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NEW EXAMPLES OF INTEGRABLE GEODESIC FLOWS*

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Abstract. The existence of a family of nilmanifolds which possess riemannian metrics with Liouville-integrable geodesic flow is demonstrated. These homogeneous spaces are of the form $D\backslash H$, where H is a connected, simply-connected two-step nilpotent Lie group and D is a discrete, cocompact subgroup of H. The metric on $D\backslash H$ is obtained from a left-invariant metric on H. The topology of these nilmanifolds is quite rich; in particular, the first example of a Liouville-integrable geodesic flow on a manifold whose fundamental group possesses no commutative subgroup of finite index is obtained. It is shown that several of the conclusions of Taĭmanov's theorems on the topology of manifolds with real-analytically Liouville-integrable geodesic flows do not obtain in the smooth category [28, 29].

1. Introduction. A manifold (L^{2k}, ω) is *symplectic* if the skew symmetric 2-form ω is closed, and non-degenerate. A vector field X on a symplectic manifold (L^{2k}, ω) is *hamiltonian* if its contraction with ω is exact: $i_X\omega = dH$ where H is a smooth, real-valued function on L^{2k}. X is often denoted by X_H in such a situation. Here *smooth* will always mean C^∞ unless explicitly stated otherwise. An especially important class of hamiltonian vector fields, which forms the starting point of Poincaré's study of problems in celestial mechanics [26] and its subsequent elaboration in KAM theory, is the class of Liouville integrable hamiltonian vector fields:

DEFINITION 1.1 (Liouville Integrability). *The hamiltonian vector field X_H is* Liouville integrable *on (L^{2k}, ω) if there exists a smooth function $F := (f_1 = H, \dots, f_k)$: $M^{2k} \to \mathbb{R}^k$ such that*

i) f_i, f_j are in involution: $\omega(X_{f_j}, X_{f_i}) = df_j(X_{f_i}) = \{f_j, f_i\} = 0$ for all $1 \leq i, j \leq k$;

ii) and functionally independent on a dense subset of L^{2k}: the rank of the map $dF_p : T_pL^{2k} \to T_{F(p)}\mathbb{R}^k$ is k on an open dense subset of L^{2k}.

If all objects in the above definition share a property P, then it is customary to say that X_H is Liouville integrable with P first integrals. Most commonly, one is concerned with smooth, real-analytic or algebraic first integrals.

2. Results of this Paper. In this paper, a family of real-analytic, 2-step nilpotent manifolds of all dimensions $k \geq 3$ is introduced. The manifolds admit real-analytic riemannian metrics whose geodesic flow is Liouville integrable with k involutive first integrals. Their fundamental groups are not almost abelian, nor is there an injection of cohomology rings $H^*(\mathbb{T}^d; \mathbb{Q}) \hookrightarrow H^*(Q^k; \mathbb{Q})$ where $d = \dim H^1(Q^k; \mathbb{Q})$. In light of work done by Taĭmanov , quoted in theorem (3.2) below, these flows do not have k independent, involutive real-analytic first integrals. Therefore, the first example of a real-analytic riemannian manifold that admits smooth but not real-analytically Liouville-integrable geodesic flows is introduced.

In the work on integrable geodesic flows by Thimm [30], Guillemin and Sternberg [16], and Paternain and Spatzier [22], the Liouville integrability of geodesic flows on certain riemannian manifolds is demonstrated in a manner that leaves many questions about the flows still unanswered. For example, it is unclear if these geodesic flows have additional, non-involutive first integrals. Bogoyavlenskij [3] has shown if

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an integrable hamiltonian vector field has a non-commutative algebra of symmetries, then it is degenerate in the sense of KAM theory. It is currently unknown if a manifold that admits such degenerate integrable geodesic flows admits non-degenerate geodesic flows [11]. In this paper, explicit action-angle coordinates are constructed that demonstrate that the manifolds Q^k admit non-degenerate geodesic flows.

In addition to the non-degeneracy of the geodesic flows on Q^k, it is possible to prove that the topological entropy of these geodesic flows vanishes. This is true, in spite of the fact that all three of the hypotheses of theorem (3.3) below are not true for these geodesic flows.

3. The Topological Implications of Integrability. A question that has been posed by many researchers is this: **what are the topological implications of Liouville integrability?** This broad question is addressed in [19, 5, 6, 18, 11, 12] to cite some recent work. The short answer is that there are none [5]. However, once suitable restrictions are placed on the behaviour of the first integrals along the singular set, many more things are known. In the case where $L^{2k} = T^*M^k$, ω is the canonical symplectic form on T^*M and H is the hamiltonian associated with a riemannian metric on M, a good deal is known:

THEOREM 3.1 (Kozlov:[17]). *Let M^2 be a smooth, compact surface and $H : T^*M \to \mathbb{R}$ be the hamiltonian of a riemannian metric on M^2. Suppose that there exists a smooth function $f : S^*M = \{H = 1\} \to \mathbb{R}$ such that*

i) f is a first integral of H;

ii) f has finitely many critical values; and

ii) for each $c \in \mathbb{R}$, the set of points $q \in M$ such that $\{f = c\} \cap S^*_qM$ is either the entire fibre or finitely many points is everywhere dense in M.

If M^2 is orientable, then it is homeomorphic to either S^2 or $\mathbb{T}^2$; otherwise M^2 is homeomorphic to either $\mathbb{R}P^2$ or K^2, where K^2 is the Klein bottle.

Kozlov's theorem relies heavily on the structure of the covering spaces of two-dimensional surfaces and their minimal geodesics [13]; a weaker form of this theorem for a Bott first integral is proven in [6] that does not rely on [13]. There have been two ways in which Kozlov's theorem has been generalized: the first directly generalizes the techniques used in his theorem, while the second is based on the notion of the topological entropy of a flow.

