

REAL HYPERSURFACES IN KÄHLER MANIFOLDS*

WILHELM KLINGENBERG†

0. Introduction. The present work is concerned with the geometry of embedded real hypersurfaces in a Kähler manifold, where isomorphisms are both holomorphic and isometric in the underlying Riemannian structure. Here we introduce their local invariants and compatibility relations, solve the local realization problem, and give a characterization of the metric sphere in \mathbb{C}^n via a maximum principle for an adapted Laplace operator. A more detailed description follows.

a) In 1.3 we define a doubly covariant tensor ℓ on the complexified tangent bundle of the hypersurface N . It describes the curvature of the maximal complex subbundle HN of TN in the ambient manifold with regard to the Kähler metric. One part of ℓ is the well-known Levi form of the hypersurface with respect to a one-form of norm one that annihilates the “horizontal” tangent bundle HN . The form ℓ may be expressed in terms of the second fundamental form of the hypersurface in an appropriate frame of the underlying Riemannian manifold, see 1.4.

b) In 2 we derive differential compatibility conditions of covariant derivatives of ℓ . In 2.3, 2.4 and 2.5 we employ a connection on the horizontal bundle that is the projection of the Levi-Civita connection of the ambient manifold. It is a one-form on the hypersurface with values in the Lie algebra of the unitary group of the horizontal subbundle and we give its structure equations.

c) In 3.2 we solve the local existence problem as follows. The data are a CR -manifold U with a metric on TU that is hermitian on HU and a real function r that prescribes the ambient curvature of the curves in N that are orthogonal to HN . We require that the composition of the defining form for HN with the complex structure is closed, the complex structure on the horizontal bundle is parallel with respect to the associated Levi-Civita connection, and that an associated unitary connection has vanishing curvature. The latter corresponds to Gauss and Codazzi compatibility conditions. The conclusion is that there exists a local isometric CR -embedding of U into \mathbb{C}^n . In 3.4 we embed nondegenerate CR manifolds U together with a prescribed hermitian metric on HU rather than on TU as in 3.2. In exchange we prescribe additional data, namely mixed horizontal and vertical coefficients of ℓ . In this case, there is an embedding under analogous requirements as before. This result has features of both the classical hypersurface existence theorem in Riemannian geometry and of Kuranishi’s CR embedding [Ku]. Namely we require compatibility conditions involving metric and complex structure, and the non-degeneracy of the Levi-form.

d) In 4 we give commutator identities for the operation of the Laplace operator of HN on ℓ . In the Riemannian case these are referred to as Simons identity, see [S] and [CdCK].

e) As an application we give in 5 the following

THEOREM 5.2. *Let N be a compact strictly pseudoconvex hypersurface of \mathbb{C}^n . Assume that the horizontal mean curvature of N is constant, and $H^{1,0}N$ is parallel in $T^{1,0}\mathbb{C}^n$. Then N is a metric sphere.*

*Received February 26, 1999; accepted for publication June 9, 1999.

†Institut für Mathematik, Humboldt - Universität zu Berlin, Unter den Linden 6, 10099 Berlin, Germany (klingenb@mathematik.hu-berlin.de).

The first assumption relates only to the CR structure, the other assumptions are that the trace of the second fundamental form on HN is constant, that the curves on N orthogonal to HN are geodesics of N , and that the symmetric part of ℓ vanishes on the horizontal bundle. This result is an analogue of Alexandrov's theorem on compact convex hypersurfaces in \mathbb{R}^n : if such have constant mean curvature, then they must be metric spheres. Our result does not follow from this since we do not make any assumption on the purely vertical coefficient of ℓ . This coefficient describes the curvature in the ambient manifold of the vertical integral curves. Therefore our assumption neither implies that the full mean curvature of the hypersurface is constant nor that it is convex. Of course we use the ambient complex structure, therefore our result does not imply the one in \mathbb{R}^n . Also, by the existence result of 3.2, our assumptions locally admit other surfaces than subsets of metric spheres. In this sense, this is a global result. It is an application of our first Simons-type identity in 4.2 and the strong maximum principle for the horizontal Laplace operator on strictly pseudoconvex hypersurfaces. The remaining identities of 4.2 are recorded here for completeness and application in [HK]. The maximum principle along horizontal curves for the horizontal Laplace operator was used in [A] to study a heat flow for contact structures.

The paper [CM] gives a local normal form for real analytic hypersurfaces in complex manifolds and [B] and [YK] present some material on real submanifolds of any codimension of Kähler manifolds. In [O], geometric properties of real hypersurfaces of complex projective space are studied. In [J1] and [J2] one finds an introduction to CR -structures. The lectures of Fefferman, [BFG], give an extensive review of the relation of function theory of a domain to CR geometry of the boundary and the analogy with Riemannian geometry. Webster defines in [W] a connection on non-degenerate CR manifolds with a distinguished one form that defines the horizontal bundle. It is characterized by the requirement that the Levi form is parallel. The resulting intrinsic curvature is different from the curvature that we have, as we use the Levi-Civita connection of the induced Kähler metric.

ACKNOWLEDGEMENTS. The results presented here are part of the author's Habilitationsschrift at the Universität Tübingen, [K]. The author is grateful to Professor G. Huiskens for advice, to Professors S.S. Chern and J.K. Moser for introducing him to this subject, and to Mrs. S. Schmidt for her scrupulous typesetting of the manuscript.

1.1. Structure Equations. On a Kähler manifold M with complex structure ${}^M J$ and metric ${}^M h$ we denote by $T^{1,0}M$ and $T^{0,1}M$ the subbundle of $\mathbb{C} \otimes TM$ whose fibres consist of the eigenspaces for $+i$ and $-i$ of the \mathbb{C} -linear extension of ${}^M J$ to $\mathbb{C} \otimes TM$. By the Kähler property, the Levi-Civita connection ${}^M \nabla$ of $\mathbb{C} \otimes TM$ preserves $T^{1,0}M$ and $T^{0,1}M$. On the principal $U(n, \mathbb{C})$ -bundle of unitary frames of $T^{1,0}M$, $n = \dim_{\mathbb{C}} M$, we have the \mathbb{C}^n -valued canonical form ${}^M \theta$, and the $u(n)$ -valued connection and curvature forms ${}^M \omega$, ${}^M \Omega$. For a unitary frame $\{e_k\}_{k=1}^n$ of $T^{1,0}M$ one has for $\theta^j = {}^M \theta^j$, $\omega_k^j = {}^M \omega_k^j \in \Lambda(\mathbb{C} \otimes TM)$, $\Omega_k^j = {}^M \Omega_k^j \in \Lambda^2(\mathbb{C} \otimes TM)$, $j, k = 1, \dots, n$, the structure equations

$$\begin{aligned} d\theta^j &= -\omega_k^j \wedge \theta^k \\ d\omega_k^j &= -\omega_p^j \wedge \omega_k^p + \Omega_k^j, \end{aligned}$$

and analogous equations in $T^{0,1}M$ for the conjugates $e_{\bar{k}}$, $\theta^{\bar{j}}$, $\omega_{\bar{k}}^{\bar{j}}$, $\Omega_{\bar{k}}^{\bar{j}}$. Recall that $\omega_k^j(e_r) = \theta^j({}^M \nabla_{e_r} e_k)$ for $r = 1, \dots, n, \bar{1}, \dots, \bar{n}$. By unitarity,

$$\omega_k^j = -\omega_{\bar{j}}^{\bar{k}}$$

$$\Omega_k^j = -\Omega_j^{\bar{k}}.$$

Let now

$$F : N \rightarrow M$$

be an immersion of a real hypersurface. The above structure equations hold for the pull-back frames $\{F^*e_k\}_{k=1}^n, \{F^*\theta^k\}_{k=1}^n$, connection $(F^*{}^M\nabla)_{(\cdot)}(\cdot) : \Gamma(\mathbb{C} \otimes TN) \times \Gamma(\mathbb{C} \otimes F^*TM) \rightarrow \Gamma(\mathbb{C} \otimes F^*TM)$, $F^*{}^M\omega$, and curvature $F^*{}^M\Omega$ of $F^*T^{1,0}M$, and exterior differentiation d on N . In the sequel we will denote these pull-back objects by $\{e_k\}_{k=1}^n, \{\theta^k\}_{k=1}^n, {}^M\nabla, {}^M\omega, {}^M\Omega$ respectively, and we write $\omega = {}^M\omega$ for economy of notation.

1.2. Adapted Frames. Since N is odd dimensional, TN does not inherit a complex structure from F^*TM . We denote by $HN \hookrightarrow TN$ the maximal MJ -invariant subbundle of $i_N : TN \hookrightarrow F^*TM$:

$$HN := TN \cap {}^MJTN,$$

called the holomorphic or horizontal tangent bundle of N . Let $H^{1,0}N, H^{0,1}N$ be the $(\pm i)$ -eigenspaces of $(\mathbb{C} \otimes HN, {}^HJ = {}^MJ|_{HN})$, then $H^{1,0}N = F^*T^{1,0}M \cap \mathbb{C} \otimes TN$. For an analysis of the curvature of N with regard to the Kähler metric we consider a unitary frame $\{e_k\}_{k=1}^n$ of $F^*T^{1,0}M$ which is adapted to TN in the following way:

$$\begin{aligned} \text{span} \{e_\alpha\}_{\alpha=1}^{n-1} &= H^{1,0}N \\ \text{span} \{e_\alpha, e_{\bar{\alpha}}, \frac{1}{2}(e_n + e_{\bar{n}})\}_{\alpha=1}^{n-1} &= \mathbb{C} \otimes TN. \end{aligned}$$

Then $\{\xi^k = \frac{1}{\sqrt{2}}(\theta^k + \theta^{\bar{k}}), \xi^{n+k} = \frac{-i}{\sqrt{2}}(\theta^k - \theta^{\bar{k}})\}_{k=1}^n$ is the dual frame of $\{X_k, X_{n+k}\}$ and $\ker \xi^{2n} = TN, \ker \xi^n = HN \oplus \mathbb{R}X_{2n}$. Finally we introduce a frame $\{f_a\}$ of $\mathbb{C} \otimes TN$ by setting $f_\xi = e_\xi, f_n = \frac{1}{2}(e_n + e_{\bar{n}})$ with its dual frame $\{\varphi^a\}, \varphi^\xi = \theta^\xi, \varphi^n = (\theta^n + \theta^{\bar{n}})$. Here and later we adopt the following convention for the ranges of the indices: $j, k, p, q \in \{1, \dots, n\}$ for frames of $F^*T^{1,0}M$ $a, b, c \in \{1, \dots, n-1, \bar{1}, \dots, \overline{n-1}, n\}$ on $\mathbb{C} \otimes TN$, $\xi, \eta \in \{1, \dots, n-1, \bar{1}, \dots, \overline{n-1}\}$ on $\mathbb{C} \otimes HN$, and $\alpha, \beta, \gamma, \delta \in \{1, \dots, n-1\}$ on $H^{1,0}N$.

