CALABI-YAU THREEFOLDS OF QUOTIENT TYPE*

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Dedicated to Professor V. A. Iskovskih on the occasion of his sixtieth birthday

Abstract. By a Calabi-Yau threefold we mean a minimal complex projective threefold X such that $\mathcal{O}_X(K_X) \simeq \mathcal{O}_X$ and $h^1(\mathcal{O}_X) = 0$. This paper consists of four parts. In Section 1 we formulate an equivariant version of Torelli Theorem of K3 surfaces with finite group action and deduce some more geometrical consequences. In Section 2, we classify Calabi-Yau threefolds with infinite fundamental group by means of their minimal splitting coverings introduced by Beauville, and deduce as its Corollary that the nef cone is a rational simplicial cone and any rational nef divisor is semi-ample provided that $c_2(X) \equiv 0$ on $\mathrm{Pic}(X)_{\mathbb{R}}$. We also derive a sufficient condition for $\pi_1(X)$ to be finite in terms of the Picard number in an optimal form. In Section 3, we give a fairly concrete structure Theorem concerning c_2-contractions of Calabi-Yau threefolds as a generalisation and also a correction of our earlier works for simply connected ones. In Section 4, applying the results in these three sections together with Kawamata's finiteness result of the relatively minimal models of a Calabi-Yau fiber space, we show the finiteness of the isomorphism classes of c_2-contractions of each Calabi-Yau threefold. As a special case, we find the finiteness of abelian pencil structures on each X up to $\mathrm{Aut}(X)$.

0. Introduction. In the light of the minimal model theory, we define *a Calabi-Yau threefold* to be a $\mathbb{Q}$-factorial terminal projective threefold X defined over $\mathbb{C}$ such that $\mathcal{O}_X(K_X) \simeq \mathcal{O}_X$ and $h^1(\mathcal{O}_X) = 0$, and regard the second Chern class $c_2(X)$ as a linear form on $\mathrm{Pic}(X)_{\mathbb{R}}$ through the intersection pairing, where $c_2(X)$ for a singular X is defined as $c_2(X) := \nu_*(c_2(\tilde{X}))$ via a resolution $\nu : \tilde{X} \to X$ and is known to be well-defined (see for example [Og1, Lemma (1.4)]).

However, as is pointed out by several authors, this preferable definition of Calabi-Yau threefold has an inevitable defect: Those Calabi-Yau threefolds, such as Igusa's example ([Ig, Page 678], [Ue, Example 16.16]), that are given as an étale quotient of an abelian threefold are then included in our category. We call them *of Type A*. Indeed, their pathological nature sometimes prevents us from studying Calabi-Yau threefolds uniformly. For instance,

(1) there are no rational curves on Calabi-Yau threefolds of Type A, while it is expected, and has been already checked in some extent, that most of Calabi-Yau threefolds contain rational curves (see [Wil], [HW] and [EJS]);

(2) $c_2(X) = 0$ for such X but $c_2(X) \neq 0$ for others.

Here, for the last statement, we recall the following result due to S. Kobayashi in the smooth case and Shepherd-Barron and Wilson in the general case:

THEOREM ([KB, CHAP. IV, COROLLARY (4.15)], [SBW, COROLLARY]). *Let X be a Calabi-Yau threefold. Then, X is of Type A if and only if $c_2(X) = 0$.* □

One of the main purposes of this paper is to compensate for this defect by revealing explicit geometric structures of Calabi-Yau threefolds of Type A. It will turn out that they are remarkably few so that one can in principle handle them separately in case. Our result is:

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THEOREM (0.1). *Let X be a Calabi-Yau threefold of Type A. Then,*

(I) $X = A/G$, where A is an abelian threefold and G is its finite automorphism group acting freely on A such that either one of the following (1) or (2) is satisfied:

(1) $G = \langle a \rangle \oplus \langle b \rangle \simeq C_2^{\oplus 2}$ *and,*

$$a_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ and } b_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(2) $G = \langle a, b | a^4 = b^2 = 1, bab = a^{-1} \rangle \simeq D_8$ *and,*

$$a_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } b_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

where a_0 and b_0 stand for the Lie parts of a and b respectively and the matrix representation is the one given under appropriate global coordinates of A.

(II) In the first case $\rho(X) = h^1(T_X) = 3$ and in the second case $\rho(X) = h^1(T_X) = 2$. In particular, a Calabi-Yau threefold with Picard number $\rho \neq 2, 3$ is not of Type A.

(III) Both cases actually occur (see (2.17) and (2.18) for explicit examples).

(IV) In each case, the nef cone $\overline{\mathcal{A}}(X)$ is a rational, simplicial cone and every rational nef divisor on X is semi-ample. In particular, X admits only finitely many contractions.

We call a pair (A, G) which falls into the cases (1) and (2) *Igusa's pair* and *refined Igusa's pair* respectively.

This Theorem is shown in the subsection (2A) based on the notion of the minimal splitting covering introduced by Beauville ([Be2, Section 3], see also (2.1)), which nicely reduces a greater part of the proof to the problem of representations of a special kind of groups G called *pre-Calabi-Yau groups of Type A* ((2.2)). Then, one of the important steps is to restrict the possible orders of such G. For this purpose, we apply the Burnside-Hall Theorem concerning commutative subgroups of p-groups ([Su, Page 90], see also (2.7)).

Let us add a few remarks about the result. First, even compared with the range $1 \leq \rho \leq 9$ of the Picard numbers of abelian threefolds, the range $\{2, 3\}$ of $\rho(X)$ is quite narrow. Secondly, our statements (II) and (III) show that there certainly exist smooth Calabi-Yau threefolds containing no rational curves if $\rho = 2$ and 3, but, on the other hand, suggest some hope to ask the following:

Question (cf. [Wi1], [MS]). Does every Calabi-Yau threefold of Picard number $\rho \neq 2, 3$ contain rational curves? □

The third remark concerns the statement (IV). Recall that any smooth anti-canonical member $X \in |-K_V|$ of a smooth Fano fourfold V is a simply connected Calabi-Yau threefold. In addition, such an X always satisfies $c_2(X) > 0$ on $\overline{\mathcal{A}}(X) - \{0\}$ ([OP, Main Theorem 2]). So, the statement (IV) can be regarded as an extreme counterpart of the following Theorem due to Kollár:

THEOREM ([BO, APPENDIX]). *Let X be a smooth member of $|-K_V|$ of a smooth Fano fourfold V and $\iota : X \to V$ the natural inclusion. Then $\iota^* : \overline{\mathcal{A}}(V) \to \overline{\mathcal{A}}(X)$ is an isomorphism. In particular, the nef cone $\overline{\mathcal{A}}(X)$ is a rational polyhedral cone, every rational nef divisor on X is semi-ample, and therefore, X admits only finitely many contractions.* □

Refer also to [Wi1, Page 146, Claim], [Wi2, 4] and [Og1, Theorem (2.1), Proposition (2.7)] for related results about semi-ampleness of nef divisors.

The last remark is concerned with fundamental group. According to the Bogomolov decomposition Theorem [Be1] and its generalisation due to Yoshinori Namikawa and Steenbrink [NS, Corollary (1.4)], there is one more class of Calabi-Yau threefolds with infinite fundamental group, namely, the class consisting of those Calabi-Yau threefolds that are given as an étale quotient of (K3 surface) $\times$ (elliptic curve). We call them Calabi-Yau threefolds *of Type K* and study them in subsection (2K) in some extent. (See Theorem (2.23) for the statement.) As an application of (0.1) and (2.23), we obtain the following criterion for $\pi_1(X)$ to be finite in terms of the Picard number:

COROLLARY (0.2). *Let X be a Calabi-Yau threefold. Then, $\pi_1(X)$ is finite if $\rho(X) = 1, 6, 8, 9, 10$ or $\rho(X) \geq 12$. Moreover, this is also optimal in the sense that for each $\rho \in \mathbb{N} - \{1, 6, 8, 9, 10, n \geq 12\} (= \{2, 3, 4, 5, 7, 11\})$, there exists a Calabi-Yau threefold X such that $\pi_1(X)$ is infinite and $\rho(X) = \rho$. In particular, the fundamental group of a Calabi-Yau threefold whose Picard number one is always finite.*

The last statement in (0.2) is also obtained by Amerik, Rovinsky and Van de Ven [ARV, Proposition (3.1)], which the first author heard from Amerik after his talk on this subject.

So far, we have concerned special kinds of Calabi-Yau threefolds called of Type A and of Type K. Another particular interest of this paper, which turns out to be related to our first problem, is the role of the second Chern class $c_2(X)$ in the geometry of contractions of X. Here, the term *contraction* means a surjective morphism onto a normal, projective variety with connected fibers, and therefore, consists of the two cases, that is, the fiber space case and the birational contraction case. Let $\varphi : X \to W$ be a contraction. Then, there is a nef divisor D on X such that $\varphi = \Phi_D$, where Φ_D stands for the morphism associated with the complete linear system $|D|$. Therefore, we may relate φ with $c_2(X)$ via the intersection number $(c_2(X).D)$. Although the value $(c_2(X).D)$ itself is not well-defined for φ, it does not depend on the choice of D such that $\varphi = \Phi_D$ whether $(c_2(X).D) = 0$ or not. This is due to the pseudo-effectivity of $c_2(X)$ ([Mi]). We call φ a *c_2-contraction* if $(c_2(X).D) = 0$. For example, a pencil $\varphi : X \to \mathbb{P}^1$ is a c_2-contraction if and only if the general fiber of φ is an abelian surface.

Our first task in this direction is to enlarge our earlier classification of c_2-contractions in the simply connected case ([Og 2, 3, 4]) to the one in the general case as in (0.3) below. For the statement, we recall the following pairs of an abelian threefold and its specific Gorenstein automorphism: the pair (A_3, g_3), where A_3 is the product threefold $A_3 := E_{\zeta_3}^3$ of the elliptic curve E_{ζ_3} of period $\zeta_3 = \exp(2\pi i/3)$ and g_3 is its automorphism $\mathrm{diag}(\zeta_3, \zeta_3, \zeta_3)$; and the pair (A_7, g_7), where A_7 is the Jacobian threefold of the Klein quartic curve $C = (x_0x_1^3 + x_1x_2^3 + x_2x_0^3 = 0) \subset \mathbb{P}^2$ and g_7 is the automorphism of A_7 induced by the automorphism of C given by $[x_0 : x_1 : x_2] \mapsto [\zeta_7 x_0 : \zeta_7^2 x_1 : \zeta_7^4 x_2]$. We call (A_3, g_3) *the Calabi pair* and (A_7, g_7) *the Klein pair.*

THEOREM (0.3). *Let X be a Calabi-Yau threefold. Assume that X admits a c_2-contraction $\varphi : X \to W$ such that $dim(W) \geq 2$. Then, X is smooth and is birational to either one of the following:*

(1) *a crepant resolution of a Gorenstein quotient $(S \times E)/G$ of the product of a normal K3 surface S and an elliptic curve E, where by a normal K3 surface we mean a normal surface whose minimal resolution is a K3 surface;*

(2) *the crepant resolution of A/G, where (A, G) is either the Calabi pair, its modification, the Klein pair, Igusa's pair or refined Igusa's pair.*

(See (3.3), (3.4), (3.6) and (3.7) for more precise statement and structures.)

The main idea of the proof of Theorem (0.3) is to modify $\varphi : X \to W$ toward one of the threefolds described in (0.3) by taking appropriate coverings, Stein factorisations, Albanese maps or by running log mimimal model program, as intrinsically as possible in order to inherite group actions biregularly. This idea itself is same as the one in the simply connected case and, indeed, most part of the proof toward the case (2) can be done by a combination of (0.1) and more or less obvious minor modification of [Og2, 3]. However, the first author should confess that his argument of [Og4, Section 3] concerning lifting of certain group actions *contains a gap*, which he noticed around August 1999, and the argument [Og4, Sections 3, 4] toward the case (1) is not available. (Claim (3.4) in [Og4] is false and the right statement is that ν in (3.4) is at best the normalisation *in a certain finite field extension*. Therefore, ν is not intrinsic so that the lifting argument there and the argument after that seem to be broken.) Unlike the argument there, which is based on the theory of quasi-product threefolds, our new idea here is to apply again the notion of the minimal splitting covering of Beauville, especially, its uniqueness property, after taking the same reduction as in [Og4, Section 2]. Fortunately, this argument also recovers the main result [Og4] as its own form and even simplifies the proof. This will be done in Section 3, especially in (3.7).

The final aim of this paper is to show the following:

THEOREM (0.4). *Each Calabi-Yau threefold X admits only finitely many different c_2-contractions up to isomorphism. In particular, X admits only finitely many different abelian pencil structures up to $Aut(X)$.*

This result is particularly motivated by the following work on the finiteness of fiber spaces coming from "the opposite side" of the cone:

THEOREM [OP, MAIN THEOREM 1 AND REMARK IN SECTION 3]. *Let X be a Calabi-Yau threefold and let H be an ample divisor on X and $\epsilon > 0$ a positive real number. Set $\overline{\mathcal{A}}_\epsilon(X) := \{x \in \overline{\mathcal{A}}(X) | (c_2(X).x) \geq \epsilon(H^2.x)\}$.*

(1) *Assume that $c_2(X) > 0$ on $\overline{\mathcal{A}}(X) - \{0\}$. Then, X admits only finitely many different fibrations.*

(2) *More generally, the cardinality of the fibrations $\varphi : X \to W$ such that $\varphi^*\overline{\mathcal{A}}(W) \subset \overline{\mathcal{A}}_\epsilon(X)$ is finite.* □

Both Theorem (0.4) and Theorem [OP] are related positively to the Cone Conjecture posed by D. Morrison [MD]. However, these two Theorems are completely different both in nature and in proof. Proof of Theorem [OP] is based on the boundedeness results of log surfaces due to Alexeev [Al] and the compactness of the domain $\{x \in \overline{\mathcal{A}}_\epsilon(X) | (c_2(X).x) \leq B\}$. Compactness, in particular, implies the finiteness of the lattice points in the domain. Main idea in [OP] is to reduce the problem to this finiteness by applying the boundedness. Therefore, the result claims the finiteness of fiber spaces in question themselves. (See [OP] for details.) However, this compactness reduction does not work any more for c_2-contractions. In addition, there actually exists a Calabi-Yau threefold which admits infinitely many different abelian pencils ([Og1, Section 4]). Therefore, contrary to [OP], the finiteness of c_2-contractions themselves are false in general and it should be the core of (0.4) that we modulo them up by automorphisms. Outline of proof of (0.4) is as follows: We first take *the maximal c_2-contraction of X* ((4.1)) and denote this by $\varphi_0 : X \to W_0$. This φ_0 has a property that any c_2-contraction factors through φ_0. Then we divide into cases according to the structure of φ_0. The essential case is the case where φ_0 falls into the case (1) of (0.3).

In this case, we divide our problem into two parts: finiteness of contractions of W_0 up to isomorphisms; and, lifting of automorphisms of W_0 to $\varphi_0 : X \to W_0$. We apply Kawamata's finiteness result of relatively minimal models of a Calabi-Yau fiber space [Kaw5, Theorem (3.6)] to the second part and an equivariant Torelli Theorem for pair (S, G) of K3 surface and its finite automorphism group ((1.8), (1.10)) to the first part. We need the finiteness of minimal models, because minimal models of X are no more unique. The role of our equivariant Torelli Theorem may be more or less apparent by the fact that W_0 is birational to S/G for some (S, G). Full verification of (0.4) will be given in Section 4. It might be also worth while noticing here that, in order to examine automorphisms toward finiteness, Kawamata [Kaw5] extended Torelli Theorem to the one over non-closed field and applied "vertically" to his finiteness result, while we extended it to the one with finite group action and applied "horizontally" to our finiteness result.

We formulate our equivariant Torelli Theorem in Section 1. Besides the present application (see also (2.23)(IV)), this Theorem has been also applied to study finite automorphism groups of K3 surfaces by [OZ1, 2, 3].

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Thoughout this paper, in addition to the notation and terminology in [Ha], [KMM] and in Introduction, we employ the following:

Notation, Terminology and Convention.

(0.1). Every variety in this paper is assumed to be normal, projective and defined over $\mathbb{C}$ unless stated otherwise. The open convex cone generated by the ample classes in $N^1(X) := (\{\text{Cartier divisors}\}/\equiv) \otimes \mathbb{R}$ is called the ample cone and is denoted by $\mathcal{A}(X)$. Its closure $\overline{\mathcal{A}}(X)$ is called *the nef cone.* A $\mathbb{Q}$-Cartier divisor D on X is said to be *semi-ample* if there exists a positive integer m such that $|mD|$ is free.

(0.2). Two contractions $\varphi : X \to W$ and $\varphi' : X' \to W'$ are said to be *isomorphic* if there exist isomorphisms $F : X \to X'$ and $f : W \to W'$ such that $\varphi' \circ F = f \circ \varphi$:

$$\begin{array}{ccc} X & \xrightarrow{F} & X' \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi'} \\ W & \xrightarrow[f]{} & W'. \end{array}$$

Two contractions of X, $\varphi : X \to W$ and $\varphi' : X \to W'$ are said to be *identically isomorphic* if there exists an isomorphism $f : W \to W'$ such that $\varphi' = f \circ \varphi$. Identically isomorphic contractions should be considered to be the same. It is important to distinguish these two notions, isomorphic and identically isomorphic, especially in the case where $X = X'$. For example, two natural projections $p_i : X \times X \to X$ are clearly isomorphic but never identically isomorphic. Their difference in terms of the nef cone is as follows: Two contractions $\varphi : X \to W$ and $\varphi' : X \to W'$ are isomorphic if and only if there exists an autmorphism $F \in \mathrm{Aut}(X)$ such that $F^*\varphi^*\overline{\mathcal{A}}(W) = (\varphi')^*\overline{\mathcal{A}}(W')$, while these two are identically isomorphic if and only if $\varphi^*\overline{\mathcal{A}}(W) = (\varphi')^*\overline{\mathcal{A}}(W')$.

(0.3). Let X be a Gorenstein variety such that $\mathcal{O}_X(K_X) \simeq \mathcal{O}_X$. We denote by ω_X a generater of $H^0(\mathcal{O}_X(K_X))$. A finite automorphism group $G \subset \mathrm{Aut}(X)$ is called *Gorenstein* if $g^*\omega_X = \omega_X$ for each $g \in G$.

(0.4). Throughout this paper, we often need to examine an object with a faithful group action $G \curvearrowright S$. We distinguish several notions concerning "fixed loci" by using the following different symbols:
$S^g := \{s \in S | g(s) = s\}$ for $g \in G$;
$S^G := \cap_{g \in G} S^g$, the set of points which are fixed by *all* the elements of G;
$S^{[G]} := \cup_{g \in G - \{1\}} S^g$, the set of points which are fixed by *some* non-trivial element of G. An action $G \curvearrowright S$ is said to be fixed point free if $S^{[G]} = \emptyset$.

