KNOT CONCORDANCE AND TORSION*

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1. Introduction. The classical knot concordance group, $\mathcal{C}_1$, was defined in 1961 by Fox [F]. He proved that it is nontrivial by finding elements of order two; details were presented in [FM]. Since then one of the most vexing questions concerning the concordance group has been whether it contains elements of finite order other than 2–torsion. Interest in this question was heightened by Levine's proof [L1, L2] that in all higher odd dimensions the knot concordance group contains an infinite summand generated by elements of order 4. In our earlier work studying this problem we proved the following [LN]:

THEOREM 1.1. *Let K be a knot in S^3 with 2–fold branched cover M_K. If the order of the first homology with integer coefficients satisfies $|H_1(M_K)| = pm$ with p a prime congruent to 3 mod 4 and gcd(p, m) = 1, then K is of infinite order in the classical knot concordance group, $\mathcal{C}_1$.*

An immediate corollary was that all of the prime knots of less than 11 crossings that are of order 4 in the algebraic concordance group are of infinite order in the concordance group. There are 11 such knots [M]. One simple case of a much deeper corollary states that if the Alexander polynomial of a knot satisfies $\Delta_K(t) = 5t^2 - 11t + 5$ then K is of infinite order in $\mathcal{C}_1$. According to Levine [L2], any higher dimensional knot with this polynomial is of order 4 in concordance.

Here our goal is to prove the following enhancement of the theorem stated above:

THEOREM 1.2. *Let K be a knot in S^3 with 2–fold branched cover M_K. If $H_1(M_K) = Z_{p^n} \oplus G$ with p a prime congruent to 3 mod 4, n odd, and p not dividing the order of G, then K is of infinite order in $\mathcal{C}_1$.*

As we will describe below, the significance of this result goes beyond its apparent technical merit; however, even in terms of computations it is an important improvement. Let H_p denote the p–primary summand of $H_1(M_K)$.

COROLLARY 1.3. *Let n be a positive integer such that some prime p congruent to 3 mod 4 has odd exponent in the prime power factorization of $4n+1$. Then a knot K with Alexander polynomial $nt^2 - (2n+1)t + n$ and H_p cyclic is of infinite order in the concordance group.*

Note that according to Levine [L2], any such knot represents an order 4 class in the algebraic concordance group. The n–twisted doubles of knots provide infinitely many examples of knots with Alexander polynomial $nt^2 - (2n+1)t + n$ and $H_1(M_K)$ cyclic. Further details and examples will be provided in the last section.

Casson and Gordon's first examples of algebraically slice knots that are not slice [CG1, CG2] were taken from the set of twisted doubles of the unknot. Our analysis

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extends theirs to a much larger class of knots and is not restricted to doubles of the unknot. Notice here the rather remarkable fact that an abelian invariant of a knot is being used to obstruct an algebraically slice knot from being slice.

Theorem 1.2 is relevant to deeper questions concerning the concordance group. The underlying conjecture is that the only torsion in the knot concordance group is 2–torsion, arising from amphicheiral knots (see related questions in [G, K1, K2]); a positive solution to this conjecture seems far beyond the tools now available to study concordance, in either the smooth or topological locally flat category. However, two weaker conjectures are possible.

CONJECTURE 1.4. *If a knot K represents 4–torsion in the algebraic concordance group, then it is of infinite order in concordance.*

A simpler conjecture is:

CONJECTURE 1.5. *There exists a class of order 4 in the algebraic concordance group that cannot be represented by a knot of order 4. In particular, Levine's homomorphism does not split.*

Theorem 1.1 provided candidates for verifying Conjecture 1.5 but there are two difficult steps to extending that result from a representative of an algebraic concordance class to the entire class. It is a consequence of Witt theory (that we won't be using elsewhere in this paper) that such an extension will have two parts: one must be able to handle the case where $H_p = Z_{p^n}$, with $n > 1$, and also the case where H_p is a direct sum of such factors.

The results of this paper deal with the first part of the extension problem. A number of special cases of direct sums have been successfully addressed by us in unpublished work, but the necessary general result for sums has not yet been achieved. When it is, that result along with Theorem 1.2 should provide a proof of Conjecture 1.5 and perhaps 1.4 .

The work of this paper is largely algebraic. In the next section we will summarize the topological results that we will be using. All the work that appears here applies in both the topological locally flat and the smooth category. In Section 3 we give a proof of Theorem 1.2. The proof is fairly technical and extends the techniques used in proving Theorem 1.1 in [LN]. Section 4 discusses examples.