1.3. Second Fundamental Form in Kähler Geometry. Let e_k be adapted to $i_N : TN \rightarrow F^*TM$ as above. Since $\mathbb{C} \otimes TN = \ker(\theta^n - \theta^{\bar{n}})$ is involutive in $\mathbb{C} \otimes F^*TM$,

$$i_N^*d(\theta^n - \theta^{\bar{n}}) = 0.$$

On the other hand, we compute, recalling 1.1,

$$\begin{aligned} i_N^*d(\theta^n - \theta^{\bar{n}}) &= i_N^*(-\omega_\alpha^n \wedge \theta^\alpha + \omega_{\bar{\alpha}}^{\bar{n}} \wedge \theta^{\bar{\alpha}} - \omega_n^n \wedge \theta^n + \omega_{\bar{n}}^{\bar{n}} \wedge \theta^{\bar{n}}) \\ &= \omega_\alpha^n(f_\xi)\varphi^\alpha \wedge \varphi^\xi + \omega_\alpha^n(f_n)\varphi^\alpha \wedge \varphi^n - \omega_{\bar{n}}^{\bar{n}}(f_\xi)\varphi^{\bar{\alpha}} \wedge \varphi^\xi - \omega_{\bar{\alpha}}^{\bar{n}}(f_n)\varphi^{\bar{\alpha}} \wedge \varphi^n \\ &\quad + \varphi^n \wedge (\omega_n^n(f_\xi)\varphi^\xi + \omega_n^n(f_n)\varphi^n) \\ &= (\omega_\alpha^n(f_\beta) + \overline{\omega_\beta^n(f_\alpha)})\varphi^\alpha \wedge \varphi^\beta + \omega_\alpha^n(f_\beta)\varphi^\alpha \wedge \varphi^\beta - \overline{\omega_\alpha^n(f_\beta)}\varphi^{\bar{\alpha}} \wedge \varphi^{\bar{\beta}} \\ &\quad + (\omega_\alpha^n(f_n) + \overline{\omega_n^n(f_\alpha)})\varphi^\alpha \wedge \varphi^n - (\overline{\omega_\alpha^n(f_n)} + \omega_n^n(f_\alpha))\varphi^{\bar{\alpha}} \wedge \varphi^n. \end{aligned}$$

This implies

$$\begin{aligned} (1) \quad & \omega_\alpha^n(f_\beta) - \omega_\beta^n(f_\alpha) = 0 \\ (2) \quad & \omega_\alpha^n(f_\beta) + \overline{\omega_\beta^n(f_\alpha)} = 0 \\ (3) \quad & \omega_\alpha^n(f_n) - \omega_n^n(f_\alpha) = 0 \end{aligned}$$

This gives, also using $i_N^* \theta^n = \frac{1}{2} \varphi^n$,

$$\begin{aligned} i_N^* d\theta^n &= i_N^* (\theta^k \wedge \omega_k^n) \\ &= \omega_\alpha^n(f_a) \varphi^\alpha \wedge \varphi^a + \frac{1}{2} \omega_n^n(f_a) \varphi^n \wedge \varphi^a \\ &= \omega_\alpha^n(f_{\bar{\beta}}) \varphi^\alpha \wedge \varphi^{\bar{\beta}} + \frac{1}{2} \omega_\alpha^n(f_n) \varphi^\alpha \wedge \varphi^n - \frac{1}{2} \overline{\omega_\alpha^n(f_n)} \varphi^{\bar{\alpha}} \wedge \varphi^n. \end{aligned}$$

We define the coefficients of a doubly covariant tensor ℓ on $\mathbb{C} \otimes TN$ as follows.

$$\begin{aligned} \ell_{ja} &:= \frac{\sqrt{2}}{i} \omega_j^n(f_a) \\ \ell_{\bar{\alpha}\bar{\beta}} &:= \bar{\ell}_{\alpha\beta} \\ \ell_{\bar{\alpha}n} &:= \ell_{n\bar{\alpha}}. \end{aligned}$$

This gives

$$d\varphi^n = i\sqrt{2} \ell_{\alpha\bar{\beta}} \varphi^\alpha \wedge \varphi^{\bar{\beta}} + \frac{i}{\sqrt{2}} (\ell_{n\gamma} \varphi^\gamma - \ell_{n\bar{\gamma}} \varphi^{\bar{\gamma}}) \wedge \varphi^n.$$

The restriction of $d\varphi^n$ to $H^{1,0}N \otimes H^{0,1}N$ is the Levi-form of the real hypersurface $F(N)$ for the choice φ^n of a defining one-form for HN . By (1), (2), (3) we have

$$\begin{aligned} \bar{\ell}_{\alpha\bar{\beta}} &= \ell_{\beta\bar{\alpha}} \\ \ell_{\alpha\beta} &= \ell_{\beta\alpha} \\ \ell_{\alpha n} &= \ell_{n\alpha} \\ \bar{\ell}_{n\alpha} &= \ell_{n\bar{\alpha}} \\ \bar{\ell}_{nn} &= \ell_{nn}. \end{aligned}$$

We summarize: *To a real hypersurface we associate a quadratic form $\ell = \{\ell_{ja}\}$ on $\mathbb{C} \otimes TN$ which decomposes into a hermitian form (the Levi form) $\{\ell_{\alpha\bar{\beta}}\}$ on $H^{1,0}N \otimes H^{0,1}N$ and a symmetric form $\{\ell_{\alpha\beta}\}$ on $H^{1,0}N \otimes H^{1,0}N$ and a real form $\{\ell_{na} = \ell_{an}\}$ on $\mathbb{C} \otimes TN$. Namely $\{\ell_{na}\}$ is real on $TN \hookrightarrow \mathbb{C} \otimes TN$. Recall that N is called strictly pseudoconvex if $\{\ell_{\alpha\bar{\beta}}\}$ is positive definite.*

1.4. Relation of ℓ to the Riemannian second fundamental form. We will express the Kählerian second fundamental form ℓ as defined above relative to the adapted $U(n, \mathbb{C})$ -frame $\{e_k\}$ of $F^*T^{1,0}M$ in terms of the Riemannian second fundamental form $-k$ with respect to the $0(2n)$ -frame $\{X_k, X_{n+k}\}$ of F^*TM . The latter is defined by

$$k_{st} := \xi^{2n}({}^M \nabla_{X_s} X_t), \quad s, t = 1, \dots, 2n-1.$$

Using $\xi^{2n}([X_s, X_t]) = 0$ and $\xi^n(X_s) = -\xi^{2n}(JX_s)$, one computes

$$\begin{aligned} \ell_{\alpha\bar{\beta}} &= k_{\alpha\beta} + k_{n+\alpha, n+\beta} + i(k_{\alpha, n+\beta} - k_{n+\alpha, \beta}) \\ \ell_{\alpha\beta} &= k_{\alpha\beta} - k_{n+\alpha, n+\beta} - i(k_{n+\alpha, \beta} + k_{\alpha, n+\beta}) \\ \ell_{n\bar{\beta}} &= k_{n\beta} + ik_{n, n+\beta} \\ \ell_{nn} &= k_{nn}. \end{aligned}$$

2.1. Gauss Equations for $(TN, {}^N\nabla) \hookrightarrow (TM, {}^M\nabla)$. Associated to the orthogonal projection

$$\pi_N : \mathbb{C} \otimes F^*TM \rightarrow \mathbb{C} \otimes TN,$$

$\pi_N(v) = \varphi^\xi(v)e_\xi + \frac{1}{2}(\theta^n + \theta^{\bar{n}})(v)(e_n + e_{\bar{n}})$, we have the connection ${}^N\nabla = \pi_N \circ {}^M\nabla_{(\cdot)}(\cdot) : \Gamma(\mathbb{C} \otimes TN) \times \Gamma(\mathbb{C} \otimes TN) \rightarrow \Gamma(\mathbb{C} \otimes TN)$ with connection form ${}^N\omega$. Relative to the frame $\{f_a\}$, its coefficients are given by ${}^N\omega_\eta^\xi = {}^M\omega_\eta^\xi$, ${}^N\omega_\alpha^n(\cdot) = \varphi^n({}^N\nabla_{(\cdot)}f_\alpha) = \varphi^n({}^M\nabla_{(\cdot)}f_\alpha) = (\theta^n + \theta^{\bar{n}})({}^M\nabla_{(\cdot)}f_\alpha) = \theta^n({}^M\nabla_{(\cdot)}f_\alpha) = {}^M\omega_\alpha^n$, ${}^N\omega_{\bar{\alpha}}^n = {}^M\omega_{\bar{\alpha}}^n$, ${}^N\omega_n^\alpha(\cdot) = \varphi^\alpha({}^N\nabla_{(\cdot)}f_n) = \frac{1}{2}\varphi^\alpha({}^M\nabla_{(\cdot)}e_n + e_{\bar{n}}) = \frac{1}{2}{}^M\omega_n^\alpha(\cdot)$, ${}^N\omega_n^{\bar{\alpha}} = \frac{1}{2}{}^M\omega_n^{\bar{\alpha}}$, ${}^N\omega_n^n = 0$. The structure equations of $\mathbb{C} \otimes TN$ then read

$$\begin{aligned} d\varphi^j + {}^N\omega_a^j \wedge \varphi^a &= 0 \\ d{}^N\omega_a^j + {}^N\omega_b^j \wedge {}^N\omega_a^b &= {}^N\Omega_a^j. \end{aligned}$$

Comparing these with the structure equations of F^*TM with connection $\omega = {}^M\omega$ from 1.1 gives the Gauss equations for $TN \rightarrow F^*TM$:

$$\begin{aligned} ({}^M\Omega - {}^N\Omega)_\beta^\alpha &= d(\omega_\beta^\alpha - {}^N\omega_\beta^\alpha) + \omega_p^\alpha \wedge \omega_\beta^p - {}^N\omega_a^\alpha \wedge {}^N\omega_\beta^a \\ &= \omega_n^\alpha \wedge \omega_\beta^n - {}^N\omega_n^\alpha \wedge {}^N\omega_\beta^n \\ &= \frac{1}{2}\omega_n^\alpha \wedge \omega_\beta^n \\ &= -\frac{1}{4}\ell_{\bar{\alpha}a}\ell_{\beta b}\varphi^a \wedge \varphi^b \\ ({}^M\Omega - {}^N\Omega)_\beta^n &= d(\omega_\beta^n - {}^N\omega_\beta^n) + \omega_p^n \wedge \omega_\beta^p - {}^N\omega_a^n \wedge {}^N\omega_\beta^a \\ &= \omega_n^n \wedge \omega_\beta^n \\ &= -\frac{1}{4}\ell_{na}\ell_{\beta b}\varphi^a \wedge \varphi^b. \end{aligned}$$

2.2. Codazzi Equations for $(TN, {}^N\nabla)$. The coefficients of the covariant derivative of ℓ with respect to ${}^N\nabla$, namely $\ell_{jab}^N = ({}^N\nabla\ell)_{jab}$, satisfy

$$d\ell_{ja} = \ell_{jab}^N \varphi^b + \ell_{pa} {}^N\omega_j^p + \ell_{jb} {}^N\omega_a^b.$$

Note that ${}^N\nabla\ell$ is a triply covariant tensor on $\mathbb{C} \otimes TN$. Then we compute

$$\begin{aligned} d\omega_j^n &= \frac{i}{\sqrt{2}}(d\ell_{ja} \wedge \varphi^a + \ell_{ja}d\varphi^a) \\ &= \frac{i}{\sqrt{2}}\ell_{jab}^N \varphi^b \wedge \varphi^a + \frac{i}{\sqrt{2}}\ell_{ba} {}^N\omega_j^b \wedge \varphi^a. \end{aligned}$$

This gives for $j = \alpha$ and $j = n$

$$\begin{aligned} d\omega_\alpha^n &= \frac{i}{\sqrt{2}}\ell_{\alpha ab}^N \varphi^b \wedge \varphi^a + \frac{1}{2}\omega_\alpha^n \wedge \omega_n^n + \omega_\alpha^\gamma \wedge \omega_n^\gamma \\ d\omega_n^n &= \ell_{nab}^N \varphi^b \wedge \varphi^a + \omega_n^\gamma \wedge \omega_n^\gamma. \end{aligned}$$

Now by the structure equations of F^*TM from 1.1,

$$\ell_{jab}^N \varphi^a \wedge \varphi^b = \frac{i}{2^{3/2}}\ell_{na}\ell_{jb}\varphi^a \wedge \varphi^b + i\sqrt{2} {}^M\Omega_j^n.$$

These are Codazzi equations for $F : N \rightarrow M$.

