_0 in the closure of $\overline{E}^0$ such that, if $\sigma \in U$ and $\overline{\Phi}^0_E(P) = \sigma$, then the geodesic segments of σ joining v_1, v_2 and v_3 and corresponding to e_1, e_2 and e_3 are edges of P.*

The proof is easy: otherwise, there would exist a sequence of metrics in $\overline{E}^0$ converging to σ_0, which are images by $\overline{\Phi}^0_E$ of a sequence of polyhedra with a combinatorics which is different from that of P_0; by lemma 4.1, some subsequence of this sequence of polyhedra should converge to a polyhedron with the same induced metric as P_0, but a different combinatorics. This is impossible, because we have supposed inductively that Φ^0_E is injective.

Now we only need a local argument to conclude:

ASSERTION 7.7. *Let T_0 be a non-degenerate space-like triangle in E^3_1, and let y_0 be a point in the interior of T_0. Let V be a neighborhood of y_0 in the half-space above the plane containing T_0. Define a mapping ϕ sending $y \in V$ to the metric m on the convex hull of T_0 and y. If V is small enough, then ϕ is injective.*

We leave the proof to the reader; the point is that the position of y_0 is uniquely determined by the lengths of the segments joining it to the vertices of T_0, which are also uniquely determined by the metric on the pyramid above T_0 in the convex hull of T_0 and m.

The proof of lemma 7.5 follows, at least concerning the addition of a vertex in D_+: by assertion 7.6, metrics close to σ_0 are obtained on polyhedra which have the edges of T as edges, while assertion 7.7 shows that each metric close in $\overline{E}^0$, close enough to σ_0, and differing only in T, is obtained only once on a polyhedron which has the edges of T as edges.

We can conclude that Φ^0_E is a homeomorphism from $\mathcal{E}^0$ to E^0. We also have to show that Φ^1_E is one-to-one from $\mathcal{E}^1$ to E^1. The fact that Φ^1_E is surjective is a consequence of lemma 4.1: a metric σ in E^1 is the limit of a sequence of a limit of metrics in E^0, which are induced on a sequence of polyhedra in $\mathcal{E}^0$, which has a converging subsequence. The limit has σ as induced metric.

The proof that Φ^1_E is injective uses the same argument as in [Sch98a], based on the following analog of lemma 10.3 of [Sch98a]:

PROPOSITION 7.8. *Let $\sigma \in E$, and let U be a neighborhood of σ in E. There exists another neighborhood $V \subset U$ of σ in E such that, if $m', m'' \in V \cap E^0$, there exists a path connecting m' to m'' in $U \cap E^0$.*

The proof can be done as for lemma 10.3 of [Sch98a] (essentially using the fact that E is defined by open conditions), so we leave it again to the reader.

Now suppose that $P', P'' \in \mathcal{E}^1$ are distinct with $\sigma = \Phi^1_E(P') = \phi^1_E(P'') \in E^1$, there would exist a neighborhood U of m in E such that P' and P'' are in different connected components U', U'' respectively of $(\Phi_E)^{-1}(U)$. We could then choose a neighborhood $V \subset U$ of σ in E as in proposition 7.8, and call V', V'' the connected components of P', P'' respectively in $(\Phi_E)^{-1}(V)$.

If $P'_0 \in V' \cap \mathcal{E}^0$ and $P'' \in V'' \cap \mathcal{E}^0$, there would exist a path γ connecting $\sigma'_0 := \Phi_E(P'_0)$ to $\sigma''_0 = \Phi_E(P''_0)$ in $U \cap E^0$; since Φ^0_E is a homeomorphism, we would obtain by deformation an inverse image of σ''_0 in U'; since σ''_0 has another inverse image in U'', Φ^0_E could not be injective, a contradiction. This finishes the proof of theorem 1.2.

To prove theorem 1.3, we use similar definitions:

Definition 7.9. *Let $p_+, p_-, q_+, q_-, r \in \mathbb{N}$ and $Q \in \mathcal{Q}(q_-, q_+, r)$ with $q_+, q_- \geq 2$. A convex, non degenerate polyhedron P in S^3_1 is in $\mathcal{C}'(p_+, p_-, q_+, q_-, r, Q)$ if it has p_+ vertices in the interior of D_+, p_- in the interior of D_-, q_+ on ∂D_+, q_- in ∂D_-, r in the interior of $\mathcal{T}$, and if Q describes the intersection of ∂D_- and ∂D_+. $\mathcal{C}(p_+, p_-, q_+, q_-, r, Q)$ is the quotient of $\mathcal{C}'(p_+, p_-, q_+, q_-, r, Q)$ by the action of* $\mathrm{Isom}(S^3_1)$, *$\mathcal{C}^0(p_+, p_-, q_+, q_-, r, Q)$ the subset of polyhedra with at most one triangular degenerate face, and $\mathcal{C}^1(p_+, p_-, q_+, q_-, r, Q) := \mathcal{C}(p_+, p_-, q_+, q_-, r, Q) \setminus \mathcal{C}^0(p_+, p_-, q_+, q_-, r, Q)$.*

We fix p_+, p_-, q_+, q_-, r, Q again, and stop writing them.

