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010

MODULI SPACES OF $PU(2)$-MONOPOLES*

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1. Introduction. The most natural way to prove the equivalence between Donaldson theory and Seiberg-Witten theory is to consider a suitable moduli space of "non-abelian monopoles". In [OT5] it was shown that an S^1-quotient of a moduli space of quaternionic monopoles should give an homological equivalence between a fibration over a union of Seiberg-Witten moduli spaces and a fibration over certain $Spin^c$-moduli spaces [PT1].

By the same method, but using moduli spaces of $PU(2)$-monopoles instead of quaternionic monopoles, one should be able to express any Donaldson invariant in terms of Seiberg-Witten invariants associated with the twisted abelian monopole equations of [OT6]. In [T1], [T2], we have shown that this idea can be further generalized to express Donaldson-type invariants associated with higher symmetry groups in terms of new Seiberg-Witten-type invariants.

The strategy has a very general algebraic-geometric analogon, which we call the "Master Space" strategy. This procedure, developed by Ch. Okonek and the author [OT7], [OST] reduces the problem of the computation of certain numerical invariants of a GIT moduli space to similar computations on simpler moduli spaces. One "couples" the given GIT problem to a simpler one (having the same symmetry group), and then studies the "Master Space" associated with the coupling as a $\mathbb{C}^*$-space. The fixed point locus of the $\mathbb{C}^*$-action consists of the original moduli space and a union of simpler ones. Then one can use the S^1-quotient of the master space to define a homological equivalence between a projective fibration over the initial moduli space and a projective fibration over the other components of the fixed point locus. In the GIT-framework, as in the gauge theoretical one, the technical difficulty is the same: the master space can be singular. The present paper deals with this difficulty in the gauge theoretical situation.

A program for proving the equivalence between Donaldson theory and Seiberg-Witten theory, which also uses moduli spaces of non-abelian monopoles, is due to Pidstrigach and Tyurin [PT2], and was already announced by Pidstrigach in a Conference at the Newton Institute in Cambridge, in December 1994.

There are, however, several important differences between Pidstrigach-Tyurin's original approach, and the strategy developed by Ch. Okonek in collaboration with the author, which is the strategy we follow in the present paper.

First, our equations have a gauge group of the form $SU(E)$ and hence the moduli spaces which we construct are S^1-spaces; in contrast, the Pidstrigach-Tyurin equations [PT2] have a gauge group of the form $U(E)$. Whereas we fix the connection in the determinant line bundle, they only fix the curvature of this connection. If $H_1(X, \mathbb{Z}) = 0$, their moduli space is the S^1-quotient of ours. On the other hand, the S^1-operation plays a very important role in our strategy: The description of the ends around the abelian locus at infinity uses in an essential way the S^1-equivariance of the local models.

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Second, we do not follow Pidstrigach-Tyurin's program to prove generic regularity results. We show (see section 3.1) that the proofs of the transversality theorems which they use [PT2] to get generic regularity are incomplete, by indicating counterexamples to one of the statements on which these proofs are based[1].

It is interesting to notice that, in fact, *any* non-abelian solution of the equations in the Kähler case gives a counterexample to their statement. This same statement was also used by the authors in the their definition of the $Spin^c$-polynomial invariants [PT1].

The transversality problem is very complicated, for the $PU(2)$-monopole equations as well as for the non-abelian $Spin^c$-equations. The difficulty is the same in both cases: in the non-abelian points with degenerate spinor component transversality cannot be proved using only perturbations with 0-order operators.

In [T1] the author tried to use perturbations with first order operators, and proved that perturbations of this type lead to transversality at least away from the solutions which are abelian on a non-empty open set. However, in order to have a complete transversality result away from the abelian locus, one would need a unique continuation theorem which seems to be difficult to get because of the perturbed symbol.

Another way to achieve transversality is to use an infinite family of "holonomy perturbations" [FL].

The present paper solves two fundamental problems concerning the moduli spaces of $PU(2)$-monopoles: *generic regularity* and *compactification.*

First we prove an S^1-equivariant generic-smoothness theorem: we define perturbations of the equations which lead to S^1-spaces which, for generic choices of the perturbing parameters are smooth, at least outside the "Donaldson locus" (the vanishing locus of the projection on the spinor component) and of the abelian locus (Theorem 3.19). The proof of the generic-smoothness theorem is not a pure transversality argument; it combines a standard transversality argument with a new method to control the exceptions to transversality.

Our result shows that one does get regularity for a generic choice of a system (g, σ, β, K)*, consisting of a metric, a compatible* $Spin^{U(2)}(4)$*-structure* σ *and an order 0- perturbation* (β, K) *of the type considered in [PT2].*

We also obtain generic regularity results for the normal bundles of the Donaldson locus and the abelian locus within the moduli space (Theorem 3.21, Proposition 3.22). Similar results, but obtained using quite different lines of reasoning, were obtained by Feehan [F] in a preprint distributed around the same time as the first version of the present paper.

Therefore one can go forward towards a proof of the Witten conjecture (see for instance [OT5] for a detailed description of the strategy) using relatively simple equations.

Note however that the generic regularity results which we prove for the ASD-$Spin^c$-equations, do not automatically solve the transversality problem needed in order to give sense to the $Spin^c$-polynomial invariants, and to use them effectively. For this purpose one needs a pure transversality argument for the ASD-$Spin^c$-equations[2].

[1]This gap as well as the difficulty of the problem was pointed out by the author during the Workshop "4-dimensional manifolds", Oberwolfach, March 1996.

[2]The point is that, in order to have *well defined* invariants, it is importants to have a smooth parameterized moduli space (see [DK], p 143, 149). Generic regularity is not enough. Moreover, the Kählerian parameters are *all* non-generic in our sense; on the other hand, all the computations needed in order to get a proof of the Van de Ven conjecture using $Spin^c$-invariants, must be done in

Therefore, The theory of $Spin^c$-polynomials and the attempt to prove the Van de Ven conjecture using this theory [PT1], should be revised.

We get our result in two steps. In a first step we prove that, using only the perturbations (β, K), one can prove the following partial transversality result: If the Seiberg-Witten map extended to the parameterized moduli space is not a submersion in a point (A, Ψ, β, K), then the spinor component Ψ must be degenerate. This is very easy to see.

In the second step we prove that, if we also let the $Spin^{U(2)}(4)$-structure (together with the metric) vary , then the moduli space $\widetilde{\mathcal{DM}}^*_X$ of solutions with non-trivial but degenerate spinor component in the enlarged parameterized moduli space $\widetilde{\mathcal{M}}^*_X$ has infinite codimension in every non-abelian point. Using this, we can show (by "weakening" locally the degeneracy equation) that every non-abelian point $[p]$ in $\widetilde{\mathcal{DM}}^*_X$ has a neighbourhood $U_{[p]}$ which is a closed analytic subspace of a manifold $V_{[p]}$ which is Fredholm of negative index over the enlarged parameter space. Taking a countable subcover $(V_{[p]_i})_{i\in\mathbb{N}}$, and using the fact that Fredholm maps are locally proper ([Sm]), we prove that the set of parameters for which there exists a non-abelian solution with non-trivial degenerate spinor component is of the first category . The desired set of "generic parameters" is then obtained by intersecting the complement of this set with the set of regular values of the projection of $\widetilde{\mathcal{M}}^*_X \setminus \widetilde{\mathcal{DM}}^*_X$ on the parameter space.

We believe that this method is in fact a very general one; it can be summarized as follows: Prove first a partial transversality result using perturbations with 0-order differential operators, and show then that the space of solutions which are exceptions to transversality has infinite codimension if one introduces new variable parameters. Such a result is to be expected provided the "exceptional solutions" , the ones which are exceptions to transversality, solve an overdetermined elliptic system.

In particular, the method can be applied to obtain generic regularity along the Donaldson and the abelian locus. More precisely, the moduli space of solutions (with non-vanishing spinor component) of the Dirac-ASD system of [PT1] becomes smooth of expected dimension for generic perturbations. The same property has the complement of the zero-section in the fibration of "normal infinitesimal deformations" over the subspace of abelian solutions associated with an abelian reduction of the $Spin^{U(2)}(4)$-bundle.

In this way we obtain perturbed moduli spaces which are smooth except in the abelian points and in the Donaldson-points. These points remain exceptions to transversality, and in general, regularity (smoothness and expected dimension) *cannot* be achieved in these points by using S^1-equivariant perturbations.

The second purpose of the paper, the existence of an "Uhlenbeck compactification" for the perturbed moduli spaces, is achieved in section 4 (see Theorem 4.24). A different proof of the "Uhlenbeck compactification" can be found in [LT].

Our arguments follow the same strategy as in the instanton case [DK], which can be summarized as follows:

Local estimates – Regularity – Removable Singularities – Compactification.

Some care must be taken, since the monopole equations are only "scale invariant", not conformal invariant as in the instanton case. On the other hand, many of the results in [DK] were obtained by cutting off the solutions and transferring the

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the Kähler case.

problem from the 4-ball to the 4-sphere, and then using the conformal invariance of the equations.

Our proof uses the same method, but endows the sphere with a metric with non-negative sectional curvature which is flat in a neighbourhood of the north pole. With this choice, the corresponding first order elliptic operators ($\not{D}$, $d^* + d^+$, ...) are still injective. For the local computations we work with pairs whose connection component is in Coulomb gauge in the sense of [DK], so that all the results in [DK] about connections in Coulomb gauge apply automatically. Therefore, we do not use the Coulomb gauge condition for pairs which follows from the elliptic complex of the $PU(2)$-monopole equations (compare with [FL]).

A short version of our proof of the Uhlenbeck compactification appeared in [OT5], and a very detailed version of it can be found in [T1]. The existence of an Uhlenbeck compactification for moduli spaces of non-abelian monopoles was predicted by Pidstrigach and Tyurin in [PT2].

Note that in order to prove the equivalence between the Donaldson and the Seiberg-Witten theories, it now remains only to give explicit descriptions of the ends of the moduli space along the abelian locus, and to calculate the corresponding contributions.

My own strategy to study the ends of the moduli spaces of $PU(2)$-monopoles is based on the analytical results in [T3]. The $PU(2)$-monopole equations are not conformally invariant, so it is difficult to use the method developed in the case of instantons [DK] (which consists of identifying the solutions concentrated in a point with the solutions on the connected sum of X with S^4). We use a new strategy [T4] which is still based on the gluing method. We obtain concentrated solutions by gluing (non-concentrated) solutions on X corresponding to lower topological data, with concentrated instantons on the *tangent spaces*, and then we deform the obtained almost-solutions into solutions. This last step makes use of the classical Fredholm L^p theory on X, as well as of the Fredholm L^p-theory on the tangent spaces (instead of S^4) which is developed in the quoted paper.

Progress on this problem, using different methods, was also announced by Feehan and Leness.

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2. $PU(2)$-monopoles.

2.1. The $Spin^{U(2)}$ group and $Spin^{U(2)}$-structures. For a more detailed presentation of the theory of $Spin^{U(2)}$-structures we refer the interested reader to [T1], [T2]. In these papers we also introduce the concept of $Spin^G$-structures and G-monopole equations for quite general compact Lie groups G.

The group $Spin^{U(2)}$ is defined by

$$Spin^{U(2)} := Spin \times_{\mathbb{Z}_2} U(2) .$$

Using the natural isomorphism $U(2)/\mathbb{Z}_2 \simeq PU(2) \times S^1$, we get the exact sequences

$$1 \longrightarrow Spin \longrightarrow Spin^{U(2)} \xrightarrow{(\bar{\delta},\det)} PU(2) \times S^1 \longrightarrow 1$$

$$1 \longrightarrow U(2) \longrightarrow Spin^{U(2)} \xrightarrow{\pi} SO \longrightarrow 1$$

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow Spin^{U(2)} \xrightarrow{(\pi,\bar{\delta},\det)} SO \times PU(2) \times S^1 \longrightarrow 1 .$$

Let X be a compact manifold and P^u a $Spin^{U(2)}$-bundle over X. We consider the following associated bundles

$$\pi(P^u) := P^u \times_\pi SO,\ \bar{\delta}(P^u) := P^u \times_{\bar{\delta}} PU(2),\ \det(P^u) := P^u \times_{\det} S^1,$$

$$\mathbb{G}_0 := P^u \times_{Ad\circ\bar{\delta}} SU(2);\ \mathfrak{g}_0 := P^u \times_{ad\circ\bar{\delta}} SU(2) ,$$

where $Ad : PU(2) \longrightarrow Aut(SU(2))$, $ad : PU(2) \longrightarrow so(su(2))$ are induced by the adjoint morphism $SU(2) \longrightarrow Aut(SU(2))$, $SU(2) \longrightarrow so(su(2))$.

The group of sections $\mathcal{G}_0 := \Gamma(X, \mathbb{G}_0)$ can be identified with the group of automorphisms of P^u over $\pi(P^u) \times_X \det(P^u)$. After suitable Sobolev completions it becomes a Lie group, whose Lie algebra is the corresponding completion of $A^0(\mathfrak{g}_0)$.

Let P be a SO bundle over X. A $Spin^{U(2)}$-structure in P is a morphism $P^u \longrightarrow P$ of type π, where P^u is a $Spin^{U(2)}$-bundle. Two $Spin^{U(2)}$-structures $P^u \longrightarrow P$, $P'^u \longrightarrow P$ in P are called equivalent if the bundles P^u, P'^u are isomorphic over P. A $Spin^{U(2)}(n)$-structure in an oriented Riemannian 4-manifold (X, g) is a $Spin^{U(2)}(n)$-structure in the bundle P_g of oriented coframes.

We refer to [T1], [T2] for the following classification result:

PROPOSITION 2.1. *Let P be a principal SO-bundle, $\bar{P}$ a $PU(2)$-bundle, and L a Hermitian line bundle over X.*

i) P admits a $Spin^{U(2)}$-structure $P^u \to P$ with

$$P^u \times_{\bar{\delta}} PU(2) \simeq \bar{P} ,\quad P^u \times_{\det} \mathbb{C} \simeq L$$

if and only if $w_2(P) = w_2(\bar{P}) + \bar{c}_1(L)$, where $\bar{c}_1(L)$ is the mod 2 reduction of $c_1(L)$.

ii) If the base X is a compact oriented 4-manifold, then the map

$$P^u \longmapsto ([P^u \times_{\bar{\delta}} PU(2)], [P^u \times_{\det} \mathbb{C}])$$

defines a 1-1 correspondence between the set of isomorphism classes of $Spin^{U(2)}$-structures in P and the set of pairs of isomorphism classes $([\bar{P}], [L])$, where $\bar{P}$ is a $PU(2)$-bundle and L an S^1-bundle with $w_2(P) = w_2(\bar{P}) + \bar{c}_1(L)$. The latter set can be identified with

$$\{(p, c) \in H^4(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) |\ p \equiv (w_2(P) + \bar{c})^2 \bmod 4\}$$

□

The group $Spin^{U(2)}(4)$ can be written as

$$Spin^{U(2)}(4) = SU(2)_+ \times SU(2)_- \times U(2)/\mathbb{Z}_2 ,$$

hence it comes with natural orthogonal representations

$$\mathrm{ad}_\pm : Spin^{U(2)}(4) \longrightarrow so(su(2)),$$

defined by the adjoint representations of $SU(2)_\pm$, and with natural unitary representations

$$\sigma_\pm : Spin^{U(2)}(4) \longrightarrow U(\mathbb{H}_\pm \otimes_{\mathbb{C}} \mathbb{C}^2)$$

obtained by coupling the canonical representations of $SU(2)_\pm$ with the canonical representation of $U(2)$.

We denote by $\mathrm{ad}_\pm(P^u)$, $\Sigma^\pm(P^u)$ the corresponding associated vector bundles. The Hermitian 4-bundles $\Sigma^\pm(P^u)$ are called the spinor bundles of P^u, and the sections in these bundles are called spinors.