(0.5). Let G be a finite group and X a variety with a group action $\rho_X : G \to \mathrm{Aut}(X)$. A contraction $\varphi : X \to W$ is said to be *G-stable* if there exists a representation $\rho_W : G \to \mathrm{Aut}(W)$ which satisfies $\varphi \circ \rho_X(g) = \rho_W(g) \circ \varphi$. Note that the representation ρ_W is uniquely determined by ρ_X and φ. The G-stability of the contraction is also equivalent to the existence of a line bundle $D \in \mathrm{Pic}(X)^G$ such that $\varphi = \Phi_D$. Two G-stable contractions $\varphi : X \to W$ and $\varphi' : X' \to W'$ are said to be *G-equivariantly isomorphic* if there exist isomorphisms $F : X \to X'$ and $f : W \to W'$ such that

$$\varphi' \circ F = f \circ \varphi,\ F \circ \rho_X(g) = \rho_{X'}(g) \circ F,\ f \circ \rho_W(g) = \rho_{W'}(g) \circ f.$$

(0.6). $\zeta_n := \exp(2\pi\sqrt{-1}/n)$, the primitive n-th root of unity in $\mathbb{C}$.

(0.7). We denote some specific groups appearing in this paper by the following fairly standard symbols:
$C_n := \langle a | a^n = 1\rangle$, the cyclic group of order n;
$D_{2n} := \langle a, b | a^n = b^2 = 1, bab = a^{-1}\rangle \simeq C_n \rtimes C_2$, the dihedral group of order $2n$;
$Q_{4n} := \langle a, b | a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1}\rangle$, the binary dihedral group of order $4n$;
$S_n := \mathrm{Aut}_{\mathrm{set}}(\{1, 2, \cdots, n\})$, the n-th symmetric group;
$A_n := \mathrm{Ker}(\mathrm{sgn} : S_n \to \{\pm 1\})$, the n-th alternative group.

(0.8). Let $A := \mathbb{C}^d/\Lambda$ be a d-dimensional complex torus. By abuse of language, we call global coordinates $(z_1, z_2, ..., z_d)$ of $\mathbb{C}^n$ *global coordinates of* A if they are obtained by an affine transformation of the natural global coordinates of $\mathbb{C}^d$ given by the i-th projections. When the origin 0 of A is specified and A is regarded as a group variety, we identify A with its translation group in the natural manner and denote by $(A)_n$ the group of n-torsion points and by t_* the translation given by $* \in A$. Under this identification, we have $\mathrm{Aut}(A) = A \rtimes \mathrm{Aut}_{\mathrm{Lie}}(A)$, where $\mathrm{Aut}_{\mathrm{Lie}}(A)$ is the subgroup consisting of the elements g such that $g(0) = 0$. We often call the second factor of $h \in \mathrm{Aut}(A)$ under this decomposition, *the Lie part of* h and denote it by h_0. We also denote by E_ζ the elliptic curve whose period is ζ in the upper half plane.

(0.9). In this paper, we often regard group actions on varieties as the so-called co-action through their coordinates. The advantage of this convention is that we may then describe its action on cohomology as if it were covariant, namely, $(ab)^* = a^*b^*$.

(0.10). We abbreviate by S_K the scalar extension $S \otimes_{\mathbb{Z}} K$ of the space S.

1. An equivariant Torelli Theorem for K3 surfaces with finite group action and its applications. Let X be a K3 surface and $G \subset \mathrm{Aut}(X)$ a finite automorphism group. Throughout this section this pair (X, G) is fixed. The aim of this section is to formulate an equivariant version of the Torelli Theorem (1.8) which describes the automorphisms of X which commute with G in terms of their actions on cohomology, and apply this to get more geometrical consequences (1.9)-(1.11). The core of the formulation is to define the G-equivariant reflection group (1.6).

(1.1). As usual, we consider the second cohomology group $H^2(X, \mathbb{Z})$ as a lattice by the non-degenerate symmetric bilinear form $(*.*)$ induced by the cup product. We denote by $O(H^2(X, \mathbb{Z}))$ the orthogonal group of $H^2(X, \mathbb{Z})$.

(1.2). For the convenience of the formulation, we introduce the following notation which in principle follows the rule that U denotes the G-invariant part of the abstract one U'. (The reason behind this usage of notation is the fact that G-invariant part plays more important roles in our formulation.) Another rule is that the symbol W^+ indicates a subgroup of a group W:

$S' := H^{1,1}(X) \cap H^2(X, \mathbb{Z})$, the Néron-Severi lattice of X;

$\mathcal{N}' := \{[E] \in S' | E \subset X, E \simeq \mathbb{P}^1\}$, the set of nodal classes;

$S := (S')^G (= \{x \in S' | g^*(x) = x \text{ for all } g \in G\})$;

$T := S^\perp = \{x \in H^2(X, \mathbb{Z}) | (x.y) = 0 \text{ for all } y \in S\}$, the orthogonal lattice of S in $H^2(X, \mathbb{Z})$ (Note that T contains the transcendental lattice of X.);

$S^* := \mathrm{Hom}(S, \mathbb{Z})$, which we always regard as an overlattice of S, $S \subset S^* \subset S_{\mathbb{Q}}$, via the non-degenerate pairing $(*.*)|S$;

$(\mathcal{C}')^\circ :=$ the positive cone of X, that is, the connected component of the space $\{x \in (S')_{\mathbb{R}} | (x.x) > 0\}$ containing the ample classes;

$\mathcal{C}' :=$ the union of $(\mathcal{C}')^\circ$ and all $\mathbb{Q}$-rational rays in the boundary $\partial(\mathcal{C}')^\circ$ of $(\mathcal{C}')^\circ (\subset (S')_{\mathbb{R}})$;

$\mathcal{C} := (\mathcal{C}')^G = \mathcal{C}' \cap S_{\mathbb{R}}$;

$\mathcal{A}' :=$ the intersection of the nef cone $\overline{\mathcal{A}}(X)$ and $\mathcal{C}'$;

$\mathcal{A} := (\mathcal{A}')^G = \mathcal{A}' \cap S_{\mathbb{R}}$;

$Q := \{f \in \mathrm{Aut}(X) | f \circ g = g \circ f \text{ for all } g \in G\}$;

$O(S) :=$ the orthogonal group of the lattice S preserving $\mathcal{C}$;

$O(S)^+ :=$ the subgroup of $O(S)$ consisting of the elements of the form $\tau|S$, where τ is a Hodge isometry of $H^2(X, \mathbb{Z})$ such that $\tau g^* = g^* \tau$ for all $g \in G$ (Note that by the last condition, such τ always satisfy $\tau(S) = S$);

$P(S) := \{\sigma \in O(S) | \sigma(\mathcal{A}) = \mathcal{A}\}$;
$P(S)^+ := \{\sigma \in O(S) | \sigma = f^*|S \text{ for some } f \in Q\}$.

We should keep in mind the following easy facts and relations:

LEMMA (1.3).

(1) *S is an even hyperbolic lattice if* $\operatorname{rank} S \geq 2$.

(2) *The interior $\mathcal{A}^\circ$ of $\mathcal{A}$ consists of the G-invariant ample classes of X and is non-empty. Moreover, $\mathcal{A}^\circ = (\mathcal{A}')^\circ \cap S_{\mathbb{R}}$.*

(3) *$f^*(S) = S$ and $f^*|S \in P(S)^+$ for all $f \in Q$. In other words, $P(S)^+$ is the image of the homomorphism $Q \to O(S)$ given by $f \mapsto f^*|S$. In particular, $\sigma(\mathcal{A}) = \mathcal{A}$ for each $\sigma \in P(S)^+$.*

(4) *Set $O(S)^{++} := \{\sigma \in O(S) \,|\, \sigma|(S^*/S) = id\}$. Then, $O(S)^{++} \subset O(S)^+ \subset O(S)$ and each of these inclusions is of finite index.*

Proof. The assertions (1) and (2) are clear. Note that a Hodge isometry σ of $H^2(X,\mathbb{Z})$ satisfies $\sigma(S) = S$ if $\sigma g^* = g^*\sigma$ for all $g \in G$. This implies the assertion (3). We show the assertion (4). Since $O(S)^{++} = \operatorname{Ker}(O(S) \to \operatorname{Aut}(S^*/S))$ and $\operatorname{Aut}(S^*/S)$ is a finite group, $O(S)^{++} \subset O(S)$ is of finite index. It remains to show that $O(S)^{++} \subset O(S)^+$. Let σ be an element of $O(S)^{++}$. Since $\sigma|(S^*/S) = id$, the pair $(\sigma, id) \in O(S) \times O(T)$ can be extended to an element $\tau \in O(H^2(X,\mathbb{Z}))$. This τ is a Hodge isometry, because $[\omega_X] \in T_{\mathbb{C}}$ and $\tau|T = id$. Moreover, given $g \in G$, we have $\tau \circ g^* = g^* \circ \tau$ on $H^2(X,\mathbb{Z})$, because $\tau \circ g^* = g^* \circ \tau (= \sigma)$ on S and $\tau \circ g^* = g^* \circ \tau (= g^*)$ on T. Hence $\sigma \in O(S)^+$. □

(1.4). Let us define the right object $\mathcal{N}$ for $\mathcal{N}'$. Unfortunately, $\mathcal{N}$ is larger than the set $(\mathcal{N}')^G$, in general. Let $[b] \in \mathcal{N}'$ and define a reduced divisor B by $B := (\sum_{g \in G} g^*(b))_{red}$ and denote by $B = \coprod_{k=1}^{n(b)} B_k$ the decomposition of B into the connected components, where and in what follows, we identify $[b]$ with the unique smooth rational curve b which represents $[b]$. Then, G acts on the set $\{B_k\}_{k=1}^{n(b)}$ transitively, and the value $(B_k.B_k)$ is then independent of k. Moreover, since B_k is connected and reduced, using the Reimann-Roch Theorem and the exact sequence $0 \to \mathcal{O}_X(-B_k) \to \mathcal{O}_X \to \mathcal{O}_{B_k} \to 0$, we see that $1 \geq 1 - p_a(B_k) = -(B_k.B_k)/2$. This implies $(B_k.B_k) = -2$ if $(B_k.B_k) < 0$. Set

$$\mathcal{N} := \{b \in \mathcal{N}' | (B_k.B_k) = -2\},$$

and define for $b \in \mathcal{N}$ and $k \in \{1, 2, ..., n(b)\}$ the reflection r_{B_k} on $H^2(X,\mathbb{Z})$ by $r_{B_k}(x) = x + (x.B_k)B_k$. It is easily seen that r_{B_k} are Hodge isometries and satisfy $r_{B_k}(\mathcal{C}') = \mathcal{C}'$, $r_{B_k} \circ r_{B_l} = r_{B_l} \circ r_{B_k}$ and $r_{B_k}^2 = id$.

By using the last two formulae in (1.4), we readily obtain

LEMMA (1.5).

(1) $(\prod_{k=1}^{n(b)} r_{B_k})(x) = x + \sum_{k=1}^{n(b)} (x.B_k)B_k$ *for* $x \in H^2(X,\mathbb{Z})$.

(2) $(\prod_{k=1}^{n(b)} r_{B_k})^2 = id$. □

Now G-equivariant reflection group is defined as follows:

DEFINITION (1.6). Set $R_b := \prod_{k=1}^{n(b)} r_{B_k}$ for $b \in \mathcal{N}$ and define a subgroup Γ of $O(H^2(X,\mathbb{Z}))$ by $\Gamma := \langle R_b \,|\, b \in \mathcal{N} \rangle$. We call this group Γ the *G-equivariant reflection group of the pair (X, G).*

LEMMA (1.7).

(1) *$\Gamma \subset O(S)^+$, or more precisely, for each $\sigma \in \Gamma$, the restriction $\sigma|S$ lies in $O(S)^+$ and satisfies $\sigma = id$ if $\sigma|S = id$.*

(2) *The cone $\mathcal{A}$ is a fundamental domain for the action Γ on $\mathcal{C}$, that is, $\mathcal{A}$ satisfies that $\sigma(\mathcal{A}^\circ) \cap \mathcal{A}^\circ \neq \emptyset$ if and only if $\sigma = id$, and that $\Gamma \cdot \mathcal{A} = \mathcal{C}$.*

Proof. Take $x \in S'$ and $g \in G$. Using (1.5), we calculate

$$\begin{aligned} g^* \circ R_b(x) &= g^*(x) + \sum_{k=1}^{n(b)} (x.B_k) g^*(B_k) \\ &= g^*(x) + \sum_{k=1}^{n(b)} (g^*(x).g^*(B_k)) g^*(B_k) \\ &= g^*(x) + \sum_{k=1}^{n(b)} (g^*(x).B_k) B_k \\ &= R_b \circ g^*(x). \end{aligned}$$

Hence $g^* \circ R_b = R_b \circ g^*$ and therefore, $R_b(S) = S$. Since $R_b(\mathcal{C}') = \mathcal{C}'$, this also gives $R_b(\mathcal{C}) = \mathcal{C}$. Moreover, R_b is a Hodge isometry, because $R_b([\omega_X]) = [\omega_X]$. Therefore $R_b \in O(S)^+$. Assume that $\sigma|S = id$ for some $\sigma \in \Gamma$. Then, $\sigma(\mathcal{A}^\circ) \cap \mathcal{A}^\circ \neq \emptyset$, and in particular, $\sigma((\mathcal{A}')^\circ) \cap (\mathcal{A}')^\circ \neq \emptyset$. This implies $\sigma = id$, because $\mathcal{A}'$ is the fundamental domain for the action $\langle r_b \mid b \in \mathcal{N}' \rangle$ on $\mathcal{C}'$ (see for example [BPV, Chap.VIII, Proposition (3.9)]). It remains to check the equality $\Gamma \cdot \mathcal{A} = \mathcal{C}$. Recall that G acts transitively on the set $\{B_k\}_{k=1}^{n(b)}$ and satisfies $(x.g^*(B_k)) = (g^*(x).g^*(B_k)) = (x.B_k)$ for $x \in S_{\mathbb{R}}$ if $g \in G$. Therefore, $(x.B_k) = (x.B_l)$ for $x \in S_{\mathbb{R}}$ and for $b \in \mathcal{N}$, and we get $R_b(x) = x + (x.B_1) \sum_{k=1}^{n(b)} B_k$. This formula shows that $R_b|S_{\mathbb{R}}$ is nothing but a reflection with respect to the hyperplane defined by $(*.B_1) = 0$. Recall that by (1.3)(1) the subgroup (of index two) of the orthogonal group $O(S_{\mathbb{R}})$ preserving $\mathcal{C}^\circ$ makes the quotient space $\mathcal{C}^\circ/\mathbb{R}_{>0}$ a Lobachevskii space. Then, we can apply the general theory on the discrete reflection group on Lobachevskii space [Vi] to see that the space

$$\tilde{\mathcal{A}} := \{x \in \mathcal{C} \mid (x.B_1) \geq 0 \,\text{for all}\, b \in \mathcal{N}\}$$

is a fundamental domain for the action Γ on $\mathcal{C}$. Therefore, it is sufficient to check $\tilde{\mathcal{A}} = \mathcal{A}$. It is clear that $\mathcal{A} \subset \tilde{\mathcal{A}}$. Let $x \in \tilde{\mathcal{A}}$ and take $b \in \mathcal{N}'$. Then, $(x.B_1) = c(x.b)$, where c is the number of the irreducible components of B_1, because x is G-invariant. If $b \notin \mathcal{N}$, then $(B_1.B_1) \geq 0$, and therefore, $(x.B_1) \geq 0$ by the Hodge index Theorem. If $b \in \mathcal{N}$, then $(x.B_1) \geq 0$ by the definition of $\tilde{\mathcal{A}}$. Hence, $(x.b) \geq 0$ for all $b \in \mathcal{N}'$. This gives $\tilde{\mathcal{A}} \subset \mathcal{A}$. □

Now we can formulate an equivariant Torelli Theorem as follows. This is a reformulation of the abstract version of the Torelli Theorem for K3 surfaces [SPP] in an equivariant setting and is also regarded as a sort of generalisation of the Torelli Theorem for Enriques surfaces due to Horikawa and Yukihiko Namikawa ([Ho], [Nm]).

THEOREM (1.8). *Γ is a normal subgroup of $O(S)^+$ and fits in with the semi-direct decomposition $O(S)^+ = \Gamma \rtimes P(S)^+$.*

Proof. Let σ be an element of $O(S)^+$ and take $y \in \mathcal{A}^\circ$. Applying (1.7)(2) for $\sigma(y)$, we find an element $r \in \Gamma$ such that $r^{-1} \circ \sigma(y) \in \mathcal{A}^\circ$. Note that $r^{-1} \circ \sigma \in O(S)^+$ by (1.7)(1). Then, there exists a Hodge isometry $\rho \in O(H^2(X, \mathbb{Z}))$ such that $\rho|S = r^{-1} \circ \sigma$ and that $\rho \circ g^* = g^* \circ \rho$ for all $g \in G$. In addition, this ρ is also effective, because $\rho(y) = r^{-1} \circ \sigma(y) \in \rho((\mathcal{A}')^\circ) \cap (\mathcal{A}')^\circ$. Hence, by the Torelli Theorem for K3 surfaces [PSS], [BPV, Chap.IIIV], there exists $f \in \mathrm{Aut}(X)$ such that $f^* = \rho$. Moreover, $f \circ g = g \circ f$ for all $g \in G$ again by the Torelli Theorem, because $f^* \circ g^* = g^* \circ f^*$. Hence

$f \in Q$ and $f^*|S \in P(S)^+$ by (1.3)(3). Since $f^*|S = r^{-1} \circ \sigma$, we get $O(S)^+ = \Gamma \cdot P(S)^+$. Assume that $r \circ \tau = r' \circ \tau'$ for some $r, r' \in \Gamma$ and $\tau, \tau' \in P(S)^+$. Since $\tau \circ (\tau')^{-1}(\mathcal{A}) = \mathcal{A}$ (1.3)(3), we have $r^{-1} \circ r'(\mathcal{A}) = \mathcal{A}$. Therefore, by (1.7)(2), we obtain $r^{-1} \circ r' = id$. This shows the uniqueness of the factorisation of elements of $O(S)^+$. It remains to show that Γ is a normal subgroup of $O(S)^+$. For this, it is now enough to check that for each $b \in \mathcal{N}$ and $\sigma \in P(S)^+$ there exists an element $b' \in \mathcal{N}$ such that $\sigma^{-1} \circ R_b \circ \sigma = R_{b'}$. Let us choose $f \in Q$ such that $\sigma = f^*|S$. Then, we calculate

$$\begin{aligned}\sigma^{-1} \circ R_b \circ \sigma(x) &= (f^{-1})^* \circ R_b \circ f^*(x) \\ &= x + \sum_{i=1}^{n(b)} (f^*(x).B_i)(f^{-1})^*(B_i) \\ &= x + \sum_{i=1}^{n(b)} (x.(f^{-1})^*(B_i))(f^{-1})^*(B_i).\end{aligned}$$

In addition, we have $(f^{-1})^*(b) \in \mathcal{N}$, because $b \in \mathcal{N}$ and $f^* \circ g^* = g^* \circ f^*$. Therefore, we may take this $(f^{-1})^*(b)$ as b'. □

Theorem (1.8) provides some more geometrical consequences. The first one is a generalisation of the main result of H. Sterk [St] (see also [Kaw5, Section 2]) to an equivariant setting:

COROLLARY (1.9). *There exists a finite rational polyhedral fundamental domain Δ for the action $P(S)^+$ on $\mathcal{A}$.*

Proof. Note that $O(S)^+$ is of finite index in the arithemetic group $O(S)$ of the self-dual homogeneous cone $\mathcal{C}$ by (1.3)(4). Then, by [AMRT, Chap.II, Pages 116-117], there exists a finite rational polyhedral fundamental domain Δ for the action $O(S)^+$ on $\mathcal{C}$. Translating Δ by an appropriate element of Γ if necessary, we take such Δ as $(\Delta)^\circ \cap \mathcal{A}^\circ \neq \emptyset$. This Δ satisfies $\Delta \subset \mathcal{A}$. (Indeed, otherwise, there would be an element $b \in \mathcal{N}$ such that the wall $H_b = \{x \in S_{\mathbb{R}} | (x.b) = 0\}$ of $\mathcal{A}$ satisfies $(\Delta)^\circ \cap H_b \neq \emptyset$. However, since $R_b(y) = y$ for $y \in (\Delta)^\circ \cap H_b$, we would then have $R_b((\Delta)^\circ) \cap (\Delta)^\circ \neq \emptyset$, a contradiction.) Now, combining $P(S)^+ = O(S)^+/\Gamma$ (1.8), $P(S)^+ \cdot \mathcal{A} = \mathcal{A}$ (1.3)(3) and the fact that $\mathcal{A}$ is a fundamental domain for the action Γ on $\mathcal{C}$ (1.7)(2), we conclude that this Δ gives a desired fundamental domain. □

COROLLARY (1.10). *Let Z be a normal K3 surface and G_Z a finite automorphism group of Z. Then Z admits only finitely many G_Z-stable contractions up to G_Z-equivariant isomorphism.*

Proof. First consider the case where Z is smooth. Let Δ be the fundamental domain found in (1.9) for (Z, G_Z) and decompose Δ into its locally closed strata, $\Delta = \Delta_1 \sqcup \Delta_2 \sqcup \cdots \sqcup \Delta_n$. Then, any two $\mathbb{Q}$-rational points H_1 and H_2 in the same strata Δ_i give G_Z-equivariantly isomorphic G_Z-stable contractions, because, as there exist positive integers m_1, m_2 and m_3 such that $m_2H_2 - m_1H_1, m_3H_1 - m_2H_2 \in \Delta_i$, by the semi-ampleness of rational nef divisors on a K3 surface [SD], the contractions given by H_1 and H_2 factor through each other and hence G_Z-equivariantly isomorphic. Let $\Phi : Z \to W$ be a G_Z-stable contraction and choose a G_Z-invariant line bundle H such that $\Phi = \Phi_H$. Then, by (1.9) and (1.3)(3), there exist an integer $i \in \{1, 2, ..., n\}$ and $f \in Q$ (for Z) such that $f^*([H]) \in \Delta_i$. Therefore, the result follows for smooth Z.