THEOREM 3.2 (Taĭmanov:[28, 29]). *Let M^n be a compact, connected, boundaryless manifold and g be a riemannian metric on M such that the geodesic flow of g is* geometrically simple. *Then:*

i) $\pi_1(M^n)$ possesses a commutative subgroup of finite index;

ii) $\dim H_1(M^n;\mathbb{Q}) \leq n$;

iii) if $d = \dim H_1(M^n;\mathbb{Q})$, then there is an injection of algebras $H^*(\mathbb{T}^d;\mathbb{Q}) \hookrightarrow H^*(M^n;\mathbb{Q})$

iv) if $d = n$, then the injection is an isomorphism.

In the same paper, Taĭmanov shows that a real-analytic metric that is Liouville integrable with n real-analytic first integrals has a geometrically simple geodesic flow.

Theorem (7.1) of this paper demonstrates that the conclusions *i* and *iii* of theorem (3.2) do not obtain when the first integrals are constrained only to be C^∞.

The topological-entropy approach to the topological implications of integrability, due to Paternain, is based on two appealing observations: (i) the flow of a Liouville-integrable vector field has zero topological entropy when restricted to an open, dense domain of invariant tori; (ii) the surfaces identified in Kozlov's theorem that do (do not) admit a Liouville-integrable geodesic flow are precisely those surfaces that do (do not) admit geodesic flows with zero topological entropy [10]. Indeed, until the work of Bolsinov and Taĭmanov [7], no integrable geodesic flows were known to have positive topological entropy.

THEOREM 3.3 (Paternain:[23, 24, 25]). *Let* (M^k, g) *be a smooth, compact riemannian manifold and* $H : T^*M \to \mathbb{R}$ *be the hamiltonian of the geodesic flow of* g *and let* $S^*M = H^{-1}(1)$. *If*

i) H *is invariant under the effective, hamiltonian action of* $\mathbb{T}^{k-1}$ *on* T^*M*; or*
ii) there is an $m \in M$ *such that every point* $p \in S^*_m M$ *has a neighbourhood in* S^*M *that admits action-angle variables with singularities; or*
iii) if H *is integrable with non-degenerate first integrals,*

then the fundamental group of M *is of subexponential growth and if* $\pi_1(M)$ *is finite,* M *is rationally elliptic.*

In each proof the main difficulty is ensuring that the behaviour of the geodesic flow on the singular set of the first integrals is "tame enough" that it has zero topological entropy. The work of Dinaburg [10], Gromov [15] and Yomdin [32] then imply the two topological results.

Although it was conjectured that smoothly integrable geodesic flows must have zero topological entropy, Bolsinov and Taĭmanov have recently shown that the suspension manifold of a linear Anosov diffeomorphism of a torus admits a geodesic flowthat is Liouville integrable and has positive topological entropy [7].

4. Continuing Questions. There are a number of questions that this paper naturally poses. Are there simply connected manifolds that are not rationally elliptic and have integrable geodesic flows? It is conjectured by Taĭmanov [29] that the Betti numbers of M^k are dominated by those of $\mathbb{T}^k$ if M^k admits a geodesic flow that is either Liouville integrable or has zero topological entropy. Is this true?

The first integrals of the geodesic flow in this paper, although non-analytic and degenerate, are relatively tame: the singular set of the first-integral mapping is a union of closed submanifolds. Is it possible to characterize the topological implications of integrability with this type of first integral?

5. A Statement of the Theorems.

5.1. The Groups $\mathcal{H}_p$. In this subsection, the groups $\mathcal{H}_p$ and the topology of their quotients is studied. Let $\mathcal{H}_p := (\mathbb{R}^1 \times \mathbb{R}^p \times \mathbb{R}^p, *)$ where the multiplication $*$ is defined by

$$(x, y, z) * (x', y', z') := (x + x', y + y', z + z' + xy'). \tag{1}$$

It is easily verified that these groups are 2-step nilpotent and the coordinates (x, y, z) are coordinates of the second kind [20].

The family of 2-step nilpotent groups, $\mathcal{H}_*$, has a rich structure. First, the group's descending central series consists of $\mathcal{H}_p, \mathcal{H}_p/Z(\mathcal{H}_p), 1$ where the centre of the group coincides with the derived group. This allows one to write $\mathcal{H}_p$ as the non-trivial central extension of $\mathbb{R}^{p+1}$ by $\mathbb{R}^p$

$$1 \to \mathbb{R}^p \to \mathcal{H}_p \to \mathbb{R}^1 \times \mathbb{R}^p \to 1. \tag{2}$$

The subgroup $\mathcal{N}_p := 0 \times \mathbb{R}^p \times \mathbb{R}^p$ is also a normal, commutative subgroup. This allows us to express $\mathcal{H}_p$ as a non-central extension of $\mathbb{R}^1$ by $\mathcal{N}_p$

$$1 \to \mathcal{N}_p \to \mathcal{H}_p \to \mathbb{R}^1 \to 1. \tag{3}$$

In this case $x \in \mathbb{R}^1$ acts on $(y, z) \in \mathcal{N}_p$ by the matrix $x \to \begin{bmatrix} I_p & 0 \\ xI_p & I_p \end{bmatrix}$ where I_p is the $p \times p$ identity matrix. The group $\mathcal{H}_p$ is therefore the semidirect product of $\mathbb{R}^1$ with $\mathbb{R}^{2p}$ in which the representation of $\mathbb{R}^1$ in $\mathrm{GL}(2p;\mathbb{R})$ is indecomposable. It is clear that $\mathcal{N}_p$ is a maximal abelian subgroup of $\mathcal{H}_p$. In the sequel, this fact will be important.

It is clear that for each p, $\mathcal{H}_p$ possesses a discrete, cocompact subgroup; take the subgroup $D := \{(x, y, z) \;:\; x \in \mathbb{Z}, y, z \in \mathbb{Z}^p\}$, for example.

