CHARACTERISTIC CLASSES FOR $GO(2N, \mathbb{C})$*

YOGISH I. HOLLA† AND NITIN NITSURE‡

The complex Lie group $GO(n)$ is by definition the closed subgroup of $GL(n)$ consisting of all matrices g such that ${}^t gg$ is a scalar matrix λI for some $\lambda \in \mathbb{C}^*$. (We write simply $GL(n)$, $SO(n)$, $O(n)$, etc. for the complex Lie groups $GL(n, \mathbb{C})$, $SO(n, \mathbb{C})$, $O(n, \mathbb{C})$, etc.)

The group $GO(1)$ is just $\mathbb{C}^*$, with classifying space $B\mathbb{C}^* = \mathbf{P}^\infty_{\mathbb{C}}$, whose cohomology ring $H^*(B\mathbb{C}^*; \mathbb{Z}/(2))$ is the polynomial ring $\mathbb{Z}/(2)[\lambda]$ where $\lambda \in H^2(B\mathbb{C}^*; \mathbb{Z}/(2))$ is the Euler class. For an odd number $2n + 1 \geq 3$, the group $GO(2n + 1)$ is isomorphic to the direct product $\mathbb{C}^* \times SO(2n + 1)$. Hence $BGO(2n + 1)$ is homotopic to the direct product $B\mathbb{C}^* \times BSO(2n + 1)$, with cohomology ring the polynomial ring $\mathbb{Z}/(2)[\lambda, w_2, w_3, \ldots, w_{2n+1}]$ where for $2 \leq i \leq 2n + 1$, the elements $w_i \in H^i(BSO(2n + 1); \mathbb{Z}/(2))$ are the Stiefel-Whitney classes.

In this paper, we consider the even case $GO(2n)$. The main result is an explicit determination, in terms of generators and relations, of the singular cohomology ring $H^*(BGO(2n); \mathbb{Z}/(2))$. This is the Theorem 3.9 below.

An outline of the argument is as follows.

To each action of the group $\mathbb{C}^*$ on any space X, we functorially associate a certain derivation $s : H^*(X) \to H^*(X)$ on the cohomology ring of X, which is graded of degree -1, with square zero (see Lemma 2.2 below). In terms of the action $\mu : \mathbb{C}^* \times X \to X$ and the projection $p : \mathbb{C}^* \times X \to X$, it is given by the formula

$$\mu^* - p^* = \eta \otimes s$$

where η is the positive generator of $H^1(\mathbb{C}^*)$.

When X, together with its given $\mathbb{C}^*$-action, is the total space of a principal $\mathbb{C}^*$-bundle $\pi : X \to Y$ over some base Y, recall that we have a long exact Gysin sequence

$$\cdots \xrightarrow{\lambda} H^i(Y) \xrightarrow{\pi^*} H^i(X) \xrightarrow{d} H^{i-1}(Y) \xrightarrow{\lambda} H^{i+1}(Y) \xrightarrow{\pi^*} \cdots$$

We show (Lemma 2.3) that in this case, the derivation s on $H^*(X)$ equals the composite

$$s = \pi^* \circ d$$

As $O(2n) \subset GO(2n)$ is normal with quotient $\mathbb{C}^*$, $BO(2n)$ can be regarded as the total space of a principal $\mathbb{C}^*$-bundle $BO(2n) \to BGO(2n)$. The resulting action of $\mathbb{C}^*$ on $BO(2n)$ gives a derivation s on the ring $H^*(BO(2n)) = \mathbb{Z}/(2)[w_1, \ldots, w_{2n}]$ with $\mu^* - p^* = \eta \otimes s$. The last equality enables us to write the following expression for s (see Lemma 3.4 below)

$$s = \sum_{i=1}^{n} w_{2i-1} \frac{\partial}{\partial w_{2i}}$$

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†School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400 005, India (yogi@math.tifr.res.in).

‡School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400 005, India (nitsure@math.tifr.res.in).

Determining the invariant subring $\ker(s)$ of $H^*(BO(2n))$ thus becomes a purely commutative algebraic problem, which we solve by using the technique of regular sequences and Koszul complex, and express the invariant subring $B = \ker(s)$ in terms of generators and relations (see Theorem 1.1 below).

As explained above, in terms of the Gysin sequence

$$\cdots \xrightarrow{\lambda} H^i(BGO(2n)) \xrightarrow{\pi^*} H^i(BO(2n)) \xrightarrow{d} H^{i-1}(BGO(2n)) \xrightarrow{\lambda} H^{i+1}(BGO(2n)) \xrightarrow{\pi^*} \cdots$$

the derivation s equals the composite $\pi^* \circ d$. This fact, together with the explicit knowledge of the kernel and image of s (in terms of the ring B), allows us to 'solve' the above long exact Gysin sequence, to obtain the main Theorem 3.9, which gives generators and relations for the ring $H^*(BGO(2n))$.

The paper is arranged as follows. The purely algebraic problem of determining the kernel ring of the derivation s is solved in Section 1. In Section 2, we describe the derivation on the cohomology of a space associated to a $\mathbb{C}^*$-action, and connect this with the Gysin sequence for principal $\mathbb{C}^*$-bundles. We also show that the Steenrod operations commute with Gysin boundary, and deduce the useful consequence that Gysin boundary kills squares. Finally, all the above material is applied to determine the cohomology ring of $BGO(2n)$ in Section 3.

1. Derivation on a polynomial algebra. Let k denote the field $\mathbb{Z}/(2)$. For any integer $n \geq 1$, consider the polynomial ring $C = k[w_1, \ldots, w_{2n}]$ where $w_1, \ldots, w_{2n}$ are independent variables (the ring C will stand for $H^*(BO(2n))$, with w_i the Stiefel-Whitney classes, when we apply this to topology).

In what follows, we will treat the odd variables w_{2i-1} differently from the even variables w_{2i}, so for the sake of clarity we introduce the notation

$$u_i = w_{2i-1} \text{ and } v_i = w_{2i} \text{ for } 1 \leq i \leq n.$$

On the ring $C = k[u_1, \ldots, u_n, v_1, \ldots, v_n]$, we introduce the derivation

$$s = \sum_{i=1}^{n} u_i \frac{\partial}{\partial v_i} : C \to C$$

Note that s is determined by the conditions $s(u_i) = 0$ and $s(v_i) = u_i$ for $1 \leq i \leq n$. As $2 = 0$ in k, it follows from the above that

$$s \circ s = 0$$

We are interested in the subring $B = \ker(s) \subset C$. Let $A \subset B \subset C$ be the subring $A = k[u_1, \ldots, u_n, v_1^2, \ldots, v_n^2]$. For any subset $T = \{p_1, \ldots, p_r\} \subset \{1, \ldots, n\}$ of cardinality $r \geq 1$, consider the monomial $v_T = v_{p_1} \cdots v_{p_r}$. We put $v_T = 1$ when T is empty. It is clear that C is a free A-module of rank 2^n over A, with A-module basis formed by the v_T. Hence as the ring A is noetherian, the submodule $B \subset C$ is also finite over A. In particular, this proves that B is a finite type k-algebra.

We are now ready to state the main result of this section, which is a description of the ring B by finitely many generators and relations.