Next consider the case where Z is singular. Let $\nu : Y \to Z$ be the minimal resolution, G_Y the unique equivariant lift of the action G_Z on Y and E the exceptional divisor of ν. Then, E is a disjoint sum of reduced divisors of Dynkin type and is stable under G_Y. Moreover, Y admits only finitely many G_Y-stable contractions up to G_Y-equivariant isomorphism by the previous argument. Let us denote the representatives of the G_Y-stable contractions of Y by $\Phi_i : Y \to W_i$ $(i = 1, 2, ..., I)$. Each of these Φ_i is either an elliptic fibration or a birational contraction. In particular, for each Φ_i, there are only finitely many effective reduced divisors E_{ij} $(j = 1, 2, ..., J_i)$ on Y satisfying that E_{ij} is a disjoint sum of reduced divisors of Dynkin type, stable under G_Y and $\dim \Phi_i(E_{ij}) = 0$. Let $\nu_{ij} : Y \to Z_{ij}$ be the contraction of E_{ij} and $\varphi_{ij} : Z_{ij} \to W_i$ the induced contraction. Then, G_Y descends equivariantly to the action on Z_{ij} and makes φ_{ij} G_Y-stable. Let $\varphi : Z \to W$ be a G_Z-stable contraction of Z and set $\Phi := \varphi \circ \nu : Y \to W$. Then, this Φ is G_Y-stable. Therefore, there exist i and isomorphisms $F : Y \simeq Y$ and $f : W \simeq W_i$ such that $\Phi : Y \to W$ and $\Phi_i : Y \to W_i$ are G_Y-equivariantly isomorphic by F and f. Since $F(E)$ satisfies all the defining properties of E_{ij}, there also exists j such that $F(E) = E_{ij}$. Therefore, F descends to give an isomorphism $F' : Z \to Z_{ij}$. This F' together with f gives a G_Z-equivariant isomorphism (with respect to the representation $G_Z \to \mathrm{Aut}(Z_{ij})$ through G_Y) between $\phi : Z \to W$ and $\varphi_{ij} : Z_{ij} \to W_i$. Now we are done. □

Next, as an application of (1.7)(2), we show the following generalisation of the result of S. Kondo [KS, Lemma(2.1)].

COROLLARY (1.11). *Let (X, G) be a pair of a K3 surface and its finite automorphism group and, as before, S the G-invariant part of the Néron-Severi group of X.*

(1) *Assume that S represents* 0. *Then X admits a G-stable elliptic fibration. In particular, if the rank of S is greater than or equal to* 5, *then X admits a G-stable elliptic fibration.*

(2) *Assume that S contains the even unimodular hyperbolic lattice U of rank* 2. *Then X admits a G-stable Jacobian fibration, that is, a G-stable elliptic fibration having at least one G-stable global sections. In particular, if* $\mathrm{rank}\, S \geq 3 + l(S)$, *where $l(S)$ is the minimal number of generaters of the finite abelian group S^*/S, then X admits a G-stable Jacobian fibration.*

Proof of (1). By the assumption, there exists a primitive point $x \in \partial\mathcal{C} \cap S$. By (1.7)(2), translating x by an appropriate element of Γ, we obtain a primitive point $y \in \mathcal{A}$ such that $(y.y) = 0$. This y gives a G-stable elliptic fibration on X. The last assertion now follows from the famous arithmetical fact due to [Se, Page 43] that every indefinite rational quadratic form of n-variables represents 0 provided that $n \geq 5$. □

Proof of (2). Choose an integral basis $\langle e, c \rangle$ of U such that $(e.e) = 0$, $(e.c) = 1$ and $(c.c) = -2$. Then as in (1), by translating e by an appropriate element of Γ, we may assume that e is the class of a smooth fiber E of an G-stable elliptic fibration $\varphi : X \to \mathbb{P}^1$. Let us also choose an divisor C such that $c = [C]$. Then, by the Riemann-Roch Theorem and the Serre duality, we obtain

$$h^0(\mathcal{O}_X(C)) + h^0(\mathcal{O}_X(-C)) \geq \chi(\mathcal{O}_X(C)) = (c^2/2) + 2 = 1.$$

Since $(C.E) = 1$, we have $h^0(\mathcal{O}_X(-C)) = 0$. Therefore $h^0(\mathcal{O}_X(C)) > 0$. Let $|C| = F + |M|$ be the decomposition of $|C|$ into the fixed component and the movable part. Since $|M|$ is free ([SD, Corollary 3.2]), by the Bertini Theorem, we may assume that M itself is a smooth divisor. Note also that the divisor F is G-stable by $g^*|C| = |C|$ for all $g \in G$ and by the uniqueness of the fixed part. In addition, we have either $(F.E) = 0$

and $(M.E) = 1$ or $(F.E) = 1$ and $(M.E) = 0$, because $1 = (C.E) = (F.E) + (M.E)$ and E is nef. Assume that the first case happens. Then, M must be a global section of φ by the smoothness of M and by $(E.M) = 1$, and would then satisfy $(M^2) = -2$. However, this contradicts the fact that $|M|$ is the movable part. Therefore, $(M.E) = 0$ and $(F.E) = 1$. Write the irreducible decomposition of F as $F = C_0 + \sum m_i C_i$, where $(C_0.E) = 1$ and $(C_i.E) = 0$ for $i \geq 1$. Then C_0 is a section and all other C_i $(i \geq 1)$ are vertical with respect to φ. Thus, C_0 is also G-stable. Therefore, this C_0 gives a desired section. The last statement now follows from the splitting Theorem due to Nikulin [Ni2, Corollary 1.13.5]. □

2. Calabi-Yau threefolds with infinite fundamental group. In this section we study Calabi-Yau threefolds with infinite fundamental group. As is already remarked in Introduction, such threefolds are smooth. Therefore, we may speak of their minimal splitting coverings introduced by Beauville, which, in the threefold case, is summarised as follows:

SUMMARY (2.1)([BE2, SECTION 3]). Let X be a smooth threefold with infinite fundamental group such that $c_1(X) = 0$ in $H^2(X, \mathbb{R})$. Then, by the Bogomolov decomposition Theorem, such an X admits an étale Galois covering either by an abelian threefold or by the product of a K3 surface and an elliptic curve. We also call X *of Type A* in the former case and *of Type K* in the latter case. Among many candidates of such coverings for a given X, there always exists the smallest one which is known to be unique for each X up to isomorphism as a covering space and is obtained by posing one additional condition on the Galois group G that G contains no non-zero translations in the former case and that G contains no elements of the form $(id, \text{non-zero translation})$ in the later case. According to Beauville, we call this smallest covering *the minimal splitting covering* of X. □

We study Calabi-Yau threefolds of Type A in the subsection (2A) and those of Type K in (2K) through their minimal aplitting coverings.

2A. Calabi-Yau threefolds of Type A. The aim of this subsection is to show Theorem (0.1) in Introduction.

Proof of (0.1)(I).

First of all, we introduce the following:

DEFINITION (2.2). A finite group G is called a Calabi-Yau group of Type A (resp. a pre-Calabi-Yau group of Type A), which, throughout this subsection, is abbreviated by a C.Y. group (resp. by a pre-C.Y. group), if there exist an abelian threefold A and a faithful representation $G \hookrightarrow \mathrm{Aut}(A)$ which satisfy the following conditions (1) - (4) (resp. (1) - (3)):

(1) G contains no non-zero translations;
(2) $g^*\omega_A = \omega_A$ for all $g \in G$;
(3) $A^{[G]} = \emptyset$;
(4) $H^0(A, \Omega^1_A)^G = \{0\}$.

In each case we call A a target abelian threefold. □

Then the proof of (I) is equivalent to classifying C.Y. groups together with their actions on target abelian threefolds. The following inductive nature of pre-C.Y. groups turns out to be useful:

LEMMA (2.3). *If G is a pre-C.Y. group, then so are all the subgroups of G. In other words, if G contains a non pre-C.Y. group, then G is not a pre-C.Y. group,* *either.* □

LEMMA (2.4). *Let G be a pre-C.Y. group, A its target abelian threefold and $\rho : G \to GL(H^0(A, \Omega^1_A))$ the natural representation. Then:*

(1) *ρ is injective.*

(2) *$Im(\rho) \subset SL(H^0(A, \Omega^1_A))$.*

(3) *Let g be an element of G and set $n = ord(g)$. Then, $n \in \{1, 2, 3, 4, 6\}$. Moreover, there exists a basis of $H^0(A, \Omega^1_A)$ under which $g^*|H^0(A, \Omega^1_A) = diag(1, \zeta_n^k, \zeta_n^{-k})$ for some k such that $(n, k) = 1$.*

Proof. The assertions (1) and (2) are clear. Let us show the assertion (3). Choose a basis of $H^0(A, \Omega^1_A)$ under which the matrix representaion of $g^*|H^0(A, \Omega^1_A)$ is diagonalised and write $g^* = \text{diag}(a, b, c)$. Then, there exist global coordinates (x, y, z) of A such that the (co-)action of g on A is of the form $g(x, y, z) = (ax+p, by+q, cz+r)$. Suppose $a \neq 1$, $b \neq 1$, and $c \neq 1$. Then, the point $P = (p/(1-a), q/(1-b), r/(1-c)) \in A$ is a fixed point of g, a contradiction. Therefore we may assume $a = 1$, $b = \zeta_n$ and $c = \zeta_n^{-1}$ (by replacing g by an appropriate generater of $\langle g \rangle$ and reordering the basis if necessary). Recall that $H^1(A, \mathbb{C}) = H^0(A, \Omega^1_A) \oplus \overline{H^0(A, \Omega^1_A)}$. Then, $g^*|H^1(A, \mathbb{C}) = \text{diag}(1, \zeta_n, \zeta_n^{-1}, 1, \zeta_n^{-1}, \zeta_n)$. Hence, $\varphi(n) \leq (6-2)/2 = 2$, and therefore, $n \in \{1, 2, 3, 4, 6\}$, because $g^*|H^1(A, \mathbb{C})$ is the scalar extension of $g^*|H^1(A, \mathbb{Z})$. □

First, we determine commutative pre-C.Y. groups.

LEMMA (2.5). *Let G be a commutative pre-C.Y. group and A a target abelian threefold. Then G is isomorphic to either C_n, where $1 \leq n \leq 6$ and $n \neq 5$, or $C_2^{\oplus 2}$. In particular, there exist no commutative pre-C.Y. groups of order ≥ 7. Moreover, if G is a commutative C.Y. group, then G is isomorphic to $C_2^{\oplus 2}$ and the action of G on $H^0(A, \Omega^1_A)$ is same as in (0.1) (I) (1).*

Proof. Set $G = \langle g_1 \rangle \oplus \cdots \oplus \langle g_r \rangle \simeq C_{n_1} \oplus C_{n_2} \oplus \cdots \oplus C_{n_r}$, where $r \geq 0$ and $2 \leq n_1 | n_2 | \cdots | n_r$. If $r \leq 1$, then we get the result by (2.4)(3). Assume that $r \geq 2$. Let i, j be two integers such that $1 \leq i < j \leq r$. Then, using $g_i g_j = g_j g_i$ and (2.4)(3), and replacing g_i and g_j by other generaters of $\langle g_i \rangle$ and $\langle g_j \rangle$ if necessary, we may find a basis of $H^0(A, \Omega^1_A)$ under which $g_i^*|H^0(A, \Omega^1_A)$ and $g_j^*|H^0(A, \Omega^1_A)$ are simulteneously diagonalised as either one of the following forms:

(1) $g_i^* = \text{diag}(1, \zeta_{n_i}, \zeta_{n_i}^{-1})$ and $g_j^* = \text{diag}(1, \zeta_{n_j}, \zeta_{n_j}^{-1})$ or

(2) $g_i^* = \text{diag}(1, \zeta_{n_i}, \zeta_{n_i}^{-1})$ and $g_j^* = \text{diag}(\zeta_{n_j}^{-1}, 1, \zeta_{n_j})$.

In the former case, we have $g_i^* = (g_j^*)^{n_j/n_i}$, whence by (2.4)(1), $g_i = (g_j)^{n_j/n_i}$, a contradiction. In the latter case, we calculate

$g_i^* g_j^* = \text{diag}(\zeta_{n_j}^{-1}, \zeta_{n_i}, \zeta_{n_i}^{-1}\zeta_{n_j})$ and $(g_i^{-1})^* g_j^* = \text{diag}(\zeta_{n_j}^{-1}, \zeta_{n_i}^{-1}, \zeta_{n_i}\zeta_{n_j})$.

Therefore, by(2.4)(3), we get $\zeta_{n_i}^{-1}\zeta_{n_j} = \zeta_{n_i}\zeta_{n_j} = 1$. This implies $\zeta_{n_i} = \zeta_{n_j} = -1$ and $n_i = 2$ for all $i = 1, 2, \cdots, r$ again by (2.4)(3). In particular,

$g_1^* = \text{diag}(1, -1, -1)$ and $g_2^* = \text{diag}(-1, 1, -1)$.

Assume that $r \geq 3$. Then, g_3^* must be of the form $\text{diag}(-1, -1, 1)$. However, then, $g_1^* g_2^* = g_3^*$, whence $g_1 g_2 = g_3$, a contradiction. Therefore, if $r \geq 2$, then $G \simeq C_2^{\oplus 2}$ and there exists a basis of $H^0(A, \Omega^1_A)$ under which $g_1^* = \text{diag}(1, -1, -1)$ and $g_2^* = \text{diag}(-1, 1, -1)$. The remaining assertions follow from this description and (2.4)(3). □

Let us examine non-comutative pre-C. Y. groups. First we estimate their orders. The following two Theorems are extremally useful:

THEOREM (2.6) (WIELANDT, EG.[KT, CHAP.2, THEOREM (2.2)]). *Let G be* *an arbitrary finite group, p a prime number and h a positive integer such that* $p^h||G|$. *Then there exists a subgroup H of G such that* $|H| = p^h$. □

THEOREM (2.7) (BURNSIDE-HALL, EG. [SU, PAGE 90, COROLLARY 2]). *Let K be an arbitrary p-group and H a maximal, normal commutative subgroup of G. Set* $|G| = p^n$ *and* $|H| = p^h$. *Then* $h(h+1) \geq 2n$. □

LEMMA (2.8). *Let G be a pre-C.Y. group. Then,* $|G|$ *is either* 2^n *or* $2^n \cdot 3$, *where n is an integer such that* $0 \leq n \leq 3$, *or more explicitly,* $|G| \in \{1, 2, 3, 4, 6, 8, 12, 24\}$.

Proof. By (2.6) and (2.4)(3), we have $|G| = 2^n \cdot 3^m$, where m and n are some non-negative integers. Assume for a contradiction that $m \geq 2$. Then, by(2.6), G contains a subgroup H of order 3^2. This H must be a pre-C.Y. group by (2.3) and is also commutative. However, this contradicts (2.5). Therefore $m = 0$ or 1. Assume to the contrary, that $n \geq 4$. Then, again by (2.6), G contains a subgroup H of order 2^4. Let K be a maximal normal commutative subgroup of H and set $|K| = 2^k$. By (2.7), this k satisfies $k(k+1) \geq 8$ and hence $k \geq 3$. However, this again contradicts (2.3) and (2.5). Therefore $n \leq 3$. □

Combining this with the classification of non-commutative finite groups of small order (eg. [Bu, Chap.4, Pages 54-55 and Chap.5, Pages 83-89]), we get:

COROLLARY (2.9). *Let G be a pre-C.Y. group. Assume that G is non-commutative and satisfies* $|G| \leq 12$. *Then G is isomorphic to either one of* $D_6 (\simeq S_3)$, D_8, Q_8, D_{12}, Q_{12} *or* A_4. □

Next we show that among the candidates in (2.9), only D_8 can be realised as a C.Y. group of Type A. For proof, let us recall the following:

PROPOSITION (2.10) (EG. [KT, CHAP.8, PAGES 273-275]). *Up to equivalence, the complex linear irreducible representaions of* D_{2n} $(3 \leq n \in \mathbb{Z})$, Q_{4n} $(1 \leq n \in \mathbb{Z})$ *and* A_4 *are given as follows:*

(D_0). $D_{2n} = \langle a, b | a^n = b^2 = 1, bab = a^{-1} \rangle$ *such that* $n \equiv 0 (mod 2)$:

(1) $\rho_{1,0} : a \mapsto 1, b \mapsto 1$; $\rho_{1,1} : a \mapsto 1, b \mapsto -1$; $\rho_{1,2} : a \mapsto -1, b \mapsto 1$; $\rho_{1,3} : a \mapsto -1, b \mapsto -1$;

(2) $\rho_{2,k} : a \mapsto \begin{pmatrix} \zeta_n^k & 0 \\ 0 & \zeta_n^{-k} \end{pmatrix}, b \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, *where k is an integer such that* $1 \leq k \leq n/2 - 1$.