2.3. Gauss Equations for $(HN, {}^H\nabla) \hookrightarrow (TM, {}^M\nabla)$. The orthogonal projection $\pi_H : \mathbb{C} \otimes F^*TM \rightarrow \mathbb{C} \otimes HN$, $\pi_H(v) = \theta^\xi(v)e_\xi$, gives rise to the connection ${}^H\nabla = \pi_H \circ {}^M\nabla : \Gamma(\mathbb{C} \otimes TN) \times \Gamma(\mathbb{C} \otimes HN) \rightarrow \Gamma(\mathbb{C} \otimes HN)$ with connection form ${}^H\omega$. Relative to the frame $\{e_k\}$, ${}^H\omega_\beta^\alpha = {}^M\omega_\beta^\alpha$, ${}^H\omega_\alpha^n = {}^H\omega_n^\alpha = 0$. The structure equations of $\mathbb{C} \otimes HN$ read

$$\begin{aligned} d\varphi^\alpha + {}^H\omega_k^\alpha \wedge \varphi^k &= ({}^H\omega_k^\alpha - {}^N\omega_k^\alpha) \wedge \varphi^k \\ &= -{}^N\omega_n^\alpha \wedge \varphi^n \\ &= -\frac{i}{2^{3/2}} \ell_{\bar{\alpha}a} \varphi^a \wedge \varphi^n \\ d{}^H\omega_\beta^\alpha + {}^H\omega_\gamma^\alpha \wedge {}^H\omega_\beta^\gamma &= {}^H\Omega_\beta^\alpha. \end{aligned}$$

From the first equation we read off that ${}^H\nabla$ has torsion. As above, we derive the Gauss equations for $HN \rightarrow F^*TM$:

$$\begin{aligned} ({}^M\Omega - {}^H\Omega)_\beta^\alpha &= d(\omega_\beta^\alpha - {}^H\omega_\beta^\alpha) + \omega_p^q \wedge \omega_\beta^p - {}^H\omega_p^q \wedge {}^H\omega_\beta^p \\ &= \omega_n^\alpha \wedge \omega_\beta^n \\ &= -\frac{1}{2} \ell_{\beta a} \ell_{\bar{\alpha}b} \varphi^a \wedge \varphi^b. \end{aligned}$$

2.4. Codazzi Equations for $(HN, {}^H\nabla)$. The coefficients of the covariant derivative $\ell_{jab}^H = ({}^H\nabla\ell)_{jab}$ of ℓ on $(HN, {}^H\nabla)$ satisfy

$$dl_{ja} = \ell_{jab}^H \varphi^b + l_{pa} {}^H\omega_j^p + l_{jb} {}^H\omega_a^b.$$

Here $\{\ell_{\alpha\xi\eta}^H\}$ is a triply covariant tensor on $\mathbb{C} \otimes HN$, and $\{\ell_{n\xi\eta}^H\}$, $\{\ell_{\alpha n\eta}^H\}$, etc. are of corresponding lower covariance on $\mathbb{C} \otimes HN$. We define

$$\ell_{ab}^2 := \ell_{a\bar{\gamma}} \ell_{\gamma b} \quad , \quad \ell_{ab}^3 := \ell_{a\bar{\gamma}} \ell_{\gamma\bar{\delta}} \ell_{\delta b}$$

and compute as in 2.2:

$$\begin{aligned} d\omega_j^n &= \frac{i}{\sqrt{2}} dl_{ja} \wedge \varphi^a + \frac{i}{\sqrt{2}} l_{ja} d\varphi^a \\ &= \frac{i}{\sqrt{2}} (\ell_{jab}^H \varphi^b + l_{pa} {}^H\omega_j^p + l_{jb} {}^H\omega_a^b) \wedge \varphi^a \\ &\quad + \frac{i}{\sqrt{2}} l_{j\bar{\gamma}} (\varphi^\delta \wedge \omega_\delta^\gamma + \frac{1}{2} \varphi^n \wedge \omega_n^\gamma) + \frac{1}{\sqrt{2}} l_{j\bar{\gamma}} (\varphi^\delta \wedge \omega_\delta^{\bar{\gamma}} + \frac{1}{2} \varphi^n \wedge \omega_n^{\bar{\gamma}}) \\ &\quad + \frac{i}{\sqrt{2}} l_{jn} (i\sqrt{2} \ell_{\gamma\bar{\delta}} \varphi^\gamma \wedge \varphi^{\bar{\delta}} + \frac{i}{\sqrt{2}} (\ell_{n\gamma} \varphi^\gamma - \ell_{n\bar{\gamma}} \varphi^{\bar{\gamma}}) \wedge \varphi^n) \\ &= \frac{i}{\sqrt{2}} \ell_{jab}^H \varphi^b \wedge \varphi^a + {}^H\omega_j^p \wedge \omega_p^n + \frac{1}{4} (\ell_{aj}^2 - \ell_{ja}^2) \varphi^a \wedge \varphi^n \\ &\quad - l_{jn} \ell_{\gamma\bar{\delta}} \varphi^\gamma \wedge \varphi^{\bar{\delta}} + \frac{1}{2} l_{jn} (\ell_{n\bar{\gamma}} \varphi^{\bar{\gamma}} - \ell_{n\gamma} \varphi^\gamma) \wedge \varphi^n. \end{aligned}$$

This, again using the structure equations of F^*TM , gives

$$\begin{aligned} \ell_{jab}^H \varphi^a \wedge \varphi^b &= i\sqrt{2} \omega_p^n \wedge ({}^H\omega_j^p - \omega_j^p) + \frac{i}{2^{3/2}} (\ell_{ja}^2 - \ell_{aj}^2) \varphi^a \wedge \varphi^n \\ &\quad + \frac{i}{\sqrt{2}} l_{jn} (\ell_{n\gamma} \varphi^\gamma - \ell_{n\bar{\gamma}} \varphi^{\bar{\gamma}}) \wedge \varphi^n + i\sqrt{2} l_{jn} \ell_{\gamma\bar{\delta}} \varphi^\gamma \wedge \varphi^{\bar{\delta}} + i\sqrt{2} \Omega_j^n. \end{aligned}$$

We evaluate the first expression on the right-hand side for $j = \alpha$ and $j = n$:

$$\begin{aligned} i\sqrt{2}\omega_p^n \wedge ({}^H\omega_\alpha^p - \omega_\alpha^p) &= \frac{i}{\sqrt{2}}\ell_{na}\ell_{\alpha b}\varphi^a \wedge \varphi^b \\ i\sqrt{2}\omega_p^n \wedge ({}^H\omega_n^p - \omega_n^p) &= \frac{i}{\sqrt{2}}\ell_{ba}^2\varphi^a \wedge \varphi^b. \end{aligned}$$

2.5. Commutator relations for the second covariant derivatives of ℓ on $(HN, {}^H\nabla)$. The coefficients of the second covariant derivative of ℓ on $(HN, {}^H\nabla)$, namely $({}^H\nabla^2\ell)_{jabc} = \ell_{jabc}^H$ satisfy

$$d\ell_{jab}^H = \ell_{jabc}^H\varphi^c + \ell_{pab}^H {}^H\omega_j^p + \ell_{jcb}^H {}^H\omega_a^c + \ell_{jac}^H {}^H\omega_b^c.$$

Then we compute

$$\begin{aligned} d\ell_{ja} &= d(\ell_{jab}^H\varphi^b + \ell_{pa} {}^H\omega_j^p + \ell_{jc} {}^H\omega_a^c) \\ &= (\ell_{jabc}^H\varphi^c + \ell_{pab}^H {}^H\omega_j^p + \ell_{jcb}^H {}^H\omega_a^c + \ell_{jac}^H {}^H\omega_b^c) \wedge \varphi^b \\ &\quad + \ell_{ja\gamma}^H(\varphi^\delta \wedge \omega_\delta^\gamma + \frac{1}{2}\varphi^n \wedge \omega_n^\gamma) + \ell_{ja\bar{\gamma}}^H(\varphi^{\bar{\delta}} \wedge \omega_{\bar{\delta}}^{\bar{\gamma}} + \frac{1}{2}\varphi^n \wedge \omega_n^{\bar{\gamma}}) \\ &\quad + \ell_{jan}^H(i\sqrt{2}\ell_{\gamma\bar{\delta}}\varphi^\gamma \wedge \varphi^{\bar{\delta}} + \frac{i}{\sqrt{2}}(\ell_{n\gamma}\varphi^\gamma - \ell_{n\bar{\gamma}}\varphi^{\bar{\gamma}}) \wedge \varphi^n) \\ &\quad + (\ell_{pab}^H\varphi^b + \ell_{qa} {}^H\omega_p^q + \ell_{pc} {}^H\omega_a^c) \wedge {}^H\omega_j^p - \ell_{pa} {}^H\omega_q^p \wedge {}^H\omega_j^q + \ell_{pa} {}^H\Omega_j^p \\ &\quad + (\ell_{jcb}^H\varphi^b + \ell_{pc} {}^H\omega_j^p + \ell_{jb} {}^H\omega_c^b) \wedge {}^H\omega_b^c - \ell_{jc} {}^H\omega_b^c \wedge {}^H\omega_a^b + \ell_{jc} {}^H\Omega_a^c \\ &= \ell_{jabc}^H\varphi^c \wedge \varphi^b + \frac{i}{2^{3/2}}(\ell_{\xi\bar{\gamma}}\ell_{ja\gamma}^H - \ell_{\gamma\xi}\ell_{ja\bar{\gamma}}^H)\varphi^n \wedge \varphi^\xi \\ &\quad + \frac{i}{\sqrt{2}}\ell_{jan}^H(\ell_{n\gamma}\varphi^\gamma - \ell_{n\bar{\gamma}}\varphi^{\bar{\gamma}}) \wedge \varphi^n + i\sqrt{2}\ell_{jan}^H\ell_{\gamma\bar{\delta}}\varphi^\gamma \wedge \varphi^{\bar{\delta}} \\ &\quad + \ell_{\delta a}({}^H\Omega_j^\delta - {}^M\Omega_j^\delta) + \ell_{j\xi}({}^H\Omega_a^\xi - {}^M\Omega_a^\xi) + \ell_{\delta a} {}^M\Omega_j^\delta + \ell_{j\xi} {}^M\Omega_a^\xi. \end{aligned}$$

Note that $d\ell_{ja} = 0$. This gives for $(ja) = (\alpha, \bar{\beta}), (\alpha, \beta), (n, \bar{\beta}), (n, n)$ the following commutator relations.

$$\begin{aligned} \ell_{\alpha\bar{\beta}c}^H\varphi^b \wedge \varphi^c &= \frac{i}{2^{3/2}}(\ell_{\gamma\xi}\ell_{\alpha\bar{\beta}\bar{\gamma}}^H - \ell_{\xi\bar{\gamma}}\ell_{\alpha\bar{\beta}\gamma}^H)\varphi^\xi \wedge \varphi^n + \frac{i}{\sqrt{2}}\ell_{\alpha\bar{\beta}n}^H(\ell_{n\gamma}\varphi^\gamma - \ell_{n\bar{\gamma}}\varphi^{\bar{\gamma}}) \wedge \varphi^n \\ &\quad + i\sqrt{2}\ell_{\alpha\bar{\beta}n}^H\ell_{\gamma\bar{\delta}}\varphi^{\bar{\gamma}} \wedge \varphi^{\bar{\delta}} + \frac{1}{2}(\ell_{\alpha\bar{\beta}}^2\ell_{\alpha b} + \ell_{\alpha a}^2\ell_{b\bar{\beta}})\varphi^a \wedge \varphi^b \\ &\quad + \ell_{\delta\bar{\beta}} {}^M\Omega_\alpha^\delta + \ell_{\alpha\bar{\delta}} {}^M\Omega_{\bar{\beta}}^{\bar{\delta}} \\ \ell_{\alpha\beta c}^H\varphi^b \wedge \varphi^c &= \frac{i}{2^{3/2}}(\ell_{\gamma\xi}\ell_{\alpha\beta\bar{\gamma}}^H - \ell_{\xi\bar{\gamma}}\ell_{\alpha\beta\gamma}^H)\varphi^\xi \wedge \varphi^n + \frac{1}{\sqrt{2}}\ell_{\alpha\beta n}^H(\ell_{n\gamma}\varphi^\gamma - \ell_{n\bar{\gamma}}\varphi^{\bar{\gamma}}) \wedge \varphi^n \\ &\quad + i\sqrt{2}\ell_{\alpha\beta n}^H\ell_{\gamma\bar{\delta}}\varphi^\gamma \wedge \varphi^{\bar{\delta}} + \frac{1}{2}(\ell_{\alpha\beta}^2\ell_{\alpha b} + \ell_{\alpha a}^2\ell_{\beta b})\varphi^a \wedge \varphi^b + \ell_{\delta\beta} {}^M\Omega_\alpha^\delta + \ell_{\alpha\delta} {}^M\Omega_\beta^\delta \\ \ell_{n\bar{\beta}c}^H\varphi^b \wedge \varphi^c &= \frac{i}{2^{3/2}}(\ell_{\gamma\xi}\ell_{n\bar{\beta}\bar{\gamma}}^H - \ell_{\xi\bar{\gamma}}\ell_{n\bar{\beta}\gamma}^H)\varphi^\xi \wedge \varphi^n + \frac{i}{\sqrt{2}}\ell_{n\bar{\beta}n}^H(\ell_{n\gamma}\varphi^\gamma - \ell_{n\bar{\gamma}}\varphi^{\bar{\gamma}}) \wedge \varphi^n \\ &\quad + i\sqrt{2}\ell_{n\bar{\beta}n}^H\ell_{\gamma\bar{\delta}}\varphi^\gamma \wedge \varphi^{\bar{\delta}} + \frac{1}{2}\ell_{na}^2\ell_{b\bar{\beta}}\varphi^a \wedge \varphi^b + \ell_{n\bar{\gamma}} {}^M\Omega_{\bar{\beta}}^{\bar{\gamma}} \\ \ell_{nbc}^H\varphi^b \wedge \varphi^c &= \frac{i}{2^{3/2}}(\ell_{\gamma\xi}\ell_{nn\bar{\gamma}}^H - \ell_{\xi\bar{\gamma}}\ell_{nn\gamma}^H)\varphi^\xi \wedge \varphi^n + \frac{i}{\sqrt{2}}\ell_{nnn}^H(\ell_{n\gamma}\varphi^\gamma - \ell_{n\bar{\gamma}}\varphi^{\bar{\gamma}}) \wedge \varphi^n \\ &\quad + i\sqrt{2}\ell_{nnn}^H\ell_{\gamma\bar{\delta}}\varphi^\gamma \wedge \varphi^{\bar{\delta}}. \end{aligned}$$

