EXISTENCE AND UNIQUENESS OF STANDARD BUBBLE CLUSTERS OF GIVEN VOLUMES IN $\mathbb{R}^N$ [*]

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Abstract. We prove the existence of a unique standard k-bubble in $\mathbb{R}^n$, $k \leq n+1$, that encloses fixed volumes.

1. Introduction. A conjecture of much recent interest [SM, Prob. 2] says that a standard soap bubble cluster consisting of spherical surfaces meeting in threes at 120 degrees provides the least-perimeter way to enclose and separate k regions of prescribed volume in $\mathbb{R}^n$ for $k \leq n + 1$. Here we provide proofs of the existence and uniqueness of such a standard soap bubble.
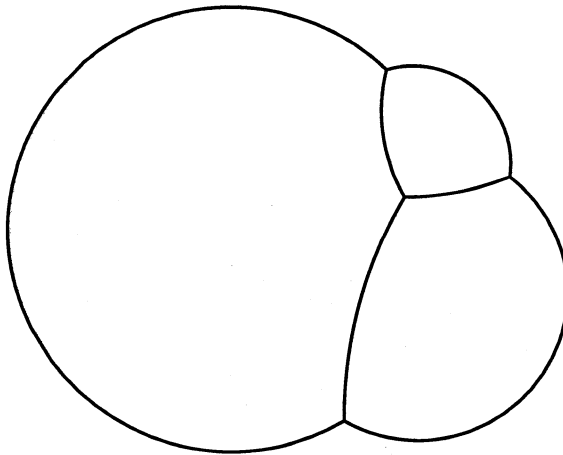

FIG. 1. *Standard 3-bubble in* $\mathbb{R}^2$

Progress on the conjecture. As for the conjecture itself, the case $k = 1$ is the classical theorem that the round sphere provides the least-perimeter way to enclose given volumes in $\mathbb{R}^n$. The case $k = 2$ is the Double Bubble Conjecture, proved in $\mathbb{R}^2$ in 1990 by a group of undergraduates [F2] and in $\mathbb{R}^3$ in 1995 for the case of equal volumes with the help of a computer by Hass, Hutchings, and Schlafly [HHS]. See [M1].

The case $k = 3$, the Triple Bubble Conjecture, was proved for connected regions in $\mathbb{R}^2$ in 1992 by another group of undergraduates (including Hutchings) [C].

General results on the existence and regularity of minimizers are due to Almgren and Taylor ([A], [T], [AT]), Morgan [M2], and Hutchings [H, 2.6, 2.9, 5.1].

Earlier results on the existence and uniqueness of the standard bubble.
Existence and uniqueness of the standard bubble has appeared for double bubbles in
$\mathbb{R}^2$ and $\mathbb{R}^3$ [F2, Thm. 2.3], [F1, Thm. 3.2], and for the equal-area triple bubble in
$\mathbb{R}^2$ [F1, Thm. 4.6]. Also Hutchings [H, p. 286], sketches a simple proof for double
bubbles in $\mathbb{R}^n$ due to Morgan.

In 1995 K. Brakke and also Hutchings proved the existence of standard k-bubbles
in $\mathbb{R}^n$ ($k \leq n + 1$). They used an idea of J. Sullivan's: obtaining a standard bubble
as a stereographic projection of a regular simplex in the sphere.

This paper. Theorem 2.1 provides Brakke's beautiful proof of the existence of
standard bubbles for given volumes. To prove uniqueness, it is more convenient to
use radii than volumes as parameters. Proposition 3.2 proves existence (and unique-
ness) for prescribed radii, using induction and an inversion which maps the spherical
surfaces to hyperplanes, making it easy to insert an additional bubble at a vertex.
Finally Theorem 4.2 deduces uniqueness for given volumes, using Lemma 4.1, which
says that as one radius increases, the other volumes decrease.

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and for insisting that I write down these ideas. Also, to the referee that helped me a
lot updating the status of the matter and suggested several improvements.