We refer to [T2] for the following simple result

PROPOSITION 2.2. *Let P be an $SO(4)$-bundle whose second Stiefel-Whitney class admits integral lifts. There is a 1-1 correspondence between isomorphism classes of $Spin^{U(2)}$-structures in P and equivalence classes of pairs consisting of a $Spin^c(4)$-structure $P^c \longrightarrow P$ in P and a $U(2)$-bundle E. Two pairs are considered equivalent if, after tensoring the first one with a line bundle, they become isomorphic over P.*

Suppose that P^u is associated with the pair (P^c, E), and let $\Sigma_c^\pm$ be the spinor bundles corresponding to P^c. Then the associated bundles $P^u \times_\pi \mathbb{R}^4$, $\Sigma^\pm(P^u)$, $\bar{\delta}(P^u)$, $\det(P^u)$, $\mathbb{G}(P^u)$, $\mathbb{G}_0(P^u)$ can be expressed in terms of the pair (P^c, E) as follows:

$$P^u \times_\pi \mathbb{R}^4 = \mathbb{R}SU(\Sigma_c^+, \Sigma_c^-) \ ,$$

$$\Sigma^\pm(P^u) = [\Sigma_c^\pm]^\vee \otimes E = \Sigma_c^\pm \otimes E^\vee \otimes [\det(P^u)], \ \bar{\delta}(P^u) \simeq {}^{P_E}/_{S^1}, \ \mathrm{ad}_\pm(P^u) = su(\Sigma_c^\pm)$$

$$\det(P^u) \simeq \det(P^c)^{-1} \otimes (\det E), \ \mathbb{G}_0(P^u) = SU(E), \ \mathfrak{g}_0(P^u) = su(E) \ .$$

Here we denoted by $\mathbb{R}SU(\Sigma_c^+, \Sigma_c^-)$ the bundle of real multiples of $\mathbb{C}$-linear isometries of determinant 1 from Σ_c^+ to Σ_c^-. The euclidean structure and the orientation in this bundle are fibrewise defined by the Pauli matrices associated with a pair of frames $(e_1^\pm, e_2^\pm)$ in $\Sigma_c^\pm$, satisfying $e_1^+ \wedge e_2^+ = e_1^- \wedge e_2^-$.

The data of a $Spin^{U(2)}(4)$-structure $P^u \longrightarrow P$ in an $SO(4)$-bundle P is equivalent to the data of an orientation preserving linear isometry

$$\gamma : P \times_{SO(4)} \mathbb{R}^4 \longrightarrow P^u \times_\pi \mathbb{R}^4 = \mathbb{R}SU(\Sigma_c^+, \Sigma_c^-) \subset \mathrm{Hom}_{\mathbb{G}_0}(\Sigma^+(P^u), \Sigma^-(P^u))$$

which will be called the Clifford map of the structure.

The Clifford map γ induces isomorphisms

$$\Gamma_\pm : \Lambda_\pm^2(P \times_{SO(4)} \mathbb{R}^4) \longrightarrow su(\Sigma_c^\pm) = \mathrm{ad}_\pm(P^u) \ ,$$

which multiply the norms by 2 ([DK] p. 77, [OT1]).

The following simple remark will play a fundamental role in this paper:

Suppose that Λ is a real oriented 4-bundle, and $\gamma : \Lambda \longrightarrow P^u \times_\pi \mathbb{R}^4$ an orientation preserving linear isomorphism. Then γ defines an Euclidean structure g_γ on Λ such that γ becomes the Clifford map of a $Spin^{U(2)}(4)$-structure in (Λ, g_γ).

2.2. The $PU(2)$-monopole equations. Let $\sigma : P^u \longrightarrow P_g$ be a $Spin^{U(2)}(4)$-structure in the oriented compact Riemannian 4-manifold (X, g). Fix a connection $a \in \mathcal{A}(\det(P^u))$. Using the third exact sequence in (1), we see that the data of a connection $A \in \mathcal{A}(\bar{\delta}(P^u))$ is equivalent to the data of a connection $B_{A,a}$ in P^u which lifts the Levi-Civita connection in P_g and the fixed connection a in $\det(P^u)$ (via the maps $P^u \longrightarrow P_g$ and $P^u \longrightarrow \det(P^u)$ respectively). The Dirac operator $\not{D}_{A,a}$ associated with the pair (A, a) is the first order elliptic operator

$$\not{D}_{A,a} : A^0(\Sigma^{\pm}(P^u)) \xrightarrow{\nabla_{B_{A,a}}} A^1(\Sigma^{\pm}(P^u)) \xrightarrow{\gamma} A^0(\Sigma^{\mp}(P^u))$$

Regarded as operator $\Sigma^+(P^u) \oplus \Sigma^-(P^u) \longrightarrow \Sigma^+(P^u) \oplus \Sigma^-(P^u)$, the Dirac operator $\not{D}_{A,a}$ is also selfadjoint.

We define the quadratic map $\Sigma^{\pm}(P^u) \longrightarrow \mathrm{ad}_+(P^u) \otimes \mathfrak{g}_0$, $\Psi \longmapsto (\Psi\bar{\Psi})_0$ by

$$(\Psi\bar{\Psi})_0 := pr_{\mathrm{ad}_+(P^u)\otimes\mathfrak{g}_0}(\Psi \otimes \bar{\Psi}) \ ,$$

where $pr_{\mathrm{ad}_+(P^u)\otimes\mathfrak{g}_0}$ denotes the orthogonal projection

$$\mathrm{Herm}(\Sigma^+(P^u)) \longrightarrow \mathrm{ad}_+(P^u) \otimes \mathfrak{g}_0 \ .$$

We introduce now the $PU(2)$-Seiberg-Witten equations SW_a^σ associated to the pair (σ, a), which are equations for a pair (A, Ψ) formed by a $PU(2)$-connection $A \in \mathcal{A}(\bar{\delta}(P^u))$ and a positive spinor $\Psi \in A^0(\Sigma^+(P^u))$:

$$(SW_a^\sigma) \qquad \begin{cases} \not{D}_{A,a}\Psi &= 0 \\ \Gamma(F_A^+) &= (\Psi\bar{\Psi})_0 \end{cases}$$

The natural symmetry group of the equations is the gauge group $\mathcal{G}_0 := \Gamma(X, \mathbb{G}_0)$. We denote by $\mathcal{M}_a^\sigma$ the moduli space

$$\mathcal{M}_a^\sigma := \left[\mathcal{A}(\bar{\delta}(P^u)) \times A^0(\Sigma^+(P^u))\right]^{SW_a^\sigma} /_{\mathcal{G}_0} \ ,$$

where $\left[\mathcal{A}(\bar{\delta}(P^u)) \times A^0(\Sigma^+(P^u))\right]^{SW_a^\sigma}$ denote the space of solutions of the equations (SW_a^σ). Using the well-known Kuranishi method one can endow $\mathcal{M}_a^\sigma$ with the structure of a ringed space, which has locally the form $Z(\theta)/_G$, where G is a closed subgroup of $SU(2)$ acting on finite dimensional vector spaces H^1, H^2, and $Z(\theta)$ is the real analytic space cut-out by a G-equivariant real analytic map $H^1 \supset U \xrightarrow{\theta} H^2$ (see [OT5], [T1], [T2] for details).

3. Smooth moduli spaces.

3.1. The difficulty. Equations for pairs (A, Ψ), where A is a unitary connection with fixed determinant connection and Ψ a non-abelian Dirac spinor have been already considered [PT1], [PT2]. For instance, the definition of $Spin^c$-polynomial invariants starts with the construction of the moduli space of solutions of the $(ASD - Spin^c)$-equations

$$\begin{cases} \not{D}_A\Psi &= 0 \ , \quad \Psi \neq 0 \\ F_A^+ &= 0 \ . \end{cases}$$

The proofs of the corresponding transversality results are incomplete. They are based on the following false statement ([PT2], [PT1]):

(A) Let $P^c \longrightarrow P_g$ be a $Spin^c(4)$-structure with spinor bundles $\Sigma^{\pm}(P^c)$ on a Riemannian 4-manifold (X, g), E a Hermitian 2-bundle on X, and A a unitary connection in E. If the $\not{D}_A$-harmonic non-vanishing positive spinor $\Psi \in A^0(\Sigma^+(P^c) \otimes E)$ is fibrewise degenerate considered as morphism $E^\vee \longrightarrow \Sigma^+$, then A is reducible.

In the proof of this assertion ([PT1] p. 277) it was used that, in the presence of a $Spin^c(4)$-structure, the Clifford pairing $(\alpha, \sigma) \longmapsto \gamma(\alpha)\sigma$ between 1-forms and positive spinors has fibrewise no divisors of zero. This is true for real 1-forms, but not for complex ones.

Counterexamples are easy to find:

Every holomorphic section in a holomorphic Hermitian 2-vector bundle $\mathcal{E}$ on a Kähler surface can be regarded as a degenerate harmonic positive spinor in $\Sigma^+_{\text{can}} \otimes \mathcal{E}$, where $\Sigma^+_{\text{can}} = \Lambda^{00} \oplus \Lambda^{02}$ is the positive spinor bundle of the canonical $Spin^c(4)$-structure in X, if we endow $\mathcal{E}$ with the Chern connection given by the holomorphic structure. Therefore any indecomposable holomorphic 2-bundle $\mathcal{E}$ with $H^0(\mathcal{E}) \neq 0$ gives a counterexample to the assertion **(A)**.

Note that these counterexamples occur precisely in the Kähler framework, where all explicit computations of moduli spaces and invariants were carried out.

3.2. Partial transversality results. Let $\sigma : P^u \longrightarrow P_g$ be a $Spin^{U(2)}(4)$-structure on (X, g), denote by

$$\gamma : \Lambda^1 \longrightarrow \text{Hom}(\Sigma^+, \Sigma^-)$$

be the associated Clifford map, and let C_0 be a fixed $SO(4)$-connection in $P^u \times_\pi SO(4) \simeq P_g$ (not necessarily the Levi-Civita connection). We fix again a connection $a \in \mathcal{A}(\det(P^u))$. For any connection $A \in \mathcal{A}(\bar{\delta}(P^u))$ we have an associated Dirac operator

$$\not{D}^0_{a,A} = \gamma \cdot \nabla_{C_0,a,A} \ ,$$

where $\nabla_{C_0,a,A} : A^0(\Sigma^+) \longrightarrow A^1(\Sigma^+)$ is the covariant derivative associated with the connection in P^u which lifts the triple (C_0, a, A).

The role and the properties of these slightly more general Dirac operators will be cleared up in the next section, where C_0 will be a fixed $\mathcal{C}^\infty$-connection in the fixed bundle $P^u \times_\pi SO(4)$, but the metric g and the Clifford map γ will be variable $\mathcal{C}^k$-parameters.

Recall that one has a canonical embedding $P^u \times_\pi \mathbb{C}^4 \subset \text{Hom}(\Sigma^+, \Sigma^-)$, and that σ defines an isomorphism $\Lambda^1_{\mathbb{C}} \xrightarrow{\sim} P^u \times_\pi \mathbb{C}^4$. We consider the following equations

$$\begin{cases} \not{D}^0_{a,A}(\Psi) + \beta(\Psi) &= 0 \\ \Gamma(F_A^+) &= K(\Psi\bar{\Psi})_0 \end{cases} ,$$

which are equations for a system

$$(A, \Psi, \beta, K) \in \mathcal{A}(\bar{\delta}(P^u)) \times A^0(\Sigma^+) \times A^0(P^u \times_\pi \mathbb{C}^4) \times \Gamma(X, GL(\text{ad}_+)).$$

Complete the configuration space $\mathcal{A} := \mathcal{A}(\bar{\delta}(P^u)) \times A^0(\Sigma^+)$ with respect to a large Sobolev norm L^2_l, and the parameter space

$$\mathcal{Q} := A^0(P^u \times_\pi \mathbb{C}^4) \times \Gamma(X, GL(\text{ad}_+))$$

with respect to the Banach norm $\mathcal{C}^k$, $k \gg l$.

The perturbations (β, K) were also considered by Pidstrigach and Tyurin in [PT2] in their attempt to get transversality for their version of non-abelian monopole equations.

An $SU(2) \times SU(2) \times SU(2)$-reduction of P^u over an open set $U \subset X$ induces isomorphisms $\Sigma^{\pm}(P^u)|_U \simeq S^{\pm} \otimes E$ where $S^{\pm}$, E are $SU(2)$-bundles. A spinor $\Psi \in \Sigma^+(P^u)$ will be called degenerate in $x \in X$ if, with respect to an $SU(2) \times SU(2) \times SU(2)$-reduction around x, $\Psi_x \in S_x \otimes E_x = S_x^{\vee} \otimes E_x$ has rank ≤ 1. Ψ will be called degenerate on $V \subset X$ if it is degenerate in every point of V.

A pair $(A, \Psi) \in \mathcal{A}(\bar{\delta}(P^u)) \times A^0(\Sigma^+)$ will be called abelian if the connection A is reducible, and the spinor Ψ is contained in one of the A-invariant summands of Σ^+.

If $(A, \Psi) \in \mathcal{A}(\bar{\delta}(P^u)) \times A^0(\Sigma^+)$ is an abelian pair, then Ψ is clearly degenerate on X. However, the counterexamples in the previous section show that there exist non-abelian pairs with non-trivial Dirac-harmonic spinor-component which is degenerate on X.

Let $sw = sw_{g,\sigma,C_0} : \mathcal{A}_l \times \mathcal{Q}^k \longrightarrow A^0(\Sigma^-)_{l-1} \times A^0(\mathrm{ad}_+ \otimes \mathfrak{g}_0)_{l-1}$ be the map defined by the left hand side of the equations above, and let

$$\mathcal{N}^* := [\mathcal{A}_l^* \times \mathcal{Q}^k] \cap sw^{-1}(0)/\mathcal{G}_{l+1}$$

be the moduli space of solutions with non-trivial spinor-component. $\mathcal{N}^*$ is the vanishing locus of the induced section $s\bar{w}$ in the Banach bundle

$$[\mathcal{A}_l^* \times \mathcal{Q}^k] \times_{\mathcal{G}_{l+1}} [A^0(\Sigma^-)_{l-1} \times A^0(\mathrm{ad}_+ \otimes \mathfrak{g}_0)_{l-1}]$$

over the Banach manifold $\mathcal{B}^* := \mathcal{A}_l \times \mathcal{Q}^k/\mathcal{G}_{l+1}$ which is defined by sw.

The purpose of this section is to prove the following partial transversality theorem

THEOREM 3.1. *If sw is not a submersion in a solution $p = (A, \Psi, \beta, K) \in \mathcal{A}_l^* \times \mathcal{Q}^k$, then Ψ must be degenerate on X. In particular, $\mathcal{N}^*$ is smooth away from the closed subspace of solutions with globally degenerate spinor component.*

Proof. Let $(\Phi, S) \in A^0(\Sigma^-)_{l-1} \times A^0(\mathrm{ad}_+ \otimes \mathfrak{g}_0)_{l-1}$ a pair which is L^2-orthogonal to $\mathrm{im}(d_p)$. Using the perturbation β we get immediately that $Re(\dot{\beta}, \Phi \otimes \bar{\Psi})$ vanishes for every variation $\dot{\beta}$ of β. With respect to any local $SU(2) \times SU(2) \times SU(2)$-reduction $(S^{\pm}, E)$ of $P^u|_U$ (U an open set) the contraction of $\Phi \otimes \bar{\Psi}$ with the Hermitian metric in E must vanish, which shows that pointwise $\Psi(v^+) \perp \Phi(v^-)$ for every $v^{\pm} \in S_u^{\pm}$, $u \in U$. If Ψ has rank 2 in a point $x \in X$, then Φ must vanish identically on a neighbourhood of x.

Also, if Ψ has rank 2 in x, then $(\Psi\bar{\Psi})_0$ has rank 3 in x as map $\mathrm{ad}_{+,x}^{\vee} \longrightarrow \mathfrak{g}_{0,x}$, hence the same argument as above shows that S must vanish on a neighbourhood of x. Therefore the pair (Φ, S) must be zero on a neighbourhood of x.

We can assume that A is the Coulomb gauge with respect to a smooth connection A_0. Therefore, by Agmon-Douglis-Nirenberg's non-linear-elliptic regularity theorems (see for instance [B], p. 467, Theorem 41), it follows that (A, Ψ) is a smooth pair (if the Clifford map $\Lambda^1 \longrightarrow P^u \times \mathbb{R}^4$ had only class C^k, we would have got a $C^{k+1-\varepsilon}$-pair, which is enough to complete the argument). Using now variations $(\dot{A}, \dot{\Psi})$, we see that (Φ, S) must satisfy an elliptic system of the form

$$\tilde{D}^1_{A,\Psi}[\tilde{D}^1_{A,\Psi}]^*(\Phi, S) = 0 \,.$$

Here $\tilde{D}^1_{A,\Psi}$ is the first derivative in (A,Ψ) of the map $\tilde{sw}$ obtained by dividing by 2 the second component of sw such that the symbol of $\tilde{D}^1_{A,\Psi}[\tilde{D}^1_{A,\Psi}]^*$ becomes a scalar, and Aronszajin's theorem applies. It follows that $(\Phi, S) = 0$, because it vanishes on a non-empty open set.