(D_1). $D_{2n} = \langle a, b | a^n = b^2 = 1, bab = a^{-1} \rangle$ *such that* $n \equiv 1 (mod 2)$:

(1) $\rho_{1,0} : a \mapsto 1, b \mapsto 1$; $\rho_{1,1} : a \mapsto 1, b \mapsto -1$;

(2) $\rho_{2,k} : a \mapsto \begin{pmatrix} \zeta_n^k & 0 \\ 0 & \zeta_n^{-k} \end{pmatrix}, b \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, *where k is an integer such that* $1 \leq k \leq (n-1)/2$.

(Q_0). $Q_{4n} = \langle a, b | a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle$ *such that* $n \equiv 0 (mod 2)$:

(1) $\rho_{1,0} : a \mapsto 1, b \mapsto 1$; $\rho_{1,1} : a \mapsto 1, b \mapsto -1$; $\rho_{1,2} : a \mapsto -1, b \mapsto 1$; $\rho_{1,3} : a \mapsto -1, b \mapsto -1$;

(2) $\rho_{2,l} : a \mapsto \begin{pmatrix} \zeta_{2n}^l & 0 \\ 0 & \zeta_{2n}^{-l} \end{pmatrix}, b \mapsto \begin{pmatrix} 0 & \zeta_4 \\ \zeta_4 & 0 \end{pmatrix}^l$, *where l is an integer such that* $1 \leq k \leq n-1$.

(Q_1). $Q_{4n} = \langle a, b | a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle$ *such that* $n \equiv 1 (mod 2)$:

(1) $\rho_{1,0} : a \mapsto 1, b \mapsto 1$; $\rho_{1,1} : a \mapsto 1, b \mapsto -1$; $\rho_{1,2} : a \mapsto -1, b \mapsto \zeta_4$; $\rho_{1,3} : a \mapsto -1, b \mapsto -\zeta_4$;

(2) $\rho_{2,l} : a \mapsto \begin{pmatrix} \zeta_{2n}^l & 0 \\ 0 & \zeta_{2n}^{-l} \end{pmatrix}, b \mapsto \begin{pmatrix} 0 & \zeta_4 \\ \zeta_4 & 0 \end{pmatrix}^l$, *where* l *is an integer such that* $1 \leq k \leq n-1$.

(A_4). $A_4 = \langle a, b \rangle (\subset S_4)$, *where* $a = (123)$ *and* $b = (12)(34)$*:*

(1) $\rho_{1,0} : a \mapsto 1, b \mapsto 1$*;* $\rho_{1,1} : a \mapsto \zeta_3, b \mapsto 1$*;* $\rho_{1,2} : a \mapsto \zeta_3^{-1}, b \mapsto 1$*;*

(2) $\rho_3 : a \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, b \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. □

LEMMA (2.11). *Let G be a pre-C.Y. group and A a target abelian threefold. Assume that G is isomorphic to either D_8, Q_8 or Q_{12}. Then, the irreducible decomposition of the natural representation $\rho : G \to SL(H^0(A, \Omega_A^1))$ is of the form $\rho = \rho_{1,1} \oplus \rho_{2,1}$ if $G \simeq D_8$ and $\rho = \rho_{1,0} \oplus \rho_{2,1}$ if $G \simeq Q_8$ or Q_{12}, where we adopt the same notation as in (2.10). In particular, if G is a C.Y. group, then $G \simeq D_8$ and the representation of G on $H^0(A, \Omega_A^1)$ is equivalent to the one given in (0.1) (I) (2).*

Proof. Note that ρ is not isomorphic to a direct sum of three 1-dimensional representations, because ρ is injective and G is non-commutative. Then, the result follows from the list in (2.10) and (2.4)(1),(2). □

The next two Lemmas are crucial and their proofs involve geometric consideration based on the non-cohomological condition $A^{[G]} = \emptyset$.

LEMMA (2.12). *Neither $D_6(\simeq S_3)$ nor D_{12} is a pre-C.Y. group.*

Proof. The assertion for D_{12} follows from the one for D_6 by (2.3), because D_6 can be embedded in D_{12}. Assume to the contrary, that $D_6 = \langle a, b | a^3 = b^2 = 1, bab = a^{-1} \rangle$ is a pre-C.Y. group. Let A be a target abelian threefold and $\rho : D_6 \to SL(H^0(A, \Omega_A^1))$ the natural representation. Then, by the same argument as in (2.11), we obtain $\rho = \rho_{1,1} \oplus \rho_{2,1}$. In other words, there exists a basis $\langle v_1, v_2, v_3 \rangle$ of $H^0(A, \Omega_A^1)$ under which $a^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3^{-1} \end{pmatrix}$ and $b^* = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Let us fix an origin 0 of A and regard A as a group variety. Set $\alpha := a(0)$ and $\beta := b(0)$. Then, we have $a = t_\alpha \circ a_0$ and $b = t_\beta \circ b_0$, where a_0, b_0 are the Lie part of a, b. Set $\tilde{E} := \mathrm{Ker}(a_0 - id_A : A \to A)$. This is a one-dimensional subgroup scheme of A. Let us take the identity component E of $\tilde{E}$ and consider the quotient homomorphism $\pi : A \to S := A/E$. Notice that E is an elliptic curve, S is an abelian surface and the fibers of π are of the form $E + s$ $(s \in A)$.

CLAIM (2.13). *G descends to an automorphism group of S, that is, there exist automorphisms $\overline{a}$ and $\overline{b}$ of S such that $\overline{a} \circ \pi = \pi \circ a$ and that $\overline{b} \circ \pi = \pi \circ b$.*

Proof of Claim. Let F be a fiber of π and write $F = E + s$. Note that for $x \in A$, we have $a(x + s) = t_\alpha(a_0(x + s)) = t_\alpha(a_0(x) + a_0(s)) = a_0(x) + (a_0(s) + \alpha)$. Since $a_0(E) = E$, this formula implies $a(E + s) = E + (a_0(s) + \alpha)$. Therefore, a descends to the automorphism $\overline{a}$ of S given by $\pi(s) \mapsto \pi(a_0(s) + \alpha)$. Similarly, we have $b(x + s) = b_0(x) + (b_0(s) + \beta)$. Moreover, using $a_0 b_0 = b_0 a_0^{-1}$, we calculate $a_0(b_0(e)) = b_0(a_0^{-1}(e)) = b_0(e)$ for $e \in E$. Therefore $b_0(E) \subset \tilde{E}$. This implies $b_0(E) = E$, because $b_0(0) = 0 \in E$. Hence, $b(E + s) = E + (b_0(s) + \beta)$, and b also descends to the automorphism $\overline{b}$ of S defined by $\pi(s) \mapsto \pi(b_0(s) + \beta)$. □

By construction, there exists a basis $\langle \overline{v}_2, \overline{v}_3 \rangle$ of $H^0(S, \Omega_S^1)$ such that $\pi^*(\overline{v}_2) = v_2$ and $\pi^*(\overline{v}_3) = v_3$. Using this basis, we have $\overline{a}^* = \begin{pmatrix} \zeta_3 & 0 \\ 0 & \zeta_3^{-1} \end{pmatrix}$ and $\overline{b}^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on

$H^0(S,\Omega_S^1)$. This expression, in particular, shows that $S^{\overline{a}}$ consists of isolated points. Therefore, using the canonical graded isomorphism

$$H^*(S,\mathbb{C}) = \oplus_{k=0}^4 \wedge^k (H^0(S,\Omega_S^1) \oplus \overline{H^0(S,\Omega_S^1)})$$

and applying topological Lefschetz fixed point formula, we readily obtain that $|S^{\overline{a}}| = 9$. On the other hand, the equality $ab = ba^{-1}$ gives an equality $\overline{a}\overline{b} = \overline{b}\overline{a}^{-1}$. Therefore, $\overline{b}$ acts on the nine point set $S^{\overline{a}}$ and has a fixed point $\overline{s} \in S^{\overline{a}}$, because $\mathrm{ord}(\overline{b}) = 2$. Put $F := \pi^{-1}(\overline{s})$. Then, $b(F) = F$ and $b^*|H^0(F,\Omega_F^1) = -1$ by the description of a and b. Since F is an elliptic curve, F^b would then be non-empty. However this contradicts $A^{[G]} = \emptyset$. □

LEMMA (2.14). *The group A_4 is not a pre-C.Y. group.*

Proof. Assume to the contrary that $A_4 = \langle a, b\rangle$ is a pre-C.Y. group, where a and b denote the elements defined in (2.10). Let A be a target abelian threefold and express A as $A = \mathbb{C}^3/\Lambda$, where Λ is a discrete sublattice of $\mathbb{C}^3$ of rank 6. (In this proof, we regard A as a three dimensional complex torus rather than an abelian variety.) Then, using the same argument as in (2.11), we readily find global coordinates (z_1, z_2, z_3) of A under which the (co-)actions of a and b on A are written as follows:

$$a(z_1,z_2,z_3) = (z_2,z_3,z_1) + (\alpha_1,\alpha_2,\alpha_3),\ b(z_1,z_2,z_3) = (z_1,-z_2,-z_3) + (\beta_1,\beta_2,\beta_3).$$

By this description, we obtain $a^3(z_1,z_2,z_3) = (z_1,z_2,z_3) + (\alpha,\alpha,\alpha)$, where we put $\alpha := \alpha_1 + \alpha_2 + \alpha_3$. Since $a^3 = id$, we have $(\alpha,\alpha,\alpha) \in \Lambda$. Set $t := t_{(\alpha,\alpha,\alpha)}$. Then, on the one hand, $b^{-1} \circ t \circ b(z_1,z_2,z_3) = (z_1,z_2,z_3) + (\alpha,-\alpha,-\alpha)$ and, on the other hand, $b^{-1} \circ t \circ b = id$, because $(\alpha,\alpha,\alpha) \in \Lambda$. Therefore, $(\alpha,-\alpha,-\alpha) \in \Lambda$ and hence, $(2\alpha,0,0) = (\alpha,\alpha,\alpha) + (\alpha,-\alpha,-\alpha) \in \Lambda$. Consider the point $P := [(0,\alpha_2+\alpha_3,\alpha_1+\alpha_2+2\alpha_3)] \in A$. Then, $a^2(P) = (2\alpha_1+2\alpha_2+2\alpha_3,\alpha_2+2\alpha_3,\alpha_1+\alpha_2+2\alpha_3) = (0,\alpha_2+\alpha_3,\alpha_1+\alpha_2+2\alpha_3) + (2\alpha,0,0) = P$, a contradiction. □

In order to complete the proof of (0.1) (I), it remains to show the following:

LEMMA (2.15). *Let G be a group of order 24. Then G is not a C.Y. group.*

Proof. Assuming to the contrary, that there exists a C.Y. group G of order 24, we shall derive a contradiction. Our argument is based on the following classification of the groups of order 24:

PROPOSITION (2.16) (EG. [BU, CHAPTER 9, PAGES 171-174]). *Let G be an (arbitrary) group of order 24, H_2 a 2-Sylow subgroup of G and $H_3 = \langle c\rangle(\simeq C_3)$ a 3-Sylow subgroup of G. Then, H_2 is isomorphic to either C_8, $C_2 \oplus C_4$, $C_2^{\oplus 3}$, D_8 or Q_8 and G is isomorphic to one of the following 15 groups according to the isomorphism class of the 2-Sylow subgroup H_2:*

(I) $H_2 = \langle a\rangle \simeq C_8$:

(I_1) $G \simeq C_3 \times C_8$;

(I_2) $G = \langle c,a\rangle \simeq C_3 \rtimes C_8$, where $a^{-1}ca = c^{-1}$.

(II) $H_2 = \langle a,b\rangle \simeq C_2 \oplus C_4$:

(II_1) $G \simeq C_3 \times (C_2 \oplus C_4)$;

(II_2) $G = \langle c,a,b\rangle \simeq C_3 \rtimes (C_2 \oplus C_4)$, where $a^{-1}ca = c$ and $b^{-1}cb = c^{-1}$.

(II_3) $G = \langle c,a,b\rangle \simeq C_3 \rtimes (C_2 \oplus C_4)$, where $a^{-1}ca = c^{-1}$ and $b^{-1}cb = c$.

(III) $H_2 = \langle a_1,a_2,a_3\rangle \simeq C_2^{\oplus 3}$:

(III_1) $G \simeq C_3 \times C_2^{\oplus 3}$;

(III_2) $G = \langle a_1,a_2,a_3,c\rangle \simeq C_2^{\oplus 3} \rtimes C_3$, where $c^{-1}a_1c = a_1$, $c^{-1}a_2c = a_3$ and

$c^{-1}a_3c = a_2a_3$;

(III_3) $G = \langle c, a, b\rangle \simeq C_3 \rtimes C_2^{\oplus 3}$, *where* $a_1^{-1}ca_1 = c$, $a_2^{-1}ca_2 = c$ *and* $a_2^{-1}ca_2 = c^{-1}$.

(IV) $H_2 = \langle a, b | a^4 = 1, a^2 = b^2, b^{-1}ab = a^{-1}\rangle \simeq Q_8$:

(IV_1) $G = \langle c\rangle \times \langle a, b\rangle \simeq C_3 \times Q_8$;

(IV_2) $G = \langle a, b, c\rangle \simeq Q_8 \rtimes C_3$, *where* $c^{-1}ac = b$, $c^{-1}bc = ab$;

(IV_3) $G = \langle c, a, b\rangle \simeq C_3 \rtimes Q_8$, *where* $a^{-1}ca = c$, $b^{-1}cb = c^{-1}$.

(V) $H_2 = \langle a, b | a^4 = 1, b^2 = 1, bab = a^{-1}\rangle \simeq D_8$:

(V_1) $G = \langle c\rangle \times \langle a, b\rangle \simeq C_3 \times D_8$;

(V_2) $G = \langle c, a, b\rangle \simeq C_3 \rtimes D_8$, *where* $a^{-1}ca = c$, $b^{-1}cb = c^{-1}$;

(V_3) $G = \langle c, a, b\rangle \simeq C_3 \rtimes D_8$, *where* $a^{-1}ca = c^{-1}$, $b^{-1}cb = c$;

(V_4) $G \simeq S_4$. □

As before, we denote by A a target abelian threefold. In the case where (I), (II), (III), H_2 is a commutative pre-C.Y. group of order 8. However, this contradicts (2.5). In the case where (IV_1), (IV_3), (V_1), and (V_2), the subgroup $\langle a, c\rangle$ of G is isomorphic to C_{12}. However, this again contradicts (2.5). In the case where (V_4), G contains a subgroup which is isomorphic to A_4. However, this contradicts (2.14). Let us consider the case (IV_2). Set $H := \langle a, b\rangle$. Then, by (2.11), the representation ρ_H of H on $H^0(A, \Omega_A^1)$ is decomposed as $\rho_H = \rho_{1,0} \oplus \rho_{2,1}$. Let us write the H-stable subspace of $H^0(A, \Omega_A^1)$ corresponding to $\rho_{1,0}$ by V_1. Then, $a(c(x)) = c(b(x)) = c(x)$ for $x \in V_1$ by $ac = cb$. Hence V_1 is G-stable. Therefore, by the Maschke Theorem, there exists a 2-dimensional G-stable subspace V_2 of $H^0(A, \Omega_A^1)$ such that $H^0(A, \Omega_A^1) = V_1 \oplus V_2$, and under an appropriate basis of V_1 and V_2, the matrix representation of G on $H^0(A, \Omega_A^1)$ is of the form; $a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_4 & 0 \\ 0 & 0 & -\zeta_4 \end{pmatrix}$, $b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \zeta_4 \\ 0 & \zeta_4 & 0 \end{pmatrix}$ and $c = \begin{pmatrix} \alpha & 0 \\ 0 & C \end{pmatrix}$, where α is a complex number and C is a 2×2 matrix. Since c is of order 3, α is either $1, \zeta_3$ or ζ_3^{-1}. If $\alpha = 1$, then $H^0(A, \Omega_A^1)^G = V_1 \neq 0$. However, this contradicts our assumption that G is a C.Y. group. Thus, we may assume that $\alpha = \zeta_3$ by replacing c by c^{-1} if necessary. Note that the eigen values of C are now $\{1, \zeta_3^{-1}\}$. Then the element a^2c does not have an eigen value 1, because $a^2c = \begin{pmatrix} \zeta_3 & 0 \\ 0 & -C \end{pmatrix}$, a contradiction to (2.4)(3). Hence the group in (IV_2) is not a C.Y. group. It remains to elminate the case (V_3). Set $V_1 := H^0(A, \Omega_A^1)^c$. Then, by (2.4)(3), $\dim V_1 = 1$. Using $ca = ac^{-1}$ and $cb = bc$, we see that V_1 is also G-stable. Then, again, by the Maschke Theorem, there exists a two-dimensional G-stable subspace V_2 of $H^0(A, \Omega_A^1)$ such that $H^0(A, \Omega_A^1) = V_1 \oplus V_2$. Note that by (2.11), this decomposition is also the irreducible decomposition of the representation of $\langle a, b\rangle (\simeq D_8)$ and there exist basis of V_1 and V_2 under which we have $a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_4 & 0 \\ 0 & 0 & -\zeta_4 \end{pmatrix}$, $b = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ and $c = \begin{pmatrix} 1 & 0 \\ 0 & C \end{pmatrix}$, where C is a 2×2 matrix. Then bc is of the form $\begin{pmatrix} -1 & 0 \\ 0 & D \end{pmatrix}$. Therefore, $\mathrm{ord}(bc) = 2$ by (2.4)(3). On the other hand, since $bc = cb$, $\mathrm{ord}(b) = 2$ and $\mathrm{ord}(c) = 3$, we have $\mathrm{ord}(bc) = 6$, a contradiction. Hence the group in (V_3) is not a C.Y. group, either. □

This completes the proof (0.1) (I). □

Proof of (0.1) (II).

Let us fix global coordinates (z_1, z_2, z_3) of A as in (0.1)(I). Recall that under the identification $H^*(A,\mathbb{C}) = H^*_{\mathrm{DR}}(A,\mathbb{C})$ we have $H^2(A,\mathbb{C}) = \wedge^2 H^1(A,\mathbb{C})$ and $H^1(A,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{C} = H^0(A,\Omega^1_A)\oplus\overline{H^0(A,\Omega^1_A)}$. Using these identities and the description given in (0.1) (I) (1) and (2), we readily calculate that

$$H^2(A,\mathbb{C})^G = \mathbb{C}\langle dz_1\wedge d\overline{z}_1, dz_2\wedge d\overline{z}_2, dz_3\wedge d\overline{z}_3\rangle \text{ if } G\simeq C_2^{\oplus 2},$$

$$H^2(A,\mathbb{C})^G = \mathbb{C}\langle dz_1\wedge d\overline{z}_1, dz_2\wedge d\overline{z}_2 + dz_3\wedge d\overline{z}_3\rangle \text{ if } G\simeq D_8.$$

In addition, we have $c_3(X) = c_3(A)/|G| = 0$. Now the result follows from these equalities together with $c_3(X) = 2(\rho(X) - h^1(T_X))$ held for a Calabi-Yau threefold. □

Proof of (0.1) (III).