THEOREM 5.1. *Let g be a left-invariant metric on $\mathcal{H}_p$ and let Γ be a discrete, cocompact subgroup of $\mathcal{H}_p$. Then the geodesic flow $\varphi_t : T^*(\Gamma\backslash\mathcal{H}_p) \to T^*(\Gamma\backslash\mathcal{H}_p)$ induced by g is Liouville integrable.*

The two following theorems are proven for the special metric g defined below (8.3):

THEOREM 5.2. *The geodesic flow φ_t is non-degenerate in the sense of KAM theory.*

THEOREM 5.3. *The topological entropy of φ_t vanishes.*

6. Liouville Integrability. Let g be a left-invariant metric on $\mathcal{H}_p$, and let $H : T^*\mathcal{H}_p \to \mathbb{R}$ be the hamiltonian associated to g, X_H the geodesic vector field and ϕ_t the geodesic flow. Because g is left invariant, all right-invariant vector fields are isometries of g; the cotangent lifts of these vector fields then provide first integrals of ϕ_t.

The left-invariant differential 1-forms $\alpha, \beta^i, \gamma^i$ provide a trivialization of $T^*\mathcal{H}_p$; let $(x, y, z, p_\alpha, p_\beta, p_\gamma)$ be the coordinates on $T^*\mathcal{H}_p$ induced by this trivialization. It is clear that the coordinates $(p_\alpha, p_\beta, p_\gamma)$ descend to coordinates on $T^*(\Gamma\backslash\mathcal{H}_p) = \Gamma\backslash\mathcal{H}_p \times \mathrm{Lie}(\mathcal{H}_p)^*$, where Γ is a discrete subgroup of $\mathcal{H}_p$.

LEMMA 6.1. *Let $Z_i := \frac{\partial}{\partial z^i}$. Then Z_i is a right-invariant vector field on $\mathcal{H}_p$ and $G_i := p_{\gamma_i}$ is a first integral of φ_t.*

Proof. Z_i is generated by the center of $\mathcal{H}_p$, and $G_i = p(Z_i) = p_{\gamma_i}$. □

LEMMA 6.2. *Let $Y_i := \frac{\partial}{\partial y^i}$. Then Y_i is a right-invariant vector field on $\mathcal{H}_p$ and $F_i := p_{\beta_i} - xp_{\gamma_i}$ is a first integral of φ_t.*

Proof. Right invariance is clear. In addition, $Y_i = \mathcal{Y}_i - x\mathcal{Z}_i$ where $\mathcal{Y}_i$ ($\mathcal{Z}_i = Z_i$) is left invariant and dual to β^i (γ^i). So: $F_i = p(Y_i) = p_{\beta_i} - xp_{\gamma_i}$ is the hamiltonian of the cotangent lift of Y_i. □

LEMMA 6.3. *Let $\phi(u) = \exp(-u^{-2})$ be a smooth function on $\mathbb{R}$, let Γ be a discrete, cocompact subgroup of $\mathcal{H}_p$, let a be a generator of the cyclic group $\Gamma/\Gamma \cap \mathcal{N}_p$ and let $T := \oint_a \alpha$ be the integral of $\alpha = dx$ over a, where the loop representing a is identified with a in the obvious way. Then the functions*

$$f_i := \phi(p_{\gamma_i}) \sin \frac{2\pi}{T} \left(\frac{p_{\beta_i}}{p_{\gamma_i}} - x \right) \tag{4}$$

are C^∞ first integrals of the geodesic flow of g on $T^*(\Gamma\backslash\mathcal{H}_p) =: T^*Q$.

Proof. The 1-form $\alpha = dx$ is closed but not exact, meaning that $x : Q \to \mathbb{R}/T\mathbb{Z}$ is a smooth function. Therefore, f_i is a well-defined, real-valued function on T^*Q, and the smoothness of f_i follows from the properties of ϕ and sin. That $f_i \circ \phi_t = f_i$ follows from the fact that $f_i = \phi(G_i) \sin \frac{2\pi F_i}{TG_i}$, where both F_i and G_i are first integrals of ϕ_t. Therefore, f_i descends to a smooth first integral of ϕ_t on T^*Q. □

LEMMA 6.4. *Let $\psi := (f_1, \ldots, f_p, G_1, \ldots, G_p, H)$ be an energy-momentum mapping for H. Then* $\operatorname{rank} d\psi_q = 2p + 1$ *for q in an open, dense subset of T^*Q.*

Proof. As usual, the cotangent bundle of Q is viewed as the cotangent bundle of $\mathcal{H}_p$ subject to the periodicity conditions imposed by Γ, and we let $\pi : T^*\mathcal{H}_p \to T^*Q$. Let $\Psi = (F_1, \ldots, F_p, G_1, \ldots, G_p, H)$ be the energy-momentum map of H on $T^*\mathcal{H}_p$. The $\operatorname{rank} d\psi_q = \operatorname{rank} d\Psi_P$, where $q = \pi(P)$, unless $df_i = 0$ for some i, that is $p_{\gamma_i} = 0$ or $\frac{1}{T}\left(\frac{p_{\beta_i}}{p_{\gamma_i}} - x\right) \in \mathbb{Z}$. So: $\operatorname{rank} d\psi_q < \operatorname{rank} d\Psi_P$ implies that for some i, $f_i(q) = 0$ or ± 1.

It is clear that $\operatorname{rank} d\Psi_P = 2p + 1$ for an open, dense subset of $P \in T^*\mathcal{H}_p$, and so $\operatorname{rank} d\psi_q = 2p + 1$ on an open, dense subset of T^*Q. □

LEMMA 6.5. *The first integrals $f_1, \ldots, f_p, G_1, \ldots, G_p, H$ Poisson commute.*

Proof. The right-invariant vector fields $Y_1, \ldots, Y_p, Z_1, \ldots, Z_p$ commute, so the hamiltonians of their cotangent lift to $T^*\mathcal{H}_p$, $F_1, \ldots, F_p, G_1, \ldots, G_p$, Poisson commute. $f_1, \ldots, f_p, G_1, \ldots, G_p$ generate an algebra that is functionally dependent on the former algebra, so this algebra is Poisson commutative, too. It has already been established that each function Poisson commutes with H. □

Proof of theorem (5.1): This completes the proof. □

7. Topology of $\Gamma\backslash\mathcal{H}_p$.

THEOREM 7.1. *Let Γ be a discrete, cocompact subgroup of $\mathcal{H}_p$, $p \geq 1$, and let $Q := \Gamma\backslash\mathcal{H}_p$. Then:*

i) $\pi_1(Q) \simeq \Gamma$ has no abelian subgroup of finite index; and

ii) $\dim H^1(Q; \mathbb{R}) = p + 1$ but there does not exist an injection of cohomology algebras $H^*(\mathbb{T}^{p+1}; \mathbb{Q}) \hookrightarrow H^*(Q; \mathbb{Q})$;

iii) the Betti numbers of Q are dominated by $\mathbb{T}^{2p+1}$.