3.1. A model bundle for the embedding problem. Let $(HU, {}^H J)$ be a CR structure on an oriented manifold U of real dimension $2n - 1$, and ${}^U h$ a metric on TU which is hermitian on $(HU, {}^H J)$. In this section we construct an associated bundle EU over U with fibre \mathbb{R}^{2n} and endowed with a complex structure ${}^E J$ and metric ${}^E h$. For this purpose let the real line bundle $RU \rightarrow TU$ be the orthogonal complement of $HU \rightarrow (TU, {}^U h)$.

We define

$$\begin{aligned} EU &:= TU \oplus RU \cong HU \oplus RU \oplus RU \\ {}^E J &:= {}^H J \oplus \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ {}^E h|_{TU} &:= {}^U h, {}^E h \text{ is hermitian on } (EU, {}^E J). \end{aligned}$$

Now choose dual unitary frames $\{e_k\}_1^n, \{\theta^k\}_1^n$ of $E^{1,0}U, (E^{1,0}U)^*$ which are adapted to

$$i_U : TU \rightarrow EU,$$

namely such that $f_\alpha := i_U^* e_\alpha \in H^{1,0}U$, $f_{\bar{n}} := \frac{1}{2}i_U^*(e_n + e_{\bar{n}}) \in TU$ are compatible with the orientation of TU . Let $\{\varphi^a\}$ be the dual frame of $\{f_a\}$.

3.2. Fundamental Existence Theorem for Real Hypersurfaces in Kähler Manifolds. We require the following data.

- a) $(HU, {}^H J)$ is a CR structure on a real oriented $(2n - 1)$ -manifold U
- b) ${}^U h$ is a metric on TU which is hermitian on $(HU, {}^H J)$
- c) $r : U \rightarrow \mathbb{R}$ a function, $v \in U(n)$.

Using the notation of 3.1 and letting ${}^U \nabla$ denote the Levi-Civita connection of $(TU, {}^U h)$, we now define a $u(n)$ -valued connection ${}^E \omega$ on $\mathbb{C} \otimes EU$:

$$\begin{aligned} {}^E \omega_\beta^\alpha &:= \varphi^\alpha({}^U \nabla_{(\cdot)} f_\beta) \\ {}^E \omega_\beta^n &:= \varphi^n({}^U \nabla_{(\cdot)} f_\beta) \\ {}^E \omega_n^n &:= \varphi^n({}^U \nabla_{f_n} f_\gamma) \varphi^\gamma - \varphi^n({}^U \nabla_{f_n} f_{\bar{\gamma}}) \varphi^{\bar{\gamma}} + ir \varphi^n \end{aligned}$$

for $s, t = 1, \dots, n, \bar{1}, \dots, \bar{n}; p, q = 1, \dots, n$. We also define the differential d_H by

$$d_H g := (e_\xi \cdot g) \theta^\xi$$

for a function g on U and in a compatible way for forms on $\mathbb{C} \otimes HU$. Note that in general $d_H^2 \neq 0$.

THEOREM 3.2. *Let $(HU, {}^H J, {}^U h, r, v)$ be given as above and assume*

$$\begin{aligned} d_H(\varphi^n \circ {}^H J) &= 0 \\ {}^H \nabla {}^H J &= 0 \\ d {}^E \omega_k^j + {}^E \omega_p^j \wedge {}^E \omega_k^p &= 0. \end{aligned}$$

Then for every $u \in U$ there exists a neighborhood U_u of u and a unique

$$F : U_u \rightarrow \mathbb{C}^n$$

with

$$\begin{aligned} F^*T^{1,0}\mathbb{C}^n &= E^{1,0}U_u \\ F^*\mathbb{C}^n h &= E h \\ F^*\mathbb{C}^n \omega &= E \omega \\ (F^*v)(u) &= id. \end{aligned}$$

REMARKS. a) Since $i_U : TU \rightarrow EU$ is involutive, we have $i_U^*d(\theta^n - \theta^{\bar{n}}) = 0$, and by $(\theta^n - \theta^{\bar{n}}) = -i(\theta^n + \theta^{\bar{n}}) \circ {}^E J$, the first assumption is an intrinsic version of the involutivity.

b) The second condition is equivalent to ${}^H\omega_\beta^\alpha = 0$, ${}^H\omega_\beta^{\bar{\alpha}} = 0$.

c) By the first conclusion and since $\mathbb{C} \otimes TU \cap E^{1,0}U = H^{1,0}U$, F is a CR -embedding. In addition by the second, F^* preserves adapted unitary frames. Therefore by the third and the definition of ${}^E\omega$, we have

$$\begin{aligned} \ell_{\alpha b} &= \frac{\sqrt{2}}{i} {}^E\omega_\alpha^n(f_b) = -i\sqrt{2}\varphi^n({}^U\nabla_{f_b}f_\alpha) \\ \ell_{n\gamma} &= \frac{\sqrt{2}}{i} {}^E\omega_n^n(f_\gamma) = i\sqrt{2}\varphi^n({}^U\nabla_{f_n}f_\gamma) \\ \ell_{nn} &= \frac{\sqrt{2}}{i} \omega_n^n(f_n) = \sqrt{2}r. \end{aligned}$$

3.3. Proof of the Existence Theorem. We prove Theorem 3.1 in two steps. First we establish the existence of functions $e_j : U \rightarrow T^{1,0}\mathbb{C}^n$, $j = 1, \dots, n$, with the property

$$de_j - \omega_j^p \cdot e_p = 0$$

where $\omega_q^p := {}^E\omega_q^p$ is defined as in 3.2. The functions $\{e_j\}$ are represented by their graph $G \subset U \times \mathbb{C}^{n^2}$. Let π_1, π_2 denote the projections to the first and second factors of the product $U \times \mathbb{C}^{n^2}$ and $z_k \in \mathbb{C}^n$, $k = 1, \dots, n$ be coordinate vectors of \mathbb{C}^{n^2} . Then $TG \subseteq TU \oplus T\mathbb{C}^{n^2}$ is characterized by

$$d(z_k \circ \pi_2) - \sum_{p=1}^n z_p \circ \pi_2 \cdot \pi_1^* \omega_k^p = 0$$

for $k = 1, \dots, n$. Let I be the ideal of differential forms on $U \times \mathbb{C}^{n^2}$ generated by the left hand sides above and their exterior derivatives. We drop the π_j and differentiate:

$$\begin{aligned} d(z_k - z_p \omega_k^p) &= dz_p \wedge \omega_k^p + z_p \cdot d\omega_k^p \\ &= (dz_p - z_p \omega_p^q) \wedge \omega_k^p \\ &\equiv 0 \text{ mod } I. \end{aligned}$$

Therefore, $\ker(z_k - z_p \omega_k^p)$ is an integrable distribution by Frobenius' Theorem, and there exists

$$e : U_u \rightarrow \mathbb{C}^{n^2}, e(p) = v$$

on a neighborhood $U_u \subset U$ of u .

We claim that the image of e is contained in $U(n) \subset \mathbb{C}^{n^2}$; let (\cdot, \cdot) denote the standard hermitian product of \mathbb{C}^n :

$$\begin{aligned} d(e_j, e_k) &= (de_j, e_k) + (e_j, de_k) \\ &= (\omega_j^p e_p, e_k) + (e_j, \omega_k^p e_p) \\ &= \omega_j^p(e_p, e_k) + \omega_k^{\bar{p}}(e_j, e_p) \\ &= \omega_j^p(e_p, e_k) - \omega_p^k(e_j, e_p). \end{aligned}$$

Define $e_{jk}(t) := (e_j, e_k)(\gamma(t))$ for a real curve in U through $u : \gamma(0) = u$. then $e_{jk}(0) = \delta_{jk}$ and $e_{jk} = \delta_{jk}$ solves the ODE

$$\dot{e}_{jk} = \omega_j^p e_{pk} - \omega_p^k e_{jp}.$$

Therefore $e_{jk} \equiv \delta_{jk}$ which proves the claim.

The second part of the proof consists of constructing $F : U_u \rightarrow \mathbb{C}^n$ with

$$dF(f_\xi) = e_\xi, \quad dF(f_n) = \frac{1}{2}(e_n + e_{\bar{n}}).$$

Again this map is represented by its graph in $U \times \mathbb{C}^n$, and its tangent distribution satisfies

$$d(z \circ \pi_2) - (e_\xi \circ \pi_1) \cdot \pi_1^* \circ \varphi^\xi - \frac{1}{2}(e_n + e_{\bar{n}}) \circ \pi_1 \cdot \pi_1^* \varphi^n = 0.$$

Here, z is the coordinate vector of \mathbb{C}^n . The exterior derivative is computed as follows. We set $\nabla = {}^U\nabla$, $\omega = {}^E\omega$.

$$\begin{aligned} d(e_\xi \varphi^\xi + \frac{1}{2}(e_n + e_{\bar{n}})\varphi^n) &= \omega_\xi^t \wedge \varphi^\xi e_t - e_\xi \varphi^\xi (\nabla_{f_a} f_b) \varphi^a \wedge \varphi^b + \frac{1}{2}(\omega_n^k e_k + \omega_{\bar{n}}^{\bar{k}} e_{\bar{k}}) \wedge \varphi^n \\ &\quad - \frac{1}{2}(e_n + e_{\bar{n}})\varphi^n (\nabla_{f_a} f_b) \varphi^a \wedge \varphi^b \\ &= \omega_\gamma^n \wedge \varphi^\gamma \cdot e_n + \omega_{\bar{\gamma}}^{\bar{n}} \wedge \varphi^{\bar{\gamma}} \cdot e_{\bar{n}} - \varphi^\xi (\nabla_{f_a} f_n) \varphi^a \wedge \varphi^n e_\xi \\ &\quad + \frac{1}{2}\omega_n^{\bar{\delta}} \wedge \varphi^n (e_n - e_{\bar{n}}) - \frac{1}{2}(\omega_{\bar{\delta}}^n e_\delta + \omega_\delta^n e_{\bar{\delta}}) \wedge \varphi^n \\ &\quad - \frac{1}{2}(e_n + e_{\bar{n}})\varphi^n (\nabla_{f_a} f_b) \varphi^a \wedge \varphi^b. \end{aligned}$$

Note that $\varphi^\xi (\nabla_{f_a} f_n) = -\frac{1}{2}\varphi^n (\nabla_{f_{\bar{a}}} f_\xi)$, here we set $\bar{n} = n$. Then by definition of $\omega = {}^E\omega$ in 3.2 we continue

$$\begin{aligned} d(e_\xi \varphi^\xi + \frac{1}{2}(e_n + e_{\bar{n}})\varphi^n) &= [\varphi^n (\nabla_{f_a} f_\gamma) \varphi^a \wedge \varphi^\gamma + \frac{1}{2}(\varphi^n (\nabla_{f_n} f_\gamma) \varphi^\gamma \\ &\quad - \varphi^n (\nabla_{f_n} f_{\bar{\gamma}}) \varphi^{\bar{\gamma}}) \wedge \varphi^n - \frac{1}{2}\varphi^n (\nabla_{f_a} f_b) \varphi^a \wedge \varphi^b] e_n \\ &\quad + [\varphi^n (\nabla_{f_a} f_{\bar{\gamma}}) \varphi^a \wedge \varphi^{\bar{\gamma}} - \frac{1}{2}(\varphi^n (\nabla_{f_n} f_\gamma) \varphi^\gamma \\ &\quad - \varphi^n (\nabla_{f_n} f_{\bar{\gamma}}) \varphi^{\bar{\gamma}}) \wedge \varphi^n - \frac{1}{2}\varphi^n (\nabla_{f_a} f_b) \varphi^a \wedge \varphi^b] e_{\bar{n}} \\ &= -\frac{1}{2}\varphi^n (\nabla_{f_{\bar{\delta}}} f_\gamma + \nabla_{f_\gamma} f_{\bar{\delta}}) \varphi^\gamma \wedge \varphi^{\bar{\delta}} (e_n - e_{\bar{n}}) \\ &= \frac{i}{2}\varphi^n ({}^H J (\nabla_{f_{\bar{\delta}}} f_\gamma - \nabla_{f_\gamma} f_{\bar{\delta}})) \varphi^\gamma \wedge \varphi^{\bar{\delta}} (e_n - e_{\bar{n}}) \\ &= 0. \end{aligned}$$

The last equation follows from the first assumption of the Fundamental Existence Theorem 3.2. We therefore conclude again by Frobenius' Theorem that $F : U_u \rightarrow \mathbb{C}^n$ exists as above and satisfies the conclusions of the Theorem by construction.