2. Background and notation.

2.1. Knots and the concordance group. We will work in the smooth category, but as just mentioned, all results carry over to the topological locally flat setting. Homology groups will always be with Z coefficients unless otherwise mentioned.

A knot is formally defined to be a smooth oriented pair, (S^3, K), with K diffeomorphic to S^1. We will denote such a pair simply by K. A knot K is called *slice* if $(S^3, K) = \partial(B^4, D)$, where D is a smooth 2–disk properly embedded in the 4–ball B^4. Knots K_1 and K_2 are called *concordant* if $K_1 \# -K_2$ is slice, where $-K$ represents the mirror image of K, formally $(-S^3, -K)$. The set of concordance classes of knots forms an abelian group under connected sum, denoted $\mathcal{C}_1$. The order of K in the knot concordance group is hence the least positive n for which the connected sum of n copies of K, nK, is slice.

Levine defined a homomorphism of $\mathcal{C}_1$ onto a group, $\mathcal{G}$, that is isomorphic to the infinite direct sum, $\mathcal{G} \cong Z^\infty \oplus Z_2^\infty \oplus Z_4^\infty$. For higher dimensions the corresponding homomorphism is an isomorphism, but in dimension 3 there is a nontrivial kernel, as first proved by Casson and Gordon [CG1, CG2]. For details concerning $\mathcal{G}$, the *algebraic concordance group*, see [L1, L2].

2.2. Casson-Gordon invariants and linking forms. Let M_K denote the 2–fold branched cover of S^3 branched over K, and let χ denote a homomorphism from $H_1(M_K)$ to Z_{p^k} for some prime p. The Casson-Gordon invariant, $\sigma(K,\chi)$ is then a well defined rational invariant of the pair (K,χ). (In Casson and Gordon's original paper, [CG1], this invariant is denoted $\sigma_1\tau(K,\chi)$, and σ is used for a closely related invariant.)

On any rational homology sphere, such as M_K, there is a nonsingular symmetric linking form, $\beta : H_1(M_K) \to Q/Z$. As before, let H_p be the p–primary summand of $H_1(M_K)$. The main result in [CG1] concerning Casson-Gordon invariants and slice knots that we will be using is the following:

THEOREM 2.3. *If K is slice there is a subgroup (or metabolizer) $L_p \subset H_p$ with $|L_p|^2 = |H_p|$, $\beta(L_p, L_p) = 0$, and $\sigma(K,\chi) = 0$ for all χ vanishing on L_p.*

We will also need the additivity theorem proved by Gilmer [Gi].

THEOREM 2.4. *If χ_1 and χ_2 are defined on M_{K_1} and M_{K_2}, respectively, then we have $\sigma(K_1 \# K_2, \chi_1 \oplus \chi_2) = \sigma(K_1,\chi_1) + \sigma(K_2,\chi_2)$.*

Any homomorphism χ from H_p to Z_{p^r} is given by linking with some element $x \in H_p$. In this situation we have the following (see Section 4 of [LN]).

THEOREM 2.5. *If $\chi\colon H_p \longrightarrow Z_{p^r}$ is a character obtained by linking with the element $x \in H_p$, then $\sigma(K,\chi) \equiv \beta(x,x)$* modulo *$Z$.*

A simple corollary, using the nonsingularity of β is:

COROLLARY 2.6. *If $H_p = Z_{p^n}$ and χ maps onto Z_{p^k} with $k > n/2$ then $\sigma(K,\chi) \neq 0$.*

Finally, we will use the result below which is a consequence of the fact that the linking form β gives a map from H_p onto $\mathrm{Hom}(L_p, \mathbf{Q}/\mathbf{Z}) \cong L_p$, with kernel equal to L_p.

THEOREM 2.7. *With H_p and L_p as in Theorem 2.3, we have $H_p/L_p \cong L_p$.*

3. Proof of Theorem 1.2. Let K be a knot in S^3 with the 2–fold branched cover M_K. Suppose that $H_1(M_K) = Z_{p^n} \oplus G$ with p a prime congruent to 3 mod 4, n odd, and p not dividing the order of G. We want to show that K is of infinite order in $\mathcal{C}_1$. The linking form of $H_1(M_K)$ represents an element of order 4 in the Witt group of Z_p linking forms. (See Corollary 23 (c) in [L2].) If K is of concordance order d, since Levine's homomorphism maps the concordance class of K to an order 4 class, we have $d = 4k$, for some positive integer k. Since p does not divide the order of G, a metabolizer for $H_1(M_{dK}) = \oplus_{4k} H_1(M_K)$ will induce a metabolizer for $(Z_{p^n})^{4k}$. We must analyze the possible metabolizers L_p for $(Z_{p^n})^{4k}$.