THEOREM 1.1. **(I) (Generators)** *Let the derivation $s : C \to C$ be defined by $s = \sum_{i=1}^{n} u_i \frac{\partial}{\partial v_i}$. Consider the subring $B = \ker(s) \subset C$. We have $s^2 = 0$, and $B = \ker(s) = \operatorname{im}(s) + A$. Consequently, B is generated as an A-module by 1 together* *with the $2^n - n - 1$ other elements $s(v_T)$, where T ranges over all subsets $T \subset \{1, \ldots, n\}$ of cardinality ≥ 2, and $v_T = \prod_{i \in T} v_i$.*

(II) (Relations) *Let $A[c_T]$ be the polynomial ring over A in the $2^n - n - 1$ algebraically independent variables c_T, indexed by all subsets $T \subset \{1, \ldots, n\}$ of cardinality ≥ 2. Let N be the ideal in $A[c_T]$ generated by*

(1) the $2^n - n(n-1)/2 - n - 1$ elements of the form

$$\sum_{i \in T} u_i \, c_{T - \{i\}}$$

where $T \subset \{1, \ldots, n\}$ is a subset of cardinality ≥ 3, and

(2) the $n(n-1)/2$ elements of the form

$$(c_{\{i,j\}})^2 + u_i^2 v_j^2 + u_j^2 v_i^2$$

where $\{i, j\} \subset \{1, \ldots, n\}$ is a subset of cardinality 2, and

(3) the $(2^n - n - 1)^2 - n(n-1)/2$ elements (not necessarily distinct) of the form

$$c_T \, c_U + \sum_{p \in T} \prod_{q \in T \cap U - \{p\}} u_p \, v_q^2 \, c_{(T - \{p\}) \Delta U}$$

where $T \neq U$ in case both T and U have cardinality 2, and Δ denotes the symmetric difference of sets $X \Delta Y = (X - Y) \cup (Y - X)$.

Then we have an isomorphism of A-algebras $A[c_T]/N \to B$, mapping $c_T \mapsto s(v_T)$.

Proof of (I) (Generators). As already seen, $s \circ s = 0$, hence we have $\operatorname{im}(s) \subset \ker(s) = B$. As $A \subset B$, we have the inclusion $A + \operatorname{im}(s) \subset B$. We next prove that in fact $B = A + \operatorname{im}(s)$. The problem translates into the exactness of a certain Koszul resolution, as follows. An exposition of the elementary commutative algebra used below can be found, for example, in §16 of Matsumura [M].

Let M be a free A-module of rank n, with basis $e_1, \ldots, e_n$. Consider the A-linear map $u : M \to A : e_i \mapsto u_i$. For any subset $T = \{p_1, \ldots, p_r\} \subset \{1, \ldots, n\}$ of cardinality r where $0 \leq r \leq n$, let $e_T = e_{p_1} \wedge \ldots \wedge e_{p_r} \in \bigwedge^r M$. These e_T form a free A-module basis of $\bigwedge^r M$. Recall that the Koszul complex for u is the complex

$$0 \longrightarrow \bigwedge^n M \xrightarrow{d_n} \bigwedge^{n-1} M \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_3} \bigwedge^2 M \xrightarrow{d_2} M \xrightarrow{d_1} A \longrightarrow 0$$

where for $1 \leq r \leq n$, the differential $d_r : \bigwedge^r M \to \bigwedge^{r-1} M$ is defined by putting

$$d_r(e_T) = \sum_{j \in T} u_j e_{T - \{j\}} \tag{1.1}$$

(note that T is non-empty as $r \geq 1$). As $u_1, \ldots, u_n$ is a regular sequence in A, the Koszul complex is exact except in degree 0 (in fact the Koszul complex gives rise to a projective resolution of the A-module $A/\operatorname{im}(d_1) = A/(u_1, \ldots, u_n)$, but we do not need this).

Let $\bigwedge(M) = \oplus_{0 \leq r \leq n} \bigwedge^r M$, which we regard as a free A-module of rank 2^n with basis e_T. (The algebra structure of $\bigwedge(M)$ is not relevant to us.) The graded module $\bigwedge(M)$ comes with an A-linear endomorphism d of degree -1, where by definition d

is d_r on the graded piece $\bigwedge^r M$. Hence we have the equality $\ker(d) = \operatorname{im}(d) + A$ as submodules of $\bigwedge(M)$.

Now consider the A-linear isomorphism $\psi : \bigwedge(M) \to C$ of free A-modules under which $e_T \mapsto v_T$. From the expression $s = \sum_i u_i(\partial/\partial v_i)$, it follows that

$$s(v_T) = \sum_{j\in T} u_j v_{T-\{j\}} \tag{1.2}$$

when T is non-empty. By comparing (1) and (2) we see that $\psi : \bigwedge(M) \to C$ takes the Koszul differential d to the differential s, hence $\ker(s) = \operatorname{im}(s) + A$.

As $\operatorname{im}(s)$ is generated by all the elements $s(v_T)$ where T is non-empty, the A-module B is generated by 1 together with the $2^n - 1$ other elements $s(v_T)$, where T ranges over all non-empty subsets $T \subset \{1, \dots, n\}$.

This completes the proof of part (I) (Generators) of the Theorem 1.1.

Proof of Theorem 1.1.(II) (Relations). As $s \circ s = 0$, the equation (2) gives $\sum_{i\in T} u_i s(v_{T-\{i\}}) = 0$. This proves that the relation II.(1) is satisfied by the assignment $c_T \mapsto s(v_T)$.

In particular, when $T = \{i, j\}$ has cardinality 2, we get $s(v_{\{i,j\}}) = u_i v_j + u_j v_i$, and hence $s(v_{\{i,j\}})^2 = u_i^2 v_j^2 + u_j^2 v_i^2$. This proves that the relation II.(2) is satisfied by the assignment $c_T \mapsto s(v_T)$.

Note that if X and Y are subsets of $\{1, \dots, n\}$, then we have

$$v_X v_Y = \prod_{i\in X\cap Y} v_i^2\, v_{X\Delta Y} \tag{1.3}$$

where $X\Delta Y$ denotes the symmetric difference $(X - Y) \cup (Y - X)$.

From the equations (2) and (3) it follows by a straight-forward calculation that

$$s(v_T)\, s(v_U) = \sum_{p\in T} \prod_{q\in T\cap U-\{p\}} u_p\, v_q^2\, s(v_{(T-\{p\})\Delta U})$$

where $T \neq U$ in case both T and U have cardinality 2. This proves that the relation II.(3) is also satisfied by the assignment $c_T \mapsto s(v_T)$.

Hence the A-algebra homomorphism $A[c_T] \to B$ defined by sending $c_T \mapsto s(v_T)$ kills the ideal N, hence defines an A-algebra homomorphism $\eta : A[c_T]/N \to B$. By part (I) of the theorem, which we have already proved, the A-algebra B is generated by the $s(v_T)$, hence η is surjective.

To prove that η is injective, we use the following lemma.

LEMMA 1.2. *Let $F \subset C$ be the A-submodule generated by all v_T where $T \subset \{1, \dots, n\}$ has cardinality ≥ 1. Let $E \subset B$ be the A-submodule generated by all $s(v_T)$ where $T \subset \{1, \dots, n\}$ has cardinality ≥ 2. Then we have $E \subset F$, and $F \cap A = 0$, and $B = A \oplus E$, as A-modules.*

Proof. Note that for each i, the v_i-degree of each non-zero monomial in any element of A is even. On the other hand, if $|T| \geq 1$ (where $|T|$ denotes the cardinality of T), then the monomial v_T has v_i-degree 1 for for all $i \in T$. Hence in any A-linear combination $w = \sum_{|T|\geq 1} a_T\, v_T$, each monomial has odd v_i-degree for some i. It follows that $A \cap F = 0$.