3.4. Existence Theorem for nondegenerate CR manifolds. Here we prescribe a different set of data from 3.2. Namely we assume, using the notation of 3.1:

- a) $(HU, {}^H J)$ is a nondegenerate oriented CR-manifold.
- b1) τ is a one-form on TU with $\ker \tau = HN$ and compatible with the orientation of U , and $s : U \rightarrow \mathbb{R}^+$
- b2) ${}^H h$ is a hermitian metric on $(HN, {}^H J)$
- b3) $\{r_\alpha\} : U \rightarrow \mathbb{C}^{n-1}$
- c) $r : U \rightarrow \mathbb{R}$, $v \in U(n, \mathbb{C})$.

PROPOSITION 3.3. *Let $(HU, {}^H J, {}^H h, \tau, s, r_\alpha)$ be as in a)-b3) and $\{f_\alpha\}$ as in 3.1. Then there exists a unique vector field $X \in \Gamma(TU)$ with*

- i) $\tau(X) = s$
- ii) $h(\nabla_X X, f_\alpha) = r_\alpha$ for the metric h on TU with $h|_H = {}^H h$, $h(X, X) = 1$, $X \perp H$, and ∇ its Levi-Civita connection.

Proof. Let $\{f_\alpha, t\}$ be dual to $\{\varphi^\alpha, \tau\}$ and make the ansatz $X = a_\eta f_\eta + st$. Then by [KN I, p. 160] and the integrability of the CR structure,

$$\begin{aligned} h(\nabla_X X, f_\alpha) &= 2h([f_{\bar{\alpha}}, X], X) \\ &= 2h([f_{\bar{\alpha}}, a_\gamma f_\gamma + a_{\bar{\gamma}} f_{\bar{\gamma}} + st], X) \\ &= 2a_\gamma \zeta([f_{\bar{\alpha}}, f_\gamma]) + 2s\zeta([f_{\bar{\alpha}}, t]) + 2s^{-1}(f_{\bar{\alpha}} \cdot s). \end{aligned}$$

Here, $\zeta = h(\cdot, X)$ and one has $\zeta = s^{-1}\tau$. Therefore,

$$h(\nabla_X X, f_\alpha) = 2s^{-1}a_\gamma \tau([f_{\bar{\alpha}}, f_\gamma]) + 2\tau([f_{\bar{\alpha}}, t]) + 2s^{-1}(f_{\bar{\alpha}} \cdot s).$$

Since $(HN, {}^H J) \hookrightarrow TN$ is a nondegenerate CR manifold, its Levi-form $\tau[f_{\bar{\alpha}}, f_\gamma]$ is invertible, and the above equation can be solved uniquely for a_γ . Then, since $a_{\bar{\gamma}} = \overline{a_\gamma}$, the vector field X is uniquely determined. \square

Let $\varphi^n := \zeta, f_n := X, {}^U h := h$ and recall the notation of 3.1. Now defining ${}^E \omega_\beta^\alpha = \varphi^\alpha(\nabla_{(\cdot)} f_\beta)$, ${}^E \omega_\beta^n = \varphi^n(\nabla_{f_\xi} f_\beta)\varphi^\xi + \frac{1}{i\sqrt{2}}r_\beta \varphi^n$, ${}^E \omega_n^n := \frac{1}{\sqrt{2}}(r_\gamma \varphi^\gamma - r_{\bar{\gamma}} \varphi^{\bar{\gamma}}) + ir\varphi^n$, we have the following

COROLLARY 3.4. *Let $(HU, {}^H J, {}^H h, \tau, s, r_\alpha, r, v)$ be as in a)-c). Then the statement of Theorem 3.2 holds for ${}^E \omega$ as defined above.*

Proof. By Proposition 3.3, the above definition of ${}^E \omega$ matches the one in 3.2. Therefore we may apply Theorem 3.2. \square

4.0. Laplace Operator on $(TN, {}^N h)$. Setting $f_n = f_{\bar{n}}$, the Laplace operator on a tensor T on $\mathbb{C} \otimes TN$ is given by

$${}^N \Delta T := \text{trace}_{TN}({}^N \nabla^2 T).$$

Note that for an immersion $F : N^{2n-1} \rightarrow M$, ${}^N \Delta F = {}^M \nabla_{f_k}^M \nabla_{f_{\bar{k}}} F - {}^M \nabla_{N \nabla_{f_k} f_{\bar{k}}} F + \text{complex conjugate} = {}^M \nabla_{f_k} f_{\bar{k}} - {}^N \nabla_{f_k} f_{\bar{k}} + \text{c.c.} = L^N X_{2n}$, where $L^N := \Sigma \ell_{k\bar{k}}$.

The latter equals the trace of the Riemannian second fundamental form oriented in such a way that $L^N X_{2n}$ points out of a convex hypersurface. Let now

$$\ell_{(\alpha\bar{\beta})}^2 := \sum_{j=1}^{k=2n-1} k_{\alpha j} k_{j\beta} + k_{n+\alpha j} k_{jn+\beta} + i(k_{\alpha j} k_{jn+\beta} - k_{\beta j} k_{jn+\alpha}).$$

Note the analogy with $\ell_{\alpha\bar{\beta}}$ as given in 1.4. For the other coefficients (pa) , $\ell_{(pa)}^2$ is defined analogously in the pattern of 1.4. The is easy to verify the following

$$\ell_{(pa)}^2 = \frac{1}{2} \ell_{p\bar{n}} \ell_{n\alpha} + \ell_{pn} \ell_{n\alpha}.$$

We also define $|\ell|^2 := \sum_{p,q=1}^{2n-1} k_{pq}^2$. Now the Simons commutator identity in \mathbb{C}^n , see [CdCK] reads

$${}^N \Delta \ell_{pa} - 2({}^N \nabla^2 L^N)_{pa} = L^N \ell_{(pa)}^2 - |\ell|^2 \ell_{pa}.$$

4.1. Laplace Operator on $(HN, {}^H h)$. We define a weakly elliptic second order differential operator

$${}^H \Delta T := \text{trace}_{\text{HN}}({}^H \nabla^2 T).$$

for a tensor T as above. It gives a tensor ${}^H \Delta T$ on $\mathbb{C} \otimes HN$ of the same type. We also define the trace of the Levi form,

$$L := \sum \ell_{\alpha\bar{\alpha}},$$

which may be viewed as a horizontal mean curvature of N . For $F : N \rightarrow M$, we have

$$\begin{aligned} {}^H \Delta F &= {}^M \nabla_{f_{\xi}}^M \nabla_{f_{\bar{\xi}}} F - {}^M \nabla_{H \nabla_{f_{\xi}} f_{\bar{\xi}}} F \\ &= {}^M \nabla_{f_{\xi}} f_{\bar{\xi}} - {}^H \nabla_{f_{\xi}} f_{\bar{\xi}} \\ &= \theta^n ({}^M \nabla_{e_{\alpha}} e_{\bar{\alpha}} + {}^M \nabla_{e_{\bar{\alpha}}} e_{\alpha}) e_n + \theta^{\bar{n}} ({}^M \nabla_{e_{\alpha}} e_{\bar{\alpha}} + {}^M \nabla_{e_{\bar{\alpha}}} e_{\alpha}) e_{\bar{n}} \\ &= \frac{i}{\sqrt{2}} \ell_{\alpha\bar{\alpha}} e_n - \frac{i}{\sqrt{2}} \ell_{\alpha\bar{\alpha}} e_{\bar{n}} \\ &= LX_{2n}. \end{aligned}$$

4.2. Commutator identities for ${}^H \Delta \ell$ in Euclidean Space. For a real hypersurface $F : N \rightarrow \mathbb{C}^n$ we compute the following commutator relations for ${}^H \Delta \ell$. We use the Codazzi and second-order relations from 2.4 and 2.5. Note that since $M = \mathbb{C}^n$ here, ${}^M \Omega \equiv 0$.

$$\begin{aligned} &{}^H \Delta \ell_{\alpha\bar{\beta}} - 2\ell_{\gamma\bar{\gamma}\alpha\bar{\beta}} \\ &= \ell_{\alpha\bar{\beta}\gamma\bar{\gamma}} - \ell_{\alpha\bar{\beta}\bar{\gamma}\gamma} + 2(\ell_{\alpha\bar{\beta}\bar{\gamma}} - \ell_{\alpha\bar{\gamma}\bar{\beta}})\gamma + 2(\ell_{\alpha\bar{\gamma}\bar{\beta}\gamma} - \ell_{\alpha\bar{\gamma}\gamma\bar{\beta}}) + 2(\ell_{\alpha\bar{\gamma}\gamma} - \ell_{\gamma\bar{\gamma}\alpha})\bar{\beta} \\ &= i\sqrt{2}L\ell_{\alpha\bar{\beta}n} + \frac{1}{2}(\ell_{\gamma\bar{\beta}}^2 \ell_{\alpha\bar{\gamma}} + \ell_{\alpha\bar{\gamma}}^2 \ell_{\gamma\bar{\beta}} - \ell_{\bar{\gamma}\bar{\beta}}^2 \ell_{\alpha\bar{\gamma}} - \ell_{\alpha\bar{\gamma}}^2 \ell_{\gamma\bar{\beta}}) \\ &\quad + i\sqrt{2}(\ell_{n\bar{\beta}} \ell_{\alpha\bar{\gamma}} - \ell_{n\bar{\gamma}} \ell_{\alpha\bar{\beta}})\gamma \\ &\quad - i\sqrt{2}^{3/2} \ell_{\gamma\bar{\beta}} \ell_{\alpha\bar{\gamma}n} + (\ell_{\beta\bar{\gamma}}^2 \ell_{\alpha\bar{\gamma}} + \ell_{\alpha\bar{\beta}}^2 L - \ell_{\gamma\bar{\gamma}}^2 \ell_{\alpha\bar{\beta}} - \ell_{\alpha\bar{\gamma}}^2 \ell_{\gamma\bar{\beta}}) \\ &\quad + i\sqrt{2}(\ell_{\alpha\bar{\gamma}} \ell_{\gamma n} - L\ell_{n\alpha})\bar{\beta} \\ &= i\sqrt{2}[L\ell_{\alpha\bar{\beta}n} + \ell_{\alpha\bar{\gamma}}(\ell_{n\bar{\beta}\gamma} - \ell_{\gamma\bar{\beta}n}) + \ell_{\alpha\bar{\gamma}} \ell_{\gamma\bar{\beta}n} + \ell_{n\bar{\beta}}(\ell_{\alpha\bar{\gamma}\gamma} - \ell_{\gamma\bar{\gamma}\alpha}) + \ell_{n\bar{\beta}} L\alpha \end{aligned}$$