A vector in L_p can be written as $x = (x_i)_{i=1\dots d} \in (Z_{p^n})^d$. Applying the Gauss-Jordan algorithm to a generating set for L_p, and perhaps reordering, we can find a generating set of a particularly simple form. The next example illustrates a possible form for one such set, where the generators appear as the rows of the matrix.

EXAMPLE 3.1. *Let $H_p = \left(Z_{p^3}\right)^8$. A generating set for a metabolizer L_p of the standard nonsingular Q/Z linking form can be written as follows:*

$$\begin{pmatrix} 1 & * & * & * & * & * & * & * \\ 0 & p & 0 & 0 & * & * & * & * \\ 0 & 0 & p & 0 & * & * & * & * \\ 0 & 0 & 0 & p & * & * & * & * \\ 0 & 0 & 0 & 0 & p^2 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & p^2 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & p^2 & * \end{pmatrix}$$

In the above matrix, there is 1 row with leading term p^0, and 3 rows each with leading terms p^1 and p^2, respectively.

We will denote the number of rows with leading term p^i by k_i, the vectors in these k_i rows by $v_{i,1}, \cdots, v_{i,k_i}$, and $\sum_{j=0}^{i} k_j$ by S_i. For notational purposes, let $S_{-1} = 0$. Then, in general, the generating set consists of $\{v_{i,j}\}_{i=0,\ldots,n-1,j=1,\ldots,k_i}$ where $0 \le k_i \le 2k$, such that the first S_i entries of $v_{i,j}$ are 0, except for the $S_{i-1}+j$ entry which is p^i, and each of the remaining entries is divisible by p^i. From 2.7 it follows that $k_i = k_{n-i}$, for $i > 0$, and $S_{(n-1)/2} = 2k$.

DEFINITION 3.2. *If $a \in H_p$, let $\chi_a \colon H_1(M_K) \to Q/Z$ be the character given by linking with a. In the case that H_p is cyclic, isomorphic to Z_{p^n}, we can fix a generator of H_p and write χ_a where a is an integer representing an element in Z_{p^n}.*

With this notation, we now see that our goal is to show that $\sigma(K, \chi_{p^{(n-1)/2}}) = 0$. Since $\chi_{p^{(n-1)/2}}$ maps onto $Z_{p^{(n+1)/2}}$ this will contradict 2.6 and it will follow that K cannot be of finite order.

As in Example 3.1, arrange the $\{v_{i,j}\}$ as rows of a $(4k-k_0) \times 4k$ matrix following the dictionary order on (i,j). We multiply the first k_0 vectors by p^{n-1}, the next k_1 vectors by p^{n-2}, and, so on, to obtain p^{n-1} on the diagonal. Clear the off-diagonal entries in the left $(4k-k_0) \times (4k-k_0)$ block. Now, adding all the rows gives us a vector in L_p with the first $4k-k_0$ entries equal to p^{n-1}. This vector corresponds to a character χ, given by linking an element with it, to Z_p on which the Casson-Gordon invariants should vanish. That is, $\sigma((4k)K, \chi) = 0$. By 2.4 this leads to a relation of the form $(4k-k_0)\sigma(K, \chi_{p^{n-1}}) + \sum_{x_i \ne 0} \sigma(K, \chi_{x_i}) = 0$, where x_i are the remaining k_0 entries, each of which is divisible by p^{n-1}.

The set of nonzero characters from Z_{p^n} to Z_p is isomorphic to the multiplicative group of units in Z_p, which is a cyclic group of order $p-1$. Denoting a generator for this group by g, each nonzero χ_{x_i} corresponds to g^{α_i} for some α_i. The correspondence can be given by $\chi_{x_i} \longrightarrow g^{x_i/p^{n-1}}$. As in [LN] we use further shorthand, setting $t^{\alpha_i} = \sigma(K, \chi_{x_i})$. Each metabolizing vector leads to a relation $\sum_{x_i \ne 0} t^{\alpha_i} = 0$. Note that at this point the symbol t^α does not represent a power of any element "t", it is purely symbolic. However it does permit us to view the relations as being elements in the ring $Z[Z_{p-1}]$. Furthermore, since $\sigma(K, \chi_{x_i}) = \sigma(K, \chi_{p^n - x_i})$, we have that $t^j = t^{j+(p-1)/2}$. (Recall that $g^{(p-1)/2} = -1$.) Hence, we can view the relations as sitting in $Z[Z_q]$, where $q = (p-1)/2$.

