THE CLASSIFICATION OF HOMOGENEOUS 2-SPHERES IN $\mathbb{C}P^n$*

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Dedicated to Professor Shiing-shen Chern at the occasion of his 90th birthday

Abstract. Let M be a connected, orientable surface. An immersion $x : M \to \mathbb{C}P^n$ is called homogeneous, if for any two points $p, q \in M$ there exists a holomorphic isometry T of $\mathbb{C}P^n$ and a diffeomorphism $\sigma : M \to M$ such that $\sigma(p) = q$ and $x \circ \sigma = T \circ x$. All such T form a subgroup G of the holomorphic isometry group $U(n+1)/S^1$ and $x(M)$ is an orbit surface of G. Such surfaces in $\mathbb{C}P^n$ have constant curvature, constant Kaehler angle, and in general are non-minimal. In this paper we show that any homogeneous surface in $\mathbb{C}P^n$ generates a sequence of homogeneous surfaces in $\mathbb{C}P^n$. In the case of a homogeneous sphere $x : S^2 \to \mathbb{C}P^n$ the sequence has to stop in the both directions, and any two different homogeneous spheres in the sequence are complex orthogonal. We give a construction of homogeneous 2-spheres in $\mathbb{C}P^n$ by using Veronese sequences in $\mathbb{C}P^m$ with $m \leq n$ as foundation stones, and prove that any homogeneous 2-sphere in $\mathbb{C}P^n$ can be obtained (up to a holomorphic isometry) by this construction.

1. Introduction.

A surface in $\mathbb{C}P^n$ is said to be homogeneous, if it is a 2-dimensional orbit of a subgroup in the holomorphic isometry group $U(n+1)/S^1$ of $\mathbb{C}P^n$. Standard examples of homogeneous 2-spheres are the so-called Veronese sequence in $\mathbb{C}P^n$, which are also minimal in $\mathbb{C}P^n$ (see, for examples, [B-J-R-W], [B-O] and [B-W]).

As is well-known, any surface in $\mathbb{C}P^n$ sits in a sequence of surfaces in $\mathbb{C}P^n$, which can be constructed by using derivatives with respect to its complex coordinate z and $\bar{z}$. Surfaces in the sequence may contain singularities. If $x : M \to \mathbb{C}P^n$ is minimal (harmonic), then all surfaces in the sequence are minimal (harmonic), and the sequence is called the harmonic sequence of x. In this paper we show that any homogeneous surface in $\mathbb{C}P^n$ generates a sequence of homogeneous surfaces. In case of homogeneous 2-sphere we show that any two different homogeneous spheres in its homogeneous sequence are complex orthogonal, thus its homogeneous sequence has to stop in both directions.

Using the Veronese sequence in $\mathbb{C}P^k, k = 0, 1, \cdots$, we can give a complete construction of homogeneous 2-spheres in $\mathbb{C}P^n$. For each integer $n \geq 0$ we denote by

$$\{[\phi_n^0], [\phi_n^1], \cdots, [\phi_n^n]\} \tag{1.1}$$

the Veronese sequence in $\mathbb{C}P^n$ (see §4). We make the convention that $[\phi_0^0] = [1]$ and the sequence $[\phi_0^1], [\phi_1^1] : \mathbb{C}P^1 \to \mathbb{C}P^1$ are defined by $[\phi_0^1] = id$ and $[\phi_1^1]([z, w]) = [-\bar{w}, \bar{z}]$. Let $\{n_1, n_2, \cdots, n_r\}$ be nonnegative integers. We take from each Veronese sequence in $\mathbb{C}P^{n_\alpha}$ a surface $[\phi_{j_\alpha}^{n_\alpha}]$ such that

$$n_1 - 2j_1 = n_2 - 2j_2 = \cdots = n_r - 2j_r\,. \tag{1.2}$$

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Then for nonzero constants $c_\alpha \in \mathbb{C}$ we construct a new homogeneous sphere $[\phi] : S^2 \to \mathbb{C}P^n$ with $n = n_1 + \cdots + n_r + r - 1$ by joining them together:

$$[\phi] = [c_1\phi^{j_1}_{n_1}, c_2\phi^{j_2}_{n_2}, \cdots, c_r\phi^{j_r}_{n_r}]. \tag{1.3}$$

These homogeneous 2-spheres in $\mathbb{C}P^n$ are in general non-minimal. Our main theorem in this paper is the following

Classification Theorem. *Let $x : S^2 \to \mathbb{C}P^n$ be a homogeneous sphere. Then there exist integers $\{n_\alpha, j_\alpha\}$ satisfying (1.2) and nonzero constants $\{c_\alpha\} \in \mathbb{C}$ such that $x = T \circ [\phi]$, where $[\phi]$ is defined by (1.3) and $T \in U(n+1)$.*

To prove the classification theorem we use operators ∂ and $\overline{\partial}$ which are non-linear analogues to the linear operators introduced by [Ca] and used in [B]. With the help of these operators we show that any homogeneous surface in $\mathbb{C}P^n$ generates a sequence of homogeneous surfaces in $\mathbb{C}P^n$. Using homogeneous sequence we show that if a homogeneous sphere $x : S^2 \to \mathbb{C}P^n$ is non-minimal, then x can split into a minimal homogeneous sphere $x_1 : S^2 \to \mathbb{C}P^{n_1}$ and a homogeneous sphere $x_2 : S^2 \to \mathbb{C}P^{n_2}$ with $n_1 + n_2 + 1 = n$, and x is holomorphically isometric to the join of x_1 and x_2. This leads to a proof of the classification theorem.

We organize this paper as follows. In §2 we give structure equations we need for $x : M \to \mathbb{C}P^n$. In §3 we study the properties of homogeneous sequence. Then we give the construction of homogeneous 2-spheres in $\mathbb{C}P^n$ in §4, and prove the classification theorem in §5.

2. Structure equations for submanifolds in $\mathbb{C}P^n$. In this section we recall the theory of submanifolds in $\mathbb{C}P^n$ briefly and give a characterization theorem for minimal submanifolds in $\mathbb{C}P^n$ which we will need in §3.

Let $\pi : S^{2n+1} \to \mathbb{C}P^n$ be the standard projection. For any local section $Z : \mathbb{B}^{2n} \to \mathbb{S}^{2n+1}$ of π defined on an open ball $\mathbb{B}^{2n}$ of $\mathbb{C}P^n$ we define

$$h_0 = (dZ - (dZ \cdot \overline{Z})Z) \otimes (d\overline{Z} - (d\overline{Z} \cdot Z)\overline{Z}). \tag{2.1}$$

It is easy to check that h_0 is independent of the choice of local section Z and thus a globally defined Hermitian metric on $\mathbb{C}P^n$, called Fubini-Study metric. We write

$$h_0 = h - i\Omega, \tag{2.2}$$

where $h = Re(h_0) = \frac{1}{2}(h_0 + \overline{h_0})$ is standard Riemannian metric h on $\mathbb{C}P^n$, and $\Omega = \frac{i}{2}(h_0 - \overline{h_0})$ is the Kaehler form on $\mathbb{C}P^n$. It is well-known that $\pi : S^{2n+1} \to \mathbb{C}P^n$ is a submersion. The horizontal space $\mathbb{H}_Z$ at Z is given by

$$\mathbb{H}_Z = \{W \mid W \cdot \bar{Z} = 0\} = \{V - (V \cdot \bar{Z})Z \mid V \in \mathbb{C}^{n+1}\}.$$