From the equality $s(v_T) = \sum_{i\in T} u_i v_{T-\{i\}}$, it follows that if $|T| \geq 2$ then $s(v_T) \in F$, so $E \subset F$. It follows that $A \cap E = 0$. By part (I) of Theorem 1.1, we already know that $A + E = B$, so the lemma follows. □

We now prove that η is injective. For this, we define a homomorphism of A-modules $\theta : C \to A[c_T]/N$ by $\theta(1) = 0$, $\theta(v_i) = u_i$, and $\theta(v_T) = c_T$ for $|T| \geq 2$. With this definition of θ, note that the composite $C \xrightarrow{\theta} A[c_T]/N \xrightarrow{\eta} B \hookrightarrow C$ equals $s : C \to C$. Moreover, note that $\theta(E) = 0$, where $E \subset C$ is the A-submodule in the previous lemma. To see the above, we must show that $\theta(s(v_T)) = 0$ for all T. For $|T| \geq 3$, this follows from the relations (1) in part (II) of Theorem 1.1, while for $|T| \leq 2$, it follows from $\theta(v_i) = u_i$ and $\theta(1) = 0$.

Now suppose $x \in A[c_T]/N$ with $\eta(x) = 0$. From the generators (3) of N, it follows that any element of $x \in A[c_T]/N$ can be written in the form

$$x = a_0 + \sum_{|T|\geq 2} a_T\, c_T \quad \text{where } a_0,\, a_T \in A$$

Hence by definition of η, we have $\eta(x) = a_0 + \sum_{|T|\geq 2} a_T\, s(v_T)$. As $\eta(x) = 0$, and as $A \cap E = 0$ by the above lemma, we have

$$a_0 = \sum_{|T|\geq 2} a_T\, s(v_T) = 0$$

Hence we get

$$x = \sum_{|T|\geq 2} a_T\, c_T = \theta(x') \quad \text{where } x' = \sum_{|T|\geq 2} a_T\, v_T$$

As $s(x') = \sum_{|T|\geq 2} a_T\, s(v_T) = 0$, we have $x' \in B$. In the notation of the above lemma, x' is in F, and as $B \cap F = E$, we see that $x' \in E$. As $\theta(E) = 0$, it finally follows that $x = \theta(x') = 0$.

This proves η is injective, and completes the proof of the Theorem 1.1. □

2. $\mathbb{C}^*$-actions, derivations, and Gysin boundary.

2.1. The derivation associated to a $\mathbb{C}^*$-action. Given an action $\mu : \mathbb{C}^* \times X \to X$ of the group $\mathbb{C}^*$ on a space X, consider the pullback $\mu^* : H^*(X) \to H^*(\mathbb{C}^* \times X)$, where the cohomology is with arbitrary coefficients. By Künneth formula, for each n we get a homomorphism $\mu^* : H^n(X) \to H^0(\mathbb{C}^*) \otimes H^n(X) \oplus H^1(\mathbb{C}^*) \otimes H^{n-1}(Y)$. As we have a section $X \to \mathbb{C}^* \times X : x \mapsto (1, x)$, it follows that for any α in $H^n(X)$, we have $\mu^*(\alpha) = 1 \otimes \alpha + \eta \otimes \alpha'$ where η is the positive generator of $H^1(\mathbb{C}^*)$, and the element $\alpha' \in H^{n-1}$ is uniquely determined by α. On the other hand, under the projection $p : \mathbb{C}^* \times X \to X$ we have $p^*(\alpha) = 1 \otimes \alpha$.

We now define a linear operator $s : H^n(X) \to H^{n-1}(X)$ on $H^*(X)$ by the equality $\mu^*(\alpha) = 1 \otimes \alpha + \eta \otimes s(\alpha)$.

Example 2.1. Let $X = \mathbb{C}^*$, and consider the action $m : \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{C}^* : (x, y) \mapsto xy$. Then we have the basic equality

$$m^*(\eta) = 1 \otimes \eta + \eta \otimes 1$$

It follows that the corresponding operator s on $H^*(\mathbb{C}^*) = k[\eta]/(\eta^2)$ is given by $s(1) = 0$ and $s(\eta) = 1$.

LEMMA 2.2. *Given an action $\mu : \mathbb{C}^* \times X \to X$ of the group $\mathbb{C}^*$ on a space X, the map $s : H^*(X) \to H^*(X)$ on the singular cohomology ring of X with arbitrary coefficients, defined by the equality*

$$\mu^* - p^* = \eta \otimes s$$

where $p : \mathbb{C}^* \times X \to X$ is the projection and η is the positive generator of $H^1(\mathbb{C}^*)$, is a graded anti-derivation of degree (-1) on the ring $H^*(X)$, that is, for all $\alpha \in H^i(X)$ and $\beta \in H^j(X)$, we have

$$s(\alpha\beta) = s(\alpha)\beta + (-1)^i \alpha\, s(\beta) \in H^{i+j-1}(X)$$

Moreover, the composite $s \circ s = 0$.

Proof. The verification of the derivation property is straight from the definition. We now prove the property $s \circ s = 0$. For this consider the commutative diagram

$$\begin{array}{ccc} \mathbb{C}^* \times \mathbb{C}^* \times X & \xrightarrow{(m,\mathrm{id}_X)} & \mathbb{C}^* \times X \\ {\scriptstyle (\mathrm{id}_{\mathbb{C}^*},\mu)}\downarrow & & \downarrow{\scriptstyle \mu} \\ \mathbb{C}^* \times X & \xrightarrow{\mu} & X \end{array}$$

where $m : \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{C}^* : (x,y) \mapsto xy$. As $m^*(\eta) = 1 \otimes \eta + \eta \otimes 1$, it follows from the definition of s that for any $\alpha \in H^i(X)$,

$$(m, \mathrm{id}_X)^* \circ \mu^*(\alpha) = 1 \otimes 1 \otimes \alpha + 1 \otimes \eta \otimes s(\alpha) + \eta \otimes 1 \otimes s(\alpha), \quad \text{and}$$
$$(\mathrm{id}_{\mathbb{C}^*}, \mu)^* \circ \mu^*(\alpha) = 1 \otimes 1 \otimes \alpha + 1 \otimes \eta \otimes s(\alpha) + \eta \otimes 1 \otimes s(\alpha) + \eta \otimes \eta \otimes s^2(\alpha).$$

Comparing, we get $\eta \otimes \eta \otimes s^2(\alpha) = 0$, which means $s^2(\alpha) = 0$. □

2.2. Derivation in the case of principal $\mathbb{C}^*$-bundles. We next express the above derivation s in the case when X is the total space of a principal $\mathbb{C}^*$-bundle $\pi_X : X \to Y$ over some base Y, in terms of the Gysin boundary map $d_X : H^*(X) \to H^*(Y)$.