If a metabolizing vector x corresponds to the relation $f = 0$, where f is represented by an element in $Z[Z_q]$, then ax corresponds to the relation $t^\alpha f$ where $g^\alpha = a$. It follows that the relations between Casson-Gordon invariants generated by the element $x \in L_p$ together with its multiples form an ideal in $Z[Z_q]$ generated by the polynomial

f.

With this in mind our relation can be written as $f = (4k - k_0) + \sum_{i=1}^{k'} t^{\alpha_i} = 0$, where $k' \leq k_0$. (Note that $4k - k_0 = S_{n-1}$.) We show that the ideal generated by f in $Z[Z_q]$ contains a nonzero integer. This will follow from the fact that f and $t^q - 1$ are relatively prime, which will be the case unless f vanishes at some q–th root of unity, say ω; however, by considering norms and the triangle inequality we see that this will be the case only if $k' = 2k$ and $\omega^{a_i} = -1$ for all i. But since q is odd, no power of ω can equal -1.

It follows that $n\sigma(K, \chi_{p^{n-1}}) = 0$, for some $n \in Z$. This implies that $\sigma(K, \chi_{p^{n-1}}) = 0$. Similarly we can show that $\sigma(K, \chi_{ap^{n-1}}) = 0$, for $0 < a < p$.

Next, let l be a nonnegative integer, and assume that $\sigma(K, \chi_{ap^s}) = 0$, for all $a \in Z$, and all s such that $l < s \leq n - 1$. We will show that $\sigma(K, \chi_{p^l}) = 0$.

For $0 \leq i \leq S_l$, we multiply the vectors from the $(S_{i-1}+1)$st to the S_ith vector by p^{l-i}, clear off-diagonal entries in the upper left $S_l \times S_l$ square block, and add the first S_l rows to get a vector in L_p with first S_l entries equal to p^l, and the remaining entries divisible by p^l. Since we have assumed that $\sigma(K, \chi_{ap^s}) = 0$, for $l < s \leq n - 1$, we can ignore the entries which are of the form ap^s, with $s > l$. Then we have a character to the multiplicative group of units in $Z_{p^{n-l}}$. Since p is odd, this is a cyclic group of order $p^{n-l-1}(p-1)$ (see [D]). Again, since $\sigma(K, \chi_{x_i}) = \sigma(K, \chi_{p^n - x_i})$, we can view the relations as sitting in $Z[Z_q]$, where $q = p^{n-l-1}(p-1)/2$. As $p^{n-l-1}(p-1)/2$ is odd, as above, it follows that the relation $f = S_l + \sum_{i=1}^{k'} t^{\alpha_i} = 0$, where $0 \leq k' \leq 4k - S_l$, is relatively prime to $t^q - 1$. It follows that $\sigma(K, \chi_{p^l}) = 0$. Similarly, $\sigma(K, \chi_{ap^l}) = 0$ for $0 < a < p$.

Thus, we have $\sigma(K, \chi_{p^{(n-1)/2}}) = 0$, which contradicts Corollary 2.6, and proves that K cannot be of finite order in the concordance group.

4. Examples. Basic examples illustrating the applicability of Theorem 1.2 are easily constructed. For instance, since the 2–fold branched cover of S^3 over an unknotting number one knot has cyclic homology, to apply Theorem 1.2 we only need to check the order of $H_1(M_K)$ which equals the Alexander polynomial evaluated at -1. We have the following.

COROLLARY 4.1. *Let K be an unknotting number one knot with Alexander polynomial Δ. If a prime p which is congruent to 3 mod 4 appears in the prime power factorization of $\Delta(-1)$ with an odd exponent, then K is of infinite order in the concordance group.*

More generally, suppose that there is a 3–ball $B \subset S^3$ intersecting the knot K in two arcs so that the pair $(B, B \cap K)$ is trivial and so that removing $(B, B \cap K)$ from S^3 and gluing it back in via a homeomorphism of the boundary yields the unknot. Since the 2–fold branched cover of the ball over two trivial arcs is a solid torus, the 2–fold branched cover of S^3 over K is formed from S^3 (the 2–fold branched cover of S^3 over the unknot) by removing a solid torus and sewing it back in via some homeomorphism. In particular, the 2–fold branched cover has cyclic homology. Such knots include all unknotting number one knots and all 2–bridge knots. In the case of a 2–bridge knot $K(p, q)$, we have $H_1(M_K) = Z_p$.