If we denote by $<,>$ the real inner product $< v, w >= 1/2(v \cdot \bar{w} + \bar{v} \cdot w)$ on $\mathbb{C}^{n+1}$, then $d\pi_Z : (\mathbb{H}_Z, <,>) \to (T_{[Z]}\mathbb{C}P^n, h)$ is an isometry. Since for any smooth curve $c(t)$ on $\mathbb{C}P^n$ we have $\pi(Z(c(t))) = c(t)$, we get for any $X \in T\mathbb{C}P^n$ the formula

$$d\pi_Z(X(Z)) = X,$$

where $X(Z)$ is the partial derivative of Z with respect to X. Since for any $\lambda \in \mathbb{C}^*$, $t\lambda Z$ is a curve in $\mathbb{C}^{n+1}$ with $\pi(t\lambda Z) = \pi(Z)$, we have $d\pi_Z(\lambda Z) = 0$. It is clear that

$$T_Z\mathbb{C}^{n+1} = \mathbb{C}Z \oplus \mathbb{H}_Z,$$

and $d\pi_Z : (H_Z, i) \to (\mathbb{C}P^n, J)$ is a complex isomorphism. Here the complex structure J of $\mathbb{C}P^n$ is then given by the simple formula

$$J(X) = d\pi_Z(iX(Z)),\ X \in T_{[Z]}\mathbb{C}P^n. \tag{2.3}$$

By (2.1) and (2.2) we get

$$h = \frac{1}{2}\{(dZ-(dZ\cdot\overline{Z})Z)\otimes(d\overline{Z}-(d\overline{Z}\cdot Z)\overline{Z})+(d\overline{Z}-(d\overline{Z}\cdot Z)\overline{Z})\otimes(dZ-(dZ\cdot\overline{Z})Z)\}; \tag{2.4}$$

$$\Omega = \frac{i}{2}(dZ - (dZ\cdot\bar{Z})Z) \wedge (d\bar{Z} - (d\bar{Z}\cdot Z)\bar{Z}) = \frac{i}{2}dZ \wedge d\bar{Z}. \tag{2.5}$$

Using (2.4) one can verify the following formula for the Levi-Civita connection $\overline{\nabla}$ of $(\mathbb{C}P^n, h)$:

$$\overline{\nabla}_X Y = d\pi_Z\{XY(Z) - (X(Z)\cdot\bar{Z})Y(Z) - (Y(Z)\cdot\bar{Z})X(Z)\}. \tag{2.6}$$

Let $x : M \to \mathbb{C}P^n$ be a m-dimensional submanifold. Then x induces a metric $g := x^*h$ on M. For any local section Z of $\pi : \mathbb{S}^{2n+1} \to \mathbb{C}P^n$ we can define a local lift $y := Z\circ x$ of the immersion $x : M \to \mathbb{C}P^n$. Such a local lift y of x exists around each point of M. Let $\{e_j, 1 \le j \le m\}$ be an orthonormal basis for g with dual basis $\{\omega_j\}$. We define

$$\xi_j = e_j(y) - (e_j(y)\cdot\bar{y})y,\ 1 \le j \le m, \tag{2.7}$$

then from (2.4) we obtain

$$g = x^*h = \sum_{j,k=1}^{m} < \xi_j, \xi_k > \omega_j \otimes \omega_k.$$

Thus $\{y, iy, \xi_1, \cdots, \xi_m\}$ is an orthonormal subbasis in $\mathbb{C}^{n+1}$ with respect to $<,>$. We add $\{\xi_\alpha, m+1 \le \alpha \le n\}$ to the subbasis such that $\{y, iy, \xi_j, \xi_\alpha\}$ forms an orthonormal basis of $\mathbb{C}^{n+1}$ with respect to $<,>$. Then we have an orthonormal basis $\{d\pi_y(\xi_j), d\pi_y(\xi_\alpha)\}$ for $(T_x\mathbb{C}P^n, h)$. Since

$$d\pi_y(\xi_j)=d\pi_y(e_j(y))=d\pi_Z(e_j(Z\circ x))=d\pi(dx(e_j)(Z))=d\pi\circ dZ(dx(e_j))dx(e_j), \tag{2.8}$$

we know that $\{d\pi_y(\xi_\alpha)\}$ is an orthonormal basis for the normal bundle of $x : M \to \mathbb{C}P^n$. Thus the mean curvature vector H of x is given by

$$H = \frac{1}{m}\sum_{j=1}^{m}\sum_{\alpha=m+1}^{n} h(\overline{\nabla}_{dx(e_j)}dx(e_j), d\pi_y(\xi_\alpha))d\pi_y(\xi_\alpha). \tag{2.9}$$

It follows from (2.6) that

$$H = \frac{1}{m}\sum_{j,\alpha} < e_j(e_j(y)) - 2(e_j(y)\cdot\bar{y})e_j(y), \xi_\alpha > d\pi_y(\xi_\alpha). \tag{2.10}$$

Since $\{y, iy, \xi_j, \xi_\alpha\}$ forms an orthonormal basis of $\mathbb{C}^{n+1}$ with respect to $<,>$, we obtain immediately from (2.10) that $H = 0$ if and only if we can find locally smooth complex function $\mu : U \to \mathbb{C}$ and real functions $\mu_j : U \to \mathbb{R}$ such that

$$\sum_j(e_j(\xi_j) - (e_j(y)\cdot\bar{y})\xi_j) = \mu y + \sum_j \mu_j\xi_j. \tag{2.11}$$

It follows that

$$\mu = \sum_j e_j(\xi_j) \cdot \bar{y} = -\sum_j \xi_j \cdot \overline{e_j(y)} = -m.$$

Furthermore, if we denote by $\{\Gamma^{\nu}_{jk}\}$ the Christoffel symbols of g with respect to $\{e_j\}$, then by (2.11) and (2.6) we get

$$\mu_k = \sum_j < e_j(\xi_j) - (e_j(y) \cdot \bar{y})\xi_j, \xi_k >= \sum_j h(\overline{\nabla}_{dx(e_j)} dx(e_j), dx(e_k)) = \sum_j \Gamma^k_{jj}. \tag{2.12}$$

Since we have

$$m = \sum_j |\xi_j|^2 = \sum_j (|e_j(y)|^2 - |e_j(y) \cdot \bar{y}|^2), \; e_j(y) \cdot \bar{y} + y \cdot e_j(\bar{y}) = 0, \tag{2.13}$$

we know from (2.11), (2.12) that $x : M \to \mathbb{C}P^n$ is minimal if and only if

$$\Delta y - 2\sum_j (e_j(y) \cdot \bar{y}) e_j(y) - \{(\Delta y - 2\sum_j (e_j(y) \cdot \bar{y}) e_j(y)) \cdot \bar{y}\} y = 0. \tag{2.14}$$

Thus we get the following charaterization theorem for minimal submanifolds in $\mathbb{C}P^n$:

THEOREM 2.1. *A submanifold $x : M^m \to \mathbb{C}P^n$ is minimal if and only if around each point of M there exists a local lift $y : U \to \mathbb{C}^{n+1}$ and local function $\lambda : U \to \mathbb{C}$ such that*

$$\Delta y - 2\sum_j (e_j(y) \cdot \bar{y}) e_j(y) = \lambda y, \tag{2.15}$$

where $\{e_j\}$ is an orthonormal basis with respect to the induced metric of x.

We note that the left hand side of (2.15) is independent of the choice of orthonormal basis $\{e_j\}$, and changes conformally $(mod\, y)$ if we take other local lift of x. Theorem 2.1 is an analogue of the Takahashi theorem of minimal submanifolds in S^n for minimal submanifolds in $\mathbb{C}P^n$ (see [Ta]).

3. Homogeneous sequence for a homogeneous surface in $\mathbb{C}P^n$. In this section we define the homogeneous sequence of a given homogeneous surface in $\mathbb{C}P^n$.

Let $x : M \to \mathbb{C}P^n$ be an immersion of oriented surface. The induced metric $g = x^*h$ defines a complex structure on M. Let U be a open set of M such that there exist a complex coordinate z on U and a local lift $y : U \to S^{2n+1}$ of x. Such open set exists around each point of M. We call such an open set U an adapted coordinate of x. For each integer $k \geq 1$ we denote by $\Gamma^k(U)$ the space of $\mathbb{C}^{n+1}$-valued complex k-form of (1,0) type, by $\Gamma^{-k}(U)$ the space of $\mathbb{C}^{n+1}$-valued complex k-form of (0,1) type, and by $\Gamma^0(U)$ the space of all smooth complex functions from U to $\mathbb{C}^{n+1}$.