LEMMA 2.3. *Let $\pi_X : X \to Y$ be a principal $\mathbb{C}^*$-bundle. Then the graded anti-derivation s_X on the singular cohomology ring $H^*(X)$ with arbitrary coefficients, which is associated to the given $\mathbb{C}^*$-action on X by Lemma 2.2, equals the composite*

$$s_X = \pi_X^* \circ d_X$$

where $d_X : H^*(X) \to H^*(Y)$ is the Gysin boundary map of degree (-1) and $\pi_X : H^*(Y) \to H^*(X)$ is induced by the projection $\pi_X : X \to Y$.

Proof. Let $\mu_X : \mathbb{C}^* \times X \to X$ be the action, and $p_X : \mathbb{C}^* \times X \to X$ be the projection. If the bundle X is trivial, then the equality $\mu_X^* - p_X^* = \eta \otimes s_X$ is obvious from the definitions. So it remains to prove this equality for a non-trivial X.

For any map $f : B \to Y$, let $\pi_M : M \to B$ denote the pullback bundle. The two projections $\pi_M : M \to B$ and $r : M \to X$, and the Gysin boundary map d_M, satisfy

$$\pi_M^* \circ f^* = r^* \circ \pi_X^* \text{ and } d_M \circ r^* = f^* \circ d_X$$

Here the second equality follows from the fact that the Euler class of X pulls back to the Euler class of M. Hence the following diagram commutes

$$\begin{array}{ccc} H^n(X) & \xrightarrow{\pi_X^* d_X} & H^{n-1}(X) \\ {\scriptstyle r^*}\downarrow & & \downarrow{\scriptstyle r^*} \\ H^n(M) & \xrightarrow{\pi_M^* d_M} & H^{n-1}(M) \end{array}$$

For the projection $(\mathrm{id}_{\mathbb{C}^*}, r) : \mathbb{C}^* \times M \to \mathbb{C}^* \times X$, we similarly have

$$p_M^* r^* = (\mathrm{id}_{\mathbb{C}^*}, r)^* p_X^* \quad \text{and} \quad \mu_M^* r^* = (\mathrm{id}_{\mathbb{C}^*}, r)^* \mu_X^*$$

Now take $f : B \to Y$ to be $\pi_X : X \to Y$. Then M is trivial and so the lemma holds for s_M. The projection $r : M \to X$ has a tautological section $\Delta : X \to M$ which is the diagonal of $M = X \times_Y X$. (Under the canonical isomorphism $(\mu_X, p_X) : \mathbb{C}^* \times X \to X \times_Y X$, the section $\Delta : X \to M$ becomes the map $u \mapsto (1, u)$.)

Hence given $\alpha \in H^i(X)$ we have

$$\begin{aligned}
\eta \otimes \pi_X^* d_X(\alpha) &= \eta \otimes \Delta^* r^* \pi_X^* d_X(\alpha) \quad \text{as } r \circ \Delta = \mathrm{id}_X . \\
&= \eta \otimes \Delta^* \pi_M^* d_M r^*(\alpha) \quad \text{as } r^* \pi_X^* d_X = \pi_M^* d_M r^* . \\
&= (\mathrm{id}_{\mathbb{C}^*}, \Delta)^* (\eta \otimes \pi_M^* d_M r^* \alpha) \\
&= (\mathrm{id}_{\mathbb{C}^*}, \Delta)^* (\mu_M^* - p_M^*)(r^* \alpha) \quad \text{as } M \text{ is trivial.} \\
&= (\mathrm{id}_{\mathbb{C}^*}, \Delta)^* (\mathrm{id}_{\mathbb{C}^*}, r)^* (\mu_X^* - p_X^*)(\alpha) \\
&= (\mu_X^* - p_X^*)(\alpha) \quad \text{as } (\mathrm{id}_{\mathbb{C}^*}, r) \circ (\mathrm{id}_{\mathbb{C}^*}, \Delta) = \mathrm{id}_{\mathbb{C}^* \times X} . \\
&= \eta \otimes s_X(\alpha) \quad \text{by definition of } s_X .
\end{aligned}$$

This proves the lemma in the general case. □

2.3. Gysin boundary commutes with Steenrod squares.

Note. From now onwards, in the rest of this paper, singular cohomology will be with coefficients $k = \mathbb{Z}/(2)$.

LEMMA 2.4. (Steenrod) *If $L \to Y$ a complex line bundle on a space Y, and $L_o = L - Y$ the complement of the zero section $Y \subset L$, then the connecting homomorphism δ for the singular cohomology of the pair (L, L_o) commutes with the Steenrod squaring operations Sq^j on cohomologies of L_o and (L, L_o), giving a commutative rectangle*

$$\begin{array}{ccc}
H^i(L_o) & \xrightarrow{\delta} & H^{i+1}(L, L_o) \\
{\scriptstyle \mathrm{Sq}^j} \downarrow & & \downarrow {\scriptstyle \mathrm{Sq}^j} \\
H^{i+j}(L_o) & \xrightarrow{\delta} & H^{i+j+1}(L, L_o)
\end{array}$$

Proof. See Chapter I, Lemma 1.2 of Steenrod [S]. □

The Gysin sequence for L and the long exact sequence of the pair (L, L_o) fit in the following commutative diagram, where the vertical maps $\psi : H^{r-1}(L) \to H^{r+1}(L, L_o)$ are the Thom isomorphisms, given by product with the Thom class $U \in H^2(L, L_o)$, and where $\lambda \in H^2(X) = H^2(L)$ the Euler class. Note that U is in the image of $H^2(L, L_o; \mathbb{Z}) \to H^2(L, L_o; \mathbb{Z}/(2))$ by the orientability of the underlying rank 2 real vector bundle of L. We have identified $H^*(Y)$ with $H^*(L)$ via the projection $L \to Y$.

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & H^r(L) & \longrightarrow & H^r(L_o) & \xrightarrow{d} & H^{r-1}(L) & \xrightarrow{\lambda} & H^{r+1}(L) & \longrightarrow \cdots \\
& \| & & \| & & {\scriptstyle \psi}\downarrow & & \| & \\
\cdots \longrightarrow & H^r(L) & \longrightarrow & H^r(L_o) & \xrightarrow{\delta} & H^{r+1}(L, L_o) & \longrightarrow & H^{r+1}(L) & \longrightarrow \cdots
\end{array}$$

We next show that Gysin boundaries and Steenrod squares commute.

LEMMA 2.5. *For any complex line bundle $L \to Y$, with $L_o = L - Y$ the complement of zero section, the Gysin boundary map $d : H^r(L_o) \to H^{r-1}(L)$ and the*

Steenrod squaring operations Sq^j *fit in the commutative rectangle*

$$\begin{array}{ccc} H^i(L_o) & \xrightarrow{d} & H^{i-1}(L) \\ {\scriptstyle \mathrm{Sq}^j}\downarrow & & \downarrow{\scriptstyle \mathrm{Sq}^j} \\ H^{i+j}(L_o) & \xrightarrow{d} & H^{i+j-1}(L) \end{array}$$

Proof. We identify $H^*(Y)$ with $H^*(L)$ using the projection $L \to Y$. We have for any $x \in H^i(L_o)$ the following sequence of equalities in $H^{i+j+3}(L, L_o)$.