2. Existence of standard bubbles with prescribed volumes. For defining
the standard bubble with more than one region we need to fix the angle at which two
of its faces meet; thus, the dimension of these faces must be at least 1 and the bubble
should live in $\mathbb{R}^n$ with $n \geq 2$. This will be assumed throughout the paper.

A standard $(n + 1)$-bubble in $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ that is homeomorphic to the
barycentric subdivision of an n-simplex; each of its faces (I mean always the $(n - 1)$-
dimensional ones) must be part of a sphere or a hyperplane and where two faces meet,
they do at 120° angles. The bubble divides $\mathbb{R}^n$ into $n + 2$ regions. An *inner interface*
separates two of the compact regions, and an *outer face* separates a compact region
from the non compact one. We call *center of the bubble* the center of the barycentric
subdivision, that is the point where all of the inner interfaces meet. A standard k-
bubble in $\mathbb{R}^n$, $k \leq n$, is a subset homeomorphic to the result of removing an outer face
from a $(k + 1)$-bubble, and which satisfies the same conditions on faces and incidence
angles as the $(n + 1)$-bubble. We call outer or inner radii to the radii of the outer or
inner faces.

Consider in $\mathbb{R}^{n+1}$ a regular $(n + 1)$-simplex, X, and let S be the sphere that
passes through its vertices. From the center of S, project X on S, and let Y be
that projection. By symmetry, where three of the projected $(n - 1)$-dimensional faces
meet, they must do at equal angles, that is at 120° angles. Hence, the stereographic
projection of Y upon $\mathbb{R}^n$ gives a standard $(n + 1)$-bubble. This is Sullivan's idea.

THEOREM 2.1. *(Brakke[B]) Let $1 \leq k \leq n + 1$ and let $V_1, \ldots, V_k > 0$. Then there
is a standard k-bubble in $\mathbb{R}^n$ that encloses and separates compact regions of volumes
$V_1, \ldots, V_k$.*

Proof. Look at a standard k-bubble in $\mathbb{R}^n$ as the stereographic projection from
S^n of the appropriate net dividing S^n into $k + 1$ regions, with the projection point
in region $k + 1$ that we denote D (if $k \leq n$, intersect D with the appropriate S^{k-1}).

Note that D is a simplex. Consider the map $f : D \to T$, where T is the simplex in $\mathbb{R}^k$ that is the hyperplane $x_1 + \cdots + x_k = 1$ in the positive orthant, defined by

$$f(P) = \frac{\left(\frac{1}{v_1}, \ldots, \frac{1}{v_k} \right)}{\frac{1}{v_1} + \cdots + \frac{1}{v_k}},$$

where v_i is the volume of region i as projected from the point P. Then each m-dimensional face of simplex D maps into a corresponding face of T, and by induction is a degree 1 map onto. □

3. Existence of the standard bubble with prescribed outer radii. The standard 1-bubble in $\mathbb{R}^n$ is a sphere. The standard 2-bubble in $\mathbb{R}^n$ is built from two spheres that meet at $120°$ angles and define the outer faces of the bubble, and a third sphere or hyperplane that meets them also at $120°$ angles, as shown in Figure 2. The region enclosed by the outer faces is divided by the third hypersurface and in this manner we have the inner interface.
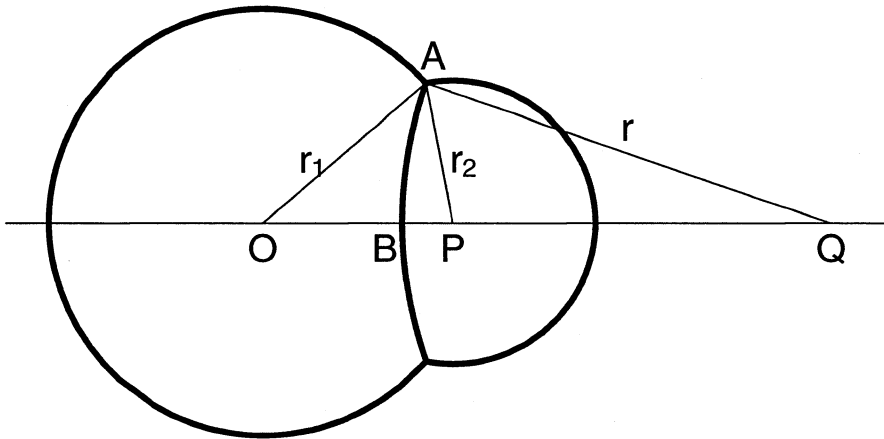

FIG. 2.