For any nowhere-vanishing $\xi \in \Gamma^k(U)$ we can write $\xi = f dz^k$ for some complex coordinate z on U and complex function $f : U \to \mathbb{C}^{n+1}$. We use the convention that $dz^0 = 1$ and $dz^{-k} = d\bar{z}^k$ for $k \geq 1$ and write

$$g = e^{2\omega}|dz|^2. \tag{3.1}$$

Then we can define an operator $\partial : \Gamma^k(U) \to \Gamma^{k+1}(U)$ by

$$\partial\xi = \partial f \cdot dz^{k+1},$$

where

$$\partial f := f_z - \frac{(f_z \cdot \bar{f})}{|f|^2} f, \text{ if } k \geq 0, \ \partial f := e^{-2\omega}\{f_z - \frac{(f_z \cdot \bar{f})}{|f|^2} f\}, \text{ if } k \leq -1. \tag{3.2}$$

Similarly we can define an operator $\overline{\partial} : \Gamma^k(U) \to \Gamma^{k-1}(U)$ by $\overline{\partial}\xi = \overline{\partial} f \cdot dz^{k-1}$, where

$$\overline{\partial} f := e^{-2\omega}\{f_{\bar{z}} - \frac{(f_{\bar{z}} \cdot \bar{f})}{|f|^2} f\}, \text{ if } k \geq 1, \ \overline{\partial} f := f_{\bar{z}} - \frac{(f_{\bar{z}} \cdot \bar{f})}{|f|^2} f, \text{ if } k \leq 0. \tag{3.3}$$

We make a convention that $\partial\xi = 0$ and $\overline{\partial}\xi = 0$ in case that $\xi \equiv 0$. Using (3.1) we can easily check that $\partial\xi$ is independent of the choice of complex coordiante z on U, and we have

$$\overline{\partial}\xi = \overline{\partial\bar{\xi}}.$$

Moreover, for any smooth map $\rho : U \to \mathbb{C}\backslash\{0\}$ and any $\xi \in \Gamma^k(U)$ we have

$$\partial(\rho\xi) = \rho\partial\xi, \ \partial(\rho f) = \rho\partial f; \tag{3.4}$$

$$\overline{\partial}(\rho\xi) = \rho\overline{\partial}\xi, \ \overline{\partial}(\rho f) = \rho\overline{\partial} f. \tag{3.5}$$

It is clear that for any $T \in U(n+1)$ and $\xi \in \Gamma^k(U)$ we have

$$\partial(\xi T) = (\partial\xi)T, \ \overline{\partial}(\xi T) = (\overline{\partial}\xi)T. \tag{3.6}$$

The operators ∂ and $\overline{\partial}$ are nonlinear analogues to the linear operators used by Calabi [Ca] and Bryant [B] for surfaces in S^n. We note that if $\xi, \eta \in \Gamma^k(U)$ is linearly dependent, i.e., $\xi = \lambda\eta$ for some function $\lambda : U \to \mathbb{C}$, then we still have

$$\partial(\xi + \eta) = \partial\xi + \partial\eta, \overline{\partial}(\xi + \eta) = \overline{\partial}\xi + \overline{\partial}\eta.$$

Now let $y : U \to S^{2n+1}$ be a local lift of $x : M \to \mathbb{C}P^n$. By (2.1) we have

$$x^* h_0 = (\partial y dz + \overline{\partial} y d\bar{z}) \otimes (\overline{\overline{\partial} y} dz + \overline{\partial y} d\bar{z}).$$

It follows that

$$g = \partial y \cdot \overline{\overline{\partial} y} dz \otimes dz + \overline{\partial} y \cdot \overline{\partial y} d\bar{z} \otimes d\bar{z} + (|\partial y|^2 + |\overline{\partial} y|^2)|dz|^2.$$

Thus we get from (3.1) that

$$\partial y \cdot \overline{\overline{\partial} y} = 0, \ \partial y \cdot \bar{y} = \overline{\partial} y \cdot \bar{y} = 0; \tag{3.7}$$

$$|\partial y|^2 + |\overline{\partial} y|^2 = e^{2\omega}. \tag{3.8}$$

Since the Kaehler angle $\theta : M \to [0, \pi]$ is given by the formula $x^*\Omega = \cos\theta dM$, where dM is the volume form of x, we get from (2.5) that

$$\cos\theta = e^{-2\omega}(|\partial y|^2 - |\overline{\partial} y|^2). \tag{3.9}$$

We define $Adz \in \Gamma^{-1}(U)$ and $Bd\bar{z} \in \Gamma^1(U)$ by

$$A = \{\frac{(\partial y)_{\bar{z}} \cdot \overline{\partial y}}{|\partial y|^2} - y_{\bar{z}} \cdot \bar{y}\}, \ B = \{\frac{(\overline{\partial} y)_z \cdot \overline{\overline{\partial} y}}{|\overline{\partial} y|^2} - y_z \cdot \bar{y}\}. \tag{3.10}$$

It is easy to check that $\mathbb{A} := Ad\bar{z}$ and $\mathbb{B} := Bdz$ are independent of the choice of y and complex coordinate z, thus globally defined invariants of x (see (3.6)).

PROPOSITION 3.1. *For any surface in $\mathbb{C}P^n$ we have the following formula*

$$(\overline{\partial}\partial - \partial\overline{\partial})(y) = -e^{-2\omega} A\partial y + e^{-2\omega} B\overline{\partial} y - \cos\theta y. \tag{3.11}$$

Proof. Since $y_z = \partial y + (y_z \cdot \bar{y})y$ and $y_{\bar{z}} = \bar{\partial} y + (y_{\bar{z}} \cdot \bar{y})y$, using the identity $y_{z\bar{z}} = y_{\bar{z}z}$we get

$$(\partial y)_{\bar{z}} + (y_z \cdot \bar{y})y_{\bar{z}} = (\bar{\partial} y)_z + (y_{\bar{z}} \cdot \bar{y})y_z, \; mod\{y\}.$$

By the definition (3.2), (3.3) and (3.10) we get

$$e^{2\omega}(\bar{\partial}\partial y - \partial\bar{\partial} y) = -A\partial y + B\bar{\partial} y + e^{2\omega}\lambda y$$

for some function $\lambda : U \to \mathbb{C}$. Using (3.7) we get

$$\begin{aligned}\lambda = (\bar{\partial}\partial y - \partial\bar{\partial} y) \cdot \bar{y} &= e^{-2\omega}\{(\partial y)_{\bar{z}} \cdot \bar{y} - (\bar{\partial} y)_z \cdot \bar{y}\} \\ &= -e^{-2\omega}\{|\partial y|^2 - |\bar{\partial} y|^2\} = -\cos\theta.\end{aligned}$$

□

PROPOSITION 3.2. *A surface $x : M \to \mathbb{C}P^n$ is minimal if and only if around each point of M there exists a local lift y such that $\bar{\partial}\partial y + e^{-2\omega}A\partial y = \lambda y$ (or equivalently $\partial\bar{\partial} y + e^{-2\omega}B\bar{\partial} y = \mu y$) for some function λ or μ.*

Proof. We define an orthonormal basis for g by

$$e_1 = e^{-\omega}(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}),\; e_2 = ie^{-\omega}(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}). \tag{3.12}$$

Then by Theorem 2.1 we know that $x : M \to \mathbb{C}P^n$ is minimal if and only if

$$4e^{-2\omega}\{y_{z\bar{z}} - (y_z \cdot \bar{y})y_{\bar{z}} - (y_{\bar{z}} \cdot \bar{y})y_z\} = \Delta y - 2\sum_j (e_j(y) \cdot \bar{y})e_j(y) = \lambda^* y$$

for some function λ^*. Since

$$y_{z\bar{z}} - (y_z \cdot \bar{y})y_{\bar{z}} - (y_{\bar{z}} \cdot \bar{y})y_z = (\partial y)_{\bar{z}} - (y_{\bar{z}} \cdot \bar{y})\partial y, \; mod\{y\},$$

by definition (3.3) and (3.10) we get

$$\bar{\partial}\partial y + e^{-2\omega}A\partial y = \lambda y \tag{3.13}$$

for some function λ. The equation $\partial\bar{\partial} y + e^{-2\omega}B\bar{\partial} y = \mu y$ follows immediately from (3.11) and (3.13). In fact we get from (3.8) and (3.9) that $\lambda = -\cos^2\frac{\theta}{2}$ and $\mu = -\sin^2\frac{\theta}{2}$. □

