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GENERIC MODULES FOR EXTENSION ALGEBRAS*

XIANNENG DU†

Abstract. Let A be a tame hereditary algebra (finite-dimensional over an algebraically closed field), R_A^m ($m \geq 1$) the extension algebra of A. A generic R-module M over an arbitrary ring R is by definition an indecomposable R-module of infinite length, such that M considered as an $\mathrm{End}(M)$-module, is of finite length (its endolength). In this paper we investigate the generic modules of $\widehat{A}$ (the repetitive algebra of A) and R_A^m. It is proved that R_A^m has at least $2m$ generic modules.

Introduction. The notion of generic module was introduced in [1] by Crawley-Boevey. The concept seems to be quite natural and important. The generic modules even have a dominating position in the category of modules. In [2], it was shown that whether a finite-dimensional algebra over an algebraically closed field is tame or wild is determined completely by the behaviour of the generic modules for that algebra.

In [3], Aronszajn and Fixman gave the concept of a divisible module for the Kronecker algebra and showed that for the Kronecker algebra there exists a unique indecomposable torsion-free divisile module. In [4], Ringel generalized the work of Aronszajn and Fixman and proved the same result for a tame hereditary algebra. Ringel's work, in fact, showed that for a tame hereditary algebra, there exists a unique generic module. In [6], we solved the existence and uniqueness of generic module for the tilted algebra determined by a tame hereditary algebra.

Following [1], A generic R-module M over an arbitrary ring R is by definition an indecomposable R-module of infinite length, such that M considered as an $\mathrm{End}(M)$-module, is of finite length (its endolength). Of course, the generic modules with endomorphism ring a division ring just, form the vertices of the (Cohn) spectrum of R. By [1], the endomorphism ring of a generic module always is a local ring.

Our purpose here is to investigate the generic module of the extension algebra R_A^m (defined below) for a tame hereditary algebra A. In section 1, we investigate the ν-orbits of generic modules for a repetitive algebra. we shall prove that $\mathrm{Mod}\,\widehat{A}$ has at least two ν-orbits of generic $\widehat{A}$-modules (Theorem 1.2). In section 2, we shall prove our main result on generic modules of R_A^m: R_A^m has at least $2m$ generic modules (Theorem 2.4 and Corollary 2.5).

Throughout this paper, we denote by k an algebraically closed field. An algebra means basic, connected and finite-dimensional k-algebra. For an algebra A we denote by $\mathrm{Mod}\,A$ the category of all right A-modules, by $\mathrm{mod}\,A$ the full subcategory of $\mathrm{Mod}\,A$ consisting of all finitely generated right A-modules and by $\underline{\mathrm{mod}}\,A$ the corresponding stable category. We shall use freely properties of the Auslander-Reiten sequences, irreducible maps, Auslander-Reiten translation $\tau = D\mathrm{Tr}$ and $\tau^{-1} = \mathrm{Tr}D$, and the Auslander-Reiten quiver Γ_A of an algebra B, for which we refer to [5].

1. ν-orbits of Generic Modules of Repetitive Algebras. Let $A = k\vec{\Delta}$ be a tame hereditary algebra over an k. We denote by $D^b(A)$ the derived category $D^b(\mathrm{mod}\,A)$ of bounded complexes over $\mathrm{mod}\,A$. For the definition of derived category we refer to [7]. By DA we denote the minimal injective cogenerator of A, where $D = \mathrm{Hom}_k(-, k)$ is the usual dual functor. Consider the repetitive algebra [7]:

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† Department of Mathematics, Anhui University, Hefei, 230039, P. R. China (xndu@mars.ahu.edu.cn). Reserch was supported by the National Science Foundation of China.