C^0 has a natural manifolds structure, defined as for E^0, but with the cosh of the (complex) lengths of the edges instead of the squares. We call $\mathcal{T}_{C^0}$ the associated topology on C^0. $\mathcal{C}^0$ also has a natural (and simple) manifold structure: it is locally a product of $3N$ copies of S^3 (each corresponding to a projective model) quotiented by an action of a 6-dimensional group.

The analog of lemma 7.2 is:

Lemma 7.10. *Both $\mathcal{C}^0$ and C^0 have dimension $3N - 6$, where N is the total number of vertices of polyhedra in $\mathcal{C}$, and of singular points of metrics in C.*

Now define $\Phi_C : \mathcal{C}^S \to C^S$, sending a convex polyhedron to the induced metric. Φ_C restricts to $\Phi^0_C : \mathcal{C}^0 \to C^0$, where C^0 is the space of metrics in C which have at most one degenerate triangular face. With the manifold structures defined above, it is not difficult to check that Φ^0_C is C^1 (it is necessary, though, to use the cosh of the edge lengths to obtain this result). Lemma 3.5 shows that:

Lemma 7.11. *Φ^0_C is locally injective, i.e. its differential is an isomorphism at each point.*

Theorem 4.2 also contains, as the special case where the sequence considered is made of polyhedra of compact type, the:

Lemma 7.12. *Φ^0_C is proper.*

Φ^0_C is therefore a covering of C^0 by $\mathcal{C}^0$. Again, C and C^0 are connected by lemma 6.4. We prove, inductively on the total number of vertices/singular points, the:

Lemma 7.13. *Φ^0_C is a homeomorphism from $\mathcal{C}^0$ to C^0.*

The proof is the same as for lemma 7.5, except that the local argument needed (equivalent to assertion 7.7) is now in S^3_1 instead of E^3_1.

The analog of proposition 7.8 is again proved as in [Sch98a]:

PROPOSITION 7.14. *Let $\sigma \in C$, and let U be a neighborhood of σ in C. There exists another neighborhood $V \subset U$ of σ in C^S such that, if $m', m'' \in V \cap C^0$, there exists a path connecting m' to m'' in $U \cap C^0$.*

The proof of theorem 1.3 is then as the proof of theorem 1.2 above.

We now give the idea of the proof of case (D.2) of theorem 1.3.

DEFINITION 7.15. *Let $p_+, p_-, r \in \mathbf{N}$ with $p_+ + p_- + r \geq 3$. A convex, non-degenerate polyhedron P in $\tilde{HS}^3$ is in $\mathcal{H}'(p_+, p_-, r)$ if it has p_+ vertices in H^3_+, p_- in H^3_-, and r in S^3_1. $\mathcal{H}(p_+, p_-, r)$ is the quotient of $\mathcal{H}'(p_+, p_-, r)$ by the action of* $\mathrm{Isom}(\tilde{HS}^3)$, *$\mathcal{H}^0(p_+, p_-, r) \subset \mathcal{H}(p_+, p_-, r)$ the subset of polyhedra with at most one triangular degenerate face, and $\mathcal{H}^1(p_+, p_-, r) := \mathcal{H}(p_+, p_-, r) \setminus \mathcal{H}^0(p_+, p_-, r)$.*

We fix p_+, p_-, r again, and stop writing them.

H^0 has a natural manifolds structure, defined as for E^0, with the cosh of the (complex) lengths of the edges. We call $\mathcal{T}_{H^0}$ the associated topology on H^0. $\mathcal{H}^0$ also has a natural (and simple) manifold structure: it is locally a product of $3N$ copies of S^3 (each corresponding to a projective model) quotiented by an action of a 6-dimensional group.