Now we explain how any surface in $\mathbb{C}P^n$ generates a sequence of surfaces in $\mathbb{C}P^n$. Let $y : U \to S^{2n+1}$ be a lift of a surface $x : M \to \mathbb{C}P^n$. If $\partial^r y \neq 0$ and $\bar{\partial}^s y \neq 0$, we can define by (3.2) and (3.3) the maps $\partial^{r+1}y, \bar{\partial}^{s+1}y : U \to \mathbb{C}^{n+1}$. Thus we get a sequence of maps $\{\partial^j y, \bar{\partial}^k y, j, k = 0, 1, 2, \cdots\}$. Since $\partial^j y\, dz^j$ and $\bar{\partial}^k y\, d\bar{z}^k$ are independent of the choice of complex coordinate z, the maps $[\partial^j y], [\bar{\partial}^k y] : U \to \mathbb{C}P^n$ are independent of the choice of complex coordinate. By (3.4) and (3.5) we know also that $\partial^j x := [\partial^j y]$ and $\bar{\partial}^k x := [\bar{\partial}^k y]$ are independent of the choice of local lift y of x. Thus any surface $x : M \to \mathbb{C}P^n$ induces a sequence of surfaces $\{\partial^j x, \bar{\partial}^k x : M \to \mathbb{C}P^n\}$, possibly with singularities on M. The following diagram indicates the embedding of a surface $x : M \to \mathbb{C}P^n$ in its sequence:

$$\cdots \xleftarrow{\bar{\partial}} \bar{\partial}^k x \xleftarrow{\bar{\partial}} \cdots \bar{\partial} x \xleftarrow{\bar{\partial}} x \xrightarrow{\partial} \partial x \cdots \xrightarrow{\partial} \partial^j x \xrightarrow{\partial} \cdots \tag{3.14}$$

Now let $x : M \to \mathbb{C}P^n$ be a homogeneous surface. We assume that M is connected and orientable. Let G be the set of all diffeomorphism $\sigma : M \to M$ such that there exists a holomorphic isometry T of $\mathbb{C}P^n$ satisfying $x \circ \sigma = T \circ x$. Then G is a transformation group acting transitively on M. We denote by G_0 the subgroup of G, consisting of all orientation preserving diffeomorphism in G.

LEMMA 3.3. *G_0 acts transitively on M.*

Proof. Fix a point $p \in M$ we denote by H the isotropic subgroup of G at p. Thus we have the standard projection $\pi : G \to G/H = M$. For any $q \in M$ we can find a smooth curve $c(t)$ on M such that $c(0) = p$ and $c(1) = q$. Let $\sigma(t) \subset G$ be the smooth lift of $\pi : G \to G/H = M$ with $\sigma(0) = id \in G$. Then we have $\sigma(1)(p) = q$. Since $\sigma(0) = id$ is orientation preserving and $\sigma(t)$ is smooth, we know that $\sigma(1)$ is orientation preserving. Thus G_0 acts transitively on M.

With the help of Lemma 3.3, now we can prove that

THEOREM 3.4. *Every homogeneous surface in $\mathbb{C}P^n$ generates a sequence of homogeneous surfaces.*

Proof. Let $x : M \to \mathbb{C}P^n$ be a homogeneous surface. Let U be an adapted open set, i.e., an open set of M such that there exists complex coordinate z of (M, g) and a lift $y : U \to S^{2n+1}$ of x. Since $\partial^r y dz^r$ and $\overline{\partial}^s y d\bar{z}^s$ are independent of the choice of complex coordinate z, we know that the functions $e^{-2r\omega}|\partial^r y|^2$ and $e^{-2s\omega}|\overline{\partial}^s y|^2$ are independent of the choice of complex coordinate z. Moreover, we know from (3.4) and (3.5) that they are also independent of the choice of local lift y of x. Thus they are globally defined functions on M, which by (3.6) are invariants of x. By the homogeneity of x we know that

$$e^{-2r\omega}|\partial^r y|^2 = constant, \; e^{-2s\omega}|\partial^s y|^2 = constant. \tag{3.15}$$

If these constants are nonzero, we can define $\partial^{r+1} y$ and $\overline{\partial}^{s+1} y$ via (3.2) and (3.3). If one or both of them are zero, then the sequence stops in one or both direction. Thus all surfaces in the sequence (3.14) have not singularity. Now we show that $\partial^r x = [\partial^r y] : M \to \mathbb{C}P^n$ is also homogeneous. Let p and q are two point on M. First we assume that they lie in the same adapted open set U. By Lemma 3.3 there exists an orientation-preserving diffeomorphism $\sigma : M \to M$ and $T \in U(n+1)$ such that $x \circ \sigma = T \circ x$ and $\sigma(p) = q$. From the fact that $\sigma^* g = g$ and σ is orientation-preserving, we know that $\sigma : M \to M$ is holomorphic. In U we write $\sigma = \sigma(z)$. Thus by (3.6) we have

$$(\partial^r y)T = \partial^r(yT) = \partial^r(y \circ \sigma) = (\sigma'(z))^r(\partial^r y) \circ \sigma,$$

which implies that

$$T \circ [\partial^r y] = [\partial^r y] \circ \sigma, \; \sigma(p) = q. \tag{3.16}$$

Now if p and q do not lie in the same adapted open set, we can find finite number of adapted open sets $\{U_j, \, 0 \le j \le m\}$ such that (i) $p \in U_0$, $q \in U_m$; $U_j \cap U_{j+1} \ne \phi$. Thus we can also find $\sigma : M \to M$ and $T \in U(n+1)$ such that (3.16) holds. Thus $\partial^r x := [\partial^r y] : M \to \mathbb{C}P^n$ is homogeneous. Similarly we can show that $\overline{\partial}^s x : M \to \mathbb{C}P^n$ is also homogeneous. We complete the proof of Theorem 3.4.□

4. Construction of homogeneous 2-spheres in $\mathbb{C}P^n$. The construction of minimal homogeneous 2-spheres in $\mathbb{C}P^n$ is given by O. Bando and Y. Ohnita in [B-O]. In this section we generalize their construction to give examples of non-minimal homogeneous 2-spheres in $\mathbb{C}P^n$. We will show in §5 that up to holomorphic isometries these examples exhaust all homogeneous 2-spheres in $\mathbb{C}P^n$.

Let $\mathbb{V}^{n+1}$ be the representation space of $SU(2)$, consisting of all complex homogeneous polynomials of degree n in two variable λ and μ. We assign to $P = \sum_j a_j\lambda^{n-j}\mu^j \in \mathbb{V}^{n+1}$ an operator

$$\tau(P) = \sum_j a_j \frac{\partial^{n-j}}{\partial\bar{\lambda}^{n-j}} \frac{\partial^j}{\partial\bar{\mu}^j}. \tag{4.1}$$

Then the standard Hermitian inner product $<>$ in $\mathbb{V}^{n+1}$ is given by

$$(P, Q) := \tau(P)(\overline{Q}) = \sum_{j=0}^{n} j!(n-j)!a_j\overline{b_j}, \tag{4.2}$$

where $Q = \sum_j b_j\lambda^{n-j}\mu^j \in \mathbb{V}^{n+1}$. It is easy to check that $\{P_k(\lambda, \mu)\}$ defined by

$$P_k(\lambda, \mu) = \frac{1}{\sqrt{k!(n-k)!}}\lambda^{n-k}\mu^k, \; 0 \le k \le n, \tag{4.3}$$

is a unitary basis for $\mathbb{V}^{n+1}$. For any $(z, w) \in \mathbb{C}^2\backslash\{0\} := \mathbb{H}^*$ we have $\mathbb{H}^*$-action on $\mathbb{V}^{n+1}$ defined by

$$\begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} \cdot P_k(\lambda, \mu) := P_k(z\lambda + w\mu, -\bar{w}\lambda + \bar{z}\mu). \tag{4.4}$$

Since $P_k(z\lambda + w\mu, -\bar{w}\lambda + \bar{z}\mu) \in \mathbb{V}^{n+1}$ we can write

$$P_k(z\lambda + w\mu, -\bar{w}\lambda + \bar{z}\mu) = \sum_{j=0}^{n} c_j^k(z, w)P_j(\lambda, \mu), \; 0 \le j, k \le n, \tag{4.5}$$

where $\{c_k^j(z, w)\}$ are polynomials of degree n in $\{z, \bar{z}, w, \bar{w}\}$. We define

$$\phi_n^k = (c_0^k, c_1^k, \cdots, c_n^k) : \mathbb{C}^2\backslash\{0\} \to \mathbb{C}^{n+1}. \tag{4.6}$$