We may construct explicit examples.

(2.17) Example for (I)(1) (Igusa's example [Ig, Page 678], [Ue, Example 16.16]). Let us first take three elliptic curves E_1, E_2 and E_3 and consider the product abelian threefold $A = E_1\times E_2\times E_3$. Let us fix points $\tau_1\in(E_1)_2-\{0\}$, $\tau_2\in(E_2)_2-\{0\}$, $\tau_3\in(E_3)_2-\{0\}$ and define automorphisms a and b of A by,

$a(z_1,z_2,z_3) = (z_1+\tau_1, -z_2, -z_3)$, and $b(z_1,z_2,z_3) = (-z_1, z_2+\tau_2, -z_3+\tau_3)$.

Then, it is easy to check that $\langle a,b\rangle$ is isomorphic to $C_2^{\oplus 2}$ and acts on A as a C. Y. group. Therefore, the quotient threefold $A/\langle a,b\rangle$ gives a desired example. □

(2.18) Example for (I)(2). Let us first take two elliptic curves E_1 and E_2 and consider the product abelian threefold $\tilde{A} = E_1\times E_2\times E_2$. Let us fix points $\tau_1\in(E_1)_4-(E_1)_2$, $\tau_2,\tau_3\in(E_2)_2-\{0\}$ such that $\tau_2\neq\tau_3$ and define automorphisms $\tilde{a}$ and $\tilde{b}$ of $\tilde{A}$ by

$\tilde{a}(z_1,z_2,z_3) = (z_1+\tau_1, -z_3, z_2)$, and $\tilde{b}(z_1,z_2,z_3) = (-z_1, z_2+\tau_2, -z_3+\tau_3)$.

Put $\tilde{G} = \langle\tilde{a},\tilde{b}\rangle$. Then $\tilde{a}^4 = \tilde{b}^2 = id$, $\tilde{a}\tilde{b}\tilde{a}\tilde{b} = t_\tau$, $\tilde{a}t_\tau\tilde{a}^{-1} = t_\tau$ and $\tilde{b}t_\tau\tilde{b}^{-1} = t_\tau$, where $\tau = (0,\tau_2+\tau_3,\tau_2+\tau_3)$. In particular, $\langle t_\tau\rangle(\simeq C_2)$ is a normal subgroup of $\tilde{G}$. Set $A := \tilde{A}/\langle t_\tau\rangle$, $G := \tilde{G}/\langle t_\tau\rangle$, $a := \tilde{a} \bmod \langle t_\tau\rangle$, and $b := \tilde{b} \bmod \langle t_\tau\rangle$. Then $G = \langle a,b\rangle$ and acts on A in the natural manner. It is now easy to check that this pair (A,G), and hence, the quotient threefold A/G, gives a desired example. □

Taking the contraposition of (II) and (III), we obtain the following criterion for the non-triviality of the second Chern class in terms of the Picard number:

COROLLARY (2.19). *Let X be a Calabi-Yau threefold. If $\rho(X) = 1$ or $\rho(X)\geq 4$, then the second Chern class satisfies $c_2(X)\neq 0$. This is also optimal in the same sense as in (0.2).* □

Proof of (0.1) (IV).

Let $A\to X$ be the minimal splitting covering of X and $G = \langle a,b\rangle$ its Galois group as in (0.1)(I). Let us fix an origin $0\in A$ and write $a = t_\alpha\circ a_0$ and $b = t_\beta\circ b_0$. We proceed our argument dividing into the cases (1) and (2) in (0.1) (I).

Case (1) in (0.1) (I).

In this case $G\simeq C_2^{\oplus 2}$. Let us take the identity component S_1 of the kernel $0\in S_1\subset \mathrm{Ker}(a_0+id_A : A\to A)$, and consider the quotient map $\pi_1 : A\to E_1 := A/S_1$. This is an abelian fibration over an elliptic curve E_1 whose fibers are S_1+p, $p\in A$. Then, a similar argument to (2.13) shows that $\pi_1 : A\to E_1$ is G-stable. Therefore, $\pi_1 : A\to E_1$ induces an abelian fibration $\overline{\pi}_1 : X = A/G\to E_1/G = \mathbb{P}^1$. Similarly, the identity components, $0\in S_2\subset\mathrm{Ker}(b_0+id_A : A\to A)$ and $0\in S_3\subset\mathrm{Ker}(a_0b_0+id_A : A\to A)$ induce abelian fibrations $\overline{\pi}_2 : X\to\mathbb{P}^1$ and $\overline{\pi}_3 : X\to\mathbb{P}^1$ respectively. Note that these three abelian fibrations $\overline{\pi}_i$ are mutually different by the shape of a and b. Let us denote general fiber of $\overline{\pi}_i$ by F_i.

CLAIM (2.20). *The classes $[F_1]$, $[F_2]$ and $[F_3]$ give a basis of $Pic(X)_{\mathbb{Q}}$.*

Proof. Note that $F_i \cap F_j \neq \emptyset$ if $i \neq j$. Let H be an ample divisor on X. Then, $(F_i.F_j.H) \neq 0$ for $i \neq j$ and $(F_i^2.H) = (F_j^2.H) = 0$. Therefore, $[F_i]$ and $[F_j]$ $(i \neq j)$ are linearly independent in $\mathrm{Pic}(X)_{\mathbb{Q}}$. Assume for a contradiction that $[F_1]$, $[F_2]$ and $[F_3]$ are linearly dependent in $\mathrm{Pic}(X)_{\mathbb{Q}}$. Then there exist rational numbers c_1 and c_2 such that $F_3 = c_1F_1 + c_2F_2$ in $\mathrm{Pic}(X)_{\mathbb{Q}}$ and satisfies $0 = (F_3^2.H) = 2c_1c_2(F_1.F_2.H)$. Therefore, either $c_1 = 0$ or $c_2 = 0$, a contradiction to the linear independence of $[F_i]$ and $[F_j]$ $(i \neq j)$. Since $\rho(X) = 3$, this gives the assertion. □

CLAIM (2.21). $\overline{\mathcal{A}}(X) = \mathbb{R}_{\geq 0}[F_1] + \mathbb{R}_{\geq 0}[F_2] + \mathbb{R}_{\geq 0}[F_3]$.

Proof. The inclusion $\overline{\mathcal{A}}(X) \supset \mathbb{R}_{\geq 0}[F_1] + \mathbb{R}_{\geq 0}[F_2] + \mathbb{R}_{\geq 0}[F_3]$ is clear. Let us show the other inclusion. Let us choose an ample class $[H] \in \mathcal{A}(X)$ and write $H = c_1F_1 + c_2F_2 + c_3F_3$ in $\mathrm{Pic}(X)_{\mathbb{R}}$. Then $0 < (H.F_1.F_2) = c_3(F_1.F_2.F_3)$. Since $(F_1.F_2.F_3) \geq 0$, we have $c_3 > 0$. Similarly, $c_1 > 0$ and $c_2 > 0$. Therefore $\mathcal{A}(X) \subset \mathbb{R}_{\geq 0}[F_1] + \mathbb{R}_{\geq 0}[F_2] + \mathbb{R}_{\geq 0}[F_3]$. This implies the result. □

Since $h^1(\mathcal{O}_X) = 0$, the set of numerically trivial classes of $\mathrm{Pic}(X)$ is a finite group. Now combining this together with Claim (2.21) and the fact that F_i are semi-ample, we also obtain the semi-ampleness assertion. □

Case (2) in (0.1) (I).

As before, take the identity components $0 \in S_1 \subset \mathrm{Ker}(a_0^2 + id_A : A \to A)$ and $0 \in E_2 \subset \mathrm{Ker}(a_0 - id_A : A \to A)$. Then, S_1 is an abelian surface and E_2 is an elliptic curve by the shape of a_0. Let us consider the quotient maps $\pi_1 : A \to E := A/S_1$ and $\pi_2 : A \to S := A/E_2$. Then as in (2.13), G descends to the actions on the base spaces E and S. Therefore π_1 and π_2 induce fibrations $\overline{\pi}_1 : X \to \mathbb{P}^1 = E/G$ and $\overline{\pi}_2 : X \to \overline{S} = S/G$. Let F_1 be a general fiber of $\overline{\pi}_1$ and F_2 the pull back of an ample divisor on $\overline{S}$. Recall that in this case $\rho(X) = 2$ and $\partial\overline{\mathcal{A}}(X)$ consists of two rays. Then, $[F_1]$ and $[F_2]$ are linearly independent in $\mathrm{Pic}(X)_{\mathbb{Q}}$ and also satisfy $\overline{\mathcal{A}}(X) = \mathbb{R}_{\geq 0}[F_1] + \mathbb{R}_{\geq 0}[F_2]$. This implies the result. □

Remark (2.22). In Case (1), X admits exactly 6 different non-trivial contractions corresponding to the 6 different strata of $\partial\overline{\mathcal{A}}(X) - \{0\}$. Each of the three 1-dimensional strata corresponds to an abelian fibration, as was observed in the proof, and each of the three 2-dimensional strata corresponds to an elliptic fibration. Moreover, we can see that the base space of this elliptic fibration is a normal Enriques surface whose singularity is of Type $8A_1$ by using the shape of a and b. In Case (2), we can also show that the base space $\overline{S}$ of $\overline{\pi}_2 : X \to \overline{S}$ is a normal Enriques surface whose singularity is of Type $2A_3 + 3A_1$. □

2K. Calabi-Yau threefolds of Type K. In this subsection, we study Calabi-Yau threefolds of Type K. The aim of this section is to show the following:

THEOREM (2.23). *Let X be a Calabi-Yau threefold of Type K. Let $S \times E \to X$ be the minimal splitting cover, where S is a K3 surface and E is an elliptic curve, and G its Galois group. Then, X is isomorphic to $(S \times E)/G$ and,*

(I) G is isomorphic to either $C_2^{\oplus n}$ $(1 \leq n \leq 3)$, D_{2n} $(3 \leq n \leq 6)$ or $C_3^{\oplus 2} \rtimes C_2$.

(II) In each case, the Picard number $\rho(X)$, which is again equal to $h^1(T_X)$, is uniquely determined by G and is calculated as in the following table:

G	C_2	$C_2^{\oplus 2}$	$C_2^{\oplus 3}$	D_6	D_8	D_{10}	D_{12}	$C_3^{\oplus 2} \rtimes C_2$
$\rho(X)$	11	7	5	5	4	3	3	3

(III) The cases $G \simeq C_2^{\oplus n}$, where $1 \le n \le 3$, and $G \simeq D_8$ really occur. (Regrettably, it has not been settled yet whether the remaining cases occur or not.)

(IV) There exists a finite rational polyhedral cone Δ such that $\overline{\mathcal{A}}'(X) = Aut(X)\cdot\Delta$, where $\overline{\mathcal{A}}'(X)$ is the rational convex hull of the ample cone in $Pic(X)_{\mathbb{R}}$. Moreover, every nef $\mathbb{Q}$-divisor on X is semi-ample. In particular, each X admits only finitely many different contractions up to isomorphisms.

Proof of (2.23)(I).

As in the subsection (2A), we define:

DEFINITION (2.24). We call a finite group G a Calabi-Yau group of Type K, which again throughout this subsection, is abbriviated by a C.Y. group, if there exist a K3 surface S, an elliptic curve E and a faithful representation $G \hookrightarrow \mathrm{Aut}(S \times E)$ such that the following conditions (1) - (4) hold:

(1) G contains no elements of the form (id_S, non-zero translation of E);
(2) $g^*\omega_{S\times E} = \omega_{S\times E}$ for all $g \in G$;
(3) $(S \times E)^{[G]} = \emptyset$;
(4) $H^0(S \times E, \Omega^1_{S\times E})^G = \{0\}$.

We call $S \times E$ a target threefold. □

As in subsection (2A), our proof is reduced to determine C. Y. group.

We first collect some easy Lemmas, whose varifications are so easy that we may omit them.

LEMMA (2.25) ([BE2, PAGE 8, PROPOSITION]). *Let S be a normal K3 surface and E an elliptic curve. Then $Aut(S \times E) = Aut(S) \times Aut(E)$. In other word, each element g of $Aut(S \times E)$ is of the form (g_S, g_E), where $g_S \in Aut(S)$ and $g_E \in Aut(E)$.* □

LEMMA (2.26). *Let S be a K3 surface and g an element of finite order of $Aut(S)$. Assume that $S^{[\langle g\rangle]} = \emptyset$. Then,*

(1) if $g^*\omega_S = \omega_S$, then $g = id$; and

(2) if $g^*\omega_S \ne \omega_S$, then g is of order 2 and $g^*\omega_S = -\omega_S$. In this case, the quotient surface $S/\langle g\rangle$ is an Enriques surface. □

LEMMA (2.27). *Let S and E be same as in (2.25) and G a finite subgroup of $Aut(S \times E)$ satisfying the same properties as (1), (2) and (3) in (2.24). Let $p_1 : G \to Aut(S)$ and $p_2 : G \to Aut(E)$ be the natural projections under the identification $Aut(S \times E) = Aut(S) \times Aut(E)$ (2.25) and put $G_S := Im(p_1)$ and $G_E := Im(p_2)$. Then, $G_S \simeq G \simeq G_E$ by p_1 and p_2.* □

Let us return back to our study of C.Y. groups.

LEMMA (2.28). *Let G be a C.Y. group and $S \times E$ its target threefold. Then, there exist a normal commutative subgroup H of G and an element ι of order 2 of G which satisfy the following properties (1) -(3):*

(1) $\iota \notin H$ and $G = H \rtimes \langle\iota\rangle$, where the semi-direct product structure is given by $\iota h \iota = h^{-1}$ for all $h \in H$;

(2) $\iota_E = -1_E$ and $H_E = \langle t_a\rangle \oplus \langle t_b\rangle$, under an appropriate origin of E, where a

and b are torsion points such that $ord(a)|ord(b)$. In particular, $H \simeq H_E \simeq C_n \oplus C_m$ for some $1 \leq n|m$; and

(3) *$S^{g_S} = \emptyset$ and $g_S^*\omega_S = -\omega_S$ for all $g \in G - H$, and $h_S^*\omega_S = \omega_S$ for all $h \in H$.*

Proof. Let us consider the natural homomorphism $G_E \to \mathrm{GL}(H^0(E, \Omega_E^1))$ and denote by H_E the kernel of this homomorphism. Then, we have $H^0(S \times E, \Omega_{S\times E}^1)^H \simeq H^0(E, \Omega_E^1)^{H_E} \simeq \mathbb{C}$. In particular, $H_E \neq G_E$. Take an arbitrary element ι_E of $G_E - H_E$ and put $\iota := (\iota_S, \iota_E) \in \mathrm{Aut}(S \times E)$. Then, there exists a complex number $\alpha \neq 1$ such that $\iota_E^* \omega_E = \alpha\omega_E$. This implies $E^{\iota_E} \neq \emptyset$ and $\iota_S\omega_S = \alpha^{-1}\omega_S$. In particular, $S^{\iota_S} = \emptyset$. Therefore ι_S is an involution and $\alpha = -1$ by (2.26). Let us fix one of such an ι. Then, for any $\iota'_E \in G_E - H_E$, we have $\iota'_E \circ \iota_E \in H_E$. Therefore, $G_E = H_E \rtimes \langle \iota_E \rangle$. Fix the origin 0 in E^{ι_E}. Then $\iota_E = -1_E$ and $-1_E \circ t_a \circ -1_E = t_{-a} = t_a^{-1}$. In particular, $\iota_E \circ h_E \circ \iota_E = h_E^{-1}$ if $h \in H$. This gives the semi-direct product structure. Moreover, since H_E consists of translations of E, there exist positive integers n and m such that $H_E \simeq C_n \oplus C_m$ and that $n|m$. □

LEMMA (2.29). *Let (n, m) be same as in (2.28)(2). Then, $(n, m) \in \{(1, k)(1 \leq k \leq 6), (2, 2), (3, 3)\}$.*

Proof. For proof, we make use of the following result due to Nikulin:

THEOREM (2.30) ([NI, PAGE 106, SECTION 5, PARAGRAPH 8]). *Let S be a K3 surface.*

(1) *Let $g \neq id$ be a Gorenstein automorphism of finite order. Then, $ord(g) \leq 8$. Moreover, S^g is a finite set and its cardinality $|S^g|$ is given as in the following table:*

$\mathrm{ord}(g)$	2	3	4	5	6	7	8
$\lvert S^g \rvert$	8	6	4	4	2	3	2

(2) *Let H be a finite, commutative, Gorenstein subgroup of $Aut(S)$. Then H is isomorphic to either one of C_k ($1 \leq k \leq 8$), $C_2^{\oplus l}$ ($2 \leq l \leq 4$), $C_2 \oplus C_4$, $C_2 \oplus C_6$, $C_3^{\oplus 2}$, or $C_4^{\oplus 2}$.* □

Recall that $G_S \simeq G_E$ and $H_S \simeq H_E$ under the isomorphisms in (2.27). Then, H_S is a Gorenstein automorphism group of S and is isomorphic to $H_S \simeq C_n \oplus C_m$. Now it is sufficient to elminate the following cases in (2.30)(2):

$$(n, m) = (1, 7),\ (1, 8),\ (2, 4),\ (2, 6),\ (4, 4).$$

We elminate all cases by a more or less similar method. So, we explain how to do this for the hardest case $(n, m) = (2, 4)$ and leave the other cases to the readers. Assume to the contrary, that $(n, m) = (2, 4)$. Then $H_S = \langle g_S \rangle \oplus \langle h_S \rangle \simeq C_2 \oplus C_4$. Note that $\langle g_S, h_S, \iota_S \rangle / \langle h_S^2 \rangle$ is isomorphic to $C_2^{\oplus 3}$ and acts on $S^{h_S^2} - S^{h_S}$. Note also that by (2.33)(1), we have $|S^{h_S^2} - S^{h_S}| = 4$. Then, this action induces a homomorphism $\varphi : C_2^{\oplus 3} \to S_4$. Since S_4 does not contain a subgroup isomorphic to $C_2^{\oplus 3}$, we have $\mathrm{Ker}(\varphi) \neq \{id\}$. Moreover, $\mathrm{Ker}(\varphi) \subset \langle g_S, h_S \rangle / \langle h_S^2 \rangle$, because $S^f = \emptyset$ for $f \in G_S - H_S$. Let $\alpha \in \langle g_S, h_S \rangle$ be a lift of a non-trivial element of $\mathrm{Ker}(\varphi)$ and take $P \in S^{h_S^2} - S^{h_S}$. Then we have a natural injection $\langle \alpha, h_S^2 \rangle \hookrightarrow \mathrm{SL}(T_{S,P}) = \mathrm{SL}(2, \mathbb{C})$. In addition, using $h_S \notin \mathrm{Ker}(\varphi)$, we see that $\langle \alpha, h_S^2 \rangle$ is isomorphic to either $C_2^{\oplus 2}$ or $C_2 \oplus C_4$. However, this contradicts the following well-known:

THEOREM (2.31) (SEE FOR EXAMPLE [SU, CHAP. III, §6]). *Let G be a finite subgroup of $SL(2, \mathbb{C})$. Then G is isomorphic to either one of C_n, Q_{4n}, T_{24}, O_{48} or I_{120}, where T_{24}, O_{48}, I_{120} are the binary polyhedral groups of indicated orders.* □

This completes the proof of (2.23)(I). □

Proof of (2.23)(II).