$$\begin{aligned} \psi d\,\mathrm{Sq}^j\, x &= \delta\,\mathrm{Sq}^j\, x, \quad \text{as } \psi d = \delta. \\ &= \mathrm{Sq}^j\,\delta x, \quad \text{as } \mathrm{Sq}^j \text{ and } \delta \text{ commute by lemma 2.4.} \\ &= \mathrm{Sq}^j\,\psi z, \quad \text{where } z = \psi^{-1}\delta x. \\ &= \mathrm{Sq}^j(zU), \quad \text{as by definition, } \psi(z) = zU \text{ where } U \text{ is the Thom class.} \\ &= \mathrm{Sq}^j(z)U + \mathrm{Sq}^{j-1}(z)\,\mathrm{Sq}^1(U) + \mathrm{Sq}^{j-2}(z)\,\mathrm{Sq}^2(U), \\ &\qquad \text{by the general formula for } \mathrm{Sq}^j(ab), \text{ and as } U \in H^2(L). \\ &= \mathrm{Sq}^j(z)U + \mathrm{Sq}^{j-2}(z)\,\mathrm{Sq}^2(U), \\ &\qquad \text{as } \mathrm{Sq}^1(U) = 0, \text{ since } U \text{ is defined over coefficients } \mathbb{Z}. \\ &= \mathrm{Sq}^j(z)U + \mathrm{Sq}^{j-2}(z)\lambda U, \quad \text{as } \mathrm{Sq}^2(U) = U^2 = \lambda U. \\ &= (\mathrm{Sq}^j(z) + \mathrm{Sq}^{j-2}(z)\lambda)U. \end{aligned}$$

Hence we get the equality $\psi d\,\mathrm{Sq}^j\, x = (\mathrm{Sq}^j(z) + \mathrm{Sq}^{j-2}(z)\lambda)U$ in $H^{i+j+3}(L, L_o)$. As ψ is given by cupping by U and is injective, we get $d\,\mathrm{Sq}^j\, x = \mathrm{Sq}^j(z) + \mathrm{Sq}^{j-2}(z)\lambda$. As we had taken $z = \psi^{-1}\delta x = dx$, we get

$$d\,\mathrm{Sq}^j\, x = \mathrm{Sq}^j\, dx + (\mathrm{Sq}^{j-2}\, dx)\lambda \ \text{ in } H^{i+j+1}(L, L_o).$$

For $j = 1$, we therefore have $d\,\mathrm{Sq}^1\, x = \mathrm{Sq}^1\, dx$ as $\mathrm{Sq}^{-1} = 0$. Also, as $\mathrm{Sq}^0 = \mathrm{id}$, the equality $d\,\mathrm{Sq}^0\, x = \mathrm{Sq}^0\, dx$ holds. Now we proceed by induction on j. Substituting $d\,\mathrm{Sq}^{j-2}\, x$ for $\mathrm{Sq}^{j-2}\, dx$ in the above displayed equality gives

$$d\,\mathrm{Sq}^j\, x = \mathrm{Sq}^j\, dx + (d\,\mathrm{Sq}^{j-2}\, x)\lambda = \mathrm{Sq}^j\, dx$$

where the final equality holds because $(dy)\lambda = 0$ for all $y \in H^*(L_o)$. Hence the lemma is proved. □

LEMMA 2.6. *As before, let* L_o *the complement of the zero section of a complex line bundle* L *on a space* X. *Let* $x \in H^i(L_o)$. *Then under the Gysin boundary* $d : H^{2i}(L_o) \to H^{2i-1}(L)$, *we have* $d(x^2) = 0$, *equivalently, there exists* $y \in H^{2i}(Y)$ *with* $\pi^*(y) = x^2$.

Proof. As $x^2 = \mathrm{Sq}^i(x)$, we have $d(x^2) = d(\mathrm{Sq}^i(x)) = \mathrm{Sq}^i(dx)$ by lemma 2.5. As $dx \in H^{i-1}(BGO(2n))$, we have $\mathrm{Sq}^i(dx) = 0$, so the lemma is proved. □

3. Cohomology of $BGO(2n)$.

3.1. Principle $GO(n)$-bundles and reductions to $O(n)$.

Principal $GO(n)$-bundles and triples (E, L, b). Recall that principal $O(n)$-bundles Q on a space X are equivalent to pairs (E, q) where E is the rank n complex bundle on X associated to Q via the defining representation of $O(n)$ on $\mathbb{C}^n$, and $q : E \otimes E \to \mathcal{O}_X$ is the everywhere non-degenerate symmetric bilinear form on E with values in the trivial line bundle $\mathcal{O}_X$ on X, such that q is induced by the standard quadratic form $\sum x_i^2$ on $\mathbb{C}^n$. The converse direction of the equivalence is obtained via a Gram-Schmidt process, applied locally.

By a similar argument, principal $GO(n)$-bundles P on X are equivalent to triples (E, L, b) where E is the vector bundle associated to P via the defining representation of $GO(n)$ on $\mathbb{C}^n$, and $b : E \otimes E \to L$ is the everywhere nondegenerate symmetric bilinear form on E induced by the standard form $\sum x_i^2$ on $\mathbb{C}^n$, which now takes values in the line bundle L on X, associated to P by the homomorphism $\sigma : GO(n) \to \mathbb{C}^*$ defined by the equality ${}^tgg = \sigma(g)I$.

Example 3.1. Let L be any line bundle on X. On the rank 2 vector bundle $F = L \oplus \mathcal{O}_X$, we define a nondegenerate symmetric bilinear form b with values in L by putting $b : (x_1, x_2) \otimes (y_1, y_2) \mapsto x_1 \otimes y_2 + y_1 \otimes x_2$. Now for any $n \geq 1$, let the triple $(F^{\oplus n}, L, b)$ be the orthogonal direct sum of n copies of (F, L, b). This shows that given any space X and any even integer $2n \geq 2$, there exists some nondegenerate symmetric bilinear triple (E, L, b) on X of rank $2n$, where L is a given line bundle.

Reductions to $O(n)$. Given a principle $GO(n)$-bundle P on X, let (E, L, b) be the corresponding triple, and let $L_o = L - X$ (complement of zero section). As the sequence

$$1 \to O(n) \to GO(n) \xrightarrow{\sigma} \mathbb{C}^* \to 1$$

is exact, $L_o \to X$ is the associated $GO(n)/O(n)$-bundle to P, and so reductions of structure group of P from $GO(n)$ to $O(n)$ are the same as global sections (trivializations) of L. This can be described in purely linear terms, by saying that given an isomorphism $v : L \xrightarrow{\sim} \mathcal{O}_X$, we get a pair $(E, v \circ b : E \otimes E \to \mathcal{O}_X)$ from the triple (E, L, b).

Let $u : L \xrightarrow{\sim} \mathcal{O}_X$ and $v : L \xrightarrow{\sim} \mathcal{O}_X$ be two such reductions. Then the two $O(n)$-bundles $(E, u \circ b)$ and $(E, v \circ b)$ are not necessarily isomorphic. In particular, the two sets of Stiefel-Whitney classes $w_i(E, u \circ b)$ and $w_i(E, v \circ b)$ need not coincide, but are related as follows.

Given any rank n vector bundle E together with an $\mathcal{O}_X$-valued nondegenerate bilinear form $q : E \otimes E \to \mathcal{O}_X$, consider the new bilinear form $yq : E \otimes E \to \mathcal{O}_X$ where $y : X \to \mathbb{C}^*$ is a nowhere vanishing function. Let $(y) \in H^1(X, \mathbb{Z}/(2))$ be the pull-back of the generator $\eta \in H^1(\mathbb{C}^*, \mathbb{Z}/(2))$ (in other words, (y) is the Kummer class of y). Note that $(y)^2 = 0$.