The following result will be useful.

PROPOSITION 3.1. *Let $\mathcal{O}, \mathcal{P}$ be two spheres in $\mathbb{R}^n$ with respective radii $r_1, r_2 > 0$. They meet at $120°$ angles iff their centers are at a distance $\sqrt{r_1^2 + r_2^2 - r_1 r_2}$. There is a unique standard 2-bubble in $\mathbb{R}^n$ whose outer faces have radii r_1 and r_2.*

Proof. Figure 2 represents the intersection of the spheres with any 2-plane that passes through their centers. We see that the circles centered in O and P with respective radii r_1 and r_2 meet at $120°$ angles iff the angle $\widehat{OAP}$ is $60°$. That is iff $\cos(\widehat{OAP}) = 1/2$, that is iff $\overline{OP}^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos(\widehat{OAP}) = r_1^2 + r_2^2 - r_1 r_2$, as claimed. Assume now that, for instance, $r_1 > r_2$, and that both circles meet at $120°$ angles. It is easy to see then that a circle that passes through A and A' and therefore is centered (say, at Q) in the line OP meets the other two circles at $120°$ angles iff its radius is $r = r_1 r_2 / (r_1 - r_2)$ and P belongs to the segment OQ. □

PROPOSITION 3.2. *Let $1 \leq k \leq n + 1$ and $r_1, \ldots, r_k > 0$. Then there is a unique (up to isometries of $\mathbb{R}^n$) standard k-bubble in $\mathbb{R}^n$ whose outer faces have radii $r_1, \ldots, r_k$.*

Proof. The uniqueness is obvious since, by 3.1, the distances among the centers of the outer faces are fixed so that these centers form a unique $(k-1)$-simplex, up to isometries; then, the positions of the centers of the inner interfaces, and their radii, are also fixed. However, for $k > 2$ it is not obvious that we can always form a $(k-1)$-simplex with distances among vertices given by $\sqrt{r_i^2 + r_j^2 - r_i r_j}$.

First, assume that $2 < k < n + 1$, that P is a $(k-1)$-plane in $\mathbb{R}^n$ and that B_k is a standard k-bubble in P; the centers of the inner and outer faces of B_k are at the appropriate distances according to their radii, so that we can define a standard k-bubble in $\mathbb{R}^n$ by 'blowing' B_k, that is the bubble built from the $(n-1)$-dimensional spheres with same centers and radii as the faces of B_k.

Now, we proceed by induction. Let H be any hyperplane of $\mathbb{R}^n$, and $\tilde{B}_n$ a standard n-bubble in H with outer radii $r_1, \ldots, r_n$. Let B_n be the n-bubble in $\mathbb{R}^n$ obtained as before by blowing $\tilde{B}_n$. The intersection of all of the inner interfaces of $\tilde{B}_n$ is a point, so that the intersection of all of the inner interfaces of B_n is an arc of circle (or line segment). Any of the ends, A and A', of that arc or line segment belongs to all
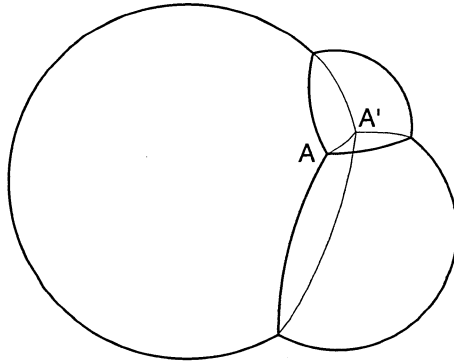