The analog of lemma 7.2 is:

LEMMA 7.16. *Both $\mathcal{H}^0$ and H^0 have dimension $3N - 6$, where N is the total number of vertices in of polyhedra in $\mathcal{H}$, and of singular points of metrics in H.*

Now define $\Phi_H : \mathcal{H} \to H$, sending a convex polyhedron to the induced metric. Φ_H restricts to $\Phi^0_H : \mathcal{H}^0 \to H^0$, where H^0 is the space of metrics in H which have at most one degenerate triangular face. With the topologies defined above, it is easy to check that Φ^0_H is C^1. Lemma 3.5 shows that:

LEMMA 7.17. *Φ^0_H is locally injective, i.e. its differential is an isomorphism at each point.*

We can also reformulate theorem 4.2, in the special case where the sequence of polyhedra considered is of bi-hyperbolic type, as:

LEMMA 7.18. *Φ^0_H is proper.*

Φ^0_H is therefore a covering of H^0 by $\mathcal{H}^0$. Again, H and H^0 are connected by lemma 6.4. We prove, inductively on the total number of vertices/singular points, the:

LEMMA 7.19. *Φ^0_H is a homeomorphism from $\mathcal{H}^0$ to H^0.*

The proof is the same as for lemma 7.5, except that the local argument needed (equivalent to assertion 7.7) is now in $\tilde{\mathrm{HS}}^3$ instead of E^3_1; it can be in H^3 for a hyperbolic face, or in S^3_1 with a time-like triangle, depending on the kind of face which is modified by adding a vertex.

The analog of proposition 7.14 is again proved as in [Sch98a]:

PROPOSITION 7.20. *Let $\sigma \in H$, and let U be a neighborhood of σ in H. There exists another neighborhood $V \subset U$ of σ in H such that, if $m', m'' \in V \cap H^0$, there exists a path connecting m' to m'' in $U \cap H^0$.*

The proof of case (D.2) of theorem 1.3 follows again as for theorem 1.2 above.

8. Complete polyhedra. This section contains the proof of theorem 1.5 from theorem 1.3.

If $P \subset S^3_1$ is a convex, complete (non compact) polyhedron, convex meaning that it lies on the boundary of its convex hull, then it is easy to see that P is the restriction to S^3_1 of a convex polyhedron in $\tilde{\mathrm{HS}}^3$, to which theorem 1.3 can be applied. We need to understand whether a complete polyhedral metric on S^2 minus M points can be extended, by adding disks with complete hyperbolic metrics, to obtain a HS metric to which theorem 1.3 can be applied to find a polyhedron. The main remark is that the only invariant at each end is the "angle" defined in section 1, right before theorem 1.5.

The first point is that the angle at infinity of a complete hyperbolic metric on D^2, with singular points of positive singular curvature, is in $[0, 2\pi] \cup i\mathbf{R}_+$.

LEMMA 8.1. *Let σ be a complete hyperbolic metric on D^2, with N singular points where the singular curvature is positive. Then the angle θ at the end is in $[0, 2\pi] \cup i\mathbf{R}$, and $\theta = 2\pi$ if and only if (D^2, σ) is isometric to H^2.*

This lemma could actually be extended to the case where the metric is hyperbolic only in a neighborhood of ∂D^2, and is only HS and convex at its singular points.

The second point is that there is no other restriction, and that all other angles can be achieved uniquely by taking a HS metric with only one singular point (but they might have a de Sitter part). There are no constraints at all for HS metrics which are de Sitter on a neighborhood of ∂D^2:

LEMMA 8.2.

1. *Let $\theta \in [0, 2\pi)$. There exists a unique complete hyperbolic metric on D^2 with one singular point where the curvature is positive (it is equal to $2\pi - \theta$);*
2. *for $\theta \in i\mathbf{R}_+$, there exists a unique complete HS metric on D^2, hyperbolic near D^2, with one convex singular point (it is in a de Sitter component of the metric if $\theta \neq 0$);*
3. *for $\theta \in \mathbf{R} \cup i\mathbf{R}$, there exists a unique complete HS metric on D^2, modeled on S^2_1 near ∂D^2, with one convex singular point.*

The proof of theorem 1.5 is now straightforward. Suppose first that (σ, Σ) is a complete marked HS metric induced on a complete convex polyhedron P in S^3_1. Since P is convex — i.e. it lies on the boundary of its convex hull — it is easy to construct a convex polyhedron $\overline{P} \subset \tilde{\mathrm{HS}}^3$ such that $\overline{P} \cap S^3_1 = P \cap S^3_1$. Conditions (A), (B), (C) and (E) of theorem 1.5 are the consequences of conditions (A), (B), (C), (D) respectively of theorem 1.3, while condition (D) of theorem 1.5 follows from lemma 8.1.