Proof. (i) Γ, because it is a discrete, cocompact subgroup of a connected, simply connected nilpotent Lie group, is a finitely generated, torsion-free 2-step nilpotent group [20]. By [31], the word growth of Γ is $3p + 1$; if $\Gamma' \leq \Gamma$ were an abelian subgroup of finite index, then the word growth of Γ would be equal to that of Γ', which as a rank $2p + 1$ free abelian group, would be $2p + 1$.

(ii) Let $\alpha = dx$, $\beta^i = dy^i$ and $\gamma^i = dz^i - xdy^i$ for $i = 1, \ldots, p$. This is a basis of $\mathrm{Lie}(\mathcal{H}_p)^*$, the vector space of left-invariant forms on $\mathcal{H}_p$ and $d\alpha = d\beta^1 = \cdots = d\beta^p = 0$ while $d\gamma^j = -\alpha \wedge \beta^j$ for $j = 1, \ldots, p$. Nomizu's theorem [21] asserts that if G is a connected, simply connected nilpotent Lie group, $\mathrm{Lie}(G)^*$ is its Lie coalgebra and D is a discrete, cocompact lattice in G then the de Rham cohomology ring of $D\backslash G$ is isomorphic to the cohomology ring of $\mathrm{Lie}(G)^*$. Moreover, the isomorphism is induced by the natural inclusion of the left-invariant alternating forms on G with the complex of differential forms on $D\backslash G$. This means that $H^1(Q; \mathbb{R}) = \mathrm{span}_{\mathbb{R}} \{\alpha, \beta^1, \ldots, \beta^p\}$; on the other hand, the cohomology class $[\alpha \wedge \beta^i] = 0$ for all i. Hence, there is no injection $H^*(\mathbb{T}^{p+1}; \mathbb{R}) \hookrightarrow H^*(Q; \mathbb{R})$. By the isomorphism between de Rham and singular cohomology, and the universal coefficients theorem, this implies that there is no injection $H^*(\mathbb{T}^{p+1}; \mathbb{Q}) \hookrightarrow H^*(Q; \mathbb{Q})$ in singular cohomology.

(iii) This is clear, since the exterior algebra $\Lambda^*(\mathrm{Lie}(\mathcal{H}_p))$ is isomorphic to $H^*(\mathbb{T}^{2p+1};\mathbb{R})$ as an $\mathbb{R}$ algebra. □

8. Non-degeneracy of the Geodesic Flow. In this section it will be shown that the geodesic flow φ_t of a left-invariant metric g on $T^*(\Gamma\backslash\mathcal{H}_p)$, $p \geq 2$, is non-degenerate in the sense of KAM theory (see below for a definition). This will be done in two steps: (i) it will be shown that the flow φ_t on an intermediate covering, $T^*(\Gamma \cap \mathcal{N}_p\backslash\mathcal{H}_p)$, is non-degenerate; (ii) this is shown to imply that the flow on $T^*(\Gamma\backslash\mathcal{H}_p)$ is non-degenerate.

8.1. A Covering Space.

LEMMA 8.1. *Let $\Gamma \leq \mathcal{H}_p$ be a discrete, cocompact subgroup of $\mathcal{H}_p$. Then there exist unique integers $1 \leq k_1|\cdots|k_p$ such that Γ is isomorphic to the group $D(k)$ generated by elements $a, b_1, \ldots, b_p, c_1, \ldots, c_p \in \mathcal{H}_p$ where $a = (1,0,0)$, $b_i = (0, e_i, 0)$ and $c_i = (0,0,k_i^{-1}e_i)$ where e_i is the i-th standard basis vector of $\mathbb{R}^p$.*

A proof of this appears in [9]. The following lemma is proven, essentially, in [14]:

LEMMA 8.2. *Let g be a left-invariant metric on $\mathcal{H}_p$ and $\Gamma \leq \mathcal{H}_p$ be a discrete, cocompact subgroup and let $f : D(k) \to \Gamma$ be an isomorphism. Then f extends to an automorphism of $\mathcal{H}_p$ and $(\Gamma\backslash\mathcal{H}_p, g)$ is isometric to $(D(k)\backslash\mathcal{H}_p, f^*g)$.*

Because all invariants of the geodesic flows are invariant under isometries, it is clear that it is only necessary to prove that there are flows on $T^*(D(k)\backslash\mathcal{H}_p) =: T^*Q(k)$ that are non-degenerate.