REMARK 3.2. *The same result holds if $\mathcal{Q}^k$ is replaced by any product $\mathcal{Q}^k \times \mathcal{R}$ of $\mathcal{Q}^k$ with a Banach manifold R, and sw by a smooth map $sw' : \mathcal{A}_l \times \mathcal{Q}^k \times \mathcal{R}$ whose restriction to any fibre $\mathcal{A}_l \times \mathcal{Q}^k \times \{r\}$ has the form sw_{g,σ,C_0} for a metric g, a $Spin^{U(2)}$-structure σ in (X,g), and an $SO(4)$-connection C_0.*

An easy way to parameterize the space of pairs consisting of a metric and a $Spin^{U(2)}(4)$-structure will be given in the next section.

3.3. $PU(2)$-monopoles with degenerate spinor component. Generic regularity. Let P^u be a $Spin^{U(2)}$-bundle. Suppose that the spinor $\Psi \in A^0(\Sigma^+)$ is degenerate on a whole neighbourhood of a point $x \in X$ but $\Psi_x \neq 0$, and let $A \in \bar{\delta}(P^u))$ be a $PU(2)$-connection. The pair (A,Ψ) will be called non-abelian in x if the second fundamental form of the line subbundle $L \subset E$ generated by Ψ around x is non-zero in x.

We recall that if P^u is associated with a pair (P^c, E), where P^c is a $Spin^c(4)$ bundle P^c of spinor bundles $\Sigma^\pm_c$ and E is a $U(2)$-bundle, then $\Sigma^\pm = [\Sigma^\pm_c]^\vee \otimes E = \Sigma^\pm_c \otimes E^\vee \otimes \det(P^u)$ and $P^u \times_\pi \mathbb{R}^4 = \mathbb{R}SU(\Sigma^+_c, \Sigma^-_c) \subset \mathrm{Hom}(\Sigma^+_c, \Sigma^-_c) \subset \mathrm{Hom}(\Sigma^+, \Sigma^-)$. The euclidean structure and the orientation in the real 4-bundle $\mathbb{R}SU(\Sigma^+_c, \Sigma^-_c)$ are fibrewise defined by the Pauli matrices associated with frames $(e^\pm_1, e^\pm_2)$ of $\Sigma^\pm_x$ satisfying $e^+_1 \wedge e^+_2 = e^-_1 \wedge e^-_2$.

DEFINITION 3.3. *Let P^u be a $Spin^{U(2)}$-bundle with $P^u \times_\pi \mathbb{R}^4 \simeq \Lambda^1$. A Clifford map is an orientation preserving linear isomorphism*

$$\gamma : \Lambda^1 \longrightarrow P^u \times_\pi \mathbb{R}^4 = \mathbb{R}SU(\Sigma^+_c, \Sigma^-_c) \subset \mathrm{Hom}(\Sigma^+, \Sigma^-) \ .$$

Every $\mathcal{C}^k$ Clifford map $\gamma : \Lambda^1 \longrightarrow P^u \times_\pi \mathbb{R}^4$ defines a $\mathcal{C}^k$ metric g_γ on X which makes γ an isometry, so that $\gamma : \Lambda^1 \longrightarrow P^u \times_\pi \mathbb{R}^4 \subset \mathrm{Hom}(\Sigma^+, \Sigma^-)$ becomes the Clifford map of a $Spin^{U(2)}$-structure σ_γ in (X, g_γ).

This formalism will play an important role in this paper. The space

$$Clif := \Gamma(X, \mathrm{Iso}_+(\Lambda^1, P^u \times_\pi \mathbb{R}^4))$$

of Clifford maps parameterizes the set of pairs consisting of a metric and a $Spin^{U(2)}(4)$-structure for that metric. Note that the metric determines a $Spin^{U(2)}$-structure with a given bundle P^u only up to an $SO(4)$-gauge transformation of the cotangent bundle.

As in the previous section fix a $\mathcal{C}^\infty$ $SO(4)$-connection C_0 in $P^u \times_\pi SO(4)$. To any pair of connections $(a, A) \in \mathcal{A}(\det(P^u)) \times \mathcal{A}(\bar{\delta}(P^u))$ we associate a Dirac operator $\not{D}^0_{\gamma,a,A}$ using the Clifford map γ and the lift $\nabla_{C_0,a,A} : A^0(\Sigma^+) \longrightarrow A^1(\Sigma^+)$ of (C_0, a, A):

$$\not{D}^0_{\gamma,a,A} = \gamma \cdot \nabla_{C_0,a,A} \ .$$

This Dirac operator does not coincide with the standard Dirac operator $\not{D}_{\gamma,a,A}$ associated with (A,a) and the $Spin^{U(2)}$-structure on (X, g_γ) defined by γ, because $\gamma^{-1}(C_0)$ may be different from the Levi-Civita connection in (Λ^1, g_γ); however, it has the same symbol as the standard one. The advantage of using these Dirac operators, is that

they depend in a very simple way on γ and that they are operators with $\mathcal{C}^k$-coefficients if γ is of class $\mathcal{C}^k$. The coefficients of the Levi-Civita connection in (Λ^1, g_γ) are in general only of class $\mathcal{C}^{k-1}$, and the coefficients of the induced Levi-Civita connection in $P^u \times_\pi \mathbb{R}^4$ are also of class $\mathcal{C}^{k-1}$, so that the coefficients of the standard Dirac operator $\not{D}_{\gamma,a,A}$ have a regularity-class smaller by 1 than the regularity class of γ.

The use of these Dirac operators, whose coefficients do not contain the derivatives of the Clifford map, is essential in our proofs.

REMARK 3.4. *There exists a section $\beta = \beta(\gamma, C_0) \in \mathcal{C}^{k-1}(P^u \times_\pi \mathbb{C}^4)$ such that $\not{D}^0_{\gamma,a,A} = \not{D}_{\gamma,a,A} + \beta$.*

To see this, let C_γ be the $SO(4)$-connection in $P^u \times_\pi \mathbb{R}^4$ induced via γ by the Levi Civita connection in (Λ^1, g_γ). The difference $\alpha := \nabla_{C_\gamma,a,A} - \nabla_{C_0,a,A}$ is an ad_+-valued 1-form of class $\mathcal{C}^{k-1}$, hence an element in

$$\mathcal{C}^{k-1}(\Lambda^1(\mathrm{ad}_+)) = \mathcal{C}^{k-1}(\Lambda^1(su(\Sigma_c^+))) \subset \mathcal{C}^{k-1}(\Lambda^1(\mathrm{End}(\Sigma^+)))$$

which does not depend on (A, a). In local coordinates, α has the form $\alpha = \sum u^i \otimes \alpha_i$, with local sections α^i in $su(\Sigma_c^+)$. Its contraction with γ has locally the form $\sum \gamma(u^i) \circ \alpha_i$, and defines a $\mathcal{C}^{k-1}$-section β in $\mathrm{Hom}(\Sigma_c^+, \Sigma_c^-) = P^u \times_\pi \mathbb{C}^4$. □

Consider the following $PU(2)$-monopole equations

$$(SW_a) \qquad \begin{cases} \not{D}^0_{\gamma,a,A}\Psi &= 0 \\ \Gamma_\gamma(F_A) &= (\Psi\bar{\Psi})_0 \,. \end{cases}$$

for a triple $(A, \Psi, \gamma) \in \mathcal{A}(\bar{\delta}(P^u)) \times A^0(\Sigma^+) \times Clif$. The map

$$\Gamma_\gamma : \Lambda^2 \longrightarrow \mathrm{End}(\Sigma_c^+) \subset \mathrm{End}(\Sigma^+)$$

is determined by γ via the formula

$$\Gamma_\gamma(u \wedge v) = \frac{1}{2}\left(-\gamma(u)^*\gamma(v) + \gamma(v)^*\gamma(u)\right)$$

and vanishes identically on Λ^2_{-,g_γ}, so that we could have written $F_A^{+_{g_\gamma}}$ instead of F_A in the second equation. In the form above it will be easier to compute the derivative with respect to γ.

Complete the configuration space $\mathcal{A} := \mathcal{A}(\bar{\delta}(P^u)) \times A^0(\Sigma^+)$ with respect to a large Sobolev norm L^2_l and the space of Clifford maps $Clif$ with respect to the Banach norm $\mathcal{C}^k$, $k \gg l$.

Before stating the main result of this section, we begin with two simple remarks

REMARK 3.5. *Let A, F be subspaces of a normed space H with F finite dimensional. Then*

$$\overline{A + F} = \bar{A} + F \,.$$

Indeed, $\overline{A+F} \supset \bar{A}$, and $\overline{A+F} \supset F$, hence $\overline{A+F} \supset \bar{A} + F$. To prove the opposite inclusion, it is enough to notice that $\bar{A}+F \supset A+F$ and to prove that $\bar{A}+F$ is closed. Let $q : H \longrightarrow {}^H\!/_{\bar{A}}$ be the canonical projection. The right hand space is also normed, hence $q(F) \subset {}^H\!/_{\bar{A}}$ is closed (being finite dimensional), and therefore $q^{-1}(q(F)) = \bar{A} + F$ is closed in H, since q is continuous. This proves the remark. □

REMARK 3.6. *Let $f : H_1 \longrightarrow H_2$ be a continuous operator with closed image and finite dimensional kernel between Banach spaces , and let $A \subset H_1$ be a closed subspace. Then $f(A)$ is closed.*

Proof. f factorizes as $H_1 \xrightarrow{p} {}^{H_1}\!/_{\ker f} \xrightarrow{\simeq} f(H_1) \hookrightarrow H_2$, where the middle arrow is an isomorphism by the Banach Theorem. Therefore it is enough to show that $p(A)$ is closed, or equivalently that $p^{-1}(p(A)) = A + \ker f$ is closed. But this follows by the remark above. □

Let $[\mathcal{A}_l \times Clif^k]^{SW}$ be the space of solutions (A, Ψ, γ) of the equations above, and let $[\mathcal{A}_l \times Clif^k]^{SW}_U$ be the subspace of solutions whose spinor component is degenerate on the open set U.

The space $[\mathcal{A}_l \times Clif^k]^{SW}_U$ is a closed real analytic subspace of the space $[\mathcal{A}_l \times Clif^k]^{SW}$, since it is the vanishing locus of the (real analytic) map

$$\mathcal{A}_l \longrightarrow A^0(\Sigma^+)_l \xrightarrow{\det} A^0(\det(P^u))_l \xrightarrow{res_U} A^0(\det(P^u)|_U)_l \ .$$

We can now state the main result of this section.

THEOREM 3.7. *Let $\theta = (A, \Psi, \gamma) \in [\mathcal{A}_l \times Clif^k]^{SW}_U$, and suppose that for a point $u \in U$, one has $\Psi_u \neq 0$, and the pair (A, Ψ) is non-abelian in u. Then the image of the Zariski tangent space $T_\theta[\mathcal{A}_l \times Clif^k]^{SW}_U$ under the projection*

$$T_\theta[\mathcal{A}_l \times Clif^k]^{SW} \longrightarrow T_\gamma(Clif^k) = \mathcal{C}^k(\mathrm{Hom}(\Lambda^1, P^u \times_\pi \mathbb{R}^4))$$

has infinite codimension.

For the proof of the theorem, we need some preparations:

Note first (using [DK], p. 135) that we may assume that the Sobolev connection A is in Coulomb gauge with respect to a smooth connection A_0 and a fixed smooth metric g_0, i.e.

$$d^{*_{g_0}}_{A_0}(A - A_0) = 0 \ .$$

Put $\alpha := A - A_0$, hence $F_A = d_{A_0}\alpha + \alpha \wedge \alpha + F_{A_0}$. The differential operator $\Gamma_\gamma \circ d_{A_0} + d^{*_{g_0}}_{A_0}$ is *elliptic* although the metrics g_0 and g_γ may be different, and it has coefficients of class $\mathcal{C}^k$. Note also that $\Gamma_\gamma \circ d_{A_0} + d^{*_{g_0}}_{A_0}$ is an operator between $\mathcal{C}^\infty$-bundles.

The Dirac operator $\not{D}^0_{\gamma,a,A_0} = \not{D}^0_{\gamma,a,A} - \gamma(\alpha)$ has coefficients of class $\mathcal{C}^k$. Therefore, the pair (α, Ψ) is a solution of the non-linear elliptic system

$$\begin{cases} \not{D}^0_{\gamma,a,A_0}\Psi + \gamma(\alpha)\Psi &= 0 \\ \Gamma_\gamma(d_{A_0}\alpha + \alpha \wedge \alpha + F_{A_0}) &= (\Psi\bar{\Psi})_0 \\ d^{*_{g_0}}_{A_0}\alpha &= 0 \quad . \end{cases}$$

Writing the left hand side as a function of x^j, α^k, Ψ^l, $\partial_j\alpha^k$, $\partial_j\Psi^l$ (with respect to a smooth chart and bundle trivializations), we see that this function has class $\mathcal{C}^k$ in this system of variables (in fact it is polynomial of degree 2 in the last four group of variables). It follows, by Agmon-Douglis-Nirenberg's non-linear-elliptic regularity theorems ([B], p. 467, Theorem 41) that α, Ψ, hence also the pair (A, Ψ), have class $\mathcal{C}^{k+1-\varepsilon}$. It would have class $\mathcal{C}^{k+1}$ if we had chosen a non-integer index $k = [k] + \varepsilon$, i.e. if we had worked with the Hölder space $\mathcal{C}^{[k],\varepsilon}$.

Let $sw : \mathcal{A}_l \times Clif^k \longrightarrow A^0(\Sigma^-)_{l-1} \times A^0(\mathrm{ad}_+ \otimes \mathbb{g}_0)_{l-1}$ be the map given by the left hand side of the equations (SW_a), and put $\det_U := res_U \circ \det$.

The tangent space $T_\theta[\mathcal{A}_l \times Clif^k]^{SW}_U$ is the space of solutions $(\dot{A}, \dot{\Psi}, \dot{\gamma})$ of the linear system

$$\begin{cases} \frac{\partial sw}{\partial(A,\Psi)}|_\theta(\dot{A},\dot{\Psi}) + \frac{\partial sw}{\partial\gamma}|_\theta(\dot{\gamma}) &= 0 \\ d_\Psi(\det_U)(\dot{\Psi}) &= 0\,. \end{cases}$$

Denote by

$$D^U_l =: \ker[d_\Psi(\det_U)] \subset A^0(\Sigma^+)_l$$

the Zariski tangent space at Ψ to the space $\mathcal{D}^U_l := \det_U^{-1}(0)$ of L^2_l positive spinors which are degenerate on U.

Theorem 3.7 can now be reformulated as follows

PROPOSITION 3.8. *The subspace*

$$\left(\frac{\partial sw}{\partial\gamma}|_\theta\right)^{-1}\left(\frac{\partial sw}{\partial(A,\Psi)}|_\theta\ (A^1(\mathfrak{g}_0)_l \times D^U_l)\right)$$

has infinite codimension in $\mathcal{C}^k(\mathrm{Hom}(\Lambda^1, P^u \times_\pi \mathbb{R}^4))$.

In order to prove Proposition 3.8 we start by giving explicit formulas for the partial derivatives above.

The derivative with respect to γ,

$$\left(\frac{\partial sw}{\partial\gamma}|_\theta\right) : \mathcal{C}^k(\mathrm{Hom}(\Lambda^1, P^u \times_\pi \mathbb{R}^4)) \longrightarrow A^0(\Sigma^-)_{l-1} \times A^0(\mathrm{ad}_+ \otimes \mathfrak{g}_0)_{l-1}\ ,$$

is given by

$$\left(\frac{\partial sw}{\partial\gamma}|_\theta\right)(\dot{\gamma}) = \begin{pmatrix} \dot{\gamma}(\nabla_{C_0,a,A}\Psi) \\ \frac{d}{d\gamma}(\Gamma_\gamma(F_A))(\dot{\gamma}) \end{pmatrix}\ . \tag{1}$$

The derivative with respect to the pair (A, Ψ),

$$\left(\frac{\partial sw}{\partial(A,\Psi)}|_\theta\right) : A^1(\mathfrak{g}_0)_l \times A^0(\Sigma^+)_l \longrightarrow A^0(\Sigma^-)_{l-1} \times A^0(\mathrm{ad}_+ \otimes \mathfrak{g}_0)_{l-1}\ ,$$

is

$$\left(\frac{\partial sw}{\partial(A,\Psi)}|_\theta\right)(\dot{A},\dot{\Psi}) = \begin{pmatrix} \not{D}^0_{\gamma,a,A}\dot{\Psi} + \gamma(\dot{A})\Psi \\ \Gamma_\gamma(d_A\dot{A}) - [(\dot{\Psi}\bar{\Psi})_0 + (\Psi\bar{\dot{\Psi}})_0] \end{pmatrix}\ . \tag{2}$$

The next two lemmata will translate the problem into a similar one which involves only Sobolev completions.