By (4.2) and (4.3) we have

$$\begin{aligned} &(P_k(z\lambda + w\mu, -\bar{w}\lambda + \bar{z}\mu), P_j(z\lambda + w\mu, -\bar{w}\lambda + \bar{z}\mu)) \\ &= P_k(z\frac{\partial}{\partial\bar{\lambda}} + w\frac{\partial}{\partial\bar{\mu}}, -\bar{w}\frac{\partial}{\partial\bar{\lambda}} + \bar{z}\frac{\partial}{\partial\bar{\mu}})(P_j(\bar{z}\bar{\lambda} + \bar{w}\bar{\mu}, -w\bar{\lambda} + z\bar{\mu})) = (|z|^2 + |w|^2)^n\delta_{kj}. \end{aligned}$$

Thus we get from (4.5) and (4.6) that

$$\phi_n^k(z, w) \cdot \overline{\phi_n^j(z, w)} = \sum_{i=0}^{n} c_i^k(z, w) \cdot \overline{c_i^j(z, w)} = (|z|^2 + |w|^2)^n\delta_{kj}. \tag{4.7}$$

Now we consider the restriction map $\phi_n^k : S^3 \to S^{2n+1}$ of (4.6). Taking transformations $(\lambda, \mu) \to (e^{i\theta}\lambda, e^{i\theta}\mu)$ in (4.5) and (4.3) we obtain

$$\phi_n^k(e^{i\theta}z, e^{i\theta}w) = e^{i\theta(n-2k)}\phi_n^k(z, w). \tag{4.8}$$

Thus we know that $[\phi_j^n] : S^2 = S^3/S^1 \to \mathbb{C}P^n$ is a well-defined map. The sequence $\{[\phi_n^0], [\phi_n^1], \cdots, [\phi_n^n]\}$ is known as Veronese sequence in $\mathbb{C}P^n$. It follows from (4.7) and (4.4) that

$$\rho_n(z, w) := (c_k^j(z, w)) = (\phi_n^0, \phi_n^1, \cdots, \phi_n^n) \in U(n+1)$$

and that $\rho_n : S^3 = SU(2) \to U(n+1)$ is a group homomorphism. Thus any surface in the Veronese sequence $\{[\phi_n^0], [\phi_n^1], \cdots, [\phi_n^n]\}$ is an orbit of the subgroup $\rho_n(S^3)$ of the holomorphic isometry group $U(n+1)/S^1$ in $\mathbb{C}P^n$, and thus a homogeneous 2-sphere in $\mathbb{C}P^n$.

Now we construct new homogeneous 2-spheres by using Veronese sequences. We make a convention that the Veronese sequence in $\mathbb{C}P^0$ is the constant map $[\phi_0^0] = [1]$, and the Veronese sequence in $\mathbb{C}P^1$ is $\{[\phi_1^0], [\phi_1^1]\}$ with $[\phi_1^0] = id$ and $[\phi_1^1]([z,w]) = [-\bar{w}, \bar{z}]$. Let $\{n_1, n_2, \cdots, n_r\}$ be nonnegative integers. We take from each Veronese sequence in $\mathbb{C}P^{n_\alpha}$ a surface $[\phi_{n_\alpha}^{j_\alpha}]$ such that

$$n_1 - 2j_1 = n_2 - 2j_2 = \cdots = n_r - 2j_r := k. \tag{4.9}$$

Then for nonzero constants $\{c_\alpha\} \in \mathbb{C}$ we can construct a join-map $\phi : S^3 \to \mathbb{C}^{n+1}$ with $n = n_1 + \cdots + n_r + r - 1$ by

$$\phi = (c_1\phi_{n_1}^{j_1}, c_2\phi_{n_2}^{j_2}, \cdots, c_r\phi_{n_r}^{j_r}). \tag{4.10}$$

Using (4.8) and (4.9) we get

$$\phi(e^{i\theta}z, e^{i\theta}w) = e^{i\theta k}\phi(z,w). \tag{4.11}$$

Thus we have a well-defined map $[\phi] : S^2 \to \mathbb{C}P^n$. It is clear that $[\phi]$ is an orbit of the subgroup

$$\mathit{diagonal}\{\rho_{n_1}(S^3), \cdots, \rho_{n_r}(S^3)\} \subset \mathit{diagonal}\{U(n_1+1), \cdots, U(n_r+1)\} \subset U(n+1).$$

Thus $[\phi] : S^2 \to \mathbb{C}P^n$ is also homogeneous.

We say surfaces $\{[\phi_{n_\alpha}^{j_\alpha}], 1 \le \alpha \le r\}$ joinable if the nonnegative integers $\{n_\alpha, j_\alpha\}$ satisfy equation (4.9). We can use the following ∇-diagram to determine joinable surfaces:

....

$$[\phi_5^0],\ [\phi_5^1],\ [\phi_5^2],\ [\phi_5^3],\ [\phi_5^4],\ [\phi_5^5]$$
$$[\phi_4^0],\ [\phi_4^1],\ [\phi_4^2],\ [\phi_4^3],\ [\phi_4^4]$$
$$[\phi_3^0],\ [\phi_3^1],\ [\phi_3^2],\ [\phi_3^3]$$
$$[\phi_2^0],\ [\phi_2^1],\ [\phi_2^2]$$
$$[\phi_1^0],\ [\phi_1^1]$$
$$[\phi_0^0]$$

Surfaces $\{[\phi_{n_\alpha}^{j_\alpha}], 1 \le \alpha \le r\}$ are joinable if and only if they lie on the same vertical line of the diagram. An easy example is the join surface of $[\phi_0^0]$ and $[\phi_2^1]$ in $\mathbb{C}P^3$. Since $[\phi_2^1]([z,w]) = [-\sqrt{2}z\bar{w}, |z|^2 - |w|^2, \sqrt{2}\bar{z}w]$, we get the join surface

$$\phi([z,w]) = [-c_1\sqrt{2}z\bar{w}, c_1(|z|^2 - |w|^2), c_1\sqrt{2}\bar{z}w, c_2],\ c_1, c_2 \in \mathbb{C}^*.$$

Another example is the join surface of $[\phi_1^0]$ and $[\phi_3^1]$ in $\mathbb{C}P^5$:

$$\phi([z,w]) = [-\sqrt{3}c_1z^2w, c_1z(|z|^2 - 2|w|^2), c_1w(2|z|^2 - |w|^2), \sqrt{3}c_1w^2\bar{z}, c_2z, c_2w].$$

These surfaces are of constant curvature, constant Kaehler angle and non-minimal.

We note that if $n_\alpha = n_\beta$ in (4.10), then by (4.9) we get $j_\alpha = j_\beta$, thus there is a nonzero vector $\nu \in \mathbb{C}^{n+1}$ such that $\phi \cdot \bar{\nu} = 0$. Thus $[\phi] : S^2 \to \mathbb{C}P^n$ is not full. If $[\phi]$ is full in $\mathbb{C}P^n$, we can arrange n_α such that $n_1 > n_2 > \cdots n_r \ge 0$.

In the rest of this section we give the following propositions which we need in §5.

PROPOSITION 4.1 (cf. Theorem 5.2 of [B-J-R-W]). *The sequence of Veronese surfaces in* $\mathbb{C}P^n$ *reads*

$$0 \xleftarrow{\overline{\partial}} [\phi_n^0] \xleftarrow{\overline{\partial}} \cdots [\phi_n^{k-1}] \xleftarrow{\overline{\partial}} [\phi_n^k] \xrightarrow{\partial} [\phi_n^{k+1}] \cdots \xrightarrow{\partial} [\phi_n^n] \xrightarrow{\partial} 0.$$

Moreover, the Gauss curvature K *and the Kaehler angle* θ *of the Veronese surface* $[\phi_n^k] : S^2 \to \mathbb{C}P^n$ *are given by*

$$K = \frac{4}{n + 2k(n-k)}, \quad \cos\theta = \frac{n-2k}{n+2k(n-k)}. \tag{4.12}$$

PROPOSITION 4.2. *Let* $y : \mathbb{C}P^1 \to S^{2n+1}$ *be a local lift of* $[\phi_n^k] : \mathbb{C}P^1 \to \mathbb{C}P^n$ *defined by*

$$y := \frac{\phi_n^k(z,1)}{|\phi_n^k(z,1)|} = \frac{\phi_n^k(z,1)}{(|z|^2+1)^{n/2}}.$$

Then z *is a complex coordinate of* $[\phi_k^n]$ *and*

$$\frac{y_z \cdot \overline{y}}{|y|^2} = (n-2k)\frac{\bar{z}}{2(1+|z|^2)}. \tag{4.13}$$

Proof. It is known (cf. [B-J-R-W]) that z is a complex coordinate of $[\phi_n^k] : S^2 \to \mathbb{C}P^n$. By taking derivative in (4.5) we get

(4.14)

$$\frac{n-k}{\sqrt{k!(n-k)!}}\lambda(z\lambda + w\mu)^{n-k-1}(-\bar{w}\lambda + \bar{z}\mu)^k = \sum_{j=0}^{n}(c_j^k(z,w))_z \frac{1}{\sqrt{j!(n-j)!}}\lambda^{n-j}\mu^j,$$

using (4.2) and (4.5) we get

$$(\phi_n^k)_z \cdot \overline{\phi_n^k} = \sum_{j=0}^{n}(c_j^k(z,w))_z \overline{c_j^k(z,w)} = (n-k)\bar{z}(|z|^2+|w|^2)^{n-1}. \tag{4.15}$$

Thus (4.13) follows immediately from (4.7) and (4.15).