FIG. 3. *Standard 3-bubble in* $\mathbb{R}^2$ *blown to* $\mathbb{R}^3$

of the faces, inner and outer (see figure 3). Thus, a central inversion Φ with center A, for instance the one that leaves the other end, A', fixed, changes all faces of B_n into parts of hyperplanes that meet at A' at equal angles. There are $n + 1$ half-lines starting from A', lying as the ones connecting the center of a regular n-simplex with its vertices, such that each subset of $n - 1$ of them determines the image of a face of B_n under Φ. Let us choose a point on each of these half-lines, all equidistant from A'. With each of these points as center we consider the piece of sphere that passes through the other n points and is contained inside the convex hull of the half-lines passing through those n points (see figure 4); one of these convex hulls contains A in its interior; let us call it R_A. Any pair of these spheres intersect at 120° angles; also if S is one of those spheres with center c and P is the image by Φ of a face of B_n such that $c \notin P$, then S and P intersect also at 120° angles. All this can easily be proved. All those spherical pieces enclose a region diffeomorphic to an n-simplex and we will assume that the choice of the centers of those spheres leaves A outside that curved n-simplex. Its exterior is divided by the images of the faces of B_n into another $n + 1$ regions, and the image of these $n + 2$ regions under Φ is thus a standard $(n+1)$-bubble. The image by Φ of the spherical piece that did lie in R_A is the new outer face, and it is evident by continuity that we can give it any positive radius, for
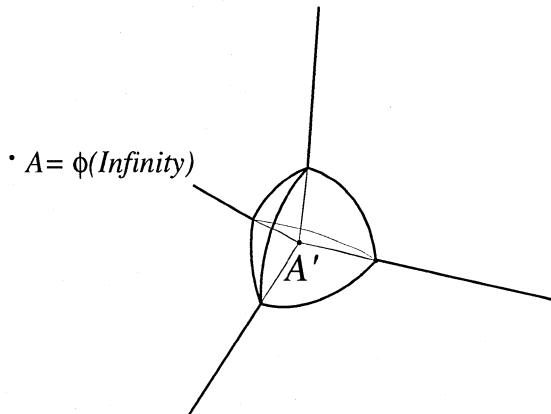
FIG. 4.

instance r_{n+1}. □

4. Uniqueness of the standard bubble with prescribed volumes. Let $\mathbf{V} : (\mathbb{R}^+)^k \to (\mathbb{R}^+)^k$ be the function that to each set $r_1, \ldots, r_k$ of positive numbers assigns the volumes of the respective regions of the standard k-bubble in $\mathbb{R}^n$, $1 \leq k \leq n+1$, whose outer faces have radii $r_1, \ldots, r_k$. In order to prove the uniqueness of the standard k-bubble that encloses given volumes we need only to prove that $\mathbf{V}$ is injective. Let $\mathbf{V}_1, \ldots, \mathbf{V}_k$ be the components of $\mathbf{V}$.
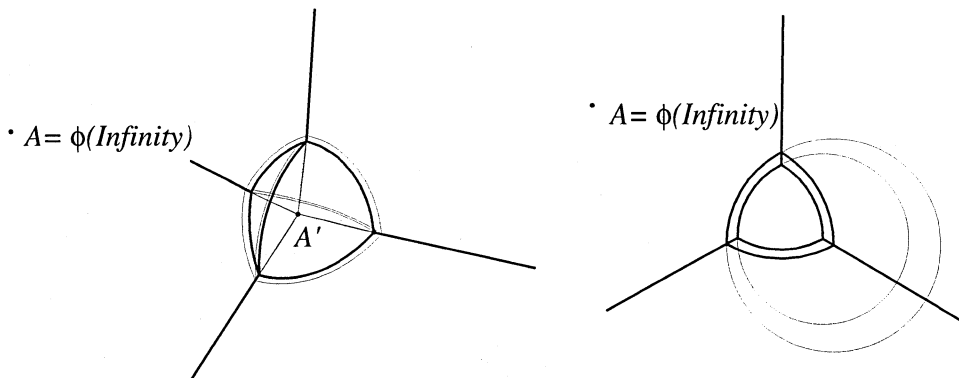