COROLLARY 4.2. *The 2-bridge knot $K(p, q)$ has infinite order in the knot concordance group if some prime congruent to 3 mod 4 has odd exponent in p.*

The following theorem, an immediate consequence of a result of Levine (Corollary 23 in [L2]), provides us with more examples of knots which represent torsion in the algebraic concordance group.

THEOREM 4.3. *If a knot K has quadratic Alexander polynomial $\Delta(t)$ then:*
(a) K is of finite order in the algebraic concordance group if and only if $\Delta(1)\Delta(-1) < 0$, in which case K is of order 1, 2 or 4.
(b) K is of order 1 if and only if $\Delta(t)$ is reducible.
(c) if K is finite order, and $\Delta(t)$ is irreducible, then K is of order 4 in the algebraic concordance group if and only if for some prime $p > 0$ with $p \equiv 3 \bmod 4$, $\Delta(1)\Delta(-1) = -p^a q$ where a is odd and $q > 0$ is relatively prime to p.

Consider the knot K_a, the a–twisted double of some knot K. The Seifert form for this knot is

$$V = \begin{pmatrix} a & 1 \\ 0 & -1 \end{pmatrix},$$

it has Alexander polynomial $\Delta(t) = at^2 - (2a+1)t + a$, and the homology of the 2–fold cyclic branched cover is $Z_{|4a+1|}$. Levine's result, Theorem 4.3, applies to determine the algebraic order of all of these knots. (In the case that K is unknotted, K_a can be described as the 2–bridge knot $K(4a+1, 2a)$.)

COROLLARY 4.4. *The a–twisted double of a knot K:*
(a) is of infinite order in the algebraic concordance group, $\mathcal{G}$, if $a < 0$.
(b) is algebraically slice if $a > 0$ and $4a + 1$ is a perfect square.
(c) is of order 2 in $\mathcal{G}$ if $a > 0$, $4a+1$ is not a perfect square, and every prime congruent to 3 mod 4 has even exponent in the prime power factorization of $4a + 1$.
(d) is of order 4 if $a > 0$ and some prime congruent to 3 mod 4 has odd exponent in $4a + 1$.

Casson and Gordon [CG1, CG2] proved that if K is unknotted, then all knots covered by case (b) above are actually of infinite order in concordance, except if $a = 2$ in which case K_2 is slice. An immediate consequence of Theorem 1.2 is:

COROLLARY 4.5. *If K_a is of order 4 in $\mathcal{G}$ then it is of infinite order in the knot concordance group.*

As in Corollary 9.5 of [LN] a simple argument using Corollary 4.5 gives an infinitely generated free subgroup of $\mathcal{C}_1$ which consists of of algebraic slice knots. (It was first shown by Jiang in [J] that the kernel of Levine's homomorphism is infinitely generated.) The extensive calculations of [CG1, CG2] are here replaced with a trivial homology calculation. Moreover, the results of [CG1, CG2] apply only in the case that K is unknotted, a restriction that does not appear in Corollary 4.5.

Recently, Tamulis [T] has proved that in the case that K is unknotted, if K_a is of order 2 in $\mathcal{G}$ and $4a + 1$ is prime, then K_a is of infinite order in concordance.

Counterexamples. Given these previous examples, it is a bit unexpected that Theorem 1.2 does not apply in all cases of order 4 knots with quadratic Alexander polynomial. The difficulty is that the conditions of Theorem 4.3 do not assure that the homology of the 2–fold cover is cyclic. The next example demonstrates this. It is the simplest possible example in terms of the coefficients of the Alexander polynomial;

its complexity illustrates the strength of Theorem 1.2. The example is obtained by letting K be a knot with Seifert form:

$$V = \begin{pmatrix} 21 & 53 \\ 52 & 21 \end{pmatrix}.$$

The Alexander polynomial for K is $\Delta(t) = 2315 - 4631t + 2315t^2$. We have that $\Delta(1) = -1$, $\Delta(-1) = 9261 = 3^3 7^3$, and hence by Theorem 4.3, K is of order 4 in the algebraic concordance group.

The homology of M_K is presented by $V + V^t$:

$$V = \begin{pmatrix} 42 & 105 \\ 105 & 42 \end{pmatrix}.$$

A simple manipulation shows that this presents the group $Z_3 \oplus Z_9 \oplus Z_7 \oplus Z_{49}$. Because this is not cyclic, Theorem 1.2 does not apply. As mentioned in the introduction, we have been able to extend our results to special cases of direct sums of cyclic groups, and one of those extensions applies to the group $Z_3 \oplus Z_9$. Hence it can actually be shown that any knot with this Seifert form is not of order 4 in concordance.

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