5. The classification of homogeneous 2-spheres in $\mathbb{C}P^n$. In this section we use the sequence of homogeneous 2-spheres to show that all of homogeneous 2-spheres can be obtained by the construction in §4.

Let $x : M \to \mathbb{C}P^n$ be a homogeneous surface. A function or form $I(x)$ on M induced by the immersion x is called invariant of x, if for any holomorphic isometry $T \in U(n+1)/S^1$ of $\mathbb{C}P^n$ we have $I(T \circ x) = x$. The basic invariants of x are known as the first and the second fundamental form. Using the operators defined by (3.2) and (3.3) we can obtain more invariant complex forms or invariant functions of x.

Let U be an open set of M such that there exists a complex coordinate z on U and a local lift $y : U \to S^{2n+1}$ of x. We write $g = e^{2\omega}|dz|^2$. Then we can define $\mathbb{C}^{n+1}$-valued forms $\{\partial^r y dz^r\}$ and $\{\overline{\partial}^s y d\bar{z}^s\}$ on U which are independent of the choice of complex coordinate z. It follows from (3.15) that each of these forms is either identically zero or nowhere zero. Thus we can define the following complex forms (or functions)

$$\xi(y) := e^{-2j\omega}\partial^k y \cdot \overline{\partial^j y} dz^{k-j}, \; k \geq j; \tag{5.1}$$

$$\eta(y) := \partial^k y \cdot \overline{\overline{\partial}^j y} dz^{k+j}. \tag{5.2}$$

It is easy to see that $\xi(y)$ and $\eta(y)$ are independent of the choice of z. Now let V be another open set of M such that there exists a complex coordinate on V and a lift $\tilde{y} : V \to S^{2n+1}$ of x. Then on $U \cap V$ we have $\tilde{y} = \rho y$ for some function $\rho : U \cap V \to S^1$. Since by (3.4) and (3.5) we have

$$\partial^r \tilde{y} = \rho \partial^r y, \overline{\partial}^s \tilde{y} = \rho \overline{\partial}^s y,$$

which imply that $\xi(\tilde{y}) = \xi(y)$ and $\eta(\tilde{y}) = \eta(y)$ at each point of $U \cap V$. Thus all complex forms in (5.1) and (5.2) are independent of the choice of complex coordinate z and local lift y, they are globally defined on M. Moreover, it follows from (3.6) that for any holomorphic isometry $T \in U(n+1)/S^1$ we have $\xi(yT) = \xi(y)$ and $\eta(yT) = \eta(y)$, we know that all these complex forms of (1,0)-type on M are invariants of x.

LEMMA 5.1. *Let $x : M \to \mathbb{C}P^n$ be a surface. Let Φ be a m-form ($m \geq 1$) of (1,0)-type on M which is an invariant of x. Then the (m-1)-form $\overline{\partial}\Phi$ and the (m+1)-form $\partial\Phi$ defined by*

$$\overline{\partial}\Phi := e^{-2\omega}\phi_{\bar{z}} dz^{m-1}, \ \partial\Phi := (\phi_z - 2\omega_z \phi) dz^{m+1} \tag{5.3}$$

are gobally defined invariant of x, where z is a complex coordinate of M and we write locally $\Phi = \phi dz^m$, $g = e^{2\omega}|dz|^2$.

Proof. It is easy to check that $\overline{\partial}\Phi$ and $\partial\Phi$ defined by (5.3) is independent of the choice of z and thus globally defined on M. Since for any holomorphic isometry $T \in U(n+1)/S^1$ of $\mathbb{C}P^n$ we have $g(T \circ x) = g(x) = g$, we can take the same complex coordinate z and thus the same function ω for both x and $T \circ x$. Since $\Phi(T \circ x) = \Phi(x)$, we have $\phi(T \circ x) = \phi(x)$, then by (5.3) we get $\overline{\partial}\Phi(T \circ x) = \overline{\partial}\Phi(x)$ and $\partial\Phi(T \circ x) = \partial\Phi(x)$. □

LEMMA 5.2. *Let $x : S^2 \to \mathbb{C}P^n$ be a homogeneous sphere. Then any complex form on S^2 which is an invariant of x vanishes.*

Proof. First we consider the case of 1-form. Let Φ be a 1-form on M which is an invariant of x. We write locally $\Phi = \phi dz$. If $\Phi \not\equiv 0$, then by the homogeneity of x we know $\Phi \neq 0$. Since $\overline{\partial}\Phi$, $\partial\Phi/\Phi^2$ and $||\Phi||^2$ are invariant function of x, by homogeneity of x they are constants. Thus we can find constants c_1, c_2 and c_3 such that

$$e^{-2\omega}\phi_{\bar{z}} = c_1, \ \phi_z - 2\omega_z\phi = c_2\phi^2, \ e^{-2\omega}|\phi|^2 = c_3. \tag{5.4}$$

From the identity $\phi_{z\bar{z}} = \phi_{\bar{z}z}$ we get $4|c_1|^2 + Kc_3 = 0$, where K is the Gaussian curvature of x. Since $K > 0$ on S^2, we get a contradiction. Thus $\Phi \equiv 0$. Now let Φ be any invariant (m+1)-form. Then $\overline{\partial}^m \Phi$ is a invariant 1-form. We get $\overline{\partial}^m \Phi \equiv 0$. Since any holomorphic form on S^2 vanishes, we get $\Phi \equiv 0$. □

LEMMA 5.3. *Let $x : S^2 \to \mathbb{C}P^n$ be a homogeneous sphere. Then the homogeneous sequence of x stops in both directions, and any two homogeneous spheres in the sequence are complex-orthogonal.*

Proof. By Lemma 5.2 we know that all complex forms defined by (5.1) and (5.2) vanish. Thus $\{\partial^j y, \overline{\partial}^k y, j \neq k\}$ are complex orthogonal in $\mathbb{C}^{n+1}$. It follows that the exists integers r and s such that $\partial^r y \neq 0$ and $\overline{\partial}^s y \neq 0$, but $\partial^{r+1} y \equiv 0$ and $\overline{\partial}^{s+1} y \equiv 0$. □

Let $\xi \in \Gamma^k(U)$ and $\eta \in \Gamma^j(U)$ with $k \geq j$. We write $\xi = fdz^k$, $\eta = hdz^j$ and define

$$\xi \cdot \overline{\eta} := e^{-2j\omega} f \cdot \overline{h} dz^{k-j}, \ ||\xi||^2 = \xi \cdot \overline{\xi}. \tag{5.5}$$

Then the (k-j)-form $\xi \cdot \overline{\eta}$ is independent of the choice of complex coordinate.