The equality $\rho(X) = h^1(T_X)$ follows from the same reason as for Type A. Since $\rho(X) = \dim H^2(S \times E, \mathbb{C})^G = \dim H^2(S, \mathbb{C})^{G_S} + 1$, the proof is reduced to the calculation of $\dim H^2(S, \mathbb{C})^{G_S}$ for each G. Our calculation is based on the topological Lefschetz fixed point formula, (2.28)(3) and (2.30)(1) and is similar for all G. So, we explain how to calculate $\rho(X)$ only for $G \simeq D_8$ and leave the other cases to the reader. Write $G_S = \langle a, b | a^4 = b^2 = 1, bab = a^{-1} \rangle (\simeq D_8)$ and write the irreducible decomposition of $G \curvearrowright H^2(S, \mathbb{C})$ as $H^2(S, \mathbb{C}) = \rho_{1,0}^{\oplus p} \oplus \rho_{1,1}^{\oplus q} \oplus \rho_{1,2}^{\oplus r} \oplus \rho_{1,3}^{\oplus s} \oplus \rho_{2,1}^{\oplus t}$, where we adopt the same notation as in (2.10). Then, by $\dim H^2(S, \mathbb{C}) = 22$, we have $22 = p + q + r + s + 2t$. Using $|S^a| = 4$ (2.30)(1), and applying the topological Lefschetz formula, we get $4 = |S^a| = 2 + \mathrm{tr}(a^* | H^2(S, \mathbb{C})) = 2 + p + q - r - s$, that is, $2 = p + q - r - s$. Similarly, from $|S^{a^2}| = 8$ by (2.30)(1), $|S^b| = 0$ and $|S^{ab}| = 0$ by (2.28)(3), we obtain $6 = p + q + r + s - 2t$, $-2 = p - q + r - s$, $-2 = p - q - r + s$. Now solving this system of equations, we readily find that $p = 3$, $q = 5$, $r = s = 3$ and $t = 4$. Therefore, $\dim H^2(S, \mathbb{C})^{G_S} = p = 3$. Hence, $\rho(X) = 3 + 1 = 4$. □

Proof of (2.23)(III).

We may construct a Calabi-Yau threefold of Type K such that the Galois group G of its minimal splitting cover is isomorphic to $C_2^{\oplus n}$ $(1 \le n \le 3)$ and D_8 respectively.

(2.32) Example for $C_2^{\oplus n}$ $(1 \le n \le 3)$. Let us first take three elliptic curves (with fixed origins) E_1, E_2 and E and denote by $S := \mathrm{Km}(E_1 \times E_2)$ the smooth Kummer surface associated with the product abelian surface $E_1 \times E_2$. Choose elements $a_i, b_i \in (E_i)_2 - \{0\}$ such that $a_i \neq b_i$ for each $i = 1, 2$. Then, the following three automorphisms of $E_1 \times E_2$ descend to those of $\mathrm{Aut}(S)$:

$(z_1, z_2) \mapsto (-z_1 + a_1, -z_2 + a_2)$, $(z_1, z_2) \mapsto (z_1 + b_1, z_2)$, $(z_1, z_2) \mapsto (z_1, z_2 + b_2)$.

We denote them by θ, t_1 and t_2 respectively. Let us choose $P_1, P_2 \in (E)_2 - \{0\}$ such that $P_1 \neq P_2$ and define the three automorphisms of $S \times E$ by $\tilde{\theta} := (\theta, -1_E)$, $\tilde{t_1} := (t_1, t_{P_1})$ and $\tilde{t_2} := (t_2, t_{P_2})$. Set $G_1 := \langle \tilde{\theta} \rangle$, $G_2 := \langle \tilde{\theta}, \tilde{t_1} \rangle$ and $G_3 := \langle \tilde{\theta}, \tilde{t_1}, \tilde{t_2} \rangle$. Then, $G_n \simeq C_2^{\oplus n}$ and act on $S \times E$ as C. Y. groups. Therefore, the quotient threefolds $(S \times E)/G_n$ give desired examples. □

Next we construct an explicit example for D_8. Let us first observe the following:

PROPOSITION (2.33). *Let $D_8 = \langle a, b | a^4 = b^2 = 1, bab = a^{-1} \rangle$ and V the regular representation of D_8 defined by $V = \rho_{2,1} \oplus \rho_{1,0} \oplus \rho_{1,2} \oplus \rho_{1,1} \oplus \rho_{1,3}$. Regard $\mathbb{P}^5 = Proj(\oplus Sym^{\cdot} V)$ and define S to be the complete intersection in $\mathbb{P}^5$ given by*

$$x_0^2 + x_1^2 + x_2 x_3 + x_4 x_5 = x_0 x_1 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = A x_2 x_3 + B x_4 x_5 = 0,$$

where A and B are sufficiently general complex numbers. Then, S is a K3 surface and is stable under the action of D_8 on $\mathbb{P}^5$. Moreover, the induced action on S is faithful and satisfies $a^* \omega_S = \omega_S$, $b^* \omega_S = -\omega_S$ and $S^{a^k b} = \emptyset$ for $k = 0, 1, 2, 3$.

Proof. Smoothness of S follows from the Jacobian criterion. The rest follows from direct calculations. □

(2.34) Example for D_8. Let S be a K3 surface in (2.33) and E be an elliptic curve. Let us consider elements of $\mathrm{Aut}(S \times E)$ defined by $g := (a, t)$ and $\iota := (b, -1_E)$, where a and b are same as in (2.33) and t is a translation automorphism of E of order 4. Then, $\langle g, \iota \rangle$ is isomorphic to D_8 and acts on $S \times E$ as a C. Y. group. Therefore, $(S \times E)/\langle g, \iota \rangle$ gives a desired example. □

Proof of (2.23)(IV).

We may identify $\mathrm{Pic}(X)_{\mathbb{Q}} = (\mathrm{Pic}(S \times E)^G)_{\mathbb{Q}}$ via the quotient map. By using $h^1(\mathcal{O}_S) = 0$, (2.25) and the Kunneth formula, we also obtain $(\mathrm{Pic}(S \times E)^G)_{\mathbb{Q}} = (\mathrm{Pic}(S)^{G_S})_{\mathbb{Q}} \oplus \mathbb{Q}$. Now the result follows from (1.9) and the semi-ampleness of rational nef divisor on K3 surface, because again the torsion part of $\mathrm{Pic}(X)$ is finite by $h^1(\mathcal{O}_X) = 0$. □

As an immediate Corollary of (0.1) and (2.23), we obtain Corollary (0.2) in Introduction and the following:

COROLLARY (2.35). *Let X be a Calabi-Yau threefold. Assume that $\pi_1(X)$ is infinite. Then $\pi_1(X)$ falls into one of the following exact sequences:*

$0 \to \mathbb{Z}^{\oplus 6} \to \pi_1(X) \to G \to 1$, *where G is isomorphic to either $C_2^{\oplus 2}$ or D_8 or,*

$0 \to \mathbb{Z}^{\oplus 2} \to \pi_1(X) \to G \to 1$, *where G is isomorphic to either $C_2^{\oplus n}$ $(1 \le n \le 3)$, D_{2n} $(3 \le n \le 6)$ or $C_3^{\oplus 2} \rtimes C_2$. In particular, $\pi_1(X)$ is always solvable.* □

3. Classification of c_2-contractions of Calabi-Yau threefolds. In this section, we give a generalisation (and a correction) of our earlier work of c_2-contractions of simply connected Calabi-Yau threefolds [Og 1-4]. First we remark the following easy facts on abelian varieties applied in Sections 3 and 4.

LEMMA (3.1). *Every contraction of an abelian variety A is of the form of the exact sequence of abelian varieties: $0 \to F \to A \to \overline{A} \to 0$.*

Proof. Let $f : A \to \overline{A}$ be a contraction and F a smooth fiber. Let us choose an origin 0 of A in F and regard A as a group variety. Since $[t_{-a}(F)] = [F] \in H^*(A, \mathbb{Z})$ for all $a \in A$, we see that $f(t_{-a}(F))$ is also a point by taking an appropriate intersection with the pull back of an ample divisor on $\overline{A}$. Therefore, $0 \in t_{-a}(F) \cap F$ and $t_{-a}(F) = F$ for all $a \in F$. Hence, F is an abelian subvariety of A and induces an isomorphism $\overline{A} \simeq A/F$. This implies the result. □

PROPOSITION (3.2).

(1) *Let (A, h) be a pair of an abelian variety of $\dim A = n$ and its automorphism h such that $h^*|H^0(A, \Omega_A^1) = \zeta_3$, the scalar multiplication by ζ_3. Then (A, g) is isomorphic to the pair $(E_{\zeta_3}^n, diag(\zeta_3, \cdots, \zeta_3))$.*

(2) *Any $diag(\zeta_3, \cdots, \zeta_3)$-satble contraction of $E_{\zeta_3}^n$ is $diag(\zeta_3, \cdots, \zeta_3)$- equivariantly isomorphic to the projection $p_{1,\cdots,m} : E_{\zeta_3}^n \to E_{\zeta_3}^m$ to the first m-factors for some m.*

Proof. The assertion (1) is shown in [CC, Proposition 5.7] and also follows from the argument of [Og3, Section 1]. Let us show the assertion (2). For the sake of simplicity, we put $A = E_{\zeta_3}^n$ and $g = \mathrm{diag}(\zeta_3, \cdots, \zeta_3)$. We regard the universal cover $\mathbb{C}^n$ as a $\mathbb{Z}[\zeta_3]$-module via the scalar action of g. Write $A = \mathbb{C}^n/\Lambda_A$. Then, the lattice Λ_A is a $\mathbb{Z}[\zeta_3]$-submodule of $\mathbb{C}^n$ and coincides with the subset $\mathbb{Z}[\zeta_3]^{\oplus n} \subset \mathbb{C}^n$. Let $\varphi : A \to B$ be a g-stable contraction of A and take the fiber F of φ which contains the origin $0 \in A$. Then, F is an abelian subvariety of A by (3.1). Let us denote by Λ_F the sublattice of Λ_A corresponding to F. Since F is g-stable, Λ_F is also a $\mathbb{Z}[\zeta_3]$-submodule of Λ_A of rank $n - m$, where $m = \dim(B)$. Moreover, Λ_F is primitive in Λ_A, because $\Lambda_F = \Lambda_A \cap V_F$, where V_F is the linear subspace of $\mathbb{C}^n$ corresponding to F. Therefore, there exists an element $h \in \mathrm{GL}(n, \mathbb{Z}[\zeta_3])$, where we regard $\mathrm{GL}(n, \mathbb{Z}[\zeta_3])$ as a subgroup of $\mathrm{Aut}(\Lambda_A)$, such that $h(\Lambda_F) = \{0\} \oplus \mathbb{Z}[\zeta_3]^{\oplus(n-m)}$. Recall that $\mathbb{Z}[\zeta_3]$ is an Euclidean domain and is also the endmorphism ring of E_{ζ_3}. Then, by the elementary divisor theory, and by the fact that g is contained in the center of $\mathrm{Aut}_{\mathrm{Lie}}(A)$, we see that the image of the natural representation $\mathrm{Aut}_{\mathrm{Lie}}(E_{\zeta_3}^n) \to \mathrm{Aut}(\Lambda_A)$ coincides with

$\mathrm{GL}(n, \mathbb{Z}[\zeta_3])$. In particular, h is the image of some Lie automorphism $\tilde{h}$ of A. It is clear that this $\tilde{h}$ gives a desired g-equivariant isomorphism. □

THEOREM (3.3). *Let $\Phi : X \to W$ be a c_2-contraction. Assume that Φ is an isomorphism. Then X is a smooth Calabi-Yau threefold of Type A (0.1).*

Proof. Since $(c_2(X).H) = 0$ for ample divisors on X, we see that $c_2(X) = 0$ as a linear form on $\mathrm{Pic}(X)_{\mathbb{R}}$. □

Next we consider non-trivial birational c_2-contraction.

THEOREM (3.4) (CF. [OG3, MAIN THEOREM]). *Let $\Phi : X \to W$ be a c_2-contraction. Assume that Φ is birational but not an isomorphism. Then, $\Phi : X \to W$ is isomorphic to either one of the following:*

(1) *The unique crepant resolution $\Phi_7 : X_7 \to \overline{X}_7 := A_7/\langle g_7 \rangle$ of $\overline{X}_7$, where (A_7, g_7) is the Klein pair. In this case $\rho(X_7) = 24$, $\rho(\overline{X}_7) = 3$ and $\pi_1(X_7) = \{1\}$.*

(2-0) *The unique crepant resolution $\Phi_3 : X_3 \to \overline{X}_3 := A_3/\langle g_3 \rangle$, where (A_3, g_3) is the Calabi pair. In this case $\rho(X_3) = 36$, $\rho(\overline{X}_3) = 9$ and $\pi_1(X_3) = \{1\}$.*

(2-1) *The unique crepant resolution $\Phi_{3,1} : X_{3,1} \to \overline{X}_{3,1} := A_3/\langle g_3, h \rangle$ of $\overline{X}_{3,1}$, where (A_3, g_3) is the Calabi pair and $\langle g_3, h \rangle \simeq C_3^{\oplus 2}$. Moreover, $\langle h \rangle$ acts on $\overline{X}_3$ freely and the representation of $\langle g_3, h \rangle$ on $H^0(\Omega^1_{A_3})$ is given by:*

$$g_3 \mapsto \begin{pmatrix} \zeta_3 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3 \end{pmatrix} \text{ and } h \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3^2 \end{pmatrix}.$$

In this case $\rho(X_{3,1}) = 12$, $\rho(\overline{X}_{3,1}) = 3$ and $\pi_1(X_3) \simeq C_3$.

(2-2) *The unique crepant resolution $\Phi_{3,2} : X_{3,2} \to \overline{X}_{3,2} := A_3/\langle g_3, h, k \rangle$, where (A_3, g_3) is again the Calabi pair and $\langle g_3, h, k \rangle$ is the unique non-commutative group of order 27 whose elements ($\neq 1$) are all of order 3 (cf. [Bu, Chap.8, Page 158]). Moreover, $\langle g_3, h, k \rangle / \langle g_3 \rangle$ is isomorphic to $C_3^{\oplus 2}$ and acts on $\overline{X}_3$ freely and the representaion of $\langle g_3, h, k \rangle$ on $H^0(\Omega^1_{A_3})$ is given by:*

$$g_3 \mapsto \begin{pmatrix} \zeta_3 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3 \end{pmatrix}, \; h \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3^2 \end{pmatrix}, \text{ and } k \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

In this case $\rho(X_{3,2}) = 4$, $\rho(\overline{X}_{3,2}) = 1$ and $\pi_1(X_{3,2}) \simeq C_3^{\oplus 2}$.

Proof. This is a refinement of [Og3], in which we have already obtained the following properties based on [SBW, Main Theorem] and the PID property of the cyclotomic integer rings $\mathbb{Z}[\zeta_I]$ of relatively small degree $\varphi(I)$ ([MM, Main Theorem]):

LEMMA (3.5) ([OG3, KEY CLAIM PAGE 334, LEMMA (2.1), (2.2), REMARK AFTER THEOREM 3]). *Under the same assumption of (3.4), $\Phi : X \to W$ is isomorphic to either $\Phi_7 : X_7 \to \overline{X}_7$ or the (necessarily unique) crepant resolution of the quotient A_3/G, where G satisfies that:*

(i) *G is a finite Gorenstein automorphism group of A_3 and $\langle g_3 \rangle \triangleleft G$;*
(ii) *G contains no non-trivial translations; and that,*
(iii) *the induced action of $G/\langle g_3 \rangle$ on $A_3/\langle g_3 \rangle$ is fixed point free.* □

Therefore, it is sufficient to determine such G. Write $A := A_3$. Then, by (i) and (ii), the natural homomorphism $G \to \mathrm{SL}(H^0(A, \Omega^1_A))$ is injective. In particular, g_3 is contained in the center of G. Take $h \in G - \langle g_3 \rangle$ and set $d := \mathrm{ord}(h)$. Then, d is either 3, 9 or 27, because h acts on the set A^{g_3} freely and $|A^{g_3}| = 27$. Moreover, at least one eigen value of $h^*|H^0(A, \Omega^1_A)$ must be one, because, otherwise h has an isolated fixed points. But this contradicts $h \in G - \langle g_3 \rangle$ if $d = 3$ and the condition *(iii)* if $d \neq 3$. Now, repeating the same argument as in (2.4)(3), we see that $\varphi(d) \leq 2 = (6-2)/2$. Therefore, $d = 3$ and $|G| = 3^n$ for some positive integer n. Let us consider a maximal normal commutative subgroup H of G and put $|H| = 3^m$. Then $g_3 \in H$ and $m(m+1) \geq 2n$ by (2.7). Assume that $n \geq 4$. Then $m \geq 3$. Therefore, H contains a subgroup L such that $L \simeq C_3^{\oplus 3}$ and $g_3 \in L$. Set $L = \langle g_3 \rangle \oplus \langle h \rangle \oplus \langle k \rangle$. Then, there exists a basis of $H^0(A, \Omega_A^1)$ under which (after replacing k by k^2 if necessarily) the matrix representation of L is of the form: $g_3^* = \mathrm{diag}(\zeta_3, \zeta_3, \zeta_3)$, $h^* = \mathrm{diag}(1, \zeta_3, \zeta_3^2)$, and $k^* = \mathrm{diag}(\zeta_3^2, 1, \zeta_3)$ or $\mathrm{diag}(\zeta_3, \zeta_3^2, 1)$. However, this implies either $g_3^2 h = k$ or $g_3 h = k$, a contradiction. Therefore $n \leq 3$ and $G \not\simeq C_3^{\oplus 3}$. This together with conditions (i) - (iii) readily implies the result. □

Let us next consider c_2-contractions $\Phi : X \to W$ such that $\dim(W) = 2$. It is known by [Nk, Corollary (0.4)] (see also [OP, Corollary (2.6)]) that $(W, 0)$ is klt and that $\mathcal{O}_W(12K_W) \simeq \mathcal{O}_W$. Let us define the global canonical index of W by $I := I(W) := \min\{n \in \mathbb{Z}_{>0} | \mathcal{O}_W(nK_W) \simeq \mathcal{O}_W\}$ and take the global index one cover of W:

$$\pi : T := Spec_{\mathcal{O}_W}(\oplus_{k=0}^{I-1} \mathcal{O}_W(-kK_W)) \to W.$$

Then, by [Kaw1, Pages 608-609] (see also [Zh, §2]), T is either a normal K3 surface or a smooth abelian surface and that π is a cyclic Galois covering étale in codimension one such that the Galois group G, which is isomorphic to C_I, acts faithfully on $H^0(\mathcal{O}_T(K_T)) = \mathbb{C}\omega_T$. Note that $I|12$ and $I \geq 2$. (Indeed, if $I = 1$, then W is either a normal K3 surface or an abelian surface. However, this contradicts $h^2(\mathcal{O}_X) = h^1(\mathcal{O}_X) = 0$.) We call $\Phi : X \to W$ of Type IIA if T is a smooth abelian surface and of Type IIK if T is a normal K3 surface. It has been shown in [Og2] the following:

THEOREM (3.6) ([OG2, MAIN THEOREM]). *Let $\Phi : X \to W$ be a c_2-contraction of Type IIA. Then $\Phi : X \to W$ is isomorphic to one of the relatively minimal models over $E_{\zeta_3}^2/\langle \overline{g}_3 \rangle$ of*

$$p_{12} : X_3 \xrightarrow{\Phi_3} \overline{X}_3 = E_{\zeta_3}^3/\langle g_3 \rangle \xrightarrow{\overline{p}_{12}} E_{\zeta_3}^2/\langle \overline{g}_3 \rangle,$$

where $\Phi_3 : X_3 \to \overline{X}_3$ is the contraction in (3.4)(2-0) and $\overline{p}_{12}$ is the natural projection. In particular, X is smooth and $I(W) = 3$. Moreover, there exist exactly 2^9 such relatively minimal models. □

The next Theorem is a generalisation and also a correction of [Og4]:

THEOREM (3.7) (CF. [OG4, MAIN THEOREM]). *Let $\Phi : X \to W$ be a c_2-contraction of Type IIK. Then $\Phi : X \to W$ is isomorphic to either:*

(1) *A fiber space of a Calabi-Yau threefold of Type A in (0.1)(I)(1) corresponding to a 2-dimensional face of its nef cone. In this case, $I(W) = 2$ and $\rho(W) = 2$. (See also (2.22).)*

(2) *The fiber space of a Calabi-Yau threefold of Type A in (0.1)(I)(2) given by the boundary of its nef cone corresponding to the elliptic fibration. In this case, $I(W) = 2$ and $\rho(W) = 1$. (See also (2.22).)*

(3) *One of the relatively minimal models over $S/\langle \overline{g}_3, \overline{h} \rangle$ of*

$$\kappa_{3,1} : X_{3,1} \xrightarrow{\Phi_{3,1}} \overline{X}_{3,1} = A_3/\langle g_3, h \rangle \xrightarrow{\overline{\kappa}} S/\langle \overline{g}_3, \overline{h} \rangle,$$

where $\Phi_{3,1} : X_{3,1} \to \overline{X}_{3,1}$ is the contraction in (3.4)(2-1) and κ is the morphism induced by the quotient map $A_3 \to S := A_3/E$ given by the identity component E of

Ker$(h_0 - id : A_3 \to A_3)$ *of the Lie part* h_0 *of* h *and* $\overline{g}_3$ *and* $\overline{h}$ *are the automorphisms of* S *induced by* g_3 *and* h. *Moreover, in this case* $I(W) = 3$ *and* $\rho(W) = 2$.