$$\widehat{A} = \begin{bmatrix} \ddots & \ddots & & & \\ & A_{i-1} & DA_{i-1} & & \\ & & A_i & DA_i & \\ & & & A_{i+1} & \ddots \\ & & & & \ddots \end{bmatrix}$$

with $A_i = A$ on the main diagonal, $DA_i = DA$, and zeros elsewhere. The elements are all matrices with only a finite number of nonzero entries, and multiplication is given by the canonical bimodule structure of DA and the zero map $DA \otimes_A DA \to 0$. It is a Frobenius algebra and always infinite-dimendional. We know that the identity maps $A_i \to A_{i+1}$ and $DA_i \to DA_{i+1}$ induce an automorphism ν (Nakayama automorphism) of $\widehat{A}$, and also an automorphism of $\operatorname{mod} A$. Since A is of finite global dimension, we have $\underline{\operatorname{mod}}\,\widehat{A} \cong D^b(A)$ [7]. We may identity $\underline{\operatorname{mod}}\,\widehat{A}$ with $D^b(A)$. By [7], $\Gamma_{D^b(A)}$ is the union :

$$\cdots \vee \mathcal{T}[i] \vee \mathcal{Q}[i] \vee \mathcal{P}[i] \vee \mathcal{T}[i+1] \vee \mathcal{Q}[i+1] \vee \mathcal{P}[i+1] \vee \mathcal{T}[i+2] \vee \ldots$$

Set $\mathcal{C}_i = \mathcal{Q}[i] \vee \mathcal{P}[i+1], i \in \mathbb{Z}$. Let Δ_0 and Δ_1 be the complete sections of $\mathcal{C}_0$ and $\mathcal{C}_1$ respectively. Let $\mathcal{S}_i = \operatorname{add}_{\widehat{A}}\Delta_i$, that is, $\mathcal{S}_i$ are the module classes in $\operatorname{mod}\widehat{A}$ generated by $\Delta_i (i = 0, 1)$. $\mathcal{S}_{j+2m} = \nu^m \mathcal{S}_j, m \in \mathbb{Z}, j = 0, 1$. The support of $\mathcal{S}_i$, denote by $\operatorname{Supp}\mathcal{S}_i$ is the set:

$$\{P(x) |\, P(x) \text{is indecomposable projective} \, \widehat{A} - \text{module with } \operatorname{Hom}(P(x), \mathcal{S}_i) \neq 0\}.$$

Let, for each $i \in \mathbb{Z}, A_i$ be the support algebra of $\mathcal{S}_i$, i.e.,

$$A_i = \operatorname{End}(\bigoplus_{P(x) \in \operatorname{Supp}\mathcal{S}_i} P(x)).$$

Then

$$A_i = \widehat{A} / \langle P(x) \notin \operatorname{Supp} \mathcal{S}_i \rangle$$

and A_i are tilted algebras of Euclidean type. Since the main purpose of the paper is to investigate the generic modules of R_A^m, we may assume that each A_i is representation-infinite. By [6], for each $i \in \mathbb{Z}$, there exists a unique generic A_i-module M_i. Of course, all M_i are also the generic $\widehat{A}$-modules.

LEMMA 1.1. *For each $i \in \mathbb{Z}$ we have*

$$Supp(\nu\mathcal{S}_i) = \nu Supp(\mathcal{S}_i).$$

Proof. Let $P(x) \in \operatorname{Supp}(\nu\mathcal{S}_i)$. Then there exists an $\widehat{A}$-module $S \in \mathcal{S}_i$ such that $\operatorname{Hom}_{\widehat{A}}(P(x), \nu S) \neq 0$. Hence we get that $\operatorname{Hom}_{\widehat{A}}(\nu^- P(x), S) \neq 0$. This means $\nu^- P(x) \in \operatorname{Supp}(\nu\mathcal{S}_i)$ and hence $P(x) \in \nu\operatorname{Supp}(\mathcal{S}_i)$. So, $\operatorname{Supp}(\nu\mathcal{S}_i) \sqsubseteq \nu\operatorname{Supp}(\mathcal{S}_i)$. Similarly, we have $\nu\operatorname{Supp}(\mathcal{S}_i) \sqsubseteq \operatorname{Supp}(\nu\mathcal{S}_i)$.

The main result of this section is the follow

THEOREM 1.2. *Let A be a tame hereditary algebra, then $Mod\widehat{A}$ has at least two ν-orbits of generic $\widehat{A}$-modules.*

Proof. Suppose A_i and M_i $(i \in \mathbb{Z})$ are as before. Then for each $i \in \mathbb{Z}$, M_i is also a generic $\widehat{A}$-module.