$$\begin{aligned}
 & -l_{\alpha\bar{\beta}}(l_{n\bar{\gamma}\gamma} - l_{\gamma\bar{\gamma}n}) - l_{\alpha\bar{\beta}}L_n - l_{n\bar{\gamma}}l_{\alpha\bar{\beta}\gamma} - 2l_{\gamma\bar{\beta}}l_{\alpha\bar{\gamma}n} + l_{n\gamma}(l_{\alpha\bar{\gamma}\bar{\beta}} - l_{\alpha\bar{\beta}\bar{\gamma}}) \\
 & + l_{n\gamma}l_{\alpha\bar{\beta}\bar{\gamma}} + l_{\alpha\bar{\gamma}}(l_{\gamma n\bar{\beta}} - l_{\gamma\bar{\beta}n}) + l_{\alpha\bar{\gamma}}l_{\gamma\bar{\beta}n} - l_{n\alpha}L_{\bar{\beta}} - L(l_{\alpha n\bar{\beta}} - l_{\alpha\bar{\beta}n}) \\
 & - L l_{\alpha\bar{\beta}n} + l_{\alpha\bar{\beta}}^2 L - l_{\alpha\bar{\beta}}l_{\gamma\bar{\gamma}}^2 + l_{\bar{\beta}\alpha}^3 - \frac{1}{2}(l_{\alpha\gamma}^2 l_{\bar{\gamma}\bar{\beta}} + l_{\alpha\gamma} l_{\bar{\gamma}\bar{\beta}}^2) \\
 = & i\sqrt{2}[l_{n\gamma}l_{\alpha\bar{\beta}\bar{\gamma}} - l_{n\bar{\gamma}}l_{\alpha\bar{\beta}\gamma} + 2(l_{\alpha\bar{\gamma}}l_{\gamma\bar{\beta}n} - l_{\gamma\bar{\beta}}l_{\alpha\bar{\gamma}n}) + l_{n\bar{\beta}}L_{\alpha} - l_{n\alpha}L_{\bar{\beta}} \\
 & - l_{\alpha\bar{\beta}}L_n] - [l_{\alpha\bar{\gamma}}(-l_{nn}l_{\gamma\bar{\beta}} + \frac{1}{2}(l_{\gamma\bar{\beta}}^2 - l_{\bar{\beta}\gamma}^2)) + l_{n\bar{\beta}}(l_{\alpha\bar{\gamma}}l_{\gamma n} - l_{n\alpha}L) \\
 & - l_{\alpha\bar{\beta}}(-l_{nn}L + \frac{1}{2}(l_{\gamma\bar{\gamma}}^2 - l_{\bar{\gamma}\gamma}^2)) + l_{n\gamma}(l_{n\bar{\gamma}}l_{\alpha\bar{\beta}} - l_{n\bar{\beta}}l_{\alpha\bar{\gamma}}) \\
 & + l_{\alpha\bar{\gamma}}(l_{nn}l_{\gamma\bar{\beta}} + \frac{1}{2}(l_{\bar{\beta}\gamma}^2 - l_{\gamma\bar{\beta}}^2)) - L(l_{nn}l_{\alpha\bar{\beta}} + \frac{1}{2}(l_{\bar{\beta}\alpha}^2 - l_{\alpha\bar{\beta}}^2))] \\
 & + l_{\alpha\bar{\beta}}^2 L - l_{\alpha\bar{\beta}}l_{\gamma\bar{\gamma}}^2 + l_{\bar{\beta}\alpha}^3 - \frac{1}{2}(l_{\alpha\gamma}^2 l_{\bar{\gamma}\bar{\beta}} + l_{\alpha\gamma} l_{\bar{\gamma}\bar{\beta}}^2) \\
 = & i\sqrt{2}[l_{n\gamma}l_{\alpha\bar{\beta}\bar{\gamma}} - l_{n\bar{\gamma}}l_{\alpha\bar{\beta}\gamma} + 2(l_{\alpha\bar{\gamma}}l_{\gamma\bar{\beta}n} - l_{\gamma\bar{\beta}}l_{\alpha\bar{\gamma}n}) + l_{n\bar{\beta}}L_{\alpha} - l_{n\alpha}L_{\bar{\beta}} - l_{\alpha\bar{\beta}}L_n] \\
 & - \frac{1}{2}l_{\alpha\bar{\beta}}(l_{\gamma\bar{\gamma}}^2 + l_{\bar{\gamma}\gamma}^2 + 2l_{nn}^2) + \frac{1}{2}L(l_{\alpha\bar{\beta}}^2 + l_{\bar{\beta}\alpha}^2 + 2l_{n\alpha}l_{n\bar{\beta}}) \\
 & + l_{\bar{\beta}\alpha}^3 - \frac{1}{2}(l_{\alpha\gamma}^2 l_{\bar{\gamma}\bar{\beta}} + l_{\alpha\gamma} l_{\bar{\gamma}\bar{\beta}}^2).
 \end{aligned}$$

$$\begin{aligned}
 & {}^H\Delta l_{\alpha\beta} - 2l_{\gamma\bar{\gamma}\alpha\beta} \\
 = & l_{\alpha\beta\gamma\bar{\gamma}} - l_{\alpha\beta\bar{\gamma}\gamma} + 2(l_{\alpha\beta\bar{\gamma}} - l_{\alpha\bar{\gamma}\beta})_{\gamma} + 2(l_{\alpha\bar{\gamma}\beta\gamma} - l_{\alpha\bar{\gamma}\gamma\beta}) + 2(l_{\alpha\bar{\gamma}\gamma} - l_{\gamma\bar{\gamma}\alpha})_{\beta} \\
 = & i\sqrt{2}L l_{\alpha\beta n} + \frac{1}{2}(l_{\gamma\beta}^2 l_{\alpha\bar{\gamma}} + l_{\gamma\alpha}^2 l_{\beta\bar{\gamma}} - l_{\bar{\gamma}\beta}^2 l_{\alpha\gamma} - l_{\bar{\gamma}\alpha}^2 l_{\beta\gamma}) \\
 & + i\sqrt{2}(l_{n\beta}l_{\alpha\bar{\gamma}} - l_{n\bar{\gamma}}l_{\alpha\beta} + 2l_{\alpha n}l_{\beta\bar{\gamma}})_{\gamma} \\
 & + l_{\beta\bar{\gamma}}^2 l_{\alpha\gamma} + l_{\alpha\beta}^2 l_{\gamma\bar{\gamma}} - l_{\gamma\bar{\gamma}}^2 l_{\alpha\beta} - l_{\alpha\gamma}^2 l_{\beta\bar{\gamma}} + i\sqrt{2}(l_{\alpha\bar{\gamma}}l_{\gamma n} - l_{n\alpha}l_{\gamma\bar{\gamma}})_{\beta} \\
 = & i\sqrt{2}[L l_{\alpha\beta n} + l_{n\beta}(l_{\alpha\bar{\gamma}\gamma} - l_{\gamma\bar{\gamma}\alpha}) + l_{n\beta}L_{\alpha} + l_{\alpha\bar{\gamma}}(l_{\beta n\gamma} - l_{\beta\gamma n}) + l_{\alpha\bar{\gamma}}l_{\gamma\beta n} \\
 & - l_{n\bar{\gamma}}l_{\alpha\beta\gamma} - l_{\alpha\beta}(l_{n\bar{\gamma}\gamma} - l_{\gamma\bar{\gamma}n}) - l_{\alpha\beta}L_n + 2l_{\alpha n}(l_{\beta\bar{\gamma}\gamma} - l_{\gamma\bar{\gamma}\beta}) + 2l_{\alpha n}L_{\beta} \\
 & + 2l_{\beta\bar{\gamma}}(l_{\alpha n\gamma} - l_{\alpha\gamma n}) + 2l_{\beta\bar{\gamma}}l_{\gamma\alpha n} + l_{\alpha\bar{\gamma}}(l_{\gamma n\beta} - l_{\gamma\beta n}) + l_{\alpha\bar{\gamma}}l_{\gamma\beta n} \\
 & + l_{n\gamma}(l_{\alpha\bar{\gamma}\beta} - l_{\alpha\beta\bar{\gamma}}) + l_{n\gamma}l_{\alpha\beta\bar{\gamma}} - l_{n\alpha}L_{\beta} - L(l_{\alpha n\beta} - l_{\alpha\beta n}) - L l_{\alpha\beta n}] \\
 & + l_{\alpha\beta}^2 L - l_{\alpha\beta}l_{\gamma\bar{\gamma}}^2 + l_{\beta\bar{\gamma}}^2 l_{\alpha\gamma} - l_{\beta\bar{\gamma}}l_{\alpha\gamma}^2 + \frac{1}{2}(l_{\alpha\beta}^3 + l_{\beta\alpha}^3) - \frac{1}{2}(l_{\alpha\gamma}^2 l_{\bar{\gamma}\beta} + l_{\bar{\gamma}\alpha}^2 l_{\beta\gamma}) \\
 = & i\sqrt{2}[l_{n\gamma}l_{\alpha\beta\bar{\gamma}} - l_{n\bar{\gamma}}l_{\alpha\beta\gamma} + 2(l_{\alpha\bar{\gamma}}l_{\gamma\beta n} + l_{\beta\bar{\gamma}}l_{\gamma\alpha n}) + l_{n\beta}L_{\alpha} + l_{n\alpha}L_{\beta} - l_{\alpha\beta}L_n] \\
 & - [l_{n\beta}(l_{\alpha\bar{\gamma}}l_{\gamma n} - l_{n\alpha}L) + l_{\alpha\bar{\gamma}}(l_{nn}l_{\beta\gamma} - 2l_{n\beta}l_{n\gamma} + \frac{1}{2}(l_{\gamma\beta}^2 - l_{\beta\gamma}^2)) \\
 & - l_{\alpha\beta}(-L l_{nn} + \frac{1}{2}(l_{\gamma\bar{\gamma}}^2 - l_{\bar{\gamma}\gamma}^2)) + 2l_{\alpha n}(l_{\beta\bar{\gamma}}l_{\gamma n} - l_{n\beta}L) \\
 & + 2l_{\beta\bar{\gamma}}(l_{nn}l_{\alpha\gamma} - 2l_{n\alpha}l_{n\gamma} + \frac{1}{2}(l_{\gamma\alpha}^2 - l_{\alpha\gamma}^2)) \\
 & + l_{\alpha\bar{\gamma}}(l_{nn}l_{\gamma\beta} - 2l_{n\gamma}l_{n\beta} + \frac{1}{2}(l_{\beta\gamma}^2 - l_{\gamma\beta}^2)) \\
 & + l_{n\gamma}(l_{n\bar{\gamma}}l_{\alpha\beta} - l_{n\beta}l_{\alpha\bar{\gamma}} - 2l_{\alpha n}l_{\beta\bar{\gamma}}) - L(l_{nn}l_{\alpha\beta} - 2l_{n\alpha}l_{n\beta} + \frac{1}{2}(l_{\beta\alpha}^2 - l_{\alpha\beta}^2))] \\
 & + l_{\alpha\beta}^2 L - l_{\alpha\beta}l_{\gamma\bar{\gamma}}^2 + l_{\beta\bar{\gamma}}^2 l_{\alpha\gamma} - l_{\beta\bar{\gamma}}l_{\alpha\gamma}^2 + \frac{1}{2}(l_{\alpha\beta}^3 + l_{\beta\alpha}^3) - \frac{1}{2}(l_{\alpha\gamma}^2 l_{\bar{\gamma}\beta} + l_{\bar{\gamma}\alpha}^2 l_{\beta\gamma}) \\
 = & i\sqrt{2}[l_{n\gamma}l_{\alpha\beta\bar{\gamma}} - l_{n\bar{\gamma}}l_{\alpha\beta\gamma} + 2(l_{\alpha\bar{\gamma}}l_{\gamma\beta n} + l_{\beta\bar{\gamma}}l_{\gamma\alpha n}) + l_{n\alpha}L_{\beta} + l_{n\beta}L_{\alpha} - l_{\alpha\beta}L_n]
 \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2}l_{\alpha\beta}(\ell_{\gamma\bar{\gamma}}^2 + \ell_{\bar{\gamma}\gamma}^2 + 2\ell_{nn}^2) + \frac{1}{2}L(\ell_{\alpha\beta}^2 + \ell_{\beta\alpha}^2 + 2l_{n\alpha}l_{n\beta}) \\
& -2l_{nn}(\ell_{\alpha\beta}^2 + \ell_{\beta\alpha}^2) + 4(\ell_{\alpha n}^2 l_{n\beta} + l_{\alpha n} \ell_{\beta n}^2) + \frac{1}{2}(\ell_{\alpha\beta}^3 + \ell_{\beta\alpha}^3) - \frac{1}{2}(l_{\alpha\gamma} \ell_{\bar{\gamma}\beta}^2 + \ell_{\bar{\alpha}}^2 l_{\beta\gamma}).
\end{aligned}$$