Let j^k_{l-1} be the compact embedding

$$j^k_{l-1} : \mathcal{C}^k(\mathrm{Hom}(\Lambda^1, P^u \times_\pi \mathbb{R}^4)) \longrightarrow A^0(\mathrm{Hom}(\Lambda^1, P^u \times_\pi \mathbb{R}^4))_{l-1}\ .$$

LEMMA 3.9.

1. The linear operator $\left(\frac{\partial sw}{\partial \gamma}|_\theta\right)$ *has a continuous extension to the Sobolev completion* $A^0(\mathrm{Hom}(\Lambda^1, P^u \times_\pi \mathbb{R}^4))_{l-1}$. *More precisely, formula (1) defines a linear continuous map*

$$a_{l-1} : A^0(\mathrm{Hom}(\Lambda^1, P^u \times_\pi \mathbb{R}^4))_{l-1} \longrightarrow A^0(\Sigma^-)_{l-1} \times A^0(\mathrm{ad}_+ \otimes \mathfrak{g}_0)_{l-1}$$

such that

$$\left(\frac{\partial sw}{\partial \gamma}|_\theta\right) = a_{l-1} \circ j^k_{l-1} \ .$$

2. The space $\frac{\partial sw}{\partial (A,\Psi)}|_\theta \ (A^1(\mathfrak{g}_0)_l \times D^U_l)$ *is closed in*

$$A^0(\Sigma^-)_{l-1} \times A^0(\mathrm{ad}_+ \otimes \mathfrak{g}_0)_{l-1}.$$

Proof. 1. The first assertion follows easily, since $\nabla_{C_0,a,A}\Psi$ and F_A have regularity class $C^{k-\varepsilon}$, and γ has regularity class C^k. Therefore, working in local C^∞-coordinates, the expression

$$(\frac{d}{d\gamma}\Gamma)(\dot\gamma)(F_A) = \frac{d}{d\gamma}\left(\frac{1}{2}[-\gamma(u^i)^*\gamma(u^j) + (\gamma(u^j)^*\gamma(u^i)] \otimes F_{A,ij}\right)(\dot\gamma)$$

is a linear operator of order 0 with $C^{k-\varepsilon}$ coefficients in the variable $\dot\gamma$.

2. Decompose $A^1(\mathfrak{g}_0)_l \times A^0(\Sigma^+)_l$ as

$$A^1(\mathfrak{g}_0)_l \times A^0(\Sigma^+)_l = D^0_{(A,\Psi)}[A^0(\mathfrak{g}_0)_{l+1}] \oplus \ker[D^0_{(A,\Psi)}]^* = \mathrm{im}D^0_{(A,\Psi)} \oplus \ker[D^0_{(A,\Psi)}]^*$$

where $D^i_{(A,\Psi)}$ are the differential operators in the fundamental elliptic complex associated with the pair (A,Ψ) and the metric g_γ. The decomposition is $L^2_{g_\gamma}$-orthogonal.

The subspace $A^1(\mathfrak{g}_0)_l \times D^U_l \subset A^1(\mathfrak{g}_0)_l \times A^0(\Sigma^+)_l$ is closed, and contains the first summand $\mathrm{im}D^0_{(A,\Psi)}$ by the gauge-invariance property of the degeneracy-condition. Using the fact that $D^1_{(A,\Psi)} \circ D^0_{(A,\Psi)} = 0$, we get

$$\frac{\partial sw}{\partial (A,\Psi)}|_\theta \ (A^1(\mathfrak{g}_0)_l \times D^U_l) = D^1_{(A,\Psi)}\left[(A^1(\mathfrak{g}_0)_l \times D^U_l) \cap \ker(D^0_{(A,\Psi)})^*\right] =$$

$$= D^1_{(A,\Psi)}|_{\ker(D^0_{(A,\Psi)})^*}\left[(A^1(\mathfrak{g}_0)_l \times D^U_l) \cap \ker(D^0_{(A,\Psi)})^*\right] \ .$$

But $D^1_{(A,\Psi)}|_{\ker(D^0_{(A,\Psi)})^*} : \ker(D^0_{(A,\Psi)})^* \longrightarrow A^0(\Sigma^-)_{l-1} \times A^0(\mathrm{ad}_+ \otimes \mathfrak{g}_0)_{l-1}$ is Fredholm and the subspace $\left[(A^1(\mathfrak{g}_0)_l \times D^U_l) \cap \ker(D^0_{(A,\Psi)})^*\right]$ of $\ker(D^0_{(A,\Psi)})^*$ is closed, so that the assertion follows from Remark 3.6.

LEMMA 3.10. *If*

$$V := \left(\frac{\partial sw}{\partial \gamma}|_\theta\right)^{-1}\left(\frac{\partial sw}{\partial (A,\Psi)}|_\theta \ (A^1(\mathfrak{g}_0)_l \times D^U_l)\right)$$

had finite codimension in $C^k(\mathrm{Hom}(\Lambda^1, P^u \times_\pi \mathbb{R}^4))$, *then*

$$V_{l-1} := a^{-1}_{l-1}\left(\frac{\partial sw}{\partial (A,\Psi)}|_\theta \ (A^1(\mathfrak{g}_0)_l \times D^U_l)\right)$$

would have finite codimension in $A^0(\mathrm{Hom}(\Lambda^1, P^u \times_\pi \mathbb{R}^4))_{l-1}$.

Proof. Suppose there exists a finite dimensional subspace F of the space $\mathcal{C}^k(\mathrm{Hom}(\Lambda^1, P^u \times_\pi \mathbb{R}^4))$, such that

$$V + F = \mathcal{C}^k(\mathrm{Hom}(\Lambda^1, P^u \times_\pi \mathbb{R}^4)) .$$

Then we have

$$j^k_{l-1}(V) + j^k_{l-1}(F) = j^k_{l-1}(\mathcal{C}^k(\mathrm{Hom}(\Lambda^1, P^u \times_\pi \mathbb{R}^4)) \subset A^0(\mathrm{Hom}(\Lambda^1, P^u \times_\pi \mathbb{R}^4))_{l-1} ,$$

hence

$$\overline{j^k_{l-1}(V) + j^k_{l-1}(F)} = A^0(\mathrm{Hom}(\Lambda^1, P^u \times_\pi \mathbb{R}^4))_{l-1} \tag{4}$$

by the density property of smooth sections in any Sobolev completion.

Therefore, under the hypothesis of the lemma, and using (4) and Remark 3.5, one gets

$$\overline{j^k_{l-1}(V)} + j^k_{l-1}(F) = A^0(\mathrm{Hom}(\Lambda^1, P^u \times_\pi \mathbb{R}^4))_{l-1} . \tag{5}$$

On the other hand, we know that $\frac{\partial sw}{\partial \gamma}|_\theta = a_{l-1} \circ j^k_{l-1}$. Therefore

$$V = [j^k_{l-1}]^{-1}(V_{l-1}) ,$$

which shows that $j^k_{l-1}(V) \subset V_{l-1}$. But V_{l-1} is closed by Lemma 3.9., hence $\overline{j^k_{l-1}(V)} \subset V_{l-1}$. From (5) it follows that

$$V_{l-1} + j^k_{l-1}(F) = A^0(\mathrm{Hom}(\Lambda^1, P^u \times_\pi \mathbb{R}^4))_{l-1}$$

which proves Lemma 3.10. □

The proof of Proposition 3.8 is now reduced to showing that V_{l-1} cannot have finite codimension in $A^0(\mathrm{Hom}(\Lambda^1, P^u \times_\pi \mathbb{R}^4))_{l-1}$. To prove this, we show that the sections in V_{l-1} must fulfill a very restrictive condition, which is not of finite codimension.

Let $v \in V_{l-1}$. Then, by definition

$$a_{l-1}(v) \in \frac{\partial sw}{\partial(A, \Psi)}|_\theta(A^1(\mathfrak{g}_0)_l \times D^U_l) ,$$

hence there exists a pair $(\dot{A}, \dot{\Psi}) \in A^1(\mathfrak{g}_0)_l \times D^U_l$ such that

$$\begin{cases} \not{D}^0_{\gamma,a,A}\dot{\Psi} + \gamma(\dot{A})\Psi & = v(\nabla_{C_0,a,A}\Psi) \\ \Gamma_\gamma(d_A\dot{A}) - [(\dot{\Psi}\bar{\Psi})_0 + (\Psi\bar{\dot{\Psi}})_0] & = \frac{d}{d\gamma}(\Gamma_\gamma(F_A))(v) . \end{cases}$$

Consider now small balls U_1, U_2 centered in u such that $U_1 \Subset U_2 \subset U$, and such that the following two conditions hold:

1. Ψ is nowhere vanishing on U_2.

Let $S^\pm$, E be the trivial $SU(2)$-bundles associated with a $SU(2) \times SU(2) \times SU(2)$-reduction of $P^u|_{U_2}$. The connection C_0 induces $\mathcal{C}^\infty$-connections in $S^\pm$, and

the pair (A, a) induces a connection B_A (with $\mathcal{C}^{k+1-\varepsilon}$-coefficients) in E which lifts the connection $A|_{U_2}$ in $\bar{\delta}(P^u)|_{U_2} = {}^{P_E}/_{S^1}$ and the connection $a|_{U_2}$ in $\det(P^u)|_{U_2} = \det(E)$. Since Ψ has rank 1 in every point of $U_2 \subset U$, it defines a $\mathcal{C}^{k+1-\varepsilon}$-splitting $E = L \oplus M$ with $\Psi|_{U_2} \in A^0(S^+ \otimes L)$.

2. The second fundamental form $b \in \mathcal{C}^{k+1-\varepsilon}(\Lambda^1_{\mathbb{C}})$ of L with respect to the unitary connection B_A (or, equivalently, with respect to A) is nowhere vanishing on U_2.

Let l, m be $\mathcal{C}^{k+1-\varepsilon}$ sections of E giving unitary frames in L and M. Then we can write $\Psi|_{U_2} = s_0^+ \otimes l$, where s_0^+ is a nowhere vanishing $\mathcal{C}^{k+1-\varepsilon}$-section of S^+. Once we have fixed this trivialization of E, we can identify the connections with the associated connection matrices, and write $B_A = A + \frac{1}{2}a\mathrm{id}$

Recall that b is defined by $b := (\nabla_{B_A} l, m)$, and for any section φl of L one has $\nabla_{B_A}(\varphi l) = \nabla_{B_L}(\varphi l) + \varphi b \otimes m$, where B_L (resp. B_M) are the connections induced by B_A in L (resp. M).

By the Dirac harmonicity condition, one has, taking the component of $\not{D}^0_{\gamma,a,A}\Psi$ in $S^- \otimes M$,

$$\gamma(b)(s_0^+) = 0 \ .$$

Denote by S_0 the rank 1 subbundle of S^+ generated by the section s_0^+, and by $S_0^\perp$ its orthogonal complement. Let Ψ_t be a path of spinors with $\Psi_0 = \Psi$ and $\det(\Psi_t) = 0$. Derivating it in 0, we get that the component of $\dot{\Psi}_0$ in $S_0^\perp \otimes M$ must vanish. Therefore, the restriction $\dot{\Psi}|_{U_1}$ of an element $\dot{\Psi} \in D_l^U = T_\Psi(\mathcal{D}_l^U)$ must have the form

$$\dot{\Psi}|_{U_1} = \dot{\sigma}^+ \otimes l + \dot{\zeta} s_0^+ \otimes m \ , \quad \dot{\sigma}^+ \in L^2_l(S^+|_{U_1}) \ , \quad \dot{\zeta} \in L^2_l(U_1, \mathbb{C}) \ .$$

Take now the component in $(S^- \otimes M)|_{U_1}$ of the restriction of the first equation to U_1. Put $\nabla_{B_M}(m) = \lambda \otimes m$, where λ is a $\mathcal{C}^{k-\varepsilon}$ pure imaginary 1-form.

One gets the following equation on U_1:

$$\not{D}^0_\gamma(\dot{\zeta} s_0^+) + \dot{\zeta}\gamma(\lambda)(s_0^+) + \gamma(b)(\dot{\sigma}^+) + \gamma(\dot{A}_1^2)(s_0^+) = v(b)(s_0^+) \ . \tag{6}$$

Here $\not{D}^0_\gamma : A^0(S^+)_{s.} \longrightarrow A^0(S^-)_{s-1}$, $s \leq k$, stands for the Dirac operator associated with the Spin(4) structure on (U_2, g_γ) defined by γ and the $SO(4)$-connection $C_0|_{U_2}$ in $\mathbb{R}SU(S^+, S^-)$. $\not{D}^0_\gamma$ is a first order elliptic operator with $\mathcal{C}^k$-coefficients. The complex 1-form $\dot{A}_1^2$ is the component of $\dot{A}$ written in the matricial form with respect to the decomposition $E = L \oplus M$.

The idea to prove Proposition 3.8 is the following:

By the properties 1., 2. above it follows that, varying v in the equation (6), one can get <u>all</u> the L^2_{l-1}-sections of the <u>rank-2</u> bundle $(S^- \otimes M)|_{U_1}$. But on the left of the same equation one has a differential operator of order 1 with $\mathcal{C}^{k-\varepsilon}$ coefficients in $(\dot{\zeta}, \dot{\sigma}^+, \dot{A}_1^2)$ which has a <u>non-surjective</u> symbol: only the complex valued function $\dot{\zeta}$, which is a section in a <u>rank-1</u> bundle on U_1, is derivated on the left.

The problem comes down to showing that the map $L^2_l \longrightarrow L^2_{l-1}$ associated with such an operator, cannot have a range of finite codimension.

We define the following operators:

$$res_{U_1} : A^0(\mathrm{Hom}(\Lambda^1, P^u \times_\pi \mathbb{R}^4))_{l-1} \longrightarrow A^0(\mathrm{Hom}(\Lambda^1, P^u \times_\pi \mathbb{R}^4)_{U_1})_{l-1} \ ,$$

$$ev_{b,s_0^+} : A^0(\mathrm{Hom}(\Lambda^1, P^u \times_\pi \mathbb{R}^4)_{U_1})_{l-1} \longrightarrow A^0(S^-|_{U_1})_{l-1} \ , \ v' \longmapsto v'(b)(s_0^+) \ ,$$

$$[\not{D}_\gamma^0]^- : A^0(S^-|_{U_1})_{l-1} \longrightarrow A^0(S^+|_{U_1})_{l-2} \ ,$$

$$pr^\perp : A^0(S^+|_{U_1})_{l-2} \longrightarrow A^0(S_0^\perp|_{U_1})_{l-2} \ .$$

Here $[\not{D}_\gamma^0]^-$ is the Dirac operator associated with the connection C_0 and the Clifford map $\gamma^- : \Lambda^1 \longrightarrow \mathbb{R}SU(S^-, S^+)$ given by

$$\gamma^-(u) = -\gamma(u)^* \ .$$

In general, the operator $[\not{D}_\gamma^0]^-$ is not the formal adjoint of $\not{D}_\gamma^0$, because $\gamma^{-1}(C_0)$ can have non-vanishing torsion, but it has the same symbol as $[\not{D}_\gamma^0]^*$ and it is an operator with $\mathcal{C}^k$-coefficients. The associated Laplacian $[\not{D}_\gamma^0]^- \circ \not{D}_\gamma^0$ has scalar symbol given by $\xi \mapsto -g_\gamma(\xi,\xi)\mathrm{id}_{S^+}$.

LEMMA 3.11.

1. The operators res_{U_1}, $pr^\perp$ are surjective.

2. The image of the operator $[\not{D}_\gamma^0]^- : A^0(S^-|_{U_1})_{l-1} \longrightarrow A^0(S^+|_{U_1})_{l-2}$ has finite codimension.

3. The operator ev_{b,s_0^+} is surjective.

Proof. 1. The surjectivity of res_{U_1} follows from the extension theorems for Sobolev spaces ([Ad], p. 83); the surjectivity of $pr^\perp$ is obvious.

2. The fact that the image of $[\not{D}_\gamma^0]^- : A^0(S^-|_{U_1})_{l-1} \longrightarrow A^0(S^+|_{U_1})_{l-2}$ has finite codimension follows from the general theory of elliptic operators (see for instance [BB]); It can also be directly verified as follows: We may suppose that X is the 4-sphere S^4 and that $S^\pm|_{U_1}$ are the restrictions to U_1 of the spinor bundles $S'^\pm$ associated with a $Spin(4)$-structure on S^4 whose Clifford map γ' extends $\gamma|_{U_1}$. We can also find a connection C_0' in the associated $SO(4)$-bundle extending $C_0|_{\bar{U}_1}$.

The image of $[\not{D}_\gamma^0]^-$ contains the image of the composition $res_{U_1} \circ [\not{D}_{\gamma'}^0]^-$, where $[\not{D}_{\gamma'}^0]^- : A^0(S'^-)_{l-1} \longrightarrow A^0(S'^+)_{l-2}$ is the Dirac operator on the sphere associated with $(\gamma')^-$ and C_0'. But res_{U_1} is surjective and $[\not{D}_{\gamma'}^0]^-$ is Fredholm.