LEMMA 8.3. *Let $N(k)$ be the rank $2p$, free abelian subgroup of $D(k)$ generated by $b_1, \ldots, c_p$. Then $T^*(N(k)\backslash\mathcal{H}_p) = T^*(\mathbb{R} \times \mathbb{T}^{2p})$ with coordinates (x, y, z, p_x, p_y, p_z), y_i is measured mod 1, z_i is measured mod k_i^{-1}. The left-invariant metric*

$$g = \alpha \otimes \alpha + \sum_{i=1}^{p} \beta^i \otimes \beta^i + \gamma^i \otimes \gamma^i,$$

induces the hamiltonian

$$2H = p_x^2 + \sum_{i=1}^{p} (p_{y^i} + x p_{z^i})^2 + p_{z^i}^2.$$

Proof. The differential forms $\alpha = dx$, $\beta^i = dy^i$, and dz^i are invariant under the left action of $N(k)$, so they descend to $T^*(N(k)\backslash\mathcal{H}_p)$. The coordinates $p_\alpha = p_x$, $p_{z^i} = p_{\gamma^i}$ and $p_{y^i} = p_{\beta^i} - x p_{\gamma^i}$, and in terms of the coordinates $p_\alpha, p_\beta, p_\gamma$, the hamiltonian is $2H = p_\alpha^2 + \sum_{i=1}^p p_{\beta^i}^2 + p_{\gamma^i}^2$. □

The geodesic vector field in the above system of coordinates on $T^*(N(k)\backslash\mathcal{H}_p)$ is

$$(5) \quad X_H = \begin{cases} \dot{p}_x = -\sum_{i=1}^p (p_{y^i} + x p_{z^i}) p_{z^i}, & \dot{x} = p_x, \\ \dot{p}_{y^i} = 0, & \dot{y}^i = p_{y^i} + x p_{z^i}, \\ \dot{p}_{z^i} = 0, & \dot{z}^i = p_{z^i} + x(p_{y^i} + x p_{z^i}). \end{cases}$$

which clearly reveals that $\Psi = (F_1 = p_{y^1}, \ldots, G_p = p_{z^p}, H)$ is a first-integral mapping that is of maximal rank on $\{\ p_x^2 + (\sum_{i=1}^p (p_{y^i} + x p_{z^i}) p_{z^i})^2 \neq 0\ \}$. There is another independent, non-involutive, first integral, in the event that $p = 1$, too: $K = \phi(p_z) \sin 2\pi \left(\frac{p_x}{p_z} + y\right)$.

LEMMA 8.4. *Let $a, b \in \mathbb{R}^p$. Suppose that $b \neq 0$. Then the level set $\{F_1 = a_1, \ldots G_p = b_p, H = E\}$ is a compact subset of $T^*(N(k)\backslash\mathcal{H}_p)$.*

Proof. The cotangent bundle $T^*(N(k)\backslash\mathcal{H}_p) = T^*(\mathbb{R} \times \mathbb{T}^{2p})$ via the chosen coordinates. It is only necessary to show that x and p_x are bounded. But $p_x^2 \leq 2E$ while $(p_{y^j} + xp_{z^j})^2 = (a_j + xb_j)^2 \leq 2E$. Since $b_j \neq 0$ for some j by hypothesis, x is bounded too. □

In the sequel it will be useful to denote by $A^2 = \sum_{i=1}^p a_i^2$, $B^2 = \sum_{i=1}^p b_i^2$, $D = \sum_{i=1}^p a_i b_i$. It will be assumed that $B^2 \neq 0$ so that $C = D/B^2$ is defined.

On the level set $\{F_1 = a_1, \ldots G_p = b_p, H = E\}$ the hamiltonian takes the form $2H - A^2 - B^2(1 - C^2) = p_x^2 + B^2(x + C)^2$. Ignoring the coordinates (y, z), this gives the equation of an ellipse with axes of length $r = \sqrt{2H - A^2 - B^2(1 - C^2)}$ and $s = r/B$. The generating function $S(x, I) = \int_{-C}^{x} p_x \, dx$ defines the implicit canonical coordinate change

$$\Phi = \begin{cases} p_x = \frac{\partial S}{\partial x}, & \theta = \frac{\partial S}{\partial I}, \\ \vartheta^i = y^i + \frac{\partial S}{\partial a_i}, & \varphi^i = z^i + \frac{\partial S}{\partial b_i}, \end{cases} \tag{6}$$

and a_i, b_i are unchanged. The action variable I is defined by $2\pi I = \oint p_x \, dx =$ area of ellipse $= \pi r^2/B$. The hamiltonian H in the canonical action-angle coordinates $(\theta, \vartheta, \varphi, I, a, b)$ is

$$2H = A^2 + B^2 - D^2/B^2 + 2IB. \tag{7}$$

The geodesic vector field in these action-angle coordinates is

$$X_H = \begin{cases} \dot\theta = B, & \dot I = 0, \\ \dot\vartheta^i = a_i - DB^{-2} b_i, & \dot a_i = 0, \\ \dot\varphi^i = (1 + IB^{-1} + D^2 B^{-4}) b_i - DB^{-2} a_i, & \dot b_i = 0. \end{cases} \tag{8}$$

In the special case of $p = 1$, the geodesic vector field is simplified to

$$X_H = \begin{cases} \dot\theta = b, & \dot I = 0, \\ \dot\vartheta = 0, & \dot a = 0, \\ \dot\varphi = I + b, & \dot b = 0. \end{cases} \tag{9}$$

The additional first integral $K = \phi(p_z) \sin 2\pi(p_x/p_z + y)$ appears here in the guise of ϑ.

8.2. Non-degeneracy of the geodesic flow.

DEFINITION 8.5 (Poincaré Non-degenerary: [26]). *Let $M = \mathbb{D}^n \times \mathbb{T}^n$ be a toroidal neighbourhood with coordinates (I, θ) and the symplectic structure $\omega = \sum_{i=1}^n d\theta^i \wedge dI_i$. The hamiltonian $H = H(I)$ is* non-degenerate *if there exists a dense subset $U \subset \mathbb{D}^n$ such that for all $I \in U$,*

$$\det \mathrm{Hess}(H(I)) \neq 0.$$

A Liouville-integrable hamiltonian vector field X_H on the symplectic manifold (L^{2n}, ω) is non-degenerate *if, for all toroidal neighbourhoods $M \hookrightarrow \mathbb{D}^n \times \mathbb{T}^n$, the hamiltonian on M is non-degenerate in the previous sense.*

Bogoyavlenskij [2, 4] has proven that this definition is independent of the choice of action-angle variables on the toroidal domain, and it is equivalent to the statement that the Lie algebra of all symmetries of X_H is abelian. It should be observed that Poincaré non-degenerary is a necessary condition for the application of KAM theory [1].