$$\begin{aligned}
& {}^H\Delta l_{n\beta} - 2l_{\gamma\bar{\gamma}n\beta} \\
& = l_{n\beta\gamma\bar{\gamma}} - l_{n\beta\bar{\gamma}\gamma} + 2(l_{n\beta\bar{\gamma}} - l_{n\bar{\gamma}\beta})_{\gamma} + 2(l_{n\bar{\gamma}\beta\gamma} - l_{n\bar{\gamma}\gamma\beta}) + 2(l_{n\bar{\gamma}\gamma} - l_{\gamma\bar{\gamma}n})_{\beta} \\
& = i\sqrt{2}Ll_{n\beta n} + \frac{1}{2}(\ell_{\beta n}^3 - \ell_{\bar{\gamma}n}^2 l_{\beta\gamma}) + i\sqrt{2}(\ell_{\bar{\gamma}\beta}^2 - \ell_{\beta\bar{\gamma}}^2 + 2l_{nn}l_{\beta\bar{\gamma}})_{\gamma} \\
& \quad + \ell_{n\beta}^2 l_{\gamma\bar{\gamma}} - \ell_{n\gamma}^2 l_{\beta\bar{\gamma}} + i\sqrt{2}(-Ll_{nn} + \frac{1}{2}(\ell_{\gamma\bar{\gamma}}^2 - \ell_{\bar{\gamma}\gamma}^2))_{\beta} \\
& = i\sqrt{2}[Ll_{n\beta n} + l_{\bar{\gamma}\delta}(\ell_{\delta\beta\gamma} - \ell_{\delta\gamma\beta}) + l_{\bar{\gamma}\delta}l_{\delta\gamma\beta} + l_{\delta\beta}(\ell_{\delta\bar{\gamma}\gamma} - l_{\gamma\bar{\gamma}\delta}) + l_{\beta\delta}L_{\delta} \\
& \quad - l_{\beta\delta}(\ell_{\delta\bar{\gamma}\gamma} - l_{\gamma\bar{\gamma}\delta}) - l_{\beta\delta}L_{\delta} - l_{\delta\bar{\gamma}}(\ell_{\beta\delta\gamma} - l_{\gamma\delta\beta}) - l_{\delta\bar{\gamma}}l_{\gamma\delta\beta} \\
& \quad + 2l_{\beta\bar{\gamma}}(l_{nn\gamma} - l_{n\gamma n}) + 2l_{\beta\bar{\gamma}}l_{n\gamma n} + 2l_{nn}(l_{\beta\bar{\gamma}\gamma} - l_{\gamma\bar{\gamma}\beta}) + 2l_{nn}L_{\beta} \\
& \quad - l_{nn}L_{\beta} - L(l_{nn\beta} - l_{n\beta n}) - Ll_{n\beta n} + \frac{1}{2}(\ell_{\gamma\bar{\gamma}}^2 - \ell_{\bar{\gamma}\gamma}^2)_{\beta}] \\
& \quad + \frac{1}{2}(\ell_{\beta n}^3 - \ell_{\bar{\gamma}n}^2 l_{\beta\gamma}) + \ell_{n\beta}^2 L - \ell_{n\gamma}^2 l_{\beta\bar{\gamma}} \\
& = i\sqrt{2}[2l_{\beta\bar{\gamma}}l_{n\gamma n} + l_{nn}L_{\beta} + l_{\beta\delta}L_{\delta} - l_{\beta\delta}L_{\delta} + \frac{1}{2}(l_{\bar{\gamma}\delta}l_{\delta\gamma\beta} - l_{\delta\gamma}l_{\delta\bar{\gamma}\beta})] \\
& \quad - [l_{\bar{\gamma}\delta}(l_{n\beta}l_{\delta\gamma} - l_{n\gamma}l_{\delta\beta}) + l_{\delta\beta}(\ell_{\bar{\gamma}n}^2 - Ll_{n\delta} - 2\ell_{n\delta}^2) \\
& \quad - l_{\beta\delta}(l_{\delta\bar{\gamma}}l_{\gamma n} - l_{n\delta}l_{\gamma\bar{\gamma}}) - l_{\delta\bar{\gamma}}(l_{n\gamma}l_{\beta\delta} - l_{\gamma\delta}l_{\beta n}) \\
& \quad + 2l_{\beta\bar{\gamma}}(\frac{3}{2}(\ell_{\gamma n}^2 - \ell_{n\gamma}^2) - l_{nn}l_{n\gamma}) + 2l_{nn}(l_{\beta\bar{\gamma}}l_{\gamma n} - l_{n\beta}L) \\
& \quad - L(\frac{3}{2}(\ell_{\beta n}^2 - \ell_{n\beta}^2) - l_{nn}l_{n\beta})] + \frac{1}{2}(\ell_{\beta n}^3 - \ell_{\bar{\gamma}n}^2 l_{\beta\gamma}) + \ell_{n\beta}^2 L - \ell_{n\gamma}^2 l_{\beta\bar{\gamma}} \\
& = i\sqrt{2}[2l_{\beta\bar{\gamma}}l_{n\gamma n} + l_{nn}L_{\beta} + l_{\beta\delta}L_{\delta} - l_{\beta\delta}L_{\delta} + \frac{1}{2}(l_{\bar{\gamma}\delta}l_{\delta\gamma\beta} - l_{\gamma\delta}l_{\delta\bar{\gamma}\beta})] \\
& \quad - l_{n\beta}(\ell_{\gamma\bar{\gamma}}^2 + \ell_{\bar{\gamma}\gamma}^2) + \frac{1}{2}L(\ell_{n\beta}^2 + \ell_{\beta n}^2 + 2l_{n\beta}l_{nn}) \\
& \quad + 2(l_{\beta\bar{\gamma}}\ell_{n\gamma}^2 + \ell_{n\beta}^3) - \frac{1}{2}(l_{n\gamma}\ell_{\bar{\gamma}\beta}^2 + \ell_{\beta n}^3).
\end{aligned}$$

$$\begin{aligned}
& {}^H\Delta l_{nn} - 2l_{\gamma\bar{\gamma}nn} \\
& = l_{nn\gamma\bar{\gamma}} - l_{nn\bar{\gamma}\gamma} + 2(l_{nn\bar{\gamma}} - l_{n\bar{\gamma}n})_{\gamma} + 2(l_{n\bar{\gamma}n\gamma} - l_{n\bar{\gamma}\gamma n}) + 2(l_{n\bar{\gamma}\gamma} - l_{\gamma\bar{\gamma}n})_n \\
& = i\sqrt{2}Ll_{nnn} + i\sqrt{2}(\frac{3}{2}(\ell_{\bar{\gamma}n}^2 - \ell_{n\bar{\gamma}}^2) + l_{nn}l_{n\bar{\gamma}})_{\gamma} \\
& \quad - i\sqrt{2}l_{n\bar{\gamma}n}l_{n\gamma} + \frac{i}{\sqrt{2}}(l_{\gamma\delta}l_{n\bar{\gamma}\delta} - l_{\delta\gamma}l_{n\bar{\gamma}\delta}) + \ell_{nn}^2 l_{\gamma\bar{\gamma}} - \ell_{n\gamma}^2 l_{n\bar{\gamma}} \\
& \quad + i\sqrt{2}(-l_{nn}L + \frac{1}{2}(\ell_{\gamma\bar{\gamma}}^2 - \ell_{\bar{\gamma}\gamma}^2))_n \\
& = i\sqrt{2}[Ll_{nnn} + \frac{3}{2}l_{\bar{\gamma}\delta}(l_{\delta n\gamma} - l_{\delta\gamma n}) + \frac{3}{2}l_{\bar{\gamma}\delta}l_{\delta\gamma n} \\
& \quad + \frac{3}{2}l_{n\delta}(l_{\delta\bar{\gamma}\gamma} - l_{\gamma\bar{\gamma}\delta}) + \frac{3}{2}l_{n\delta}L_{\delta} - \frac{3}{2}l_{n\delta}(l_{\delta\bar{\gamma}\gamma} - l_{\gamma\bar{\gamma}\delta}) - \frac{3}{2}l_{n\delta}L_{\delta} \\
& \quad - \frac{3}{2}l_{\delta\bar{\gamma}}(l_{n\delta\gamma} - l_{\gamma\delta n}) - \frac{3}{2}l_{\delta\bar{\gamma}}l_{\gamma\delta n} + l_{n\bar{\gamma}}l_{nn\gamma} + l_{nn}(l_{n\bar{\gamma}\gamma} - l_{\gamma\bar{\gamma}n}) + l_{nn}L_n]
\end{aligned}$$

$$\begin{aligned}
 & -l_{n\gamma}(l_{n\bar{\gamma}n} - l_{nn\bar{\gamma}}) - l_{n\gamma}l_{nn\bar{\gamma}} + \frac{1}{2}l_{\gamma\bar{\delta}}(l_{n\bar{\gamma}\delta} - l_{\delta\bar{\gamma}n}) + \frac{1}{2}l_{\gamma\bar{\delta}}l_{\delta\bar{\gamma}n} \\
 & - \frac{1}{2}l_{\delta\gamma}(l_{n\bar{\gamma}\delta} - l_{\delta\bar{\gamma}n}) - \frac{1}{2}l_{\delta\gamma}l_{\delta\bar{\gamma}n} - l_{nn}L_n - Ll_{nnn} + \frac{1}{2}(l_{\gamma\bar{\gamma}}^2 - l_{\bar{\gamma}\gamma}^2)n] \\
 & + l_{nn}^2L - l_{n\gamma}^2l_{n\bar{\gamma}} \\
 = & i\sqrt{2}[l_{n\bar{\gamma}}l_{nn\gamma} - l_{n\gamma}l_{nn\bar{\gamma}} + \frac{3}{2}(l_{n\delta}L_{\bar{\delta}} - l_{n\bar{\delta}}L_{\delta}) + l_{\bar{\gamma}\delta}l_{\gamma\delta n} - l_{\gamma\delta}l_{\bar{\gamma}\delta n}] \\
 & - \frac{3}{2}l_{nn}(l_{\gamma\bar{\gamma}}^2 + l_{\bar{\gamma}\gamma}^2 + \frac{2}{3}l_{nn}^2) + L(l_{nn}^2 + (l_{nn})^2) + 6l_{nn}^3 + \frac{1}{2}(l_{\gamma\bar{\gamma}}^3 - l_{\gamma\bar{\delta}}l_{\bar{\gamma}\delta}^2).
 \end{aligned}$$