For each $P(x) \notin \operatorname{Supp}(\mathcal{S}_{i+2})$, we show $\operatorname{Hom}_{\widehat{A}}(P(x), \nu M_i) = 0$. If not, we have $\operatorname{Hom}_{\widehat{A}}(\nu^- P(x), M_i) \neq 0$. This gives $\nu^- P(x) \in \operatorname{Supp}(M_i) \sqsubseteq \operatorname{Supp}(\mathcal{S}_i)$. By Lemma 1.1,

$P(x) \in \nu\text{Supp}(\mathcal{S}_i) = \text{Supp}(\nu\mathcal{S}_i) = \text{Supp}(\mathcal{S}_{i+2})$. This is a contradiction. Thus νM_i is a A_{i+2}-module. Since νM_i is a generic A_{i+2}-module and A_{i+2} has a unique generic module, we get $\nu M_i = M_{i+2}$. In general, we have $\nu^m M_i = M_{i+2m}$, $i \in \mathbb{Z}, m \in \mathbb{Z}$. By the structure of Mod $\widehat{A}$ we know that $M_i \neq M_j\,(i \neq j)$ as $\widehat{A}$-modules. We get two distinct ν-orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ of generic $\widehat{A}$-modues:

$$\mathcal{O}_0 = \{\nu^m M_0 | \, m \in \mathbb{Z}\}, \mathcal{O}_1 = \{\nu^n M_1 | \, n \in \mathbb{Z}\}.$$

2. Generic Modules for Extension Algebra $R_A{}^m$. Let $A = k\vec{\Delta}$ be a tame hereditary algebra over k. $\widehat{A}$ the repetitive algebra of A. We consider, for each $m \geq 1$, the algebra R_A^m:

$$R_A^m = \left\{ \left(\begin{array}{cccccc} \lambda_1 & x_1 & & & & \\ & \lambda_2 & x_2 & & & \\ & & \ddots & \ddots & & \\ & & & \lambda_m & x_m \\ & & & & \lambda_1 \end{array} \right) \middle| \, \lambda_i \in A, \; x_i \in DA \right\}.$$

As above, the multiplication is given by the bimodule structure of DA and zero map $DA \otimes_A DA \to 0$. In particular, R_A^1 is the trivial extension $A \ltimes DA$. The category R_A^m is just the quotient category $\widehat{A}/(\nu^m)$.

For a fixed $m \geq 1$, we consider the canonical Galois covering functor $F^m : \widehat{A} \to \widehat{A}/(\nu^m) = R_A^m$, and the associated pushdown functor F_λ^m : Mod $\widehat{A} \to$ Mod R_A^m and the pull-up functor $F_.{}^m$: Mod $R_A^m \to$ Mod $\widehat{A}$ [8].

From now on we fix some m. In this section we show that R_A^m has at least $2m$ generic modules.

We first prove some lemmas

LEMMA 2.1 [8]. *For each $N \in Mod\,\widehat{A}$ and each $r \in \mathbb{Z}$, we have*

$$F_\lambda^m((\nu^m)^r N) \cong F_\lambda^m(N).$$

LEMMA 2.2. *Let M be a generic $\widehat{A}$-module, N an indecomposable $\widehat{A}$-module. If $F_\lambda^m N \cong F_\lambda^m M$, then $N \cong \nu^{mr} M$ for some $r \in \mathbb{Z}$.*

Proof. Assume that $F_\lambda^m N \cong F_\lambda^m M$. Then by [8], we have

$$\bigoplus_{r \in \mathbb{Z}} \nu^{mr} N \cong F_.{}^m F_\lambda^m N \cong F_.{}^m F_\lambda^m M \cong \bigoplus_{l \in \mathbb{Z}} \nu^{ml} M.$$

Since M is a generic $\widehat{A}$-module, $\nu^{ms} M\,(s \in \mathbb{Z})$ are also generic $\widehat{A}$-modules, it follows from [1] that every ring $\text{End}(\nu^{mt} M)$ is local, we infer that $N = \nu^{mr} M$ for some $r \in \mathbb{Z}$.