LEMMA 5.4. *Let $\xi \in \Gamma^k(U)$ and $\eta \in \Gamma^j(U)$ with $k > j$. We assume that $\xi \cdot \overline{\eta} = 0$, then we have*

$$\overline{\partial}\xi \cdot \overline{\eta} + \xi \cdot \overline{\partial\eta} = 0, \ \ \partial\xi \cdot \overline{\eta} + \xi \cdot \overline{\overline{\partial}\eta} = 0. \tag{5.6}$$

Proof. We denote $\xi = fdz^k$, $\eta = hdz^j$. Then (5.6) follows immediately from the equations $(e^{-2j\omega} f \cdot \overline{h})_{\bar{z}} = 0$ and $(e^{-2j\omega} f \cdot \overline{h})_z = 0$. □

LEMMA 5.5. *A homogeneous surface $x : S^2 \to \mathbb{C}P^n$ is minimal if and only if around each point of M there exists a lift y of x and a constant λ such that $\overline{\partial}\partial y + \lambda y = 0$ holds (or equivalently $\partial\overline{\partial} y + \mu y = 0$ holds for some constant μ).*

Proof. It follows immediately from Proposition 3.2 and the fact that $Ad\bar{z}$ and Bdz defined by (3.10) are globally defined invariant of x, which by Lemma 5.2 vanish. □

LEMMA 5.6. *Let $x : S^2 \to \mathbb{C}P^n$ be a homogeneous minimal surface with the homogeneous sequence*

$$0 \xleftarrow{\overline{\partial}} \overline{\partial}^s x \xleftarrow{\overline{\partial}} \cdots \overline{\partial} x \xleftarrow{\overline{\partial}} x \xrightarrow{\partial} \partial x \cdots \xrightarrow{\partial} \partial^r x \xrightarrow{\partial} 0 \cdot \tag{5.7}$$

Let $y : U \to S^{2n+1}$ be any lift of x. Then we have

$$(y_z \cdot \bar{y})_{\bar{z}} + \overline{(y_z \cdot \bar{y})}_z = \frac{1}{4}(r - s)Ke^{2\omega}. \tag{5.8}$$

Proof. We get from (2.5) and the formulas $(|y|^2)_z = 0$, $x^*\Omega = \cos\theta dM$ that

$$\begin{aligned}
(y_z \cdot \bar{y})_{\bar{z}} + \overline{(y_z \cdot \bar{y})}_z &= (y_z \cdot \bar{y})_{\bar{z}} - (y_{\bar{z}} \cdot \bar{y})_z \\
&= y_z \cdot \overline{y_z} - y_{\bar{z}} \cdot \overline{y_{\bar{z}}} = dy \wedge d\bar{y}(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}) \\
&= -2i \cos\theta dM(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}) = \cos\theta e^{2\omega}.
\end{aligned}$$

Since x is equivalent to the Veronese surface $[\phi^s_{r+s}] : S^2 \to \mathbb{C}P^n$, we get from Proposition 4.2 that $\cos\theta = \frac{1}{4}(r - s)K$. □

LEMMA 5.7. *Let $x : S^2 \to \mathbb{C}P^n$ be a homogeneous sphere with the homogeneous sequence (5.7). Then $\overline{\partial}^r \partial^r y \neq 0$ and $x_1 = [\overline{\partial}^r \partial^r y] : S^2 \to \mathbb{C}P^n$ is a minimal homogeneous sphere.*

Proof. Since for any $0 \leq j \leq r-1$ the globally defined 1-form $e^{-2j\omega}\overline{\partial}^{r-j-1}\partial^r y \cdot \overline{\partial^j y} dz$ is an invariant of x and thus vanishes on S^2, we get from Lemma 5.4 that

$$\overline{\partial}^r \partial^r y \cdot \bar{y} = (-1)^r e^{-2r\omega} |\partial^r y|^2 \neq 0.$$

Therefore $y^1 := \overline{\partial}^r \partial^r y \neq 0$. The induced metric g_1 of $x_1 = [y^1] : S^2 \to \mathbb{C}P^n$ is given by

$$g_1 = \frac{1}{2|y^1|^2}\{(\partial y^1 dz + \overline{\partial} y^1 d\bar{z}) \otimes (\overline{\overline{\partial} y^1} dz + \overline{\partial y^1} d\bar{z})$$

$$+(\overline{\overline{\partial}y^1}dz + \overline{\partial y^1}d\bar{z}) \otimes (\partial y^1 dz + \overline{\partial} y^1 d\bar{z})\}$$

$$= \frac{1}{|y^1|^2} e^{-2\omega}(|\partial y^1|^2 + |\overline{\partial} y^1|^2) g := cg.$$

Here we have used the vanishing of the global form $\partial y^1 \cdot \overline{\overline{\partial} y^1} dz^2$ on S^2. Since c is independent of the choice of local lift y and complex coordinate z, and is an invariant of x under the holomorphic isometry in $\mathbb{C}P^n$, by the homogeneity of x we know that c is a constant. Let $\overline{\partial}_1$ and ∂_1 be the operator with respect to $g_1 = cg$, then we have

$$\overline{\partial}_1 = \overline{\partial} : \Gamma^0(U) \to \Gamma^{-1}(U), \partial_1 = \partial : \Gamma^0(U) \to \Gamma^1(U);$$

$$\overline{\partial}_1 = \frac{1}{c}\overline{\partial} : \Gamma^1(U) \to \Gamma^0(U), \partial_1 = \frac{1}{c}\partial : \Gamma^{-1}(U) \to \Gamma^0(U).$$

It follows that $\overline{\partial}_1\partial_1 = \frac{1}{c}\overline{\partial}\partial$ and $\partial_1\overline{\partial}_1 = \frac{1}{c}\partial\overline{\partial}$. Thus to show that $x_1 = [\overline{\partial}^r \partial^r y]$ is minimal we need only to show that $\overline{\partial}\partial(\overline{\partial}^r \partial^r y) = \lambda \overline{\partial}^r \partial^r y$ for some constant λ. Since $\overline{\partial}\partial(\partial^r y) = 0$, thus by Lemma 5.5 we know that $[\partial^r y]$ is minimal. This implies also by Lemma 5.5 that $\partial\overline{\partial}(\partial^r y) = \lambda_1 \partial^r y$ for some constant λ_1. It follows that $\overline{\partial}\partial(\overline{\partial}\partial^r y) = \lambda_1 \overline{\partial}\partial^r y$. Thus by Lemma 5.5 $[\overline{\partial}\partial^r y]$ is also minimal. Therefore we have $\partial\overline{\partial}(\overline{\partial}\partial^r y) = \lambda_2(\overline{\partial}\partial^r y)$ for some constant λ_2. It follows that $\overline{\partial}\partial(\overline{\partial}^2 \partial^r y) = \lambda_2 \overline{\partial}^2 \partial^r y$. Thus $[\overline{\partial}^2 \partial^r y]$ is minimal. By this way we know that if $\overline{\partial}^j \partial^r y \neq 0$, then $\{[\overline{\partial}^j \partial^r y]\}$ is minimal. In particular, $x_1 = [\overline{\partial}^r \partial^r y] : S^2 \to \mathbb{C}P^n$ is minimal. It follows from Theorem 3.4 that $[\partial^r y]$ and $[\overline{\partial}^r \partial^r y]$ are homogeneous. □

COROLLARY 5.8. *The induced metric* $g_1 = cg$ *of* $x_1 = [\overline{\partial}^r \partial^r y]$ *has constant curvature. Moreover,* x_1 *and* x *have the same complex coordinate.*

Now let

$$(5.9) \quad 0 \xleftarrow{\overline{\partial}} \overline{\partial}^\beta x_1 \xleftarrow{\overline{\partial}} \cdots \xleftarrow{\overline{\partial}} x_1 \xrightarrow{\partial} \cdots \xrightarrow{\partial} \partial^\alpha x_1 \xrightarrow{\partial} 0$$

be the homogeneous sequence of $x_1 = [y^1]$, where $y^1 = \overline{\partial}^r \partial^r y$. Then x_1 is holomorphically isometric to the Veronese surface $[\phi^\beta_{\alpha+\beta}]$. By making a holomorphic isometry in $\mathbb{C}P^n$ if necessary we may assume that

$$\{\overline{\partial}^\beta y^1, \cdots, \overline{\partial} y^1, y^1, \partial y^1, \cdots, \partial^\alpha y^1\}$$

is a complex orthogonal basis for $\mathbb{C}^{m+1} \subset C^{n+1}$. Let $\pi_1 : \mathbb{C}^{n+1} \to \mathbb{C}^{m+1}$ be the standard projection. Then we can find functions c_1 and $\{f_j, h_k\}$ such that

$$(5.10) \qquad \pi_1(y) = c_1 y^1 + \sum_{j=1}^{\beta} f_j \overline{\partial}^j y^1 + \sum_{k=1}^{\alpha} h_k \partial^k y^1.$$