4.3. Computation of ${}^H\Delta(l_{\alpha\bar{\alpha}}^2)$.

$$\begin{aligned}
 & {}^H\Delta(l_{\alpha\bar{\beta}}l_{\beta\bar{\alpha}}) \\
 = & 2l_{\beta\bar{\alpha}}{}^H\Delta l_{\alpha\bar{\beta}} + 4l_{\alpha\bar{\beta}\gamma}l_{\beta\bar{\alpha}\bar{\gamma}} \\
 = & 4l_{\beta\bar{\alpha}}L_{\alpha\bar{\beta}} + i2^{3/2}[l_{\beta\bar{\alpha}}l_{n\gamma}l_{\alpha\bar{\beta}\bar{\gamma}} - l_{\beta\bar{\alpha}}l_{n\bar{\gamma}}l_{\alpha\bar{\beta}\gamma} + 2(l_{\beta\bar{\gamma}}^2l_{\gamma\bar{\beta}n} - l_{\gamma\bar{\alpha}}^2l_{\alpha\bar{\gamma}n}) \\
 & + l_{n\bar{\alpha}}^2L_{\alpha} - l_{\beta n}^2L_{\bar{\beta}} - l_{\beta\bar{\beta}}^2L_n] \\
 & - l_{\beta\bar{\beta}}^2(l_{\gamma\bar{\gamma}}^2 + l_{\bar{\gamma}\gamma}^2 + 2l_{nn}^2) + L(l_{\beta\bar{\beta}}^3 + l_{\beta\bar{\alpha}}l_{\beta\bar{\alpha}}^2 + 2l_{nn}^3) \\
 & + 2l_{\beta\bar{\alpha}}l_{\beta\bar{\alpha}}^3 - (l_{\beta\bar{\gamma}}^3l_{\gamma\bar{\beta}} + l_{\beta\bar{\gamma}}^2l_{\bar{\gamma}\beta}^2) + 4l_{\alpha\bar{\beta}\gamma}l_{\beta\bar{\alpha}\bar{\gamma}} \\
 = & 4l_{\beta\bar{\alpha}}L_{\alpha\bar{\beta}} - i2^{3/2}l_{\gamma\bar{\gamma}}^2L_n + i2^{3/2}(l_{n\bar{\gamma}}^2e_{\gamma} - l_{\gamma n}^2e_{\bar{\gamma}}) \cdot L \\
 & + i\sqrt{2}(l_{n\gamma}e_{\bar{\gamma}} - l_{n\bar{\gamma}}e_{\gamma}) \cdot l_{\beta\bar{\beta}}^2 + 4l_{\alpha\bar{\beta}\gamma}l_{\beta\bar{\alpha}\bar{\gamma}} + (Ll_{\gamma\bar{\gamma}}^3 - (l_{\gamma\bar{\gamma}}^2)^2) + Ll_{\beta\bar{\alpha}}l_{\beta\bar{\alpha}}^2 - l_{\beta\bar{\beta}}^2l_{\bar{\gamma}\gamma}^2 \\
 & + 2(l_{\beta\bar{\alpha}}l_{\beta\bar{\alpha}}^3 - l_{\bar{\gamma}\gamma}^4) + 2(Ll_{nn}^3 - l_{\beta\bar{\beta}}^2l_{nn}^2).
 \end{aligned}$$

5. Maximum principle and an application. An embedded curve in N is called horizontal if its tangent is contained in $HN \hookrightarrow TN$. The Laplace operator ${}^H\Delta$ of $(HN, {}^Hh)$ satisfies a strong maximum principle along horizontal curves of N . This was proved earlier in [A] in three dimensions for contact manifolds and the arguments apply in the case of a strictly pseudoconvex N .

THEOREM 5.1. *Let ${}^H\Delta g \geq 0$ for a smooth real function g on N and $g \leq K$ on N . If $g(a) = K$ for some $a \in N$, then $g|_{\gamma} \equiv g(a)$ for all horizontal curves γ of N through a .*

If the horizontal distribution HN in TN defines a contact structure on N , that is if N inherits a nondegenerate CR -structure, then the horizontal curves are also called Legendre curves and by a well-known result, see [AG], they connect arbitrary points on N . The above now immediately gives the following strong maximum principle: Let $N \hookrightarrow (M, J, h)$ be a compact embedded real hypersurface that is strictly pseudoconvex. If ${}^H\Delta g \geq 0$ (or ≤ 0) for a smooth real function g on N , then g is constant.

THEOREM 5.2. *Let $F : N \hookrightarrow \mathbb{C}^n$ be a compact, connected, strictly pseudoconvex real hypersurface. Assume that the horizontal mean curvature L of N is constant and that ${}^M\nabla|(F^*T^{1,0}M \times H^{1,0}N) \subset H^{1,0}N$. Then N is a metric sphere.*

Proof. Let $\{e_k\}$ be frame adapted to $N \hookrightarrow \mathbb{C}^n$ as in 1.2. Then by the assumption on the Levi-Civita connection, $\theta^n(\nabla_{e_k}e_{\alpha}) = 0$, and therefore $l_{\alpha k} = 0$. Now by 4.3 and since $L = \text{const}$, we have

$$\left({}^H\Delta - i\sqrt{2}(l_{n\bar{\alpha}}e_{\alpha} - l_{\alpha n}e_{\bar{\alpha}}) \right) \cdot l_{\alpha\bar{\alpha}}^2 \geq (Ll_{\alpha\bar{\alpha}}^3 - (l_{\alpha\bar{\alpha}}^2)^2).$$

Since N is strictly pseudoconvex, we have $\ell_{\alpha\bar{\alpha}} > 0$ for every α , and therefore the right hand side is nonnegative. To see this choose a frame in which $\ell_{\alpha\bar{\beta}}$ is diagonal. Then

$$\left(\sum_{\alpha} \ell_{\alpha\bar{\alpha}} \right) \left(\sum_{\beta} (\ell_{\beta\bar{\beta}})^3 \right) - \left(\sum_{\alpha} (\ell_{\alpha\bar{\alpha}})^2 \right)^2 = \frac{1}{2} \sum_{\alpha\beta} \ell_{\alpha\bar{\alpha}} \ell_{\beta\bar{\beta}} (\ell_{\alpha\bar{\alpha}} - \ell_{\beta\bar{\beta}})^2 \geq 0.$$

The differential operator acting on $\ell_{\alpha\bar{\alpha}}^2$ satisfies the strong maximum principle, therefore $\ell_{\alpha\bar{\alpha}}^2 = \text{const}$. This implies $L\ell_{\alpha\bar{\alpha}}^3 - (\ell_{\alpha\bar{\alpha}}^2)^2 = 0$ and $\ell_{\beta\bar{\beta}} = c_1$ for all β . Next consider

$$\begin{aligned} \nabla_{e_{\alpha}} X_{2n} &= \frac{i}{\sqrt{2}} \nabla_{e_{\alpha}} (e_n - e_{\bar{n}}) \\ &= \frac{i}{\sqrt{2}} \left(\omega_n^k(e_{\alpha}) e_k - \omega_{\bar{n}}^{\bar{k}}(e_{\alpha}) e_{\bar{k}} \right) \\ &= -\frac{1}{2} (\ell_{\alpha\bar{k}} e_k + \ell_{\alpha k} e_{\bar{k}}) \\ &= -\frac{1}{2} \ell_{\alpha\bar{\beta}} e_{\beta} \\ &= -\frac{1}{2} c_1 e_{\alpha}. \end{aligned}$$

Now on $N \subset \mathbb{C}^n$ we have, if z denotes the coordinate vector of \mathbb{C}^n ,

$$\begin{aligned} \nabla_{e_{\alpha}} \left(z + \frac{2}{c_1} X_{2n} \right) &= e_{\alpha} + \frac{2}{c_1} \left(-\frac{1}{2} \right) c_1 e_{\alpha} \\ &= 0. \end{aligned}$$

This implies, since also $\nabla_{e_{\bar{\alpha}}} \left(z + \frac{2}{c_1} X_{2n} \right) = 0$, and the curvature of \mathbb{C}^n vanishes, that on N

$$\begin{aligned} 0 &= \nabla_{[e_{\alpha}, e_{\bar{\alpha}}]} \left(z + \frac{2}{c_1} X_{2n} \right) \\ &= \nabla_{\xi^n([e_{\alpha}, e_{\bar{\alpha}}])} X_n \left(z + \frac{2}{c_1} X_{2n} \right) \\ &= -\frac{i}{2^{3/2}} \ell_{\alpha\bar{\alpha}} \nabla_{X_n} \left(z + \frac{2}{c_1} X_{2n} \right). \end{aligned}$$

Therefore, since $\ell_{\alpha\bar{\alpha}} \neq 0$, we conclude that $\left(z + \frac{2}{c_1} X_{2n} \right)$ vanishes identically on N , and for any $v \in TN$, $v \cdot |z|^2 = 2\text{Re}(z, v) = -2\frac{2}{c_1}(X_{2n}, v) = 0$ which implies that $|z|^2 = c_3$ on N . Therefore N is a metric sphere. \square

REMARK. The assumption that $H^{1,0}N$ is parallel in $(F^*T^{1,0}M, {}^M\nabla)$ is equivalent to $\ell_{k\beta} = 0$ on the hypersurface. Since

$$\begin{aligned} \nabla_{X_n} X_n &= \frac{1}{2} \left(\theta^k(\nabla_{e_n}(e_n + e_{\bar{n}})) e_k + \theta^{\bar{k}}(\nabla_{e_n}(e_n + e_{\bar{n}})) e_{\bar{k}} \right) \\ &= \frac{1}{2} \left(\omega_n^k(e_n) e_k + \omega_{\bar{n}}^{\bar{k}}(e_n) e_{\bar{k}} \right) \\ &= \frac{i}{2^{3/2}} (\ell_{n\bar{k}} e_k - \ell_{kn} e_{\bar{k}}), \end{aligned}$$

we see that the condition $\ell_{\alpha n} = 0$ implies that the integral curves of the vertical line bundle are geodesics of N . This, together with the condition $\ell_{\alpha\beta} = 0$, implies that the second fundamental form for the underlying Riemannian metric, as related in 1.4, has

the form $A \oplus l_{n\bar{n}} \oplus A$. Here we refer to the real basis $X_1, \dots, X_{n-1}, X_n, X_{n+1}, \dots, X_{2n-1}$ that corresponds to the frame e_k as in 1.2 and in which $l_{\alpha\bar{\beta}}$ is diagonal, and A is the $(n-1) \times (n-1)$ matrix with the coefficients $k_{\alpha\beta}$ as in 1.4. The assumption of strict pseudoconvexity says that A is positive definite, and constant horizontal mean curvature says that the trace of A is constant in this setting. We have no assumption on the real coefficient l_{nn} .

REFERENCES

- [A] S. ALTSCHULER, *A geometric heat flow for one-forms on three dimensional manifolds*, Illinois Jour. Math., 39 (1995), pp. 98–118.
- [AG] V. I. ARNOLD AND A. B. GIVENTAL, *Symplectic Geometry*, in Dynamical Systems IV, Encyclopaedia of Mathematical Sciences 4, Springer, Berlin, 1990, pp. 71–86.
- [BFG] M. BEALS, C. FEFFERMAN, AND R. GROSSMAN, *Strictly pseudoconvex domains in \mathbb{C}^n* , Bull. AMS, 8 (1983), pp. 125–322.
- [B] A. BEJANCU, *Geometry of CR-Submanifolds*, D. Reidel Publ. Comp. Dordrecht, 1986.
- [CdCK] S. S. CHERN, M. DO CARMO, AND S. KOBAYASHI, *Minimal submanifolds of a sphere with second fundamental form of constant length*, in Functional Analysis and Related Fields, F. Browder, ed., Springer, Berlin, 1970, pp. 59–75.
- [CM] S. S. CHERN AND J. MOSER, *Real hypersurfaces in complex manifolds*, Acta Math., 133 (1974), pp. 219–271.
- [HK] G. HUISKEN AND W. KLINGENBERG, *Flow of real hypersurfaces in complex space by the trace of the Levi form*, Mathematical Research Letters, 6 (1999), pp. 645–661.
- [J1] H. JACOBOWITZ, *An Introduction to CR Structures*, Mathematical Surveys and Monographs 32, AMS, Providence, 1990.
- [J2] H. JACOBOWITZ, *Real hypersurfaces and complex analysis*, Notices of the AMS, 42:12 (1995), pp. 1480–1488.
- [K] W. KLINGENBERG, *Real Hypersurfaces in Kähler Manifolds*, Habilitationsschrift, Universität Tübingen, 1997.
- [KN] S. KOBAYASHI AND K. NOMIZU, *Foundations of differential geometry*, Vols. I, II, Interscience Publishers, 1963 and 1969.
- [Ku] M. KURANISHI, *Strongly pseudoconvex CR structures over small balls, Part III*, Ann. of Math., 116 (1982), pp. 249–330.
- [O] M. OKUMURA, *Compact real hypersurfaces with constant mean curvature of a complex projective space*, J. Diff. Geo., 13:1 (1978), pp. 43–50.
- [S] J. SIMONS, *Minimal varieties in Riemannian manifolds*, Ann. of Math., 88 (1968), pp. 62–105.
- [W] S. WEBSTER, *Pseudo-hermitian structures on real hypersurfaces*, J. Diff. Geo., 13:1 (1978), pp. 25–42.
- [YK] K. YANO AND M. KON, *CR submanifolds of Kaehlerian and Sasakian Manifolds*, Progress in Math 30, Birkhäuser, Boston, 1983.