Note that $[\not{D}_\gamma^0]^-$ is in fact surjective, if U_1 is sufficiently small.

3. The surjectivity of ev_{b,s_0^+} is the crucial point in which the fact that s_0^+ and b are nowhere vanishing on U_2 is used in an essential way.

We begin by choosing a smooth Clifford map

$$\gamma_0 : \Lambda^1_{U_2} \longrightarrow P^u|_{U_2} \times_\pi \mathbb{R}^4$$

such that $\gamma_0(b) : S^+ \longrightarrow S^-$ is an isomorphism in every point $u \in U_2$.

This can be achieved as follows: We know that $\gamma(b)(s_0^+) = 0$, so the determinant $\det(\gamma(b))$ of the induced morphism $\gamma(b) : S^+ \longrightarrow S^-$ must vanish. Therefore $g_\gamma^{\mathbb{C}}(b) = \det(\gamma(b)) = 0$, hence the real forms $\mathrm{Re}(b)$, $\mathrm{Im}(b)$ have pointwise in U_2 the same (non-zero !) g_γ-norm and are pointwise g_γ-orthogonal. It suffices to choose γ_0 such that $\mathrm{Re}(b)$, $\mathrm{Im}(b)$ are nowhere g_{γ_0}-orthogonal on U_2. With this choice $\gamma_0(b)(s_0^+)$ will be a nowhere vanishing section of S^- on U_2.

Let now $s' \in A^0(S^-|_{U_1})_{l-1}$ be an arbitrary L^2_{l-1}-negative spinor.

One can find a unique L^2_{l-1} section $\delta \in A^0(\mathbb{R}SU(S^-, S^-)|_{U_1}))_{l-1}$, such that $\delta(\gamma_0(b)(s_0^+)) = s'$: To see this, one uses the bilinear bundle map

$$\mathbb{R}SU(S^-, S^-) \times S^- \longrightarrow S^- .$$

The section δ is obtained by fibrewise dividing (in the quaternionic sense) s' by the smooth nowhere vanishing spinor $\gamma_0(b)(s_0^+)$ which is a $\mathcal{C}^{k-\varepsilon}$-section on $U_2 \supset \bar{U}_1$.

One also has a bilinear bundle map

$$\mathbb{R}SU(S^+, S^-) \times \mathbb{R}SU(S^-, S^-) \longrightarrow \mathbb{R}SU(S^+, S^-)$$

which in local coordinates looks like quaternionic multiplication.

Now define the L^2_{l-1}-morphism $v' : \Lambda^1_{U_1} \longrightarrow \mathbb{R}SU(S^+|_{U_1}, S^-|_{U_1})$ by

$$v'(\alpha) := \delta \cdot [\gamma_0(\alpha)] \ , \ \forall\, \alpha \in \Lambda^1_{U_1} .$$

This morphism defines a section in

$$A^0(\mathrm{Hom}(\Lambda^1_{U_1}, P^u|_{U_1} \times_\pi \mathbb{R}^4)_{l-1} = A^0(\mathrm{Hom}(\Lambda^1_{U_1}, \mathbb{R}SU(S^+, S^-)|_{U_1})_{l-1}$$

which acts on complex 1-forms α by

$$v'(\alpha)(\cdot) = \delta[\gamma_0(\alpha)(\cdot)] .$$

In particular, $v'(b)(s_0^+) = \delta[\gamma_0(b)(s_0^+)] = s'$. □

After these preparations we can finally prove Proposition 3.8.

Proof. We have to show that V_{l-1} has infinite codimension in

$$A^0(\mathrm{Hom}(\Lambda^1, P^u \times_\pi \mathbb{R}^4))_{l-1} .$$

Take $v \in V_{l-1}$ and apply $[pr^\perp \circ [\not{D}^0_\gamma]^-]$ to both sides of (6).

On the left, the only term containing second order derivatives of the sections $(\dot{\zeta}, \dot{\sigma}^+, \dot{A}^2_1)$ is

$$[pr^\perp \circ [\not{D}^0_\gamma]^-](\not{D}^0_\gamma(\dot{\zeta} s_0^+)) .$$

But, denoting by i_0 the bundle inclusion $U_1 \times \mathbb{C} \longrightarrow S^+|_{U_1}$, $z \longmapsto z s_0^+$, one sees that the 2-symbol of the composition

$$pr^\perp \circ [[\not{D}^0_\gamma]^- \circ \not{D}^0_\gamma] \circ i_0$$

vanishes, since the symbol of the Laplacian $[\not{D}^0_\gamma]^- \circ \not{D}^0_\gamma$ is scalar.

Therefore, applying $[pr^\perp \circ [\not{D}^0_\gamma]^-]$ on the left, one gets an expression containing only first order derivatives of the Sobolev L^2_l sections $(\dot{\zeta}, \dot{\sigma}^+, \dot{A}^2_1)$, hence an L^2_{l-1}-section of $S_0^\perp$.

On the other hand applying $[pr^\perp \circ [\not{D}^0_\gamma]^-]$ on the right of (6), one gets precisely

$$\left[pr^\perp \circ [\not{D}^0_\gamma]^- \circ ev_{b, s_0^+} \circ res_{U_1}\right](v) .$$

Now consider the operator

$$P := \left[pr^\perp \circ [\not{D}^0_\gamma]^- \circ ev_{b, s_0^+} \circ res_{U_1}\right] : A^0(\mathrm{Hom}(\Lambda^1, P^u \times_\pi \mathbb{R}^4))_{l-1} \longrightarrow A^0(S_0^\perp|_{U_1})_{l-2}$$

and the following exact sequence

$$0 \to {}^{\mathrm{im}(P)}/_{P(V_{l-1})} \to {}^{A^0(S_0^\perp)_{l-2}}/_{P(V_{l-1})} \to \mathrm{coker}(P) \to 0 \ .$$

We have seen that $P(V_{l-1})$ is contained in $A^0(S_0^\perp)_{l-1}$, which has infinite codimension in $A^0(S_0^\perp)_{l-2}$.[3]

Therefore ${}^{A^0(S_0^\perp)_{l-2}}/_{P(V_{l-1})}$ has infinite dimension. By Lemma 3.11 $\mathrm{coker}(P)$ has finite dimension, so that ${}^{\mathrm{im}(P)}/_{P(V_{l-1})}$ must have infinite dimension. But ${}^{\mathrm{im}(P)}/_{P(V_{l-1})}$ is a quotient of

$${}^{A^0(\mathrm{Hom}(\Lambda^1, P^u \times_\pi \mathbb{R}^4))_{l-1}}/_{V_{l-1}} \ ,$$

so that the latter must also have infinite dimension. □

Let $\mathcal{M}^*$, $\mathcal{DM}^*_U$ be the moduli spaces

$$\mathcal{M}^* := {}^{[\mathcal{A}^*_l \times Clif^k]^{SW_a}}/_{\mathcal{G}_{l+1}} \ , \ \mathcal{DM}^*_U := {}^{[\mathcal{A}^*_l \times Clif^k]^{SW_a}_U}/_{\mathcal{G}_{l+1}} \ ,$$

where the upper script $(\ )^*$ denotes the subspace with non-zero spinor component.

COROLLARY 3.12. *Let $p = (A, \Psi, \gamma) \in [\mathcal{A}^*_l \times Clif^k]^{SW_a}_U$ such that for some $u \in U$, $\Psi_u \neq 0$ and (A, Ψ) is non-abelian in u. Then the Zariski tangent space $T_{[p]}\mathcal{DM}^*_U$ has infinite codimension in $T_{[p]}\mathcal{M}^*$. In particular, $T_{[p]}\mathcal{DM}^*_X$ has infinite codimension in $T_{[p]}\mathcal{M}^*$ for every solution p with non-abelian (A, Ψ)-component.*

Proof. We have

$$pr_{T_\gamma(Clif^k)}(T_{[p]}\mathcal{M}^*) = \frac{\partial sw}{\partial \gamma}^{-1} \left[D^1_{(A,\Psi)}(A^1(\mathfrak{g}_0)_l \times A^0(\Sigma^+)_l) \right] \ ,$$

and the vector space $D^1_{(A,\Psi)}(A^1(\mathfrak{g}_0)_l \times A^0(\Sigma^+)_l)$ has finite codimension in $A^0(\Sigma^-)_{l-1} \times A^0(\mathrm{ad}_+ \otimes \mathfrak{g}_0)_{l-1}$.

Therefore, also the image of $T_{[p]}\mathcal{M}^*$ under the projection to $T_\gamma(Clif^k)$ has finite codimension.

But, by Theorem 3.7, the image of $T_{[p]}\mathcal{DM}^*_X$ under the same projection has infinite codimension. This proves the first assertion.

The second assertion follows from Aronszajin's unique continuation theorem and the fact that the vanishing locus of an harmonic spinor cannot separate domains [FU]. Alternatively, one can use the Unique Continuation Theorem for monopoles [FL] to see that a mnopole with non-vanishing spinor component, and which is abelian on a non-empty open set, must be globally abelian.

Therefore in the condition of the proposition we can find a point $x \in X$ with $\Psi_x \neq 0$ such that (A, Ψ) is non-abelian in x. □

Using this result we can prove that for a generic Clifford map γ, the only degenerate solutions in the moduli space $\mathcal{M}^* \cap p^{-1}_{Clif^k}(\gamma)$ are the abelian ones. The idea is the following:

[3]We used here the following simple remark: The space of L^2_{l-1}-sections in the space of L^2_{l-2} sections in a bundle has infinite codimension. Note that L^2_{l-1} is nonetheless dense in L^2_{l-2}.

Let $\mathcal{DM}_X^\circ \subset \mathcal{DM}_X^*$ be the subspace of $\mathcal{DM}_X^*$ consisting of solutions with non-abelian (A, Ψ)-component. We have proven that $\mathcal{DM}_X^\circ$ has infinite codimension in $\mathcal{M}^*$. Since the projection $\mathcal{DM}_X^\circ \longrightarrow Clif^k$ has "index $-\infty$", the generic fibre should be empty. There are of course two serious problems with this argument:

1. $\mathcal{DM}_X^\circ$ is not smooth.

2. The restriction of the projection $\mathcal{DM}_X^\circ \longrightarrow Clif^k$ to the smooth part is not Fredholm.

The idea to proceed is to weaken locally the equation defining $\mathcal{DM}_X^\circ$, such that the resulting spaces of solutions become smooth manifolds which are Fredholm of negative index over $Clif^k$. This can be achieved, since $\mathcal{DM}_X^\circ$ is embedded in the space $\mathcal{M}^*$, which, though possibly singular, is Fredholm over $Clif^k$.

In order to carry out this idea, we will need the following two general lemmata.

Let f be a smooth map taking values in a Banach space, and denote by $Z(f)$ its vanishing locus. For a point $p \in Z(f)$ define the Zariski tangent space to $Z(f)$ in p by

$$T_p(Z(f)) := \ker(d_p f) \ .$$

LEMMA 3.13. *Let Σ be a Banach manifold, $p \in \Sigma$, E a Banach space, and $s : \Sigma \longrightarrow E$ a smooth map such that $s(p) = 0$. Suppose*

i) $\ker d_p s$ *has a topological complement.*

ii) $\mathrm{im} d_p s$ *is closed and has a topological complement.*

Then there exists an open neighbourhood Σ' of p in Σ and a submanifold W of Σ containing p, such that

1. $\Sigma' \cap Z(s)$ is a closed subset of W.

2. $T_p(Z(s)) = T_p(W)$.

Proof. Put $T := \mathrm{im} d_p s$, and denote by pr_T the projection on T associated with a topological complement of T.

The composition $pr_T \circ s$ is a submersion in p, since its derivative in p is surjective and $\ker(d_p(pr_T \circ s)) = \ker(d_p s)$ has a topological complement by assumption. Let Σ' be an open neighbourhood of p such that $pr_T \circ s$ is a submersion in every point of Σ'.

Then

$$\Sigma' \cap Z(s) = \Sigma' \cap Z(pr_T \circ s) \cap Z(s) = Z(pr_T \circ s|_{\Sigma'})) \cap Z(s) \ .$$

Therefore, taking $W := Z(pr_T \circ s|_{\Sigma'}))$, claim 1. follows. Clearly

$$T_p(W) = \ker(d_p(pr_T \circ s)) = \ker(d_p s) = T_p(Z(s)) \ .$$

□

LEMMA 3.14. *Let W be a Banach manifold, E a Banach space, $p \in W$, and $\delta : W \longrightarrow E$ a smooth map such that $\ker(d_p \delta)$ has infinite codimension in $T_p(W)$. Then, for every $n \in \mathbb{N}$ there exists an open neighbourhood W_n' of p in W and a codimension n submanifold V_n of W such that $W_n' \cap Z(\delta)$ is a closed subset of V_n.*

Proof. Since $\ker(\delta_p d)$ has infinite codimension in $T_p(W)$, it follows that $\mathrm{im}(d_p \delta)$ has infinite dimension. Let $F_n \subset \mathrm{im}(d_p \delta)$ be a subspace of dimension n, and pr_{F_n} the projection associated with a topological complement of F_n in E. The composition $pr_{F_n} \circ \delta$ is a submersion in p. Indeed, the derivative in p is surjective and the kernel of

the derivative is closed of finite codimension, hence it has a topological complement. Let W_n' be an open neighbourhood of p such that $pr_{F_n} \circ \delta$ is a submersion in every point of W_n'. Then

$$W_n' \cap Z(\delta) = W_n' \cap Z(pr_{F_n} \circ \delta) \cap Z(\delta) = Z(pr_{F_n} \circ \delta|_{W_n'}) \cap Z(\delta) \ .$$

Take $V_n := Z(pr_{F_n} \circ \delta|_{W_n'})$. □

LEMMA 3.15. *Every non-abelian point $[p] \in \mathcal{DM}_X^*$ has a neighbourhood $U_{[p]}$ which is a closed analytic subspace of a submanifold $V_{[p]} \subset {}^{[\mathcal{A}_l^* \times Clif^k]}/_{\mathcal{G}_{l+1}}$ such that the projection $V_{[p]} \longrightarrow Clif^k$ is Fredholm of negative index.*

Proof. Put $p = (\theta, \gamma)$ with $\theta \in \mathcal{A}_l^*$ and $\gamma \in Clif^k$. Consider a slice $S_\theta \subset \theta + \ker(D_\theta^0)^* \subset \mathcal{A}_l^*$ through θ to the orbits of the $\mathcal{G}_{l+1}$-action, such that the restriction of the canonical projection to S_θ defines a parameterization of the quotient ${}^{\mathcal{A}_l^*}/_{\mathcal{G}_{l+1}}$ around $[\theta]$.

Note first, that the image T of the differential $d_p(sw|_{S_\theta \times Clif^k})$ is closed and has finite codimension in the Hilbert space $A^0(\Sigma^-)_{l-1} \times A^0(\mathrm{ad}_+ \otimes \mathfrak{g}_0)_{l-1}$.

Indeed, T contains the image of $\frac{\partial sw}{\partial(A,\Psi)}|_p$, which is the operator D_θ^1 associated with the deformation elliptic complex of the solution $\theta = (A, \Psi)$, and the image of D_θ^1 is already closed of finite codimension.

Now put $\Sigma := S_\theta \times Clif^k$, and note that the restriction

$$q : \Sigma \longrightarrow {}^{[\mathcal{A}_l^* \times Clif^k]}/_{\mathcal{G}_{l+1}}$$

of the canonical projection is a parametrisation of the Banach manifold ${}^{[\mathcal{A}_l^* \times Clif^k]}/_{\mathcal{G}_{l+1}}$ around $[p]$.

Claim: Put $s := sw|_\Sigma$. Then the projection

$$T_\theta S_\theta \times T_\gamma(Clif^k) \supset \ker(d_p s) \longrightarrow T_\gamma(Clif^k)$$

is Fredholm. In particular $\ker(d_p s)$ has a topological complement in the tangent space $T_p(\Sigma) = T_\theta S_\theta \times T_\gamma(Clif^k)$.

Indeed, the kernel of this map is $\mathbb{H}_\theta^1$ and its image can be identified with the subspace $\left(\frac{\partial sw}{\partial \gamma}\right)^{-1}[\mathrm{im} D_\theta^1]$, whose codimension is at most $\dim \mathbb{H}_\theta^2$. If Λ is a topological complement of $\mathbb{H}_\theta^1$ in $T_\theta S_\theta = \ker(D_\theta^0)^*$ and F is a topological complement of $\left(\frac{\partial sw}{\partial \gamma}\right)^{-1}[\mathrm{im} D_\theta^1]$ in $T_\gamma(Clif^k)$, then $(\Lambda \times \{0\}) \oplus (\{0\} \times F)$ is a topological complement of $\ker(d_p s)$ in $T_\theta S_\theta \times T_\gamma(Clif^k)$.