A simple calculation using the splitting principle shows the following.

LEMMA 3.2. *If $w_i(E, q)$ are the Stiefel-Whitney classes of the $O(n)$-bundle (E, q), then the Stiefel-Whitney classes of the $O(n)$-bundle (E, yq) are given by the formula*

$$w_i(E, yq) = w_i(E, q) + (n - i + 1)(y) \cdot w_{i-1}(E, q)$$

Remark 3.3. It can be seen that the homomorphism $GO(2n) \to \{\pm 1\} : g \mapsto \sigma(g)^n / \det(g)$ is surjective, with connected kernel $GSO(2n)$. This implies that $\pi_0(GO(2n)) = \mathbb{Z}/(2)$, and hence $\pi_1(BGO(2n)) = \mathbb{Z}/(2)$.

3.2. The natural derivation on the ring $H^*(BO(n))$. Let G be a Lie group and H a closed subgroup. If $P \to BG$ is the universal bundle on the classifying space BG of G, then the quotient P/H can be taken to be BH, so that $BH \to BG$ is the bundle associated to P by the action of G on G/H. In particular, taking $G = GO(n)$ and $H = O(n)$, we see that $BO(n) \to BGO(n)$ is the fibration $L_o \to BGO(n)$, where L_o is the complement of the zero section of the line bundle L on $BGO(n)$ occuring in the universal triple (E, L, b) on $BGO(n)$.

We denote by $\pi : BO(n) \to BGO(n)$ the projection and we denote by $\lambda \in H^2(BGO(n))$ the Euler class of L. This gives us the long exact Gysin sequence

$$\cdots \xrightarrow{\lambda} H^r(BGO(n)) \xrightarrow{\pi^*} H^r(BO(n)) \xrightarrow{d} H^{r-1}(BGO(n)) \xrightarrow{\lambda} H^{r+1}(BGO(n)) \xrightarrow{\pi^*} \cdots$$

By Lemma 2.3, the composite maps $s = \pi^* \circ d : H^r(BO(n)) \to H^{r-1}(BO(n))$ define a derivation s on the graded ring $H^*(BO(n))$. We now identify this derivation.

Recall that the singular cohomology ring $H^*(BO(n))$ with $k = \mathbb{Z}/(2)$ coefficients is the polynomial ring $H^*(BO(n)) = k[w_1, \ldots, w_n]$ in the Stiefel-Whitney classes w_i. We use the convention that $w_0 = 1$.

PROPOSITION 3.4. *Let $s : H^*(BO(n)) \to H^*(BO(n))$ be the composite $s = \pi^* \circ d$ as above. Then s is a derivation of degree (-1) on the graded ring $H^*(BO(n))$, with $s \circ s = 0$. In terms of the universal Stiefel-Whitney classes w_i, we have $H^*(BO(n)) = k[w_1, \ldots, w_n]$, and the derivation s is given in terms of these generators by*

$$s = \sum_{i=1}^{n} (n-i+1) w_{i-1} \frac{\partial}{\partial w_i} \; : \; w_i \mapsto (n-i+1) w_{i-1}$$

In particular, for $n = 2m$ even, let $u_i = w_{2i-1}$ and $v_i = w_{2i}$, where $1 \leq i \leq m$ be the generators of the polynomial ring $H^*(BO(2m))$. Then the derivation s on the polynomial ring $k[u_1, \ldots u_m, v_1, \ldots, v_m]$ is given by $s = \sum u_i \frac{\partial}{\partial v_i}$, with kernel ring explicitly given in terms of generators and relations by the Theorem 1.1.

Proof. Let (E, L, b) denote the universal triple on $BGO(n)$ and $\pi : L_o \to BGO(n)$ the projection, where L_o is the complement of the zero section of L. The pullback $\pi^*(L)$ has a tautological trivialization $\tau : \pi^*(L) \xrightarrow{\sim} \mathcal{O}_{L_o}$, which gives the universal pair $(\pi^* E, \tau \circ \pi^* q)$ on $L_o = BO(n)$. Let $f = \pi \circ p : \mathbb{C}^* \times L_o \to BGO(2n)$ be the composite of the projection $p : \mathbb{C}^* \times L_o \to L_o$ with $\pi : L_o \to BGO(n)$.

Under the projection $p : \mathbb{C}^* \times L_o \to L_o$, the pair $(\pi^* E, \tau \circ \pi^* b)$ pulls back to the pair $(f^* E, p^*(\tau \circ \pi^* q))$ on $\mathbb{C}^* \times L_o$. Under the scalar multiplication $\mu : \mathbb{C}^* \times L_o \to L_o$, the pair $(\pi^* E, \tau \circ \pi^* q)$ pulls back to the pair $(f^* E, y p^*(\tau \circ \pi^* b))$ on $\mathbb{C}^* \times L_o$, where $y : \mathbb{C}^* \times L_o \to \mathbb{C}^*$ is the projection. By Lemma 3.2, we have

$$w_i[\mu^*(\pi^* E, \tau \circ \pi^* b)] = w_i[p^*(\pi^* E, \tau \circ \pi^* b)] + (n-i+1)(y) \cdot w_{i-1}[p^*(\pi^* E, \tau \circ \pi^* b)]$$

in the cohomology ring $H^*(\mathbb{C}^* \times L_o)$.

Note that the Stiefel-Whitney classes $w_i(\pi^* E, \tau \circ \pi^* b)$ are simply the universal classes w_i. The class (y) becomes $\eta \otimes 1$ under the Künneth isomorphism. Hence the above formula reads

$$\mu^* w_i = p^* w_i + (n-i+1) \eta \otimes w_{i-1}$$

Hence the proposition follows by lemmas 2.2 and 2.3. □

3.3. Generators and relations for the ring $H^*(BGO(2n))$.

The elements λ, a_{2i-1}, b_{4i} and d_T of $H^*(BGO(2n))$. Let (E, L, b) be the universal triple on $BGO(2n)$. Recall that we denote by $\lambda \in H^2(BGO(2n))$ the Euler class of L. By Example 3.1 applied to $X = \mathbf{P}^\infty_{\mathbb{C}}$ with $L = \mathcal{O}_X(1)$ the universal line bundle on X, together with the universal property of $BGO(2n)$, it follows that $\lambda^n \neq 0$ for all $n \geq 1$. For each $1 \leq j \leq n$, we define elements

$$a_{2j-1} = dw_{2j} \in H^{2j-1}(BGO(2n)).$$

Note that we therefore have

$$\pi^*(a_{2j-1}) = s(w_{2j}) = w_{2j-1} \in H^{2j-1}(BO(2n)).$$

More generally, for any subset $T = \{i_1, \ldots, i_r\} \subset \{1, \ldots, n\}$ of cardinality $r \geq 2$, let $v_T = w_{2i_1} \cdots w_{2i_r} \in H^{2\deg(T)}(BO(2n))$ where $\deg(T) = i_1 + \cdots + i_r$. We put

$$d_T = d(v_T) \in H^{2\deg(T)-1}(BGO(2n)).$$

Next, by Lemma 2.6, the element w_{2j}^2 lies in the image of $\pi^* : H^{4j}(BGO(2n)) \to H^{4j}(BO(2n))$. We fix once and for all elements

$$b_{4j} \in H^{4j}(BGO(2n)) \text{ with } \pi^*(b_{4j}) = w_{2j}^2 \text{ for each } 1 \leq j \leq n.$$

In terms of the universal triple (E, L, b), a canonical choice for the element b_{4j} is the image of the Chern class $c_{2j}(E) \in H^{4j}(BGO(2n); \mathbb{Z})$ under the change of coefficients from $\mathbb{Z}$ to $\mathbb{Z}/(2)$.