LEMMA 2.3. *Suppose that M is a generic $\widehat{A}$-modole. Then $F_\lambda^m M$, as a left $End_{\widehat{A}}(M)$-module, is of finite length.*

Proof. Since we have an imbeding map $\text{End}_{\widehat{A}}(M) \to \text{End}_{R_A^m}(F_\lambda^m M)$, we infer that $F_\lambda^m M$ is also a left $\text{End}_{\widehat{A}}(M)$-module. Suppose that

$$(*) \qquad 0 = N_0 \leq N_1 \leq N_2 \leq \cdots \leq N_i \leq \ldots$$

be a composition series for left $\text{End}_{\widehat{A}}(M)$-module $F_\lambda^m M$. Since $F_\lambda^m M = M$ as a k vector space, every N_i in (*) is a subspace of M. For each $f \in \text{End}_{\widehat{A}}(M)$, we have $fN_i \sqsubseteq N_i$ and hence each N_i is a left $\text{End}_{\widehat{A}}(M)$-submodule of M. Hence we may regard (*) as a composition series for left $\mathrm{End}_{\widehat{A}}(M)$-module M. Since M is a generic $\widehat{A}$-module, it follows that (*) has only finite terms. Therefore $F_\lambda^m M$, as a left $\mathrm{End}_{\widehat{A}}(M)$-module, is of finite length.

We can now prove our main result

THEOREM 2.4. *Let M be a generic $\widehat{A}$-module. Then $F_\lambda^m M$ is a generic R_A^m-module.*

Proof. Since F_λ^m is left adjoint to $F_\cdot^m$, it follows from [8] that

$$\begin{aligned}\mathrm{End}_{R_A^m}(F_\lambda^m M) &= \mathrm{Hom}_{R_A^m}(F_\lambda^m M,\, F_\lambda^m M)\\ &\cong \mathrm{Hom}_{\widehat{A}}(M,\, F_\cdot^m F_\lambda^m M)\\ &\cong \mathrm{Hom}_{\widehat{A}}(M,\, \underset{l\in\mathbb{Z}}{\oplus}\nu^{ml} M).\end{aligned}$$

If $m = 1$, then, for $s \neq 0, 1$, we have $\mathrm{Hom}_{\widehat{A}}(M,\, \nu^s M) = 0$, and hence

$$\mathrm{End}_{R_A^1}(F_\lambda^1 M) \cong \mathrm{Hom}_{\widehat{A}}(M,\, M \oplus \nu M) = \mathrm{End}_{\widehat{A}}(M) \oplus \mathrm{Hom}_{\widehat{A}}(M,\, \nu M).$$

Since $\mathrm{Hom}_{\widehat{A}}(M,\, \nu M)$ is an $\mathrm{End}_{\widehat{A}}(M)$-bimodule: $g\circ f = gf$ is the ordinary composition and $(f\circ g)(x) = \nu f(g(x))$ for $f \in \mathrm{End}_{\widehat{A}}(M), g \in \mathrm{Hom}_{\widehat{A}}(M,\, \nu M)$ and $x \in M$. It follows from [9] that we have the following ring isomorphism

$$\mathrm{End}_{R_A^1}(F_\lambda^1 M) \cong \mathrm{End}_{\widehat{A}}(M) \ltimes \mathrm{Hom}_{\widehat{A}}(M,\, \nu M).$$

From the definition of trivial extension of algebra we know

$$\mathrm{End}_{R_A^1}(F_\lambda^1 M)/\mathrm{rad}\,\mathrm{End}_{R_A^1}(F_\lambda^1 M) \cong \mathrm{End}_{\widehat{A}}(M)/\mathrm{rad}\,\mathrm{End}_{\widehat{A}}(M).$$

Suppose that $m \geq 2$. By the structure of $\mathrm{Mod}\,\widehat{A}$ we have that $\mathrm{Hom}_{\widehat{A}}(M,\, \nu^{ms} M) = 0$ for $s \neq 0$. Hence

$$\mathrm{End}_{R_A^m}(F_\lambda^m M)/\mathrm{rad}\,\mathrm{End}_{R_A^m}(F_\lambda^m M) \cong \mathrm{End}_{\widehat{A}}(M)/\mathrm{rad}\,\mathrm{End}_{\widehat{A}}(M).$$

Since M is a generic $\widehat{A}$-module, we infer that $\mathrm{End}_{\widehat{A}}(M)/\mathrm{rad}\,\mathrm{End}_{\widehat{A}}(M)$ is a division ring and hence $\mathrm{End}_{R_A^m}(F_\lambda^m M)$ is local for $m \geq 1$. Therefore, $F_\lambda^m M$ is an indecomposable R_A^m-module.