Applying Lemma 3.13 to the Banach manifold Σ and the map s, we get a neighbourhood Σ' of p and a submanifold W such that $\Sigma' \cap Z(s)$ is a closed subset of W and

$$T_p(W) = T_p(Z(s)) \simeq T_{[p]}(\mathcal{M}^*) \ .$$

The restriction $\det|_W$ of the determinant map $\det : \Sigma \longrightarrow A^0(\det(P^u))_l$ satisfies the hypothesis of Lemma 3.14.

Indeed,

$$\begin{aligned} ker d_p(\det|_W) &= \ker(d_p(\det|_\Sigma)) \cap T_p(W) \\ &= \ker(d_p(\det|_\Sigma)) \cap \ker d_p(sw|_\Sigma) \simeq T[p](\mathcal{DM}_X^*), \end{aligned}$$

which has infinite codimension in $T_{[p]}(\mathcal{M}^*) \simeq T_p(W)$ by Corollary 3.12.

Using now Lemma 3.14 we get, for any $n \in \mathbb{N}$, an open neighbourhood W_n' of p in W and a codimension n submanifold V_n of W such that $W_n' \cap Z(\det|_W)$ is a closed subspace of V_n.

Let $\Sigma_n' \subset \Sigma'$ be an open neighbourhood of p in Σ such that

$$W_n' = \Sigma_n' \cap W \ .$$

Then we have

$$\begin{aligned} \Sigma_n' \cap q^{-1}((\mathcal{DM}_X^*) &= Z(sw|_{\Sigma_n'}) \cap Z(\det|_{\Sigma_n'}) \\ &= Z(pr_T \circ sw|_{\Sigma_n'}) \cap Z(sw|_{\Sigma_n'}) \cap Z(\det|_{\Sigma_n'}) \\ &= W_n' \cap Z(sw|_{\Sigma_n'}) \cap Z(\det|_{\Sigma_n'}) = [W_n' \cap Z(\det|_{W_n'})] \cap Z(sw|_{\Sigma_n'}) \ . \end{aligned}$$

Therefore $\Sigma_n' \cap q^{-1}((\mathcal{DM}_X^*)$ is a closed subspace of $[W_n' \cap Z(\det|_{W_n'})]$, which is closed in V_n.

On the other hand we know that the projection

$$T_p(W) = \ker(d_p s) \longrightarrow T_\gamma Clif^k$$

is Fredholm. Since being Fredholm is an open property, we may assume (taking Σ' small) that the projection of W on $Clif^k$ is Fredholm of constant index.

Now choose n larger than the index of this projection, and put

$$V_{[p]} := q(V_n) \ , U_{[p]} := q(\Sigma_n' \cap q^{-1}((\mathcal{DM}_X^*)) = q(\Sigma_n') \cap \mathcal{DM}_X^* \ .$$

□

COROLLARY 3.16. *The set*

$$\{\gamma \in Clif^k |\ \mathcal{DM}_X^* \cap \mathrm{pr}_{Clif^k}^{-1}(\gamma) \text{ contains a non} - \text{abelian pair}\}$$

is a set of the first category in $Clif^k$.

Proof. Indeed, let again $\mathcal{DM}_X^\circ$ be the open subspace of $\mathcal{DM}_X^*$ consisting of solutions with non-abelian (A, Ψ)-component. By Lemma 3.15 and the Lindelöf Theorem ([Ke], p. 49) we can find a countable cover $(U_i)_i$ of $\mathcal{DM}_X^\circ$ such that every U_i is a closed analytic subspace of a smooth submanifold $V_i \subset {}^{[\mathcal{A}_l^* \times Clif^k]}/_{\mathcal{G}_{l+1}}$ which projects on the parameter space $Clif^k$ via a Fredholm map of negative index. Since Fredholm maps are locally proper [Sm], it follows that $\mathrm{pr}_{Clif^k}(\mathcal{DM}_X^\circ)$ is a countable union of closed sets; each of these closed sets is contained in a set of the form $\mathrm{pr}_{Clif^k}(V_i)$, which is of the first category, by the Sard-Smale theorem. □

Corollary 3.12, Lemma 3.15, Corollary 3.16 hold for <u>every</u> family of order 0-perturbations of the equations which contains the perturbations of the Clifford map which we have studied above. We need the following particular case:

Define the space of parameters $\mathcal{P}^k$ by

$$\mathcal{P}^k := \mathcal{C}^k(P^u \times_\pi \mathbb{C}^4) \times \mathcal{C}^k(GL(\mathrm{ad}_+)) \times Clif^k \ .$$

Recall that a section β in the bundle

$$P^u \times_\pi \mathbb{C}^4 = \mathrm{Hom}(\Sigma_c^+, \Sigma_c^-) \subset \mathrm{Hom}(\Sigma^+, \Sigma^-)$$

defines an order 0-operator $A^0(\Sigma^+) \longrightarrow A^0(\Sigma^-)$, commuting with the gauge action.

Consider now the equations

$$(S\tilde{W}_a) \qquad \begin{cases} \not{D}^0_{\gamma,a,A}\Psi + \beta(\Psi) &= 0 \\ \Gamma_\gamma(F_A) &= K(\Psi\bar{\Psi})_0 \end{cases}$$

for a system

$$(A, \Psi, \beta, K, \gamma) \in \tilde{\mathcal{A}} := \mathcal{A}(\bar{\delta}(P^u))_l \times A^0(\Sigma^+)_l \times \mathcal{P}^k \ .$$

Let $[\mathcal{A}_l \times \mathcal{P}^k]^{S\tilde{W}_a}$ ($[\mathcal{A}_l \times \mathcal{P}^k]^{S\tilde{W}_a}_U$) be the space of solutions of the equations $(S\tilde{W}_a)$ (whose spinor component is degenerate on U), and denote also by $\widetilde{\mathcal{M}}^*$ ($\widetilde{\mathcal{DM}}^*_U$) the moduli space of solutions (whose spinor component is degenerate on U) with non-vanishing spinor component.

PROPOSITION 3.17. *Let $p = (A, \Psi, \beta, K, \gamma) \in [\mathcal{A}_l \times \mathcal{P}^k]^{S\tilde{W}_a}_U$ such that for some $u \in U$, $\Psi \neq 0$ and (A, Ψ) non-abelian in u. Then the Zariski tangent space $T_{[p]}\widetilde{\mathcal{DM}}^*_U$ has infinite codimension in $T_{[p]}\widetilde{\mathcal{M}}^*$.*

Proof. Consider the image of $T_p([\mathcal{A}_l \times \mathcal{P}^k]^{S\tilde{W}_a}_U)$ under the projection to the tangent space $T_{(\beta,K,\gamma)}\mathcal{P}^k$. This image has again infinite codimension. To see this it is enough to notice that the intersection of this image with the subspace $\{0\} \times \{0\} \times T_\gamma Clif^k$ has infinite codimension in $\{0\} \times \{0\} \times T_\gamma Clif^k$. But this follows by precisely the same arguments as in Theorem 3.7; one just has to replace the equations (SW_a) by their (β, K)-perturbations. The left hand side in the crucial identity (6) will only be modified by the 0-order term $\dot{\zeta}\beta(s_0^+)$. □

Using this result and the same arguments as above, we get

COROLLARY 3.18. *The set*

$$\{\mathfrak{p} \in \mathcal{P}^k |\ \widetilde{\mathcal{DM}}^*_X \cap \mathrm{pr}^{-1}_{\mathcal{P}^k}(\mathfrak{p}) \text{ contains a non} - \text{abelian pair}\}$$

is a set of the first category in $\mathcal{P}^k$.

We can state now our generic regularity result:

THEOREM 3.19. *There is a dense second category subset $\mathcal{P}^k_0$ of $\mathcal{P}^k$ such that for every $\mathfrak{p} \in \mathcal{P}^k_0$ the moduli space $\mathcal{M}^*_\mathfrak{p} := \widetilde{\mathcal{M}}^* \cap \mathrm{pr}^{-1}_{\mathcal{P}^k}(\mathfrak{p})$ is smooth away from the abelian locus.*

Proof. We know by Theorem 3.1 and Remark 3.2 that $\tilde{\mathcal{M}}^* \setminus \widetilde{\mathcal{DM}}^*_X$ is a smooth manifold. Applying the Sard-Smale theorem to the Fredholm map

$$\widetilde{\mathcal{M}}^* \setminus \widetilde{\mathcal{DM}}^*_X \longrightarrow \mathcal{P}^k$$

it follows that there exists a first category subset $\mathcal{P}_1^k \subset \mathcal{P}^k$ such that the moduli space $[\widetilde{\mathcal{M}}^* \setminus \widetilde{\mathcal{DM}}_X^*] \cap \mathrm{pr}_{\mathcal{P}^k}^{-1}(\mathfrak{p})$ is smooth for every $\mathfrak{p} \in \mathcal{P}^k \setminus \mathcal{P}_1^k$. Let $\mathcal{P}_k^2$ be the first category set given by Corollary 3.18, and take $\mathcal{P}_0^k := \mathcal{P}^k \setminus (\mathcal{P}_k^1 \bigcup \mathcal{P}_k^2)$. □

Finally consider the following parameterized ASD-$Spin^c$- equations

$$\begin{cases} \not{D}^0_{\gamma,a,A}\Psi + \beta(\Psi) &= 0 \\ \Gamma_\gamma(F_A) &= 0 \end{cases}$$

for a system $(A, \Psi, \beta, \gamma) \in \mathcal{A}(\bar{\delta}(P^u))_l \times A^0(\Sigma^+)_l \times \mathcal{C}^k(P^u \times_\pi \mathbb{C}^4) \times Clif^k$.

Let $\mathcal{M}'^*$ be the moduli space of solutions with non-trivial spinor component, and let $\mathcal{P}'^k$ be the parameter space $\mathcal{P}'^k := \mathcal{C}^k(P^u \times_\pi \mathbb{C}^4) \times Clif^k$. Denote also by $\mathcal{DM}_X'^*$ the subspace of solutions with degenerate spinor component, and by $\mathcal{M}'^*_{\mathrm{red}}$ the subspace of solution with reducible connection-component.

Using the methods of section 3.2, one can prove the following partial transversality result

PROPOSITION 3.20. *Suppose that the base manifold is simply connected. Then the moduli space $\mathcal{M}'^*$ is smooth away from the union $\mathcal{DM}_X'^* \bigcup \mathcal{M}'^*_{\mathrm{red}}$.*

Proof. Indeed, let $p = (A, \Psi, \beta, \gamma)$ be a solution with non-degenerate spinor component and non-reducible connection component, and suppose as in the proof of Theorem 3.1 that (Φ, S) is $L^2_{g_\gamma}$-orthogonal on the image of the differential in p of the map cutting out the space of solutions. Using variations $\dot{\beta}$ of β one sees that Φ must vanish on a non-empty open set. But using variations of Ψ, it follows that Φ must solve a Dirac equation, hence by Aronszajin's unique continuation theorem, it must vanish on X. Then using variations $\dot{\gamma}$ of γ we get as in [DK], p. 154 that $S = 0$. It is enough to notice that A is g_γ-ASD, and that any variation of the metric g_γ is induced by a variation of the Clifford map γ. □

In the proof of Theorem 3.7 we have only used the Dirac equation and the ellipticity (modulo the gauge group) of the system . Therefore the same arguments as above give the following important

THEOREM 3.21.

1. There exists a first category subset $\mathcal{P}_2'^k \subset \mathcal{P}'^k$ such that for every $\mathfrak{p} \in \mathcal{P}'^k \setminus \mathcal{P}_2'^k$ the only solutions with degenerate spinor component in the moduli space $\mathcal{M}'^* \cap p^{-1}_{\mathcal{P}'^k}(\mathfrak{p})$ are the abelian ones.

2. If the base manifold X is simply connected, there exists a dense second category subset $\mathcal{P}_0'^k \subset \mathcal{P}'^k$ such that for every $\mathfrak{p} \in \mathcal{P}_0'^k$ the $Spin^c$-moduli space $\mathcal{M}'^* \cap p^{-1}_{\mathcal{P}'^k}(\mathfrak{p})$ is smooth away from $\mathcal{M}'^*_{\mathrm{red}} \cap p^{-1}_{\mathcal{P}'^k}(\mathfrak{p})$.

The results above are sufficient to go forward towards a complete proof of the Witten conjecture. Moreover, one can use the same method to prove a generic regularity theorem along the abelian part of the moduli space. More precisely, let $\mathcal{M}^{\mathrm{ab}}_{\mathfrak{p}} \subset \mathcal{M}^*_{\mathfrak{p}}$ be the abelian part of the moduli space $\mathcal{M}^*_{\mathfrak{p}}$ of solutions of the monopole equations associated with the perturbation parameter $\mathfrak{p}$. The space $\mathcal{M}^{\mathrm{ab}}_{\mathfrak{p}}$ can be identified with the disjoint union of the $Spin^c$-Seiberg-Witten moduli spaces associated with the abelian reductions of P^u ([OT5], [OT7], [T1]).

Let $[p] \in \mathcal{M}^{\mathrm{ab}}_{\mathfrak{p}}$ be an abelian solution. The elliptic deformation complex $\mathcal{C}_p$ of p splits as the sum

$$\mathcal{C}_p = \mathcal{C}_p^{\mathrm{ab}} \oplus \mathcal{N}_p$$

where the first summand $\mathcal{C}_p^{ab}$ can be identified with the elliptic deformation complex of p regarded as solution of the abelian monopole equations, and $\mathcal{N}_p$ is the so called normal elliptic complex of p.

The union $\mathcal{H}^1_{\mathfrak{p}} := \bigcup_{[p]\in\mathcal{M}^{ab}_{\mathfrak{p}}} \mathbb{H}^1(\mathcal{N}_p)$ is a real analytic space which fibres over $\mathcal{M}^{ab}_{\mathfrak{p}}$, but in general is not locally trivial over $\mathcal{M}^{ab}_{\mathfrak{p}}$, and local triviality cannot be achieved in the class of S^1-equivariant perturbations.

Using the method from above one can prove

PROPOSITION 3.22. *For a generic parameter $\mathfrak{p} \in \mathcal{P}^k$, the complement of the zero section in $\mathcal{H}^1_{\mathfrak{p}}$ is smooth of the expected dimension in every point.*

4. The Uhlenbeck compactification.

4.1. Local estimates. The essential difference between the anti-self-dual and the monopole equations is that the latter are not conformal invariant. Under a conformal rescaling of a metric $g \mapsto \tilde{g} = \rho^2 g$ on a 4-manifold X, the associated objects change as follows

$$\tilde{g^*} = \rho^{-2} g^* \text{ on } 1-\text{forms}; \quad vol_{\tilde{g}} = \rho^4 vol_g \ ; \ s_{\tilde{g}} = \rho^{-2} s_g + 2\rho^{-2}\Delta\rho$$
$$\Sigma^{\pm}_{\tilde{g}} = \Sigma^{\pm}_g \text{ (as Hermitian bundles)}, \ \tilde{\gamma} = \rho^{-1}\gamma \ ; \ \tilde{\Gamma} = \rho^{-2}\Gamma \ ; \ \not{D}_{\tilde{g}} = \rho^{-\frac{5}{2}} \not{D}_g \rho^{\frac{3}{2}} \ .$$

A standard procedure used in proving regularity and compactness theorems for instantons is the following: restrict the equations on small balls in the base manifold, and then rescale the metric. In this way, using the conformal invariance of the equations, one can reduce the local computations to the unit ball endowed with a metric close to the euclidean one.

A similar procedure will be used in the case of $PU(2)$-monopoles. The problem here is that the perturbed equations depend on a much larger system of parameters (data). Using constant rescalings of the Clifford map (and hence of the metric), we show first that one can reduce the local computations to computations on the unit ball endowed with a system close to a system of "standard data" (see Definition 4.4).

First of all notice that if $(A, \Psi) \in \mathcal{A}(\bar{\delta}(P^u)) \times A^0(\Sigma^+)$ is a solution of the non-perturbed $PU(2)$-monopole equations SW^{σ}_a for the metric g with respect to the $Spin^{U(2)}(4)$-structure σ, and if ρ is a constant, then $(A, \rho^{-1}\Psi)$ is a solution of the monopole equations $SW^{\tilde{\sigma}}_a$ for $\tilde{g} = \rho^2 g$ with respect to the $Spin^{U(2)}(4)$-structure $\tilde{\sigma}$ defined by the correspondingly rescaled Clifford map $\tilde{\gamma} = \rho^{-1}\gamma$.