LEMMA 8.6. *Let $p \geq 2$. The determinant of the hessian of H in the action coordinates (I, a, b) is*

$$\det \text{Hess}(H) = \begin{cases} \frac{(I+B)^{p-2}(A^2B^2-D^2)}{B^{p+2}} & \text{if } a - DB^{-2}b \neq 0. \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

In particular, the hamiltonian H is non-degenerate on an open, dense subset of $\mathbb{R}^+ \times \mathbb{R}^p \times \mathbb{R}^p$, and the geodesic flow is a dense, quasi-periodic winding on almost all invariant tori $\{I = cst., a = cst., b = cst.\}$.

Proof. Let $a, b \in \mathbb{R}^p$, $V := \text{span}\{(0, b, 0), (0, a, 0), (0, 0, b), (0, 0, a), (1, 0, 0)\}$, and $V^\perp$ is the orthogonal complement of V in $\mathbb{R} \times \mathbb{R}^p \times \mathbb{R}^p$. The hessian of H in the action coordinates (I, a, b) is computed to be

$$\text{Hess}(H) = \begin{bmatrix} 0 & B^{-1}b' & 0 \\ B^{-1}b & I_p - D^{-2}bb' & \begin{array}{c} 2DB^{-4}bb' \\ -DB^{-2}I_p - B^{-2}ba' \end{array} \\ 0 & \begin{array}{c} 2DB^{-4}bb' \\ -DB^{-2}I_p - B^{-2}ab' \end{array} & \begin{array}{c} (1 + IB^{-1} + D^2B^{-4})I_p \\ -(IB^{-3} + 4D^2B^{-6})bb' \\ +2DB^{-4}(ab' + ba') - B^{-2}aa' \end{array} \end{bmatrix}, \tag{11}$$

where a', b' are the transposes of a, b and I_p is the $p \times p$ identity matrix. Both V and $V^\perp$ are invariant under the linear transformation $\text{Hess}(H)$ so its determinant is the product of the determinants on each subspace. On the subspace $V^\perp$ this transformation has the matrix representation

$$\text{Hess}(H)|_{V^\perp} \simeq \begin{bmatrix} I_k & -DB^{-2}I_k \\ -DB^{-2}I_k & (1 + IB^{-1} + D^2B^{-4})I_k \end{bmatrix}, \tag{12}$$

where $k = p - 2$ if $a \neq DB^{-2}b$ and $k = p - 1$ if $a = DB^{-2}b$. Therefore

$$\det\left(\text{Hess}(H)|_{V^\perp}\right) = (1 + IB^{-1})^k. \tag{13}$$

On the subspace V one computes the determinant of the restriction of the hessian to be

$$\det\left(\text{Hess}(H)|_V\right) = (A^2B^{-2} - D^2)B^{-4}. \tag{14}$$

These two facts prove the claims above. □

LEMMA 8.7. *Suppose that $\pi : (M^{2k}, \omega) \to (N^{2k}, \theta)$ is a symplectic covering, $F : N^{2k} \to \mathbb{R}$ is a smooth hamiltonian and $G = F \circ \pi$. If X_G is Liouville integrable and non-degenerate and X_F is Liouville integrable, then X_F is non-degenerate.*

Proof. Let $U \subset N^{2k}$ be an open toroidal domain. Then there exist action-angle variables $(I, \phi) : U \to \mathbb{D}^n \times \mathbb{T}^n$. Let $V = \pi^{-1}(U)$ and $J = I \circ \pi : V \to \mathbb{D}^n$. Then $\text{Hess}(F)(I) = \text{Hess}(G)(J)$. □

COROLLARY 8.8. *The geodesic flow induced by the metric g of lemma (8.3) is Poincaré non-degenerate on $T^*Q(k)$ for all compact quotients $Q(k)$ of $\mathcal{H}_p$.*

9. Topological Entropy. It is recalled that if M is a compact topological space, the *topological entropy* of the open covering U is the natural logarithm of infimum of the cardinality of all subcoverings of M:

$$h(U) := \log \inf\{|U'| \ : \ U' \subseteq U, \ U' \text{ covers } M \ \}.$$

If U and V are coverings of M then $U \vee V$ is the covering of M given by $u \cap v$ for all $u \in U$, $v \in V$. The *topological entropy* of the continuous mapping $T : M \to M$ with respect to the covering U is defined by

$$h(T|U) := \lim_{n\to\infty} h(U \vee T^{-1}U \vee \ldots \vee T^{-n}U).$$

DEFINITION 9.1 (Topological Entropy:[27]). The topological entropy of T is then

$$h(T) := \sup\{h(T|U) \ : \ U \text{ covers } M \ \}.$$

The topological entropy of a one-parameter group of transformations, or flow, is defined to be the topological entropy of the time-1 map.

A nice corollary of the Liouville-integrability of the geodesic flow is that

COROLLARY 9.2. *Let $\Gamma \leq \mathcal{H}_p$ be a discrete, cocompact subgroup of $\mathcal{H}_p$, $Q = \Gamma\backslash\mathcal{H}_p$, g be the left-invariant metric on $\mathcal{H}_p$ from lemma (8.3), and the geodesic flow of the left-invariant metric on $T^*(\Gamma\backslash\mathcal{H}_p)$ be φ_t. Then φ_t has zero topological entropy.*

Proof. Let L be the set of regular points of the energy-momentum map ψ, and let S be its set of singular points. The topological entropy of φ_t is equal to the maximum of the topological entropy of $\varphi_t|_S$ and $\varphi_t|_L$. The geodesic flow on the invariant set L is zero because this flow is smoothly conjugate to a straightline translation on a $2p+1$ torus.

Indeed, it is clear that the critical-point subset $S_m := \cup_{i=1}^{p} \{f_i = 0, p_{\gamma_i} \neq 0\}$ is "movable": Let $\tilde{f}_i = \phi(p_{\gamma_i}) \cos\frac{2\pi}{T}\left(\frac{p_{\beta_i}}{p_{\gamma_i}} - x\right)$. Then $\tilde{f}_i$ are first integrals that are independent on a neighbourhood of S_m, so on S_m the geodesic flow is smoothly congjugate to a translation flow on a torus, too.