In the ring $H^*(BGO(2n))$, we have $\lambda a_{2i-1} = 0$ for all $1 \leq i \leq n$, and $\lambda d_T = 0$ for every subset $T \subset \{1, \ldots, n\}$ of cardinality $|T| \geq 2$, as follows from the fact that $\lambda \circ d = 0$ in the Gysin sequence.

The image of π^*. We now come to a crucial lemma, one which allows us to write down the image of the ring homomorphism $\pi^* : H^*(BGO(2n)) \to H^*(BO(2n))$.

LEMMA 3.5. *We have the equality* $\ker(d) = \operatorname{im}(\pi^*) = \ker(s) \subset H^*(BO(2n))$, *in other words, the sequence* $H^i(BGO(2n)) \xrightarrow{\pi^*} H^i(BO(2n)) \xrightarrow{s} H^{i-1}(BO(2n))$ *is exact.*

Proof. We have $\operatorname{im}(s) \subset \ker(d) = \operatorname{im}(\pi^*) \subset \ker(s)$ by the exactness of the Gysin sequence. We have $w_{2i-1} = s(w_{2i}) = \pi^* d(w_{2i}) \in \operatorname{im}(\pi^*)$ and as $d(w_{2i}^2) = 0$, $w_{2i}^2 \in \ker(d) = \operatorname{im}(\pi^*)$. Hence we get the inclusion $A \subset \operatorname{im}(\pi^*)$, where A is the polynomial ring in variables w_{2i-1} and w_{2i}^2. As already seen, $\operatorname{im}(s) \subset \operatorname{im}(\pi^*)$, so $\operatorname{im}(s) + A \subset \operatorname{im}(\pi^*)$. This completes the proof, as $\ker(s) = \operatorname{im}(s) + A$ by Theorem 1.1. □

Generators for the ring $H^*(BGO(2n))$.

LEMMA 3.6. *The ring* $H^*(BGO(2n))$ *is generated by* λ, $(a_{2i-1})_i$, $(b_{4i})_i$, $(d_T)_T \in H^*(BGO(2n))$.

Proof. Let $S \subset H^*(BGO(2n))$ be the subring generated by the elements λ, $(a_{2i-1})_i$, $(b_{4i})_i$, $(d_T)_T$. By Theorem 1.1, the ring B is generated by the elements $w_{2i-1} = \pi^*(a_{2i-1})$ and $w_{2i}^2 = \pi^*(b_{4i})$ where $1 \leq i \leq n$, together with the elements $s(v_T) = \pi^*(d_T)$ where $T \subset \{1, \ldots, n\}$ with $|T| \geq 2$, which shows that $\pi^*(S) = B$. As $1 \in S$, we have $H^0(BGO(2n)) \subset S$. Moreover, $H^1(BGO(2n)) = \{0, a_1\} \subset S$. We now proceed by induction. Suppose that $H^j(BGO(2n)) \subset S$ for all $j < i$. From $\pi^*(S) = B$, and the fact that π^* is a graded homomorphism, it follows that given $x \in H^i(BGO(2n))$ there exists $x' \in S \cap H^i(BGO(2n))$ such that $\pi^*(x) = \pi^*(x')$. Hence $(x - x') \in \ker(\pi^*) = \operatorname{im}(\lambda)$, so let $x = x' + \lambda y$ where $y \in H^{i-1}(BGO(2n))$. By induction, $y \in S$, therefore $x \in S$. This proves the lemma. □

LEMMA 3.7. *In the k-algebra $H^*(BGO(2n))$, the $n+1$ elements λ and b_{4i} (where $1 \leq i \leq n$) are algebraically independent over k.*

Proof. We recall the following commutative diagram in which the top row is exact.

$$\begin{array}{ccccccc} H^j(BGO(2n)) & \stackrel{\pi^*}{\to} & H^j(BO(2n)) & \stackrel{d}{\to} & H^{j-1}(BGO(2n)) & \stackrel{\lambda}{\to} & H^{j+1}(BGO(2n)) \\ & & & s\searrow & \downarrow\pi^* & & \\ & & & & H^{j-1}(BO(2n)) & & \end{array}$$

Let $k[x_1,\ldots,x_n,y]$ be a polynomial ring in the $n+1$ variables $x_1,\ldots,x_n,y$. Let $f \in k[x_1,\ldots,x_n,y]$ be a non-constant polynomial of the lowest possible total degree, with $f(b_4,\ldots,b_{4n},\lambda) = 0$. Let $f = f_0 + yf_1$ where $f_0 \in k[x_1,\ldots,x_n]$. Then as $\pi^*(\lambda) = 0$, we get

$$0 = \pi^* f(b_4,\ldots,b_{4n},\lambda) = \pi^* f_0(b_4,\ldots,b_{4n}) = f_0(w_2^2,\ldots,w_{2n}^2)$$

As w_{2i}^2 are algebraically independent elements of the k-algebra $H^*(BO(2n))$, the equality $\pi^*(b_{4i}) = w_{2i}^2$ implies that the n elements $b_{4i} \in H^*(BGO(2n))$ are algebraically independent over k. It follows that $f_0 = 0 \in k[x_1,\ldots,x_n]$. Hence we have $f = yf_1$, and so $\lambda f_1(b_4,\ldots,b_{4n},\lambda) = f(b_4,\ldots,b_{4n},\lambda) = 0$. Hence by exactness of the Gysin sequence, there exists $z \in H^*(BO(2n))$ with $f_1(b_4,\ldots,b_{4n},\lambda) = dz$. Applying π^* to both sides, this gives $\pi^* f_1(b_4,\ldots,b_{4n},\lambda) = s(z)$. Now let $f_1 = f_2 + yf_3 \in k[x_1,\ldots,x_n,y]$, where $f_2 \in k[x_1,\ldots,x_n]$. As $\pi^* f_1(b_4,\ldots,b_{4n},\lambda) = \pi^* f_2(b_4,\ldots,b_{4n})$, we get the equality $\pi^* f_2(b_4,\ldots,b_{4n}) = s(z)$, that is,

$$f_2(w_2^2,\ldots,w_{2n}^2) \in \mathrm{im}(s)$$

Now, from the formula $s = \sum_i u_i \frac{\partial}{\partial v_i}$, it follows that $\mathrm{im}(s)$ is contained in the ideal generated by the $u_i = w_{2i-1}$. Hence the above means that $f_2(w_2^2,\ldots,w_{2n}^2) = 0$. Hence as before, $f_2 = 0 \in k[x_1,\ldots,x_n]$. Hence we get $f = yf_1 = y^2 f_3$.