Conversely, if (σ, Σ) satisfies condition (D) of theorem 1.5, then, by lemma 8.2, we can add at each end a disk containing a single singular point, so as to obtain a convex HS metric $(\overline{\Sigma}, \overline{\sigma})$. If conditions (A), (B), (C), (E) and (F) of theorem 1.5 are true, then theorem 1.3 can be applied to $(\overline{\Sigma}, \overline{\sigma})$, which is induced on a unique convex polyhedron $\overline{P} \subset \tilde{\mathrm{HS}}^3$. Moreover, $\overline{P}$ can be decomposed into a complete polyhedron $P \subset S^3_1$ and M disks, each containing a single vertex, so that P is cylindrical — and (σ, Σ) is induced on P. This proves theorem 1.5.

Here are the proofs of lemmas 8.1 and 8.2.

Proof of lemma 8.1. The first remark is that the metric has angle θ if and only if some neighborhood of ∂D^2 is isometric to the a neighborhood of ∂D^2 in (D^2, σ_N), where σ_n is a complete HS metric on D^2 with a single singular point p where σ_N is convex. p might be hyperbolic (then $\theta \in [0, 2\pi]$) or in the de Sitter part of $\tilde{\mathrm{HS}}^3$ (and $\theta \in i\mathbf{R}$).

We will prove recursively an assertion slightly more precise than the lemma: if g is a hyperbolic metric with singular points on D^2, $p_1, \cdots, p_k$ are singular points where the singular curvature of g is positive, and Ω is a convex domain with $p_1, \cdots, p_k$ in its interior, then $D^2 \setminus \Omega$ is isometric to the complement of a compact subset containing a single singular point q_k in (D^2, g_k), where g_k is a complete HS metric on D^2. Moreover, g_k is convex at q_k.

For $k = 1$, this is obvious, because g doesn't need to be modified. Suppose this assertion is established for k. Let $p_1, \cdots, p_{k+1}$ be singular points of g in a convex domain Ω'. Choose a convex domain $\Omega \subset \Omega'$ containing k of the points $p_1, \cdots, p_{k+1}$; for simplicity, suppose that those points are $p_1, \cdots, p_k$. Apply the assertion to Ω; $D^2 \setminus \Omega$ is isometric to the complement of a convex domain $\overline{\Omega}$ in (D^2, g_k), and Ω' corresponds to a convex domain $\overline{\Omega}' \subset (D^2, g_k)$ (with $\overline{\Omega}' \setminus \overline{\Omega}$ isometric to $\Omega' \setminus \Omega$). $\overline{\Omega}'$ contains 2 singular points, q_k and p_{k+1}, where g_k is convex, and p_{k+1} is a hyperbolic point.

Now g_{k+1} can be built from g_k by a simple surgery. Call $2\theta_{q_k}$ and $2\theta_{p_{k+1}}$ the singular curvatures at q_k and p_{k+1} respectively. Let $T = (p, q, r)$ be the (unique) triangle in $\tilde{\mathrm{HS}}$ such that:

- the length of (p, q) is the distance between p_{k+1} and q_k;
- the angle at p is $\theta_{p_{k+1}}$;
- the angle at q is θ_{q_k}.

Then cut g_k open along the segment joining p_{k+1} to q_k, and glue in two copies of T, with the two copies of (p, q) going to the two edges of the cut, and the two copies of (p, r) and of (q, r) respectively glued to each other. The resulting metric g_{k+1} has the properties needed. □

Proof of lemma 8.2. The proof is elementary: in each case, one only needs to check that there is a unique metric with a single singular point as described, and that the angle at the end, which is 2π minus the singular curvature at that point, takes the necessary values. □

9. Smooth analogues. Most of the polyhedral theory of isometric embeddings of Riemannian surfaces has a smooth counterpart. The basic point in this area is the following classical result:

THEOREM 9.1 (Nirenberg [Nir53], Aleksandrov [Ale51], Pogorelov [Pog73], Labourie [Lab89], etc...). *Let g be a C^∞ Riemannian metric on S^2 with curvature $K > -1$ (resp. $K > 0$, $K > 1$). Then (S^2, g) admits a unique C^∞ isometric embedding in H^3 (resp. E^3, S^3).*

This result was originally proved (in a slightly weaker version with respect to smoothness) by approximating Riemannian metrics by polyhedral ones, applying theorem 1.1, and taking the limit of a sequence of polyhedra to obtain a (not so) smooth surface.