It is therefore essential to show that the topological entropy of φ_t restricted to the critical-point set

$$\{p_{\gamma_i} = 0 : \text{ some } i\} \cup \{p_\alpha = 0, \sum_{i=1}^{p} p_{\beta_i} p_{\gamma_i} = 0\}$$

is zero. To do this, recall

THEOREM 9.3 (Bowen: [8]). *Suppose that (X, d) is a compact separable metric space, the compact Lie group G acts continuously on X and let $Y = X/G$ and π is the projection onto Y. Suppose that $S_t : X \to X$ and $T_t : Y \to Y$ are 1-parameter groups of homeomorphisms such that*

$$\pi \circ S_t = T_t \circ \pi,$$

and S_t commutes with the action of G on X. Then

$$h(S_t|X) = h(T_t|Y).$$

In the present situation, we will examine the flow generated by the geodesic flow on a symplectic quotient and show that the flow on this symplectic quotient is smoothly conjugate to a natural geodesic flow on the symplectic quotient. An induction argument on p will then prove the theorem.

For the case where $p = 0$, the space $T^*Q = T^*\mathbb{T}^1$ and the hamiltonian of the geodesic flow is $H = p_\alpha^2$ whose flow is a straightline flow everywhere. The topological entropy of this flow is therefore zero.

Assume that for $q = 0, \dots, p-1$ the geodesic flow on all the cotangent bundles T^*Q', $Q' = \Gamma' \backslash \mathcal{H}_q$, has zero topological entropy. It will be shown that the topological entropy of the geodesic flow on T^*Q must be zero, too.

First, suppose that $< p_\beta, p_\gamma > \neq 0$.

If $(x, y, z, p_\alpha, p_\beta, p_\gamma)$ are coordinates on T^*Q then the canonical 1-form

$$\theta = p_\alpha \alpha + \sum_{i=1}^{p} p_{\beta_i} \beta_i + p_{\gamma_i} \gamma_i$$

and so the symplectic form in these coordinates is

$$\Omega = -d\theta = \alpha \wedge p_\alpha + \sum_{i=1}^{p} \beta_i \wedge dp_{\beta_i} + \gamma_i \wedge dp_{\gamma_i} + p_{\gamma_i} \alpha \wedge \beta_i.$$

Let $J \subset \{1, \dots, p\}$ be a set of indices and suppose $p_{\gamma_j} = 0$ iff $j \in J$. Let I be the complement of J, and $\mathbb{T}^{J,z}$ be the torus whose action generates the momenta p_{γ_j}. The torus $\mathbb{T}^{J,z}$ acts freely on the zero momentum level set $S_J := \cap_{j \in J} \{p_{\gamma_j} = 0\}$ so it is possible to form the symplectic quotient $S_J / \mathbb{T}^{J,z}$. On this reduced space the symplectic form and hamiltonian of the geodesic vector field become

$$\Omega_J = \alpha \wedge p_\alpha + \sum_{i=1}^{p} \beta_i \wedge dp_{\beta_i} + \sum_{i \in I} \gamma_i \wedge dp_{\gamma_i} + p_{\gamma_i} \alpha \wedge \beta_i,$$

and

$$H_J = p_\alpha^2 + \sum_{i=1}^{p} p_{\beta_i}^2 + \sum_{i \in I} p_{\gamma_i}^2,$$

where notation is abused and the coordinates on the symplectic quotient $S_J / \mathbb{T}^{J,z}$ are identified with the coordinates on the unreduced cotangent bundle.

An inspection of the reduced geodesic vector field X_{H_J} reveals that the functions p_{β_j} for $j \in J$ are all first integrals of the reduced geodesic vector field. The torus $\mathbb{T}^{J,y}$ whose action generates these momenta acts freely on the common level set $S_{J,J} := \cap_{j \in J} \{p_{\beta_j} = cst\}$ and so it is possible to symplectically reduce the level $S_{J,J}$ by means of the torus $\mathbb{T}^{J,y}$.

By repeating the above computations it is clear that the second symplectic quotient $S_{J,J} / \mathbb{T}^{J,y}$ is symplectomorphic to $T^*(\Gamma' \backslash \mathcal{H}_{p'})$ with its standard symplectic form where $p' = \# I$ and $\Gamma' = \Gamma / G_{J,J} \cap \Gamma$ where $G_{J,J}$ is the subgroup of $\mathcal{H}_p$ whose right action on T^*Q has the momemtum map $(p_{\beta_j}, p_{\gamma_j})_{j \in J}$. The reduced hamiltonian $H_{J,J}$ is equal (up to a constant) to the hamiltonian of the natural geodesic vector field on this cotangent bundle. By the induction hypothesis, then, the reduced flow on this symplectic quotient has zero topological entropy. By Bowen's theorem, the topological entropy of $\varphi_t|_{S_J}$ is zero. Because the collection of such J's is finite the topological entropy of φ_t on the critical point set $\{p_{\gamma_i} = 0 : \text{ some } i\}$ is zero.

The only case that remains to be considered is $< p_\beta, p_\gamma >= 0, p_\alpha = 0$. On this set $\dot{p}_\alpha$ is identically zero along the geodesic flow and so, therefore, is $\dot{x}$ and all $\dot{p}_{\beta_j}$. Reduce the submanifold $\{< p_\beta, p_\gamma >= 0, p_\alpha = 0\}$ in T^*Q by the action of the torus $\mathbb{T}^p \simeq Z(\mathcal{H}_p)/Z(\Gamma)$ on T^*Q. On this reduced space the geodesic vector field is projected to

$$X_{H_{red}} = \begin{cases} \dot{p}_\alpha &= 0, \quad \dot{x} = 0, \\ \dot{p}_{\beta_i} &= 0, \quad \dot{y}_i = p_{\beta_i}, \\ \dot{p}_{\gamma_i} &= 0. \end{cases} \tag{15}$$

whose flow is a translation on a torus. Consequently, its topological entropy vanishes and so therefore does the topological entropy of the geodesic flow on the unreduced space. □

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REFERENCES

[1] V. I. Arnol'd and A. Avez, *Ergodic Problems of Classical Mechanics*, W. A. Benjamin, New York, 1968.