As $\lambda(\lambda f_3(b_4,\ldots,b_{4n},\lambda)) = \lambda^2 f_3(b_4,\ldots,b_{4n},\lambda) = 0$, by exactness of the Gysin sequence there is some $q \in H^*(BO(2n))$ with $d(q) = \lambda f_3(b_4,\ldots,b_{4n},\lambda)$. Hence $s(q) = \pi^* d(q) = \pi^*(\lambda f_3(b_4,\ldots,b_{4n},\lambda)) = 0$ as $\pi^*(\lambda) = 0$. Hence $q \in \ker(s)$. The Lemma 3.5 showed that $\ker(s) = \mathrm{im}(\pi^*)$, hence $q = \pi^*(h)$ for some $h \in H^*(BGO(2n))$. But then we have

$$\lambda f_3(b_4,\ldots,b_{4n},\lambda) = d(q) = d\pi^*(h) = 0$$

as $d\pi^* = 0$ in the Gysin complex. Hence the polynomial $g = yf_3 \in k[x_1,\ldots,x_n,y]$, which is non-constant with degree less than that of $f = yg$, has the property that $g(b_4,\ldots,b_{4n},\lambda) = 0$. This contradicts the choice of f, proving the lemma. □

Remark 3.8. Let $k[b_4,\ldots,b_{4n},\lambda]$ denote the polynomial ring in the variables $b_4,\ldots,b_{4n},\lambda$. We have a short exact sequence of k-modules

$$0 \to k[b_4,\ldots,b_{4n},\lambda] \stackrel{\lambda}{\to} H^*(BGO(2n)) \stackrel{\pi^*}{\to} B \to 0$$

where injectivity of λ is by Lemma 3.7, exactness in the middle is by exactness of the Gysin sequence, and surjectivity of π^* is by Lemma 3.5.

Now we state and prove the main result.

Consider the $2^n + n$ algebraically independent indeterminates λ, a_{2i-1} and b_{4i} where $1 \leq i \leq n$, and d_T where T varies over subsets of $\{1, \ldots, n\}$ of cardinality $|T| \geq 2$. Let $k[\lambda, (a_{2i-1})_i, (b_{4i})_i, (d_T)_T]$ be the polynomial rings in these variables.

THEOREM 3.9. *For any $n \geq 1$, the cohomology ring of $BGO(2n)$ with coefficients $k = \mathbb{Z}/(2)$ is isomorphic to the quotient*

$$H^*(BGO(2n)) = \frac{k[\lambda, (a_{2i-1})_i, (b_{4i})_i, (d_T)_T]}{I}$$

where I is the ideal generated by

(1) *the n elements λa_{2i-1} for $1 \leq i \leq n$, and*

(2) *the $2^n - n - 1$ elements λd_T where $T \subset \{1, \ldots, n\}$ is a subset of cardinality ≥ 2, and*

(3) *the $2^n - n(n-1)/2 - n - 1$ elements of the form*

$$\sum_{i \in T} a_{2i-1}\, d_{T-\{i\}}$$

where $T \subset \{1, \ldots, n\}$ is a subset of cardinality ≥ 3, and

(4) *the $n(n-1)/2$ elements of the form*

$$(d_{\{i,j\}})^2 + a_{2i-1}^2 b_{4j} + a_{2j-1}^2 b_{4i}$$

where $\{i, j\} \subset \{1, \ldots, n\}$ is a subset of cardinality 2, and finally

(5) *the $(2^n - n - 1)^2 - n(n-1)/2$ elements (not necessarily distinct) of the form*

$$d_T\, d_U + \sum_{p \in T} \prod_{q \in T \cap U - \{p\}} a_{2p-1}\, b_{4q}\, d_{(T-\{p\})\Delta U}$$

where $T \neq U$ in case both T and U have cardinality 2, and where Δ denotes the symmetric difference of sets $X \Delta Y = (X - Y) \cup (Y - X)$.

Proof. Let $R = k[\lambda, (a_{2i-1})_i, (b_{4i})_i, (d_T)_T]$ denote the polynomial ring. Consider the homomorphism $R \to H^*(BGO(2n))$ which maps each of the variables in this polynomial ring to the corresponding element of $H^*(BGO(2n))$. This map is surjective by Lemma 3.6.

From $\pi^*(\lambda) = 0$ and from the description of B in terms of generators and relations given in Theorem 1.1, it follows that all the generators of the ideal $I \subset R$, which are listed above, map to 0 under $\pi^* : H^*(BGO(2n)) \to B$. Hence we get an induced surjective homomorphism

$$\varphi : R/I \to H^*(BGO(2n))$$

From its definition, we see that the ideal $I \subset R$ satisfies $I \cap \lambda k[\lambda, b_4, \ldots, b_{4n}] = 0$. Hence we get an inclusion $\lambda k[\lambda, b_4, \ldots, b_{4n}] \hookrightarrow R/I$. Next, consider the map $R \to B$ which sends $\lambda \mapsto 0$, $a_{2i-1} \mapsto w_{2i-1}$, $b_{4i} \mapsto w_{2i}^2$, and $d_T \mapsto c_T$. From $\pi^*(\lambda) = 0$ and from the description of B in terms of generators and relations given in Theorem 1.1, it again follows that this map is surjective, and all the generators of the ideal $I \subset R$ map to 0, inducing a surjective homomorphism $\psi : R/I \to B$. Hence we get a short exact sequence $0 \to \lambda k[\lambda, b_4, \ldots, b_{4n}] \hookrightarrow R/I \xrightarrow{\psi} B \to 0$.

The above short exact sequence and the short exact sequence of Remark 3.8 fit in the following commutative diagram.

$$\begin{array}{ccccccc} 0 \to & \lambda k[\lambda, b_4, \ldots, b_{4n}] & \to & R/I & \stackrel{\psi}{\to} & B & \to 0 \\ & \| & & \varphi\downarrow & & \| & \\ 0 \to & \lambda k[\lambda, b_4, \ldots, b_{4n}] & \to & H^*(BGO(2n)) & \stackrel{\pi^*}{\to} & B & \to 0 \end{array}$$

Hence the theorem follows by five lemma. □

Example 3.10. In particular, the small dimensional cohomology vector spaces of the $BGO(2n)$ are as follows, in terms of linear bases over the coefficients $k = \mathbb{Z}/(2)$.

$$\begin{aligned}
H^0(BGO(2n)) &= < 1 > \\
H^1(BGO(2n)) &= < a_1 > \\
H^2(BGO(2n)) &= < \lambda,\, a_1^2 > \\
H^3(BGO(2n)) &= \begin{cases} < a_1^3 > & \text{for } n = 1, \\ < a_1^3,\, a_3 > & \text{for } n \geq 2. \end{cases} \\
H^4(BGO(2n)) &= \begin{cases} < \lambda^2,\, a_1^4,\, b_4 > & \text{for } n = 1, \\ < \lambda^2,\, a_1^4,\, a_1 a_3,\, b_4 > & \text{for } n \geq 2. \end{cases} \\
H^5(BGO(2n)) &= \begin{cases} < a_1^5,\, a_1 b_4 > & \text{for } n = 1, \\ < a_1^5,\, a_1^2 a_3,\, a_1 b_4,\, d_{\{1,2\}} > & \text{for } n = 2, \\ < a_1^5,\, a_1^2 a_3,\, a_1 b_4,\, a_5,\, d_{\{1,2\}} > & \text{for } n \geq 3. \end{cases}
\end{aligned}$$

REMARK. A determination of the ring homomorphism $H^*(BGL(2n, \mathbb{C})) \to H^*(BGO(2n, \mathbb{C}))$, together with an appliction of the results of this paper, is contained in the authors' paper 'Topology of Quadric Bundles' which will appear elsewhere (e-print math.AG/0008104 available from the site xxx.lanl.gov).

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