FIG. 5. *Increasing the spherical caps in $\mathbb{R}^3$ and in $\mathbb{R}^2$*

LEMMA 4.1. *Let $i \in \{1, \ldots, k\}$ and $r_1, \ldots, r_k \in \mathbb{R}^+$. Assume that $r_i' > r_i$. Then*

$$\mathbf{V}_j(r_1, \ldots, r_i, \ldots, r_k) > \mathbf{V}_j(r_1, \ldots, r_i', \ldots, r_k),$$

for $j \neq i$.

Proof. Let B_k be a standard k-bubble in $\mathbb{R}^{k-1}$ with outer radii $r_1, \ldots, r_k$. According to the proof of 3.2 there is a point A such that a central inversion Φ with center A converts B_k into a subdivision C_k of $\mathbb{R}^{k-1}$ as described there (see figure 4).

Since $\Phi^2 = \mathrm{id}$ and by Φ the point A goes to infinity, the outer faces of B_k are the images by Φ of the faces of C_k that bound the region R_A containing A. We can also assume that the ith outer face of B_k is mapped into the spherical cap that forms part of the boundary of R_A. Let us increase the radius of all of these spherical pieces of C_k (figure 5).

From the fact that the center of any of those spherical pieces is the opposite vertex we see that each of the complete spheres defining the new spherical caps of C_k enclose completely the old corresponding complete spheres. The same must occur after mapping them by Φ. Therefore, the new i^{th} outer face of B_k has greater radius that the former, and it is clear that this process can produce any previously specified radius r_i'. Also, the remaining outer faces, being the images of the plane pieces bounding R_A, will have same radius as initially. The net result is, thus, a standard k-bubble with same outer radii except for the ith one, that has increased. But all of the regions of C_k, except the image of the ith region, have been reduced. The same will occur after inversion and this proves our claim, because obviously these inequalities are conserved after blowing B_k to a standard k-bubble in $\mathbb{R}^n$. $\square$

THEOREM 4.2. *The map* $\mathbf{V}$ *is injective. Thus, given k positive volumes, there is (up to isometries of $\mathbb{R}^n$) a unique standard k-bubble in $\mathbb{R}^n$ that encloses those volumes.*

Proof. It is clear that if $a \in \mathbb{R}^+$, then $\mathbf{V}(ar_1, \ldots, ar_k) = a^n \mathbf{V}(r_1, \ldots, r_k)$. Now, suppose that $\mathbf{V}(r_1, \ldots, r_k) = \mathbf{V}(r_1', \ldots, r_k')$. Then, let

$$m = \max\left(\frac{r_1'}{r_1}, \ldots, \frac{r_k'}{r_k}, \frac{r_1}{r_1'}, \ldots, \frac{r_k}{r_k'}\right).$$

Clearly $m \geq 1$ and the equality sign implies $(r_1, \ldots, r_k) = (r_1', \ldots, r_k')$. Without loss of generality we can assume that

$$m = \frac{r_1'}{r_1}.$$

Then, we have

$$m^n \mathbf{V}_1(r_1', \ldots, r_k') = m^n \mathbf{V}_1(r_1, \ldots, r_k) = \mathbf{V}_1(mr_1, \ldots, mr_k)$$
$$= \mathbf{V}_1(r_1', mr_2, \ldots, mr_k) \leq \mathbf{V}_1(r_1', r_2', mr_3, \ldots, mr_k),$$

by the Lemma, because

$$mr_2 \geq \frac{r_2'}{r_2}r_2 = r_2'.$$

By iteration, we have

$$m^n \mathbf{V}_1(r_1', \ldots, r_k') \leq \mathbf{V}_1(r_1', \ldots, r_k'),$$

Hence, $m \leq 1$ and $\mathbf{V}$ is injective, as claimed. $\square$

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