From the fact that

$$\pi_1(y) \cdot \overline{\overline{\partial}^j y^1 dz^j} = y \cdot \overline{\overline{\partial}^j y^1 dz^j}, \partial^k y^1 \cdot \overline{\pi_1(y)} dz^k = \partial^k y^1 \cdot \overline{y} dz^k$$

are globally defined invariants of x which must vanish on S^2, we get from (5.10) that $f_j = h_k = 0$. Moreover, by Lemma 5.4 we have

$$c_1 |y^1|^2 = \pi_1(y) \cdot \overline{y^1} = y \cdot \overline{\overline{\partial}^r \partial^r y} = (-1)^r e^{-2r\omega} |\partial^r y|^2.$$

Thus by homogeneity we know that c_1 is a nonzero constant. We define

$$\eta = y - c_1 y^1 = (id - c_1 \overline{\partial}^r \partial^r) y : U \to (\mathbb{C}^{m+1})^{\perp} = \mathbb{C}^{n-m} \subset \mathbb{C}^{n+1}. \tag{5.11}$$

Since $\pi_1(\eta) = \pi_1(y) - c_1 y^1 = 0$, we can write $y = (c_1 y^1, \eta)$. If $\eta \equiv 0$, we know that $x = [y] = [y^1] = x_1$ is a Veronese surface . Now if $\eta \neq 0$, we know from the proof of Theorem 3.4 and (5.11) that $[\eta] : S^2 \to \mathbb{C}P^{n-m-1} \subset \mathbb{C}P^n$ is also homogeneous. We write the homogeneous sequence of $[\eta]$ as

$$0 \xleftarrow{\overline{\partial}} [\overline{\partial}^{\delta} \eta] \xleftarrow{\overline{\partial}} \cdots \xleftarrow{\overline{\partial}} [\eta] \xrightarrow{\partial} \cdots \xrightarrow{\partial} [\partial^{\gamma} \eta] \xrightarrow{\partial} 0. \tag{5.12}$$

Now we define

$$y^2 = \overline{\partial}^{\gamma} \partial^{\gamma} \eta = \overline{\partial}^{\gamma} \partial^{\gamma} (y - c_1 \overline{\partial}^r \partial^r y).$$

By the same argument of Lemma 5.7 we know that $x_2 = [y^2] : S^2 \to \mathbb{C}P^{n-m-1}$ is a homogeneous minimal sphere (cf. Theorem 3.4). We write the homogeneous sequence of x_2 by

$$0 \xleftarrow{\overline{\partial}} [\overline{\partial}^{\beta_1} \eta] \xleftarrow{\overline{\partial}} \cdots \xleftarrow{\overline{\partial}} [\eta] \xrightarrow{\partial} \cdots \xrightarrow{\partial} [\partial^{\alpha_1} \eta] \xrightarrow{\partial} 0,$$

then x_2 is holomorphically isometric to the Veronese surface $[\phi^{\beta_1}_{\beta_1 + \alpha_1}]$. Similarly we can show that $\eta = (c_2 y^2, \eta_2)$ for some constant $c_2 \neq 0$. By continuing this procedure we finally get that

$$y = (c_1 y^1, c_2 y^2, \cdots, c_r y^r) : U \to S^{2n+1}, \tag{5.13}$$

where $[y^k] : S^2 \to \mathbb{C}P^{n_k}$ is a minimal homogeneous sphere which is equivalent to $[\phi^{j_k}_{n_k}]$.

Lemma 5.9. *We have $\alpha - \beta = \alpha_1 - \beta_1$. That is, if we write $[y_k] = [\phi^{j_k}_{n_k}]$, then we have $n_1 - 2j_1 = n_2 - 2j_2 = \cdots n_r - 2j_r$.*

Proof. Let K_1 and K_2 be the Gauss curvature of the induced metric g_1 and g_2 of $[y^1]$, $[y^2]$. As in the proof of Lemma 5.7 we can easily prove that

$$g = c_1^* g_1 = c_2^* g_2 = \cdots = c_r^* g_r$$

for some positive constants c_k^*, where g_k is the induced metric of $[y^k] : S^2 \to \mathbb{C}P^{n_k}$. Let K_j be the curvature of g_j, then we have

$$Kg = K_1 g_1 = K_2 g_2 = \cdots K_r g_r. \tag{5.14}$$

Since both $[y_1]$ and $[y_2]$ are minimal homogeneous sphere and $|y^1|$, $|y^2|$ are constants, by taking the same complex coordinate we get from Lemma 5.6 and (5.14) we have

$$\left(\frac{y^1_z \cdot \overline{y^1}}{|y^1|^2}\right)_{\bar z} + \left(\overline{\frac{y^1_z \cdot \overline{y^1}}{|y_1|^2}}\right)_z = (\alpha - \beta) K e^{2\omega}, \tag{5.15}$$

$$\left(\frac{y^2_z \cdot \overline{y^2}}{|y^2|^2}\right)_{\bar z} + \left(\overline{\frac{y^2_z \cdot \overline{y^2}}{|y^2|^2}}\right)_z = (\alpha_1 - \beta_1) K e^{2\omega}. \tag{5.16}$$

Since the 1-form

$$\left(\frac{y^1_z \cdot \overline{y^1}}{|y^1|^2} - \frac{y^2_z \cdot \overline{y^2}}{|y^2|^2}\right) dz \tag{5.17}$$

is independent of the choice of y and z, thus a globally invarant of x. By Lemma 5.2 it vanishes. Thus we get from (5.15) and (5.16) that $\alpha - \beta = \alpha_1 - \beta_1$. □

Now we come to the proof of the classification theorem of homogeneous 2- spheres in $\mathbb{C}P^n$. Since $[y_k]$ is equivalent to $[\phi^{j_k}_{n_k}]$, we can find $T_k \in U(n_k+1)$ and locally defined nonzero functions ρ_k such that

$$y^k = \rho_k \xi_k T_k,\ \xi_k := \frac{\phi^{j_k}_{n_k}(z,1)}{|\phi^{j_k}_{n_k}(z,1)|},\ k = 1, 2, \cdots, r. \tag{5.18}$$

Since $|y^k| = d_k$ is a constant we can write $\rho_k = d_k e^{i\theta^k}$ for some real function θ^k, $k = 1, \cdots, r$. It follows from (5.18) and (4.13) that

$$\frac{y^k_z \cdot \overline{y^k}}{|y^k|^2} = i(\theta^k)_z + (n_k - 2j_k)\frac{\bar{z}}{2(1+|z|^2)}. \tag{5.19}$$

Thus we get from Lemma 5.9 that

$$\left(\frac{y^k_z \cdot \overline{y^k}}{|y^k|^2} - \frac{y^1_z \cdot \overline{y^1}}{|y^1|^2}\right)dz = i(\theta^k - \theta^1)_z dz. \tag{5.20}$$

Since the left hand side of (5.20) is a globally defined invariant of x, by Lemma 5.2 it vanishes identically. Thus we have constant θ^k_0 such that

$$\theta^k = \theta^1 + \theta^k_0,\ k = 2, \cdots, r.$$

Thus up to a holomorphic transformation $T = diag(T_1, \cdots, T_r)$ we get

$$\begin{aligned} y &= e^{i\theta^1}(c_1 d_1 \xi_1, c_2 d_2 e^{i\theta^2_0}\xi_2, \cdots, c_r d_r e^{i\theta^r_0}\xi_r) \\ &= e^{i\theta^1}(c_1 d_1 \phi^{j_1}_{n_1}, c_2 d_2 e^{i\theta^2_0}\phi^{j_2}_{n_2}, \cdots, c_r d_r e^{i\theta^r_0}\phi^{j_r}_{n_r}) \circ \tau, \end{aligned}$$

where $\tau(z) = (\frac{z}{\sqrt{1+|z|^2}}, \frac{1}{\sqrt{1+|z|^2}})$ is a local section of Hopf-fibration $S^3 \to S^2$. It follows that

$$x = [c_1 d_1 \phi^{j_1}_{n_1}, c_2 d_2 e^{i\theta^2_0}\phi^{j_2}_{n_2}, \cdots, c_r d_r e^{i\theta^r_0}\phi^{j_r}_{n_r}]$$

for some constants $\{c_j\}$, $\{d_j\}$, $\{\theta^0_j\}$ and integers $\{n_k, j_k\}$ with $n_1 - 2j_1 = \cdots = n_r - 2j_r$. Thus we complete the proof of the classification theorem.

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