(4) *One of the relatively minimal models over* S/G *of*

$$p_1 : \widetilde{(S \times E)}/G \overset{\nu}{\to} (S \times E)/G \overset{\overline{p}_1}{\to} S/G,$$

where S *is a normal K3 surface,* E *is an elliptic curve,* G *is a finite Gorenstein automorphism group of* $S \times E$ *whose element is of the form* $(g_S, g_E) \in Aut(S) \times Aut(E)$ *and* ν *is a crepant resolution of* $(S \times E)/G$. *Slightly more precisely,* G *is of the form* $G = H \rtimes \langle g \rangle$, *where* H *is a commutative group consisting of elements like* $h = (h_S, t_*)$ *such that* $ord(h_S) = ord(t_*) = ord(h)$ *and* g *is an element of the form* (g_S, ζ_I^{-1}) *such that* $g_S^* \omega_S = \zeta_I \omega_S$, *where* I *is the global canonical index of the base space* W. *Moreover* $I \in \{2, 3, 4, 6\}$.

Remark 1. The cases (1), (2) and (3) do not appear in the classification in [Og4]. Indeed, $\pi_1(X)$ is infinite in the cases (1) and (2) and $\pi_1(X) \simeq C_3$ in the case (3).

Remark 2. In the case (4), the minimal resolution $S' \to S$ induces a birational morphism $(S' \times E)/G \to (S \times E)/G$. Since $(S' \times E)/G$ has only Gorenstein quotient singularities, $(S' \times E)/G$, hence $(S \times E)/G$, admits a crepant resolution [Ro, Main Result]. Moreover, it is well known that any two three-dimensional birational minimal models are connected by a sequence of flop and that three-dimensional flop does not affect singularities of minimal models [Kaw3] and [Kol]. Therefore, X is smooth even in the cases (3) and (4).

Proof. As in [Og4, Section 2], let us consider the following commutative diagram:

$$\begin{array}{ccccc} X & \longleftarrow & X \times_W T & \overset{\nu}{\longleftarrow} & Y \\ \Phi \downarrow & & \varphi' \downarrow & & \downarrow \varphi_Y \\ W & \overset{\pi}{\longleftarrow} & T & = & T \end{array}$$

where ν is a resolution of singularities of $X \times_W T$. Recall from [Og4, Pages 434-435] that $\mathrm{Sing}(X \times_W T)$ is supported only in fibers of $\pi \circ \varphi'$. Let $\{E_j | j \in J\}$ be the set of the two dimensional irreducible components in the fibers of φ_Y. Then $K_Y + \epsilon \sum_{j \in J} E_j$ is klt for small $\epsilon > 0$ [Og4, Claim(2.10)] and we may therefore run the log Minimal Model Program for Y with respect to $K_Y + \epsilon \sum_{j \in J} E_j$. By repeating the same procedure as in [Og4, Proposition (2.2)], we get, as its output, a contraction $\varphi_Z : Z \to T$ such that

(1) $\mathcal{O}_Z(K_Z) \simeq \mathcal{O}_Z$ and $h^1(\mathcal{O}_Z) = 1$ [Og4, Pages 435 - 436 (1) and (6)];

(2) $\mathrm{Sing}(Z)$ is purely one dimensional compound Du Val singularities along some fibers and is also equi-singular along each of such fibers (and therefore, Z adimits at worst Gorenstein quotient singularities) [Og4, Lemma (2.7) and its proof];

(3) the natural birational action of the Galois group $\langle g \rangle \simeq C_I$ on Z is a biregular Gorenstein action and induces a birational map $\alpha : Z/\langle g \rangle \cdots \to X$ over W [Og4, Page 436 (8) and (9)].

From now on, our argument differs to the one in [Og4]. By (2), Z admits a unique crepant resolution $\mu : V \to Z$ (cf. [Re, Main Theorem (II)]). This V is a smooth threefold such that $\mathcal{O}_V(K_V) \simeq \mathcal{O}_V$ and $h^1(\mathcal{O}_V) = 1$. Moreover, the biregular action $\langle g \rangle \to \mathrm{Aut}(Z)$ lifts to the one on V again biregularly, because of the uniqueness of V. Denote by $a_V : V \to A$ the Albanese map of V and set $\varphi_V := \varphi_Z \circ \mu : V \to T$. Then A is an elliptic curve and a_V is an étale fiber bundle over A by [Kaw2, Theorem 8.3]. Moreover, by the uniqueness of the Albanese map, the action $\langle g \rangle \to \mathrm{Aut}(V)$ descends equivariantly to the one on A. These actions make both $a_V : V \to A$ and $\varphi_V : V \to T$ $\langle g \rangle$-stable. Note that $\pi_1(V)$ is infinite by $h^1(\mathcal{O}_V) > 0$. Therefore, by (2.1), V admits the minimal splitting covering $\gamma : U \to V$ from either an abelian threefold or the product of a K3 surface and an elliptic curve. Denote by H the Galois group of γ. Since $\mathcal{O}_V(K_V) \simeq \mathcal{O}_V$, this H is a Gorenstein automorphism group of U. Let us take the Stein factorisations of $a_V \circ \gamma$ and $\varphi_V \circ \gamma$ and denote them by:

$$\begin{array}{ccccc} S & \xleftarrow{\varphi_U} & U & \xrightarrow{a_U} & E \\ {\scriptstyle \gamma_S}\downarrow & & {\scriptstyle \gamma}\downarrow & & \downarrow{\scriptstyle \gamma_E} \\ T & \xleftarrow{\varphi_V} & V & \xrightarrow{a_V} & A. \end{array}$$

Then, by the uniqueness of the minimal splitting covering and by the uniqueness of the Stein factorisation, the action $\langle g \rangle \to \mathrm{Aut}(V)$ again lifts to the one $\langle g \rangle \to \mathrm{Aut}(U)$ equivariantly and also descends to both $\langle g \rangle \to \mathrm{Aut}(S)$ and $\langle g \rangle \to \mathrm{Aut}(E)$ and makes the diagram above $\langle g \rangle$-equivariant. Note that the action of g on U is also Gorensteion, because γ is étale, and that the order of g as an element of $\mathrm{Aut}(U)$ is still I, because $\langle g \rangle \to \mathrm{Aut}(V)$, and hence $\langle g \rangle \to \mathrm{Aut}(U)$, is faithful. Similarly, $H \to \mathrm{Aut}(U)$ descends to $H \to \mathrm{Aut}(S)$ and $H \to \mathrm{Aut}(E)$ to make the above diagram H-equivariant (where we define the actions of H on the varieties in the bottom line as the trivial action). Moreover, by the connectedness of fibers of φ_V and a_V and by the finiteness of γ, γ_S, γ_E, we see that $\varphi_V : V \to T$ and $a_V : V \to A$ are isomorphic to the induced morphisms $\overline{\varphi}_U : U/H \to S/H$ and $\overline{a}_U : U/H \to E/H$ respectively. Set $G := \langle H, g \rangle$ as a subgroup of $\mathrm{Aut}(U)$ and denote by $\rho_{G,S} : G \to \mathrm{Aut}(S)$ and $\rho_{G,E} : G \to \mathrm{Aut}(E)$ the equivariant actions found above. (Note that in apriori there is no reason why $\rho_{G,S}$ and $\rho_{G,E}$ are faithful.)

Let us first treat the case where $U = S' \times E'$, the product of a K3 surface S' and an ellptic curve E'. Recall by (2.25) that each element h of G is of the form $(h_{S'}, h_{E'}) \in \mathrm{Aut}(S') \times \mathrm{Aut}(E')$. In particular, the natural projections $p_1 : U \to S'$ and $p_2 : U \to E'$ are G-stable. We show that this case falls into the case (4) of (3.7).

CLAIM (3.8).

(1) *The contractions $a_U : U \to E$ and $p_2 : U \to E'$ are identically isomorphic. In particular, E is an elliptic curve.*

(2) *The contraction $\varphi_U : U \to S$ factors through p_1, or more precisely, there exists a birational morphism $\tau : S' \to S$ such that $\varphi_U = \tau \circ p_1$.*

In particular, the G-stable morphism $\varphi_U \times a_U : U \to S \times E$ is a crepant birational morphism.

Proof. Since $h^1(\mathcal{O}_S) = 0$, we have $\mathrm{Pic}(S' \times E') = p_1^*\mathrm{Pic}(S') \otimes p_2^*\mathrm{Pic}(E')$. Therefore, any divisor on X is linearly equivalent to a divisor of the form $D := p_1^*C + p_2^*L$, where C and L are divisors on S' and E' respectively. Note also that D is nef if and only if both C and L are nef. The morphism a_U is given by such a nef divisor D that $\nu(X, D) = 1$, because $\dim(E) = 1$. Here and in what follows, we denote by $\nu(X, D)$ the numerical Kodaira dimension. Therefore, $(\nu(S', C), \nu(E', L))$ is either $(0, 1)$ or $(1, 0)$, where C and L are same as above. However, in the latter case we have $\Phi_D = \Phi_C \circ p_1$ and the base space must then be $\mathbb{P}^1$, a contradiction. Hence, $(\nu(S', C), \nu(E', L)) = (0, 1)$ and $a_U = \Phi_D$ factors through p_2. Since both a_U and p_2 have connected fibers, we get the assertion (1). Similarly, by $\dim(S) = 2$, the contraction φ_U is given by a nef divisor D whose numerical Kodaira dimension is two. Therefore, $(\nu(S', C), \nu(E', L))$ is either $(2, 0)$ or $(1, 1)$ in this case. However, in the latter case we would have $S \simeq \mathbb{P}^1 \times E'$, a contradiction. Hence, the first case occurs and therefore $\varphi_U = \Phi_D$ factors through p_1. Again, since both φ_U and p_1 have connected fibers, the induced morphism $S' \to S$ is birational. Now the last assertion follows from (1) and (2). $\square$

By (3.8) and (2.27), the restrictions $\rho_{G,E}|H$ and $\rho_{G,S}|H$ are both injective. Moreover, $\rho_{G,E}(H)$ is a translation subgroup of E, because $h^0(\Omega^1_V) = h^0(\Omega^1_U) = 1$. In particular, $\rho_{G,E}(H)$ is isomorphic to $C_n \oplus C_m$ for some $1 \le n|m$. Since $\langle g\rangle \to \mathrm{Aut}(S)$ is also a lift of the original $C_I \simeq \langle g\rangle \hookrightarrow \mathrm{Aut}(T)$ by the equivariantness, $\rho_{G,S}|\langle g\rangle$ is also injective. On the other hand, since both S and T are normal K3 surfaces, we have $\omega_S = \gamma_S^*\omega_T$. Therefore, by equivariantness and by $g^*\omega_T = \zeta_I\omega_T$, we have $g^*\omega_S = \zeta_I\omega_S$. Recall that $g^*\omega_U = \omega_U$. Then, by (3.8), $g^*\omega_{S\times E} = \omega_{S\times E}$. Therefore, $g^*\omega_E = \zeta_I^{-1}\omega_E$. This implies that $\langle g\rangle \to \mathrm{Aut}(E)$ is also injective and that the image of g is a Lie automorphism of E of order I under appropriate origin of E. Combining these together with the structure of automorphism group of an elliptic curve, we obtain $I \in \{2,3,4,6\}$, a semi-direct decomposition $\rho_{G,E}(G) = \rho_{G,E}(H) \rtimes \rho_{G,E}(\langle g\rangle)$ and the injectivity $\rho_{G,E} : G \to \mathrm{Aut}(E)$. This also implies $G = H \rtimes \langle g\rangle$ and gives an isomorphism $H \simeq C_n \oplus C_m$. Let us take $\tau \in \mathrm{Ker}(\rho_{G,S})$ and write $\tau = h \circ g^i$ ($h \in H$). Since $\tau^*\omega_S = \omega_S$, we have $I|i$ and hence, $\tau = h$. Therefore $\tau = 1$ by the injectivity of $\rho_{G,S}|H$. This shows the injectivity of $\rho_{G,S}$. Now combining $G = H \rtimes \langle g\rangle$ with the construction, we readily see that the induced morphism $\overline{\pi \circ \gamma_S} : S/G \to W$ is an isomorphism and that the original $\Phi : X \to W$ and the induced contraction $\overline{p}_1 : (S\times E)/G \to S/G$ are birationally isomorphic through the isomorphism $\overline{\pi\circ\gamma_S}$ and the composition of birational maps, $\overline{\mu \circ \gamma \circ (\varphi_U \times a_U)^{-1}} : (S \times E)/G \cdots \to Z/\langle g\rangle$ and $\alpha : Z/\langle g\rangle \cdots \to X$. Hence, $\Phi : X \to W$ falls in the Case (4) in (3.7).

Next we treat the case where U is an abelian threefold. Our goal is to show that in this case $\Phi : X \to W$ falls into either one of cases (1), (2), (3) of (3.7).

By (3.1), S is an abelian surface and E is an elliptic curve. Note that a_V does not factor through φ_V, because $h^1(\mathcal{O}_T) = 0$. Then, G-stable map $\varphi_U \times a_U : U \to S \times E$ is surjective, and therefore, is an isogeny. This, in particular, implies $g^*\omega_E = \zeta_I^{-1}\omega_E$, because $g^*\omega_V = \omega_V$ and $g^*\omega_S = \zeta_I\omega_S$. Therefore, $\rho_{G,E}|\langle g\rangle$ is injective. Assume that $\rho_{G,E}|H : H \to \mathrm{Aut}(E)$ is not injective. Then, there exists $1 \ne h \in \mathrm{Ker}(\rho_{G,E}|H)$. Let F be a fiber of a_U. Then F is an abelian surface and h acts on F. Since $h^*\omega_U = \omega_U$, we have $h^*\omega_F = \omega_F$. Moreover, h has no fixed points. Therefore, $h|F$ must be a translation by the classification of automorphisms of abelian surfaces (eg. [Kat]). Thus, $h^*|H^0(U,\Omega^1_U) = id$ and h must be also a translation of U. However, this contradicts the fact that $U \to V$ is the minimal splitting covering. Therefore, $\rho_{G,E}|H$ is injective. Note also that $\rho_{G,E}(H)$ is a translation group of E, because both $A = E/H$ and A are elliptic curves. Now we may repeat the same argument as in the previous case to conclude that $I \in \{2,3,4,6\}$, $\rho_{G,E}$ is injective, $H \simeq C_n \oplus C_m$ for some $1 \le n|m$ and $G = H \rtimes \langle g\rangle$. Repeating the same argument for $\varphi_U : U \to S$, we also get the injectivity of $\rho_{G,S}|H$. Now the injectivity of $\rho_{G,S}$ again follows from the same argument as in the previous case.

CLAIM (3.9). *$H \simeq C_J$, where $J \in \{2,3,4,6\}$.*

Proof. Put $H = \langle h_1\rangle \oplus \langle h_2\rangle (\simeq C_n \oplus C_m)$. Since H is Gorenstein and acts on E as a translation, there exists a basis of $H^0(U,\Omega^1_U)$ such that the matrix representation of H is of the form, $h_1^* = \mathrm{diag}(1,\zeta_n,\zeta_n^{-1})$ and $h_2^* = \mathrm{diag}(1,\zeta_m,\zeta_m^{-1})$ (by changing the generaters if necessary). This implies $n, m \in \{1,2,3,4,6\}$ as in (2.4)(3), and also $(h_1 \circ h_2^{-m/n})^* = id$. Therefore, $h_1 = h_2^{-m/n}$ by the definition of the minimal splitting covering. Thus $n = 1$, and hence, $H \simeq C_m$. Since $h^1(\mathcal{O}_U) = 3$ and $h^1(\mathcal{O}_V) = 1$, we also see that $m \neq 1$. □

From now we argue dividing into cases according to the value J in (3.9). Set $H = \langle h \rangle$. The basic idea of proof is to play "fixed point game". For proof, we also recall here that $\mathrm{ord}(g) = I \in \{2, 3, 4, 6\}$.

CLAIM (3.10). $J \neq 6$.