In Lorentzian space-forms, the polyhedral and smooth results for Riemannian surfaces are also parallel; [Sch96] gives a smooth analog of the polyhedral result of Rivin and Hodgson [RH93]. It should be pointed out that it is not as easy to go from the polyhedral case to the smooth situation (or the other way round) as in the Riemannian setting, because of some convergence problems: a sequence of polyhedra could diverge (even if, say, the tangent space at a point is fixed) while the induced metrics converge.

More recent proofs of theorem 9.1 ([Ham82], [Lab89] or [Sch96] in the de Sitter setting) use a Nash-Moser inverse function theorem between spaces of smooth metrics and embeddings. On the other hand, Pogorelov [Pog73] proved that analogs of theorem 1.1 and 9.1 can be proved with very minimal smoothness assumptions, in the setting of Aleksandrov spaces.

There are some results for the existence and uniqueness of isometric embeddings of smoooth convex surfaces in H^3 and in S^3_1, related to theorem 1.4 or to theorem 2.7 applied to space-like metrics. But they only deal so far with metrics of constant curvature [Sch98b]. The "cylindrical" condition at infinity does not seem to make much sense in this smooth setting, but it can be replaced by the hypothesis that the boundary at infinity of the surface is a disjoint union of circles — a well-defined condition since the circles of $\partial_\infty H^3$ are the traces of the totally geodesic 2-planes.

Pogorelov [Pog73] also investigated convex caps in Riemannian space-forms. Again, some smooth results can be obtained by approximating smooth metrics by polyhedral ones. On the other hand, a direct proof of the existence of smooth embeddings leads to the study of a PDE problem on a convex domain (the unknown function is the distance to a totally geodesic 2-plane, and the PDE is on the disk with the metrics which one wants to embed). Since the PDE is elliptic of Monge-Ampère type, the methods developed by Caffarelli, Nirenberg and Spruck [CNS84] apply, leading to the existence of solutions. Other methods then show that the solutions are unique. It should be pointed out that the smoothness up to the boundary of the resulting embeddings is a subtle point, which was treated by Delanoë [Del88].

It is interesting to consider how the existence of solutions of those Monge-Ampere solutions is proved in e.g. [GS93] (see also [CNS84, HRS92, RS94, Spr95]). The basic point is a notion of sub-solutions, and a maximization argument to prove that a maximal such sub-solution exists, and that it is actually a solution. This is completely parallel to the method developed by Volkov [Vol60] to prove the existence of polyhedral convex caps, and used (along the lines of [Mil86]) in section 4. The height of the vertices as used by Volkov is the analog of the function appearing in the Monge-Ampère equations of [GS93], and it ends up being the distance to the plane containing the boundary. The Volkov method of [Vol60] could be considered as a polyhedral version of the method of [GS93] (or the other way round). The polyhedral situation involves some subtleties of a combinatorial nature absent from the smooth problem (one has to change the triangulations in an adequate way), but the results in the smooth case involve difficult questions on C^2 estimates at the boundary (one should however be aware of the very geometric approach of [Lab97] concerning those questions).

It might be interesting to understand whether the kind of results developed in this text, about convex polyhedra in $\tilde{\mathrm{HS}}^3$ with both a hyperbolic and a de Sitter part, can be extended to the smooth case. This would mean understanding the metrics induced on smooth (strictly) convex surfaces in S^3, when S^3 is considered with a projective model of $\tilde{\mathrm{HS}}^3$. Those surfaces might then have both a hyperbolic and a de Sitter component, both complete. The hyperbolic part of the metric should then be Riemannian and asymptotically hyperbolic, while the de Sitter part would be Lorentzian near the end, and in some way "asymptotically de Sitter".

One might also wonder whether theorem 1.2, for instance, has a smooth analog. In other terms, is there a geometrically significant description of the metrics induced on smooth, convex surfaces in E^3_1 ? Such metrics should again have both a Riemannian and a Lorentzian part, but not complete. I do not know whether this leads anywhere.

On the other hand, the works of Il'khamov and Sokolov [IS90] and of Gajdalovich and Sokolov [GS86] can be used to prove similar existence results for surfaces which are smooth, except along a line (for the analog of theorem 2.2) or on two points (as in corollary 2.5), by approximating a smooth metric (again except along a line or at two points) by a sequence of polyhedral metrics, and then taking a limit of the corresponding sequence of polyhedra (some care is needed to prove the convergence). The same could be done in the de Sitter setting, starting from the polyhedral theorem 2.2 and corollary 2.5. A direct approach might lead to better results in term of smoothness, and might also lead to uniqueness results.

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