The case of the perturbed equations is more delicate. Fix a $Spin^{U(2)}(4)$-bundle P^u. To write down the general perturbed $PU(2)$-monopole equations we considered, one also needs *a system of data* of the form $\mathfrak{p} = (\gamma, C, a, \beta, K)$, where γ is a Clifford map (see Definition 3.3), C is an $SO(4)$-connection in $P^u \times_{\pi} \mathbb{R}^4$, a is a connection in the line bundle $\det(P^u)$, β is a section in $P^u \times_{\pi} \mathbb{C}^4$, and K is a section in $\mathrm{End}(\mathrm{ad}_+)$.

The rescaling rule is:

REMARK 4.1. *If $(A, \Psi) \in \mathcal{A}(\bar{\delta}(P^u)) \times A^0(\Sigma^+)$ solves the perturbed $PU(2)$-monopole equations associated with the data (γ, C, a, β, K). Then $(A, \rho^{-1}\Psi)$ solves the perturbed $PU(2)$-monopole equations associated with the data*

$$(\rho^{-1}\gamma, C, a, \rho^{-1}\beta, K).$$

Let $\bar{B}$ be the standard closed 4-ball with interior B. Fix two copies $\mathbb{H}_{\pm}$ of the quaternionic skew-field $\mathbb{H}$ regarded as right complex and quaternionic vector spaces

and consider the two trivial $SU(2)$-bundles $S_0^{\pm} := \bar{B} \times \mathbb{H}_{\pm}$. Let also $E_0 = \bar{B} \times \mathbb{C}^2$ be the trivial Hermitian rank 2-vector bundle on $\bar{B}$.

Let P_0^u be the trivial $Spin^{U(2)}(4)$-bundle associated with $S_0^{\pm}$, E_0 via the morphism $SU(2) \times SU(2) \times U(2) \to Spin^{U(2)}(4)$ (section 2.1, Prop. 2.2).

A Clifford map for P^u is an orientation preserving linear isomorphism $\gamma : \Lambda^1_{\bar{B}} \longrightarrow \mathrm{Hom}_{\mathbb{H}}(S_0^+, S_0^-) = \bar{B} \times \mathbb{H}$. To every such a Clifford map γ, we can associate the constant Clifford γ^c given by the composition

$$\Lambda^1_{\bar{B}} \longrightarrow \bar{B} \times \Lambda^1_0 \xrightarrow{\mathrm{id} \times \gamma|_{\Lambda^1_0}} B \times \mathbb{H}$$

Note that the corresponding metric g_{γ^c} is flat.

Denote by $h_r : \bar{B} \longrightarrow \bar{B}_r \subset \bar{B}$ the homothety of slope $r < 1$.

REMARK 4.2. *The Clifford maps $\gamma_r := r h_r^*(\gamma|_{B_r})$ converge in the C^∞-topology to γ_0, which is a Clifford map for the flat metric g_{γ^c}. In particular the metrics $g_r := r^{-2} h_r^*(g)$ converge to the flat metric g_{γ^c}.*

Indeed, one has

$$\gamma_r(x, \lambda) = r\gamma((h_r)_*(x, \lambda)) = r\gamma(rx, r^{-1}\lambda) = \gamma(rx, \lambda)$$

The data of a $PU(2)$-connection $A \in \mathcal{A}(\bar{\delta}(P_0^u))$ is equivalent to the data of a connection matrix, i.e. an element in $A^1(\bar{B}, su(2))$. Similarly, the data of a $U(1)$-connection in $\det(P_0^u)$ is equivalent to the data of a 1-form in $A^1(\bar{B}, u(1))$.

REMARK 4.3. *Let $(A, \Psi) \in \mathcal{A}(\bar{\delta}(P_0^u)) \times A^0(\Sigma^+(P_0^u))$ be a pair which solves the monopole equations for the data (γ, C, a, β, K). Then $(h_r^*(A), r h_r^*(\Psi))$ solves the $PU(2)$-monopole equations for the data $(\gamma_r, h_r^*(C), h_r^*(a), r h_r^*(\beta), h_r^*(K))$.*

Note that, as $r \to 0$,

1. $\gamma_r \to \gamma^c$ (which is a Clifford map for the flat metric g_{γ^c}),
2. $r h_r^*(\beta) \to 0$, $h_r^*(K) \to K(0)$, $h_r^*(a)$ converges to the flat connection in $B \times \mathbb{C} = \det(P_0^u)$,
3. $h_r^*(C)$ converges to the flat connection in $B \times \mathbb{H}$, and
4. $\gamma_r^{-1}(h_r^*(C))$ converges to the flat connection in $(\Lambda^1_B = B \times \mathbb{R}^4, g_{\gamma^c})$, which is precisely the Levi-Civita connection for g_{γ^c}).

DEFINITION 4.4. *A system of data for the bundle P_0^u will be called a standard system, if it has the form $(\gamma_0, C_0, 0, 0, K_0)$, where: γ_0 is the standard identification $\Lambda^1_B = B \times \mathbb{R}^4 \longrightarrow B \times \mathbb{H}$, C_0 the flat $SO(4)$-connection in $B \times \mathbb{H}$, and K_0 is a constant automorphism of the trivial bundle $su(S_0^+) = B \times su(2)_+$.*

The metric associated with the standard identification $\Lambda^1_B = B \times \mathbb{R}^4 \to B \times \mathbb{H}$ is the standard Euclidean metric g_0 on the ball.

For any $K_0 \in \mathrm{End}(su(2))$, let $\mathfrak{p}_{K_0}$ be the standard system of data on $\bar{B}$ defined by K_0.

Let X now be 4-manifold, and P^u a $Spin^{U(2)}(4)$-bundle on it. Let x_0 be a point in X and U an open neighbourhood of x_0. Fix an identification of $P^u|_U$ with the the trivial $Spin^{U(2)}(4)$-bundle on U, i. e. with the $Spin^{U(2)}(4)$-bundle associated with the triple $U \times \mathbb{H}_{\pm}$, $U \times \mathbb{C}^2$ (see section 2.1).

Given a system of data (γ, C, a, β, K) for P^u, we consider a parameterization $B_{r_0} \xrightarrow{f} U \subset X$ around x_0 such that $f(0) = x_0$ and $\gamma|_{\Lambda^1_{x_0}} \circ [f_*]_{\Lambda^1_0}$ is the standard identification $\Lambda^1_0 = \mathbb{R}^4 \longrightarrow \mathbb{H}$.

REMARK 4.5. *For any pair (A,Ψ) solving the monopole equations for the data (γ,C,a,β,K), the pair $((f\circ h_r)^*(A),r(f\circ h_r)^*(\Psi))$ solve the monopole equations associated with the system*

$$(f^*(\gamma)_r,(f\circ h_r)^*(C),(f\circ h_r)^*(a),r(f\circ h_r)^*(\beta),(f\circ h_r)^*(K)) \ .$$

This system converges to a system of standard data on the ball, as $r\to 0$.

Therefore, as long as we are interested only in local computations, we can work on the standard ball and assume (via the transformation defined in Remark 4.5) that our system of data belongs to a small neighbourhood of a standard system.

We recall now the following important "gauge fixing" theorem (see Theorem 2.3.7 in [DK]).

THEOREM 4.6. *(Gauge-fixing) There are constants ε_1, $M>0$ such that the following holds:*

Any connection A on the trivial bundle E_0 over $\bar{B}$ with $\| F_A \|_{L^2}<\varepsilon_1$ is gauge equivalent to a connection $\tilde{A}$ over B with

(i) $d_0^*\tilde{A}=0$, where d_0^* is the normal adjoint of d with respect to the standard flat metric g_0.

(ii) $\lim_{r\to 1} A_r = 0$ on S^3,

(iii) $\| \tilde{A} \|_{L^2_1}\leq M \| F_{\tilde{A}} \|_{L^2}$.

The corresponding gauge transformation is unique up to a constant matrix.

Using this result we can prove the following

THEOREM 4.7. *(Local estimates for data close to the standard data) There is a positive constant $\varepsilon_2=\varepsilon_2(K_0)>0$ such that for any system of data $\mathfrak{p}'$ on $\bar{B}$ which is sufficiently C^2-close to the standard system $\mathfrak{p}_{K_0}$, the following holds:*

For any solution (A,Ψ) of the $PU(2)$-monopole equation for the monopole equations associated with $\mathfrak{p}$ over the open ball B satisfying the conditions $d_0^*A=0$, $\| (A,\Psi) \|_{L^4}\leq \varepsilon_2$, and any interior domain $D\Subset B$, one has estimates of the form :

$$\| (A,\Psi) \|_{L^2_l(D)}\leq C_{D,l,\mathfrak{p}'} \| (A,\Psi) \|_{L^4} \ ,$$

with positive constants $C_{D,l,\mathfrak{p}'}$, for all $l\geq 1$.

Proof. First of all we identify the ball with the upper semi-sphere of $S:=S^4$ and we endow the sphere with a metric g_s which extends the standard flat metric g_0 on the ball, and which has non-negative sectional curvature[4].

We fix a $Spin(4)$-structure on the sphere with spinor bundles $S_s^{\pm}$ given by a Clifford map $\gamma_s:\Lambda^1_S\longrightarrow \mathrm{Hom}_{\mathbb{H}}(S_s^+,S_s^-)$, which, with respect to fixed trivializations $S^{\pm}|_{\bar{B}}=\bar{B}\times\mathbb{H}_{\pm}$, extends the standard Clifford map γ_0 on the ball. Let also C_s be the Levi-Civita connection induced by γ_s in $\mathrm{Hom}_{\mathbb{H}}(S_s^+,S_s^-)$. Its restriction to the ball is the standard flat connection C_0 in $\bar{B}\times\mathbb{H}$. Let finally K_s be an extension of K_0 to an endomorphism $K_s\in A^0(\mathrm{End}(su(S_s^+)))$.

[4]Such a metric can be obtained as follows: consider a plane *convex* curve with symmetry axis Oy, which is horizontal in a neighbourhood of its upper intersection point with Oy. Then rotate this curve around the Oy-axis in the 5-dimensional space $\mathbb{R}^4\times Oy$. The hypersurface obtained in this way is also conformally flat, by a theorem of E. Cartan (see [GHL], p. 157, [Ch], Th. 4.2, p. 162)

We denote by E_s the trivial $U(2)$-bundle over S, and by P_s^u the $Spin^{U(2}(4)$-associated with the triple $(S_s^{\pm}, E_s)$. P_s^u comes with an identification $P_s^u|_{\bar{B}} = P_0^u$, induced by the fixed trivializations of $S_s^{\pm}$.

The system $(\gamma_s, C_s, 0, 0, K_s)$ is an extension on the sphere of the standard system $\mathfrak{p}_s := (\gamma_0, C_0, 0, 0, K_0)$. The point is now that any system $\mathfrak{p}'$ of data which is close to $\mathfrak{p}_{K_0}$ has an extension $\mathfrak{p}$ which is close to $\mathfrak{p}_s$.

Put $\mathfrak{p} = (\gamma, C, a, \beta, K) = (\mathfrak{q}, K)$. The system $\mathfrak{q}$ defines two first order elliptic operators on the sphere

$$\begin{array}{rcccc} \not{D}_{\mathfrak{q}} & : & A^0(S_s^+ \otimes E_s) & \longrightarrow & A^0(S_s^- \otimes E_s) \\ \delta_\gamma := d_s^* + \Gamma_\gamma \circ d & : & A^1(su(2)) & \longrightarrow & A^0(su(2))^\perp \oplus A^0(su(S_s^+) \otimes su(E_s)) \end{array}$$

The symbol d_s^* means the adjoint of $d : A^0(su(2)) \longrightarrow A^1(su(2))$ with respect to the fixed metric g_s, and $\not{D}_{\mathfrak{q}} := \not{D}_\gamma^C + \beta + \gamma(\frac{a}{2})$. $A^0(su(2))^\perp$ denotes the $L^2_{g_s}$-orthogonal complement of the 3-dimensional space of constant sections.

These operators are injective in the special case $\mathfrak{q} = \mathfrak{q}_s := (\gamma_s, C_s, 0, 0)$, by the Weitzenböck formula for the Dirac operator and because the cohomology group $H^1_{\mathrm{DR}}(S)$ vanishes. Since the coefficients of both operators in local coordinates are algebraic expressions in the components of $\mathfrak{q}$, it follows by elliptic semicontinuity that the two operators remain injective if $\mathfrak{q}$ is sufficiently $\mathcal{C}^0$-close to $\mathfrak{q}_s$. Denote by $D_{\mathfrak{q}}$ the direct sum of these operators. We get operator valued maps

$$\mathfrak{q} \mapsto D_{\mathfrak{q}} \in$$
$$\mathrm{Iso}\left[A^0(S_s^+ \otimes E_s \oplus \Lambda^1(su(2)))_{k+1}, A^0(S_s^- \otimes E_s \oplus su(S_s^+) \otimes su(E_s)_k \oplus A^0(su(2))_k^\perp\right]$$

which are continuous with respect the $\mathcal{C}^k$-topology on the space of data $\mathfrak{q}$ on the sphere.

Therefore one has elliptic estimates

$$(el_k) \qquad \| u \|_{L^2_{k+1}} \leq const(\mathfrak{q}) \, \| D_{\mathfrak{q}} u \|_{L^2_k}$$

where $const(\mathfrak{q})$ depends continuously on $\mathfrak{q}$ w. r. t. the $\mathcal{C}^k$-topology. In a sufficiently small $\mathcal{C}^2$-neighbourhood of $\mathfrak{q}_s$ one has the following estimates with $\mathfrak{q}$-independent constants

$$(el) \qquad \| u \|_{L^2_{k+1}} \leq const \, \| D_{\mathfrak{q}} u \|_{L^2_k}$$

Since $D_{\mathfrak{q}}$ is a first order operator, we have an identity of the form:

$$(*) \qquad D_{\mathfrak{q}}(\varphi v) = \varphi D_{\mathfrak{q}}(v) + A_{\mathfrak{q}, \partial\varphi}(v)$$

where $A_{\mathfrak{q}, \partial\varphi}$ is an operator of order 0 depending on $\mathfrak{q}$ and depending linearly on the first order derivatives of φ.

The first step is an input-estimate for the $L^2_1(D)$-norms:

Denote by u the pair (A, Ψ). Let φ_1 be a cut-off function supported in the open ball B which is identically 1 in a neighbourhood of $\bar{D}$. Then $u_1 := \varphi_1 u$ extends as section in the bundle $\Lambda^1(su(2)) \oplus S_s^+ \otimes E_s$ on the sphere.

Taking into account that u solves the monopole equations associated with the data $\mathfrak{p}'$, its connection component is in Coulomb gauge, and that $\mathfrak{p} = (\mathfrak{q}, K)$ extends $\mathfrak{p}'$ one gets by $(*)$

$$(1) \qquad \begin{aligned} D_{\mathfrak{q}}(u_1) &= A_{q, \partial\varphi_1}(u) + \varphi_1 \begin{bmatrix} -\gamma(A)\Psi \\ -\Gamma_\gamma(A \wedge A) + K(\Psi\bar{\Psi})_0 \end{bmatrix} \\ &= A_{q, \partial\varphi_1}(u) + \varphi_1 B_{\gamma, K}(u) \end{aligned}$$

where $B_{\gamma,K}$ is a quadratic map.

Then by (el) we obtain an elliptic estimate of the form

$$\begin{aligned} \| u_1 \|_{L_1^2} &\leq c \| D_{\mathfrak{q}} u_1 \|_{L^2} \leq c'(\| u \|_{L^4}^2 + \| d\varphi \|_{L^4} \| u \|_{L^4}) \\ &\leq c''(\| u \|_{L^4} \| u \|_{L_1^2} + \| d\varphi \|_{L^4} \| u \|_{L^4}) \end{aligned}$$

where, for the second inequality we have used on the right the bounded Sobolev embedding $L_1^2 \subset L^4$. The constants c, c' can be chosen to depend continuously on $\mathfrak{p}$, so that we can assume that they are independent of $\mathfrak{p}$ on a small neighbourhood of $\mathfrak{p}_s$. We use now the standard rearrangement procedure described in [DK], p. 60, 62. For a sufficiently small (independent of D) apriori bound $\varepsilon(K_0)$ of the norm $\| u \|_{L^4}$, we get an estimate of the type

$$\| u_1 \|_{L_1^2} \leq const_D \| u \|_{L^4} \ .$$

The constant $const_D$ in this estimate is independent of $\mathfrak{p}$ in a sufficiently small neighbourhood of $\mathfrak{p}_s$, but it depends on D via $\| d\varphi_1 \|_{L^4}$.