[2] O. I. Bogoyavlenskij, *Theory of tensor invariants of integrable hamiltonian systems. II. Theorem on symmetries and its applications*, Comm. Math. Phys., 184:2 (1997), pp. 301–365.

[3] O. I. Bogoyavlenskij, *Canonical forms for the invariant tensors and a-b-c cohomologies of integrable hamiltonian systems*, Math. USSR-Sb., 189:3 (1998), pp. 315–357.

[4] O. I. Bogoyavlenskij, *Conformal symmetries of dynamical systems and Poincaré 1892 concept of iso-energetic non-degeneracy*, C. R. Acad. Sci. Paris Sér. I Math, 326:2 (1998), pp. 213–218.

[5] A. V. Bolsinov, *Fomenko invariants in the theory of integrable hamiltonian systems*, Russian Math. Surveys, 52:5 (1997), pp. 997–1015.

[6] A. V. Bolsinov and A. T. Fomenko, *Integrable Geodesic Flows on Two-Dimensional Surfaces*, preprint, 1997.

[7] A. V. Bolsinov and I. A. Taĭmanov, *Integrable geodesic flow with positive topological entropy*, Invent. Math., 140:3 (2000), pp. 639 – 650.

[8] R. Bowen, *Entropy for group endomorphisms and homogeneous spaces*, Trans. Amer. Math. Soc., 153 (1971), pp. 401–414.

[9] L. Butler, *A new class of homogeneous manifolds with Liouville-integrable geodesic flows*, C. R. Math. Rep. Acad. Sci. Canada, 21 (1999), no. 43, pp. 7509–7522.

[10] E. I. Dinaburg, *On the relations among various entropy characteristics of dynamical systems*, Math. USSR Izv., 5:2 (1971), pp. 337–378.

[11] A. T. Fomenko, *Symplectic Geometry*, Gordon and Breach, New York, 1988.

[12] A. T. Fomenko, *Topological classification of all integrable hamiltonian differential equations with degrees of freedom*, in The Geometry of Hamiltonian Systems (Proceedings of a Workshop, June 5-16, 1989, Berkeley, USA), T. S. Ratiu, ed., Springer-Verlag, New York, 1989, pp. 131–339.

[13] E. V. Gaĭdukov, *Asymptotic geodesics on a riemannian manifold nonhomeomorphic to a sphere*, Soviet Math. Dokl., 7:4 (1966), pp. 1033–1035.

[14] C. S. Gordon and E. N. Wilson, *The spectrum of the laplacian on riemannian Heisenberg manifolds*, Michigan Math. J., 33:2 (1986), pp. 253–271.

[15] M. Gromov, *Groups of polynomial growth and expanding maps*, Inst. Hautes Études Sci. Publ. Math., 53 (1981), pp. 53–73.

[16] V. Guillemin and S. Sternberg, *On collective complete integrability according to the method of Thimm*, Ergodic Theory Dynamical Systems, 3 (1983), pp. 219–230.

[17] V. V. Kozlov, *Topological obstructions to the integrability of natural mechanical systems*, Soviet Math. Dokl., 20:6 (1979), pp. 1413–1415.

[18] V. V. Kozlov, *Symmetries, Topology, and Resonances in Hamiltonian Mechanics*, Springer, Berlin, 1996.

[19] L. M. LERMAN AND YA. L. UMANSKIY, *Four-Dimensional Integrable Hamiltonian Systems with Simple Singular Points*, Translation of Mathematical Monographs 176, AMS, Providence, R.I., 1998.

[20] A. I. MAL'CEV, *On a Class of Homogeneous Spaces*, AMS Translation 39, AMS, Providence, R.I., 1951.

[21] K. NOMIZU, *On the cohomology of compact homogeneous spaces of nilpotent Lie groups*, Ann. of Math., 59 (1954), pp. 531–538.

[22] G. P. PATERNAIN AND R. J. SPATZIER, *New examples of manifolds with completely integrable geodesic flows*, Advances in Mathematics, 108:2 (1994), pp. 346–366.

[23] G. P. PATERNAIN, *On the topology of manifolds with completely integrable geodesic flows*, Ergodic Theory Dynamical Systems, 12 (1992), pp. 109–121.

[24] G. P. PATERNAIN, *Multiplicity two actions and loop space homology*, Ergodic Theory Dynamical Systems, 13 (1993), pp. 143–151.

[25] G. P. PATERNAIN, *On the topology of manifolds with completely integrable geodesic flows ii*, J. Geom. Phys., 13 (1994), pp. 289–298.

[26] H. POINCARÉ, *Les mèthodes Nouvelles de La Mécanique Celeste*, Gauthier-Villars, Paris, 1891, 1893, 1899.

[27] R. L. ADLER, A. G. KONHEIM, AND M. H. MCANDREW, *Topological entropy*, Trans. Amer. Math. Soc., 114 (1965), pp. 309–319.

[28] I. A. TAĬMANOV, *Topological obstructions to integrability of geodesic flows on non-simply-connected manifolds*, Math. USSR Izv., 30:2 (1988), pp. 403–409.

[29] I. A. TAĬMANOV, *The topology of riemannian manifolds with integrable geodesic flows*, Proc. Steklov Inst. Math., 205:6 (1995), pp. 139–150.

[30] A. THIMM, *Integrable geodesic flows on homogeneous spaces*, Ergodic Theory Dynamical Systems, 1 (1981), pp. 495–517.

[31] J. A. WOLF, *Growth of finitely generated solvable groups and curvature of riemannian manifolds*, J. Differential Geom., 2 (1968), pp. 421–446.

[32] Y. YOMDIN, *Volume growth and entropy*, Israel J. Mathematics, 57 (1987), pp. 287–300.