Proof. Assume to the contrary, that $J = 6$. Take the origin 0 of E in E^g and choose a global coordinate z around 0 of E. Then, we have $g(z) = \zeta_I^{-1} z$, $h(z) = z + p$, and $g^{-1}hg(z) = z + \zeta_I p$, where p is a torsion point of oder 6. In addition, since $\langle h \rangle$ is a normal group of G, we have either $g^{-1}hg = h$ or $g^{-1}hg = h^{-1}$. According to these two cases, we have $\zeta_I p = p$ and $-\zeta_I p = p$ respectively. Note that $E^{\zeta_6} = E^{-\zeta_3} = \{0\}$, $E^{-\zeta_6} = E^{\zeta_3} \subset (E)_3$, $E^{\zeta_4} = E^{-\zeta_4} \subset E^{-1} = (E)_2$. Then, $I = 2$ and $g^{-1}hg = h^{-1}$, because p is a point of order 6. Let us consider the action of G on S. Then $h^*|H^0(S, \Omega_S^1)$ is of the form $\mathrm{diag}(\zeta_6, \zeta_6^{-1})$ by the injectivity of the action and by $h^*\omega_S = \omega_S$. By this description and by the topological Lefschetz fixed point formula, we also see that S^h is a one point set. Set $S^h = \{Q\}$. Then, $g(Q) = Q$ and there exist global coordinates (x_Q, y_Q) around Q such that the (co-)action of g is written as $g(x_Q, y_Q) = (-x_Q, y_Q)$, because g is an involution with $g^*\omega_S = -\omega_S$. Therefore, $(x_Q = 0)$ is a fixed curve of g. However, this contradicts the fact that $W = S/G$. Indeed, the quotient map $S \to W$ has no ramification curves, because $K_S \equiv 0$ and $K_W \equiv 0$. □

CLAIM (3.11). *Assume that $J = 4$. Then $\Phi : X \to W$ falls into the case (2).*

Proof. As in (3.10), we may write the actions of g and h on E as $g(z) = \zeta_I^{-1} z$ and $h(z) = z + p$, where $p \in E$ is a torsion point of order 4. Then, by the same argument as the first part of the proof of (3.10), we see that $I = 2$ and $g^{-1}hg = h^{-1}$. In particular, $\langle h \rangle \rtimes \langle g \rangle \simeq D_8$. Let us consider the action of G on S. Note that $\mathrm{ord}(g^i h) = 2$ and $(gh^i)^*\omega_S = -\omega_S$. Then, by the same argument as in the last part of the proof (3.10), we see that $g^i h$ has no fixed points on S. In particular, $g^i h$ has no fixed points on U. Note also that h has also no fixed points on U, because h is fixed point free on E. Therefore, the action of G on U has no fixed points and U/G is then a Calabi-Yau threefold of Type A (0.1) (I) (2). Moreover, since U/G is birational to X and contains no rational curves, U/G is isomorphic to X. Now the rest of assertion follows from Remark (2.22). □

CLAIM (3.12). *Assume that $J = 3$. Then $\Phi : X \to W$ falls into the case (3).*

Proof. Again same as before, we may write the actions of g and h on E as $g(z) = \zeta_I^{-1} z$ and $h(z) = z + p$, where $p \in (E)_3 - \{0\}$. Then, again by the same argument as the first part of the proof of (3.10), we see that either $I = 2, 6$ and $g^{-1}hg = h^{-1}$, or $I = 3$ and $g^{-1}hg = h$ hold. Note that $h^*|H^0(S, \Omega_S^1) = \mathrm{diag}(\zeta_3, \zeta_3^2)$ under an appropriate basis $\langle v_1, v_2 \rangle$. In particular, $|S^h| = 9$ by the Lefschetz fixed point formula. First consider the case $I = 2$ or 6 and $g^{-1}hg = h^{-1}$. In this case, g^3 has a fixed point in S^h, because $\mathrm{ord}(g^3) = 2$. Then, as in (3.10), g^3 have a fixed curve and gives the same contradiction. Therefore, $I = 3$ and $g^{-1}hg = h$, that is, $G = \langle h \rangle \oplus \langle g \rangle \simeq C_3^{\oplus 2}$ and $g^*|H^0(S, \Omega_S^1) = \mathrm{diag}(\zeta_3^2, \zeta_3^2)$ or $\mathrm{diag}(1, \zeta_3)$ under the same basis $\langle v_1, v_2 \rangle$. By replacing g by gh^2 if necessary, we may assume from the first that $g^*|H^0(S, \Omega_S^1) = \mathrm{diag}(\zeta_3^2, \zeta_3^2)$. Recall that $g^*\omega_U = \omega_U$ and $h^*\omega_U = \omega_U$. Then, g acts on U as a scalar multiplication by ζ_3^2 and $h^*|H^0(U, \Omega_U^1)$ is of the form $\mathrm{diag}(1, \zeta_3, \zeta_3^2)$. In particular, $U \simeq A_3$ by (3.2)(1). It remains to observe that the induced action of h on $A_3/\langle g \rangle$ has no fixed points. Assume to the contrary, that h has a fixed point $\overline{Q}$ on $A_3/\langle g \rangle$. Then there exists a point $Q \in A_3$ such that $g^i h(Q) = Q$ for some $i \in \{0,1,2\}$. Then $\varphi_U(Q) \in S$ is also a fixed point of $g^i h$. Here, we have $i \neq 0$, because h has no fixed points on E and hence on U. However, then the action $g^i h$ on S has a fixed curve passing through $\varphi_U(Q)$, because $g^i h|H^0(S, \Omega_S^1)$ has an eigen value 1 if $i \in \{1,2\}$, a contradiction. Therefore h has no fixed points on $A_3/\langle g \rangle$. □

CLAIM (3.13). *Assume that $J = 2$. Then $\Phi : X \to W$ falls into the case (1).*

Proof. In this case $h^{-1} = h$. So, in apriori, $g^{-1}hg = h$. Again, applying the same argument as the first part of the proof (3.10), we see that either $I = 2$ or 4. Let us consider first the case where $I = 4$. Then, there exists a basis of $H^0(S, \Omega_S^1)$ under which $h^* = \mathrm{diag}(-1,-1)$ and $g^* = \mathrm{diag}(1, \zeta_4)$ or $\mathrm{diag}(-1, -\zeta_4)$. Replacing g by gh in the first case, we may assume from the first that $g^* = \mathrm{diag}(-1, -\zeta_4)$. Then g has a fixed point on S and therefore so does g^2. However, g^2 has then a fixed curve, a contradiction. Therefore $I = 2$ and $G \simeq C_2^{\oplus 2}$. In this case neither g nor gh has a fixed points on S, because they are non-Gorenstein involution on S. In addition, h has no fixed points on U. Therefore G acts freely on U. Now the same argument as the last part of (3.11) gives the result. □

Now we are done. Q.E.D. of (3.7).

4. Finiteness of c_2-contractions of a Calabi-Yau threefold. In this section, we prove Theorem (0.4) in Introduction. For proof, it is convenient to introduce the notion of the maximal c_2-contraction:

LEMMA-DEFINITION (4.1). *There exists a c_2-contraction $\varphi_0 : X \to W_0$ such that every c_2-contraction $\Phi : X \to W$ of X factors through φ_0, that is, there is a morphism $\mu : W_0 \to W$ such that $\Phi = \mu \circ \varphi_0$. Moreover, such $\varphi_0 : X \to W_0$ is unique up to identical isomorphism. We call this $\varphi_0 : X \to W_0$ the maximal c_2-contraction of X.*

Proof. Let us choose a c_2-contraction $\varphi_0 : X \to W_0$ such that $\rho(W_0)$ is maximal among all c_2-contractions of X. We show that this φ_0 is the desired one. Let $\Phi : X \to W$ be any c_2-contraction. Take divisors D_0 and D such that $\varphi_0 = \Phi_{D_0}$ and $\Phi = \Phi_D$. Let us consider the contraction given by $m(D_0 + D)$ for suitably large $m > 0$ and denote this contraction by $\Phi' := \Phi_{m(D_0+D)} : X \to W'$. Since $(c_2(X).D_0 + D) = (c_2(X).D_0) + (c_2(X).D) = 0$, we see that Φ' is also a c_2-contraction. Moreover, by the construction, this Φ' factors through both Φ and φ_0, that is, there exist morphisms $p_0 : W' \to W_0$ and $p : W' \to W$ such that $\Phi = p \circ \Phi'$ and $\varphi_0 = p_0 \circ \Phi'$. Note also that W_0 and W' are both $\mathbb{Q}$-factorial by the classification of c_2-contractions. Indeed, they have at most quotient singularities (See Section 3). Hence, by the maximality of $\rho(W_0)$, the morphism p_0 must be an isomorphism. Then, $\mu := p \circ p_0^{-1} : W_0 \to W$ gives a desired factorisation. This argument also implies the last assertion. □

We proceed our proof of (0.4) dividing into cases according to the structure of the maximal c_2-contraction. In apriori, there are six possible cases:

Case I. φ_0 is an isomorphism (3.3) (= (0.1));

Case B. φ_0 is a birational contraction but not isomorphism (3.4);

Case K. φ_0 is of Type IIK (3.7);

Case A. φ_0 is of Type IIA (3.6);

Case P. $\dim(W_0) = 1$; and

Case T. $\dim(W_0) = 0$.

In Case I, the result follows from (0.1)(IV). In Case P, φ_0 is an abelian fibration over $\mathbb{P}^1$ and this is the only (non-trivial) c_2-contraction of X. Case T is nothing but the case where X admits no (non-trivial) c_2-contractions. It remains to consider Cases B, K, A.

Proof of (0.4) in Case B. In this case $\varphi_0 : X \to W_0$ is isomorphic to one of the contractions given in (3.4). First we treat the case where $\varphi_0 : X \to W_0$ is isomorphic to $\Phi_7 : X_7 \to \overline{X}_7$ (3.4)(1). Recall that Φ_7 is the unique crepant resolution of $\overline{X}_7$. Therefore, it is sufficient to show the following:

LEMMA (4.2). *$\overline{X}_7$ admits no non-trivial contractions.*

Proof. Let $f : \overline{X}_7 \to W$ be a contraction and consider the Stein factorisation of the map $f \circ q$, where $q : A_7 \to \overline{X}_7$ the quotient map:

$$\begin{array}{ccc} A_7 & \xrightarrow{f'} & V \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q'} \\ \overline{X}_7 & \xrightarrow[f]{} & W. \end{array}$$

Then V is an abelian variety (3.1). Moreover, by the uniqueness of the Stein factorisation, $\langle g_7 \rangle$ acts on V equivariantly with respect to f' and the induced map $\overline{f}' : \overline{X}_7 \to V/\langle g_7 \rangle$ coincides with $f : \overline{X}_7 \to W$. In addition, the action g_7 on V is of order 7, has only isolated fixed points, and satisfies $(g_7)^* \omega_V \neq \omega_V$ if $\dim V < 3$, because $\overline{f}' \circ \Phi_7$ is also a c_2-contraction. However, since $\varphi(7) = 6$, there are no elliptic curves and no abelian surfaces which admit such an automorphism. Therefore V is an abelian threefold and f' must be an isomorphism. □

Next consider the case where $\varphi_0 : X \to W_0$ is isomorphic to $\Phi_3 : X_3 \to \overline{X}_3$ in (3.4)(2-0). Since Φ_3 is the unique crepant resolution, the same argument as the first half part of (4.2) reduces our proof to the finiteness of g_3-stable contractions of A_3 up to g_3-equivariant isomorphisms. Therefore, the result follows from (3.2)(2).

Let us consider the case where $\varphi_0 : X \to W_0$ is isomorphic to the unique crepant resolution $\Phi_{3,1} : X_{3,1} \to \overline{X}_{3,1}$ (3.4)(2-1). Again as before, it is sufficient to show the finiteness of contractions of $\overline{X}_{3,1}$. In this case, we can say more:

LEMMA (4.3). *The nef cone $\overline{\mathcal{A}}(\overline{X}_{3,1})$ is a rational simplicial cone and $\overline{X}_{3,1}$ admits exactly 6 different non-trivial contractions.*

Proof. Our proof is quite similar to the one for (0.1) (IV) and we give just a sketch. Let us consider the elliptic curves E_i $(1 \leq i \leq 3)$ given as the identity components of the kernel of the endmorphisms, $\mathrm{Ker}(h_0 \circ g_3^{3-i} - id : A_3 \to A_3)$. Let $q_i : A_3 \to S_i := A_3/E_i$ be the quotient map. Then the action of $\langle h, g_3 \rangle$ descends equivariantly to the one on S_i, which we denote by $\langle \overline{h}, \overline{g}_3 \rangle$. Then again taking the quotient of S_i by the identity component K_i of the kernel of $\overline{h} \circ (\overline{g}_3)^{3-i} - id$, we finally obtain three different abelian fibrations $A_3 \to B_i := S_i/K_i$. Moreover, these fibrations are $\langle h, g_3 \rangle$-stable and therefore induce three differnet abelian fibrations $\varphi_i : \overline{X}_{3,1} \to \mathbb{P}^1$. Since $\rho(\overline{X}_{3,1}) = 3$, the rest of the proof is same as in (0.1)(IV). □

Finally, we consider the case where $\varphi_0 : X \to W_0$ is isomorphic to the crepant resolution $\Phi_{3,2} : X_{3,2} \to \overline{X}_{3,2}$ (3.4)(2-2). However, in this case $\overline{X}_{3,2}$ admits no non-trivial contractions, because $\rho(\overline{X}_{3,2}) = 1$, and we are done. □

Proof of (0.4) in Case K. By the case assumption, $\varphi_0 : X \to W_0$ is isomorphic to one of the contractions in (3.7). In the first three cases of (3.7), we have $\rho(W_0) \leq 2$ so that W_0 admits at most two contractions and we are done. Let us consider the last case in (3.7). This is the essential case. In this case, $\varphi_0 : X \to W_0$ is isomorphic to one of the relatively minimal models of $p_1 : (S \times E)/G \to S/G$. We denote this model by $f_0 : Y \to B_0 := S/G$ and fix a birational map $\rho_0 : Y \cdots \to (S \times E)/G$ over B_0. Let $y_i : Y_i \to B_0$ $(i = 1, 2, ..., I)$ be the complete representatives of the set of the relatively minimal models of $f_0 : Y \to B_0$ modulo isomorphism over B_0. These are finite in number by virtue of the result of Kawamata [Kaw5, Theorem 3.6]. Indeed, since Y itself is a Calabi-Yau threefold, $f_0 : Y \to B_0$ is, in particular, a Calabi-Yau fiber space. Let $s_j : S \to S_j$ $(j = 1, 2, ..., J)$ be the complete representatives of the set of G-stable contractions of S modulo G-equivariant isomorphism. These are also finite in number as was shown in (1.10) (= (0.5)). We denote by $b_j : B_0 \to B_j := S_j/G$ the contraction induced by s_j. In order to complete the proof, it is enough to show the following:

LEMMA (4.4). *Every c_2-contraction of Y is isomorphic to either one of $b_j \circ y_i : Y_i \to B_j$.*

Proof. Let $f : Y \to B$ be a c_2-contraction and $b : B_0 \to B$ the factorisation of f and write $f = b \circ f_0$. Let us denote by $q : S \to B_0 = S/G$ the quotient map and consider the Stein factorisation of the map $b \circ q$:

$$\begin{array}{ccc} S & \xrightarrow{b'} & C \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q'} \\ B_0 & \xrightarrow[b]{} & B. \end{array}$$

As in (4.2), by the uniqueness of the Stein factorisation, G acts on C equivariantly and makes $b' : S \to C$ a G-stable contraction. Moreover, the induced morphism $\overline{b'} : S/G \to C/G$ coincides with $b : B_0 \to B$. Choose $j \in \{1, 2, ..., J\}$ such that $b' : S \to C$ is G-equivariantly isomorphic to $s_j : S \to S_j$ and denote this isomorphism by:

$$\begin{array}{ccc} S & \xrightarrow{\sigma} & S \\ {\scriptstyle b'}\downarrow & & \downarrow{\scriptstyle s_j} \\ C & \xrightarrow[\sigma_C]{} & S_j. \end{array}$$

This pair (σ, σ_C) descends to a pair of isomorphisms, $\sigma_{B_0} : B_0 \to B_0$ and $\sigma_B : B \to B_j$ which give an isomorphism between $b : B_0 \to B$ and $b_j : B_0 \to B_j$:

$$\begin{array}{ccc} B_0 & \xrightarrow{\sigma_{B_0}} & B_0 \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle b_j} \\ B & \xrightarrow[\sigma_B]{} & B_j. \end{array}$$

Let us consider an automorphism $\tilde{\sigma} = (\sigma, id)$ of $S \times E$. By the description of the elements of G (3.7)(4), the pair $(\tilde{\sigma}, \sigma)$ gives a G-equivariant isomorphism of p_1 :

$S \times E \to S$:

$$\begin{array}{ccc} S \times E & \xrightarrow{\tilde{\sigma}} & S \times E \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_1} \\ S & \xrightarrow[\sigma]{} & S. \end{array}$$

Therefore $\tilde{\sigma}$ induces an isomorphism, $\tau : (S \times E)/G \to (S \times E)/G$ such that the pair (τ, σ_{B_0}) gives an isomorphism on $p_1 : (S \times E)/G \to B_0$:

$$\begin{array}{ccc} (S \times E)/G & \xrightarrow{\tau} & (S \times E)/G \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_1} \\ B_0 & \xrightarrow[\sigma_{B_0}]{} & B_0. \end{array}$$

Note that $\sigma_{B_0} \circ f_0 : Y \to B_0$ is a relatively minimal model of $f_0 : Y \to B_0$ via the birational map $(\rho_0)^{-1} \circ \tau \circ \rho_0$. Then, there exists $i \in \{1, 2, ..., I\}$ such that $\sigma_{B_0} \circ f_0 : Y \to B_0$ and $y_i : Y_i \to B_0$ are isomorphic over B_0. Let us choose one of such isomorphisms and denote it by $\tau_i : Y \to Y_i$. Then, the pair (τ_i, σ_{B_0}) gives an isomorphism between $f_0 : Y \to B_0$ and $y_i : Y_i \to B_0$:

$$\begin{array}{ccc} Y & \xrightarrow{\tau_i} & Y_i \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle y_i} \\ B_0 & \xrightarrow[\sigma_{B_0}]{} & B_0. \end{array}$$

Composing (τ_i, σ_{B_0}) with the pair (σ_{B_0}, σ_B), we get an isomorphism between $f = b \circ f_0 : Y \to B$ and $b_j \circ y_i : Y_i \to B_j$. □

This completes the proof in Case K. □

Proof of (0.4) in Case A. Finally we consider the case where $\varphi_0 : X \to W_0$ is of the form $f_0 : Y \to B_0 := E^2_{\zeta_3}/\langle \mathrm{diag}(\zeta_3, \zeta_3)\rangle$ described in (3.6). Recall by (3.6) that the number of the relatively minimal models of $f_0 : Y \to B_0$ is just 2^9. We denote them by $y_i : Y_i \to B_0$ $(i = 1, 2, ..., 2^9)$. Let $p_1 : B_0 \to E_{\zeta_3}/\langle\zeta_3\rangle = \mathbb{P}^1$ be the natural projection to the first factor. In order to conclude the result, it is sufficient to show the following:

LEMMA (4.5). *Every c_2-contraction of Y is isomorphic to one of $y_i : Y_i \to B_0$ and $p_1 \circ y_i : Y_i \to \mathbb{P}^1$.*

Proof. By (3.2)(2), we see that each $\langle \mathrm{diag}(\zeta_3, \zeta_3)\rangle$-stable (non-trivial) contraction of $E^2_{\zeta_3}$ is $\langle \mathrm{diag}(\zeta_3, \zeta_3)\rangle$-equivariantly isomorphic to either $id : E^2_{\zeta_3} \to E^2_{\zeta_3}$ or $p_1 : E^2_{\zeta_3} \to E_{\zeta_3}$. Now we may repeat the same argument as in (4.4) to obtain the result. □

This completes the proof of (0.4). □

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