In a next step we estimate the L_2^2-norms:

Put $u_2 = \varphi_2 u$, where φ_2 is identically 1 on D, but the support $\mathrm{supp}\varphi_2$ is contained in the interior of $\varphi_1^{-1}(1)$. Then we can also write $u_2 = \varphi_2 u_1$, and we have $A_{\mathfrak{q},\partial\varphi_2}(u) = A_{\mathfrak{q},\partial\varphi_2}(u_1)$.

We estimate first the L_1^2-norm of the right hand side of the formula obtained by replacing φ_1 with φ_2 in (1) . We find

$$\| D_{\mathfrak{q}}(u_2) \|_{L_1^2} \leq const \| \varphi_2 B_{\gamma,K}(u_1) \|_{L_1^2} + const_D \| u_1 \|_{L_1^2} \ , \tag{2}$$

and again we can assume that the constants do not depend on $\mathfrak{q}$. The term $\varphi_2 B_{\gamma,K}(u_1)$ can be written as $\tilde{B}_{\gamma,K}(\varphi_2 u_1 \otimes u_1)$, where $\tilde{B}_{\gamma,K}$ is the linear map defined on the tensor product $\left(\Lambda^1(su(2)) \oplus S_s^+ \otimes E_s\right)^{\otimes 2}$ associated with the quadratic map $B_{\gamma,K}$.

In local coordinates we can write:

$$\begin{aligned} \partial_i[\tilde{B}_{\gamma,K}(\varphi_2 u_1 \otimes u_1)] &= \partial_i(\tilde{B}_{\gamma,K})(\varphi_2 \otimes u_1) \otimes u_1 + \tilde{B}_{\gamma,K}[\partial_i(\varphi_2 u_1) \otimes u_1 + u_1 \otimes (\varphi_2 \partial_i u_1)] \\ &= \partial_i(\tilde{B}_{\gamma,K})(\varphi_2 \otimes u_1) \otimes u_1 + \tilde{B}_{\gamma,K}[\partial_i(\varphi_2 u_1) \otimes u_1 + u_1 \otimes \partial_i(\varphi_2 u_1) - \partial_i(\varphi_2) u_1 \otimes u_1] \end{aligned}$$

This gives an estimate of the form

$$\| \tilde{B}_{\gamma,K}(\varphi_2 u_1 \otimes u_1) \|_{L_1^2} \leq const \| u_2 \|_{L_1^4} \| u_1 \|_{L^4} + const_D \| u_1 \|_{L^4}^4 \ ,$$

which together with (2) and (el) gives

$$\| u_2 \|_{L_2^2} \leq const \| u_2 \|_{L_1^4} \| u_1 \|_{L^4} + const_D(\| u_1 \|_{L_1^2} + \| u_1 \|_{L^4}) \ .$$

By the same rearrangement argument and using the existence of a bounded inclusion $L_2^2 \subset L_1^4$, we get, for a sufficiently small, independent of D, apriori bound of $\| u \|_{L^4}$, an estimate of the form

$$\| u_2 \|_{L_2^2} \leq const_D \| u \|_{L^4} \ .$$

The estimates for the third step can be proved by the same algorithm, using the existence of a bounded inclusion $L_3^2 \subset L_2^4$.

Since L^2_3 is already a Banach algebra, the estimates for the higher Sobolev norms follow by the usual bootstrapping procedure using the estimates (el_k). Note in particular that we no longer need to use the rearrangement argument, so we do not have to take smaller bounds for $\| u \|_{L^4}$ to get estimates of the higher Sobolev norms, so that a positive number $\varepsilon_2 = \varepsilon(K_0)$ (independent of l and D !) with the required property does exist. □

Let V_+, F Hermitian vector spaces of rang 2. One can easily check that there exists a universal constants $\mathfrak{e} > 0$, C, $C_1 > 0$, $C_2 > 0$ such that for every $K \in \mathrm{End}(su(V_+))$ with $|K - \mathrm{id}| < \mathfrak{e}$, and every $\Psi \in V_+ \otimes F$ the following inequalities hold

$$C_1|\Psi|^2 \leq |K(\Psi\bar{\Psi})_0| \leq C_2|\Psi|^2 \tag{3}$$

$$C|\Psi|^4 \leq \big(K(\Psi\bar{\Psi})_0, (\Psi\bar{\Psi})_0\big) = \big(K(\Psi\bar{\Psi})_0(\Psi), \Psi\big) \tag{4}$$

From now on we'll always assume the last component K of a system of data $(\gamma, C_0, a, \beta, K)$ satisfies in every point x the inequality $|K(x) - \mathrm{id}_{\mathrm{ad}_+}| < \mathfrak{e}$.

COROLLARY 4.8. *(Estimates in terms of the curvature) There exists a constant $\varepsilon > 0$, such that for any system $\mathfrak{p}'$ of data on the closed ball which is sufficiently C^2-close to a system of standard data $(\gamma_0, C_0, 0, 0, K_0)$ with $|K_0 - \mathrm{id}| < \mathfrak{e}$ the following holds:*

For any interior ball $D \Subset B$ and any $l \geq 1$ there exist a positive constants $C_{D,l,\mathfrak{p}'}$, $C'_{D,l,\mathfrak{p}'}$ such that every solution (A, Ψ) of the $PU(2)$-monopole equations on $\bar{B}$ associated with $\mathfrak{p}'$ satisfying $\| F_A \|_{L^2} \leq \varepsilon$, is gauge equivalent on B to a pair $(\tilde{A}, \tilde{\Psi})$ satisfying the estimates

$$\| \tilde{A} \|_{L^2_l(D)} \leq C_{D,l,\mathfrak{p}'} \| F_A \|_{L^2} \ , \quad \| \tilde{\Psi} \|_{L^2_l(D)} \leq C'_{D,l,\mathfrak{p}'} \| F_A \|^{\frac{1}{2}}_{L^2} \ .$$

Proof. Note first that all the pairs (A, Ψ) with $\| F_A \|_{L^2} \leq \varepsilon_1$ are gauge equivalent to pair $(\tilde{A}, \tilde{\Psi})$ whose connection component is in the Coulomb gauge with respect to the trivial connection and such that

$$\| \tilde{A} \|_{L^2_1} \leq M \| F_{\tilde{A}} \|_{L^2} \tag{5}$$

Since now the constant K_0 is supposed to belong to the bounded set $B(\mathrm{id}, \mathfrak{e})$ the conclusion of Theorem 4.7 holds for a constant ε_2 which can be chosen *independently* of K_0.

On the other hand, by the estimate (3) and the second monopole equation, one has

$$\| \tilde{\Psi} \|_{L^4_{g_{\gamma'}}} \leq \frac{1}{C_1^{\frac{1}{2}}} \| \Gamma_{\gamma'}(F_{\tilde{A}}) \|^{\frac{1}{2}}_{L^2_{g_{\gamma'}}} = \frac{\sqrt{2}}{C_1^{\frac{1}{2}}} \| F^{+_{g_{\gamma'}}}_{\tilde{A}} \|^{\frac{1}{2}}_{L^2_{g_{\gamma'}}} \leq \frac{\sqrt{2}}{C_1^{\frac{1}{2}}} \| F_{\tilde{A}} \|^{\frac{1}{2}}_{L^2_{g_{\gamma'}}} \tag{6}$$

Since γ' is supposed to belong to a small neighbourhood of γ_0 this gives an uniform estimate of $\| \tilde{\Psi} \|_{L^4}$ in terms of $\| F_{\tilde{A}} \|^{\frac{1}{2}}_{L^2}$. Using the bounded inclusion $L^2_1 \subset L^4$, and the estimates (5), (6) we see now that the L^4 norm of the pair $(\tilde{A}, \tilde{\Psi})$ can be made as small as we please by choosing ε small, in particular smaller than the constant ε_2. With this choice the conclusion of Theorem 4.7 holds, and we get estimates of the Sobolev norms of the restrictions on smaller disks $D \Subset B$ in terms of $\| (\tilde{A}, \tilde{\Psi}) \|_{L^4}$, hence in terms of $\| F_{\tilde{A}} \|^{\frac{1}{2}}_{L^2}$.

On the other hand, the same cutting off procedure as in the proof of Theorem 4.7, gives on the sphere an identity of the form

$$(d_s^* + \Gamma_\gamma d)(\varphi_1 \tilde{A}) = A'_{\mathfrak{q},\partial\varphi}(\tilde{A}) + \varphi[-\Gamma_\gamma(\tilde{A} \wedge \tilde{A}) + (\tilde{\Psi}\bar{\tilde{\Psi}})_0] \ ,$$

which is similar to the identity (1) in the proof of the theorem. Using Theorem 4.7 to estimate the quadratic term on the right, it follows that the L^2_l-norm of $\tilde{A}|_D$ can be estimated in terms of the L^2_{l-1}-norm of the restriction of $\tilde{A}$ to a slightly larger disk $D_l \Subset B$ and $\| (\tilde{A}, \tilde{\Psi}) \|^2_{L^4}$. Inductively we get an estimate of the L^2_l-norm of $\tilde{A}|_D$ in terms of the L^2_1-norm of $\tilde{A}$ and of $\| (\tilde{A}, \tilde{\Psi}) \|^2_{L^4}$. But both terms can be estimated now in terms of $\| F_{\tilde{A}} \|_{L^2}$. □

Note that the estimate in terms of $\| F_{\tilde{A}} \|^{\frac{1}{2}}_{L^2}$ which we obtained by applying directly Theorem 4.7, is in fact fully sufficient for our purposes. However it is interesting to notice that the Sobolev norms of the connection component $\tilde{A}$ can be estimated as in the instanton case in terms of $\| F_{\tilde{A}} \|_{L^2}$.

COROLLARY 4.9. *(Local compactness) There exists a constant $\varepsilon > 0$ such that the following holds:*

For any pair system of data $\mathfrak{p}$ which is sufficiently close to a system of standard data $\mathfrak{p}_{K_0}$ on the ball with $|K_0 - \mathrm{id}| < \mathfrak{e}$, and any sequence (A_n, Ψ_n) of solutions of the $PU(2)$-monopole equations for $\mathfrak{p}$ with $\| F_{A_n} \|_{L^2} \leq \varepsilon$, there is a subsequence m_n of $\mathbb{N}$ and gauge equivalent solutions $(\tilde{A}_{m_n}, \tilde{\Psi}_{m_n})$ converging in the C^∞-topology on the open ball B.

□

We can prove now the following result, which is the analogon of Proposition 4.4.9 p. 161 [DK].

COROLLARY 4.10. *(Global compactness) Let Y be a compact 4-manifold (with or without boundary), set $\Omega = Y \setminus \partial Y$ and let P^u be a $Spin^{U(2)}(4)$-bundle on Ω such that $\Lambda^1_\Omega \simeq P^u \times_\pi \mathbb{R}^4$ as oriented 4-bundles. Let $\mathfrak{p} = (\gamma, C, a, \beta, K)$ be an arbitrary system of data for (Ω, P^u) satisfying the condition $|K(x) - \mathrm{id}_{\mathrm{ad}_+}| < \mathfrak{e}$ in every point $x \in \Omega$.*

Let (A_n, Ψ_n) be a sequence of solutions of the $PU(2)$-monopole equations associated with $\mathfrak{p}$ such that every point $x \in \Omega$ has a geodesic ball neighbourhood D_x such that for all large enough n,

$$\int_{D_x} |F_{A_n}|^2_{g_\gamma} vol_{g_\gamma} < \varepsilon^2$$

where ε is the constant in Corollary 4.9. Then there is a subsequence $(m_n) \subset \mathbb{N}$ and gauge transformations $u_n \in \mathcal{G}_0$ such that $u_n(A_{m_n}, \Psi_{m_n})$ converges in the C^∞-topology on Ω.

Proof. First of all note that every point has a geodesic ball neighbourhood $D'_x \subset D_x$ such that for a suitable subsequence $(m^x_n)_n \subset \mathbb{N}$ and suitable gauge transformations u^x_n over D'_x the sequence $(u^x_n(A_{m^x_n}|_{D'_x}, \Psi_{m^x_n}|_{D'_x})_n$ converges in the C^∞ topology on D'_x. This follows from Remark 4.5, Corollary 4.9 and the conformal invariance of the L^2-norm of 2-forms.

Using now Corollary 4.4.8 p. 160 [DK] we get a subsequence $(m_n)_n$ of $\mathbb{N}$ and gauge transformations u_n such that $u_n(A_{m_n})$ converges in the C^∞-topology on Ω to a connection A. But using the first monopole equation we see that the convergence of the connection component together with the local L^4-bound of the spinor component

(provided by the local L^2-boundedness of the curvature) implies the local boundedness of the spinor component in any L^2_l-norm. □

PROPOSITION 4.11. *(Apriori C^0-boundedness of the spinor) Let X be a compact oriented 4-manifold, P^u a $Spin^{U(2)}(4)$-bundle on X with $P^u \times_\pi \mathbb{R}^4 \simeq \Lambda^1_X$ as oriented 4-bundles, and $\mathfrak{p} = (\gamma, C, a, \beta, K)$ a system of data for the pair (X, P^u) satisfying the condition $|K(x) - \mathrm{id}_{\mathrm{ad}_+}| < \mathfrak{e}$ in every point $x \in X$.*

1. If $\beta = 0$, and C is induced via γ by the Levi-Civita connection in (Λ^1, g_γ), then for any solution $(A, \Psi) \in \mathcal{A}(\bar{\delta}(P^u)) \times A^0(\Sigma^+(P^u))$ of the $PU(2)$-monopole equations associated with $\mathfrak{p}$, the following apriori estimate holds:

$$\sup_X |\Psi|^2_{g_\gamma} \leq \max\left(0, C^{-1} \sup_X (-\frac{s}{4} + c|F_a^+|_{g_\gamma})\right)$$

Here s stands for the scalar curvature of g_γ, c is a universal positive constant, and C is the universal positive constant in (4) .

2. In the general case one has an apriori estimate of the form

$$\sup_X |\Psi|^2_{g_\gamma} \leq \max\left(0, C^{-1}\left[\sup_X (-\frac{s}{4} + c|F_a^+|_{g_\gamma}) + \sigma(\gamma, C, \beta)\right]\right) ,$$

where $\sigma(C, \beta, \gamma)$ depends continuously on the coefficients of γ, C, β with respect to the $C^2 \times C^1 \times C^1$-topology.

Proof. We prove the second assertion. Using Remark 3.4, it follows that, modifying β if necessary, we may assume that C is induced via γ by the Levi-Civita connection in (Λ^1, g_γ), so that the Dirac operator $\not{D}^C_{\gamma,a,A}$ associated with C coincides with the standard Dirac operator $\not{D}_{\gamma,a,A}$.

The Weitzenböck formula for coupled Dirac operators gives for any triple (A, a, Ψ) $\in \mathcal{A}(\bar{\delta}(P^u)) \times \mathcal{A}(\det(P^u)) \times A^0(\Sigma^+(P^u))$

$$\not{D}^2_{\gamma,A,a}\Psi = \nabla^*_{A,a}\nabla_{A,a}\Psi + \Gamma_\gamma[(F_A + \frac{1}{2}F_a)^{+_{g_\gamma}}]\Psi + \frac{s}{4}\,\Psi.$$

On the other hand

$$\not{D}_{\gamma,a,A}(\not{D}_{\gamma,a,A} + \beta) = \not{D}^2_{\gamma,a,A} + \gamma \cdot \nabla_{a,A} \circ \beta$$

If (A, Ψ) solves the $PU(2)$-monopole equations for the system of data $\mathfrak{p}$, it most hold pointwise

$$(\nabla^*_{A,a}\nabla_{A,a}\Psi, \Psi) + (K(\Psi\bar{\Psi})_0(\Psi), \Psi) + \frac{1}{2}(\Gamma_\gamma(F_a)(\Psi), \Psi) + \frac{s}{4}|\Psi|^2 + (\gamma \cdot \nabla_{a,A} \circ \beta(\Psi), \Psi) = 0.$$

Using the inequality (4), we get

$$(7)\quad \frac{1}{2}\Delta|\Psi|^2 = (\Delta_{A,a}\Psi, \Psi) - |\nabla_{A,a}\Psi|^2 \leq -C|\Psi|^4 + (c|F_a^+| - \frac{s}{4})|\Psi|^2 + |(\gamma \cdot \nabla_{a,A} \circ \beta(\Psi), \Psi)| - |\nabla_{A,a}\Psi|^2 .$$

On the other hand

$$\gamma \cdot \nabla_{a,A} \circ \beta(\Psi) = \gamma \cdot [(\nabla_C \beta)(\Psi) + \beta\nabla_{A,a}\Psi] .$$

Therefore the term $(\gamma \cdot \nabla_{a,A} \circ \beta(\Psi), \Psi)$ can be estimated as follows

$$(8) \qquad |(\gamma \