Write $C = \mathrm{End}_{\widehat{A}}(M)$, $D = \mathrm{End}_{R_A^m}(F_\lambda^m M)$, $\overline{C} = C/\mathrm{rad}C$, $\overline{D} = D/\mathrm{rad}D$. Let $l_A(M)$ denote the lenght of A-module M. We have

$$\begin{aligned}l_D(F_\lambda^m M) = {}& l_D(F_\lambda^m M/(\mathrm{rad}D)\, F_\lambda^m M) + l_D((\mathrm{rad}\, D)F_\lambda^m M/(\mathrm{rad}^2\, D)\, F_\lambda^m M) + \dots\\ &+ l_D((\mathrm{rad}^i\, D)\, F_\lambda^m M/(\mathrm{rad}^{i+1}\, D)\, F_\lambda^m M) + \dots.\end{aligned}$$

Since each $(\mathrm{rad}^i\, D)\, F_\lambda^m M/(\mathrm{rad}^{i+1}\, D)\, F_\lambda^m M$ is a $\overline{D}$-module and

$$l_D((\mathrm{rad}^i\, D)\, F_\lambda^m M/(\mathrm{rad}^{i+1}\, D)\, F_\lambda^m M) = l_{\overline{D}}((\mathrm{rad}^i\, D)\, F_\lambda^m M/(\mathrm{rad}^{i+1}\, D)\, F_\lambda^m M).$$

We have

$$\begin{aligned}
l_D(F_\lambda^m M) &= l_{\overline{D}}(F_\lambda^m M/(\operatorname{rad} D)\, F_\lambda^m M) + l_{\overline{D}}((\operatorname{rad} D) F_\lambda^m M/(\operatorname{rad}^2 D)\, F_\lambda^m M) \\
&+ \dots + l_{\overline{D}}((\operatorname{rad}^i D)\, F_\lambda^m M/(\operatorname{rad}^{i+1} D)\, F_\lambda^m M) + \dots \\
&\overset{\overline{C}\cong\overline{D}}{=} l_{\overline{C}}(F_\lambda^m M/(\operatorname{rad} D)\, F_\lambda^m M) + l_{\overline{C}}((\operatorname{rad} D)\, F_\lambda^m M/(\operatorname{rad}^2 D)\, F_\lambda^m M) \\
&+ \dots + l_{\overline{C}}((\operatorname{rad}^i D)\, F_\lambda^m M/(\operatorname{rad}^{i+1} D)\, F_\lambda^m M) + \dots \\
&= l_C(F_\lambda^m M/(\operatorname{rad} D)\, F_\lambda^m M) + l_C((\operatorname{rad} D)\, F_\lambda^m M/(\operatorname{rad}^2 D)\, F_\lambda^m M) \\
&+ \dots + l_C((\operatorname{rad}^i D)\, F_\lambda^m M/(\operatorname{rad}^{i+1} D)\, F_\lambda^m M) + \dots \\
&= l_C(F_\lambda^m M).
\end{aligned}$$

By lemma 2.3, we have $l_C(F_\lambda^m M) \leq \infty$ and hence $l_D(F_\lambda^m M) \leq \infty$. $F_\lambda^m M$ is clearly of infinite -dimension since M is so.

Therefore $F_\lambda^m M$ is a generic R_A^m-module.

COROLLARY 2.5. *R_A^m has at least $2m$ generic modules.*

Proof. By Theorem 1.2, Mod $\widehat{A}$ has two ν-orbits $\mathcal{O}_0$ and $\mathcal{O}_1$ of generic $\widehat{A}$-modules.

$$\mathcal{O}_0 = \{\nu^m M_0 \mid m \in \mathbb{Z}\}, \mathcal{O}_1 = \{\nu^n M_1 \mid n \in \mathbb{Z}\}.$$

From Lemma 2.1 and 2.2, it is easy to know that $F_\lambda^m(\nu^l N)$, $F_\lambda^m(\nu^t N)$ $(l, t = 0, 1, 2, \dots, m-1)$ are different generic R_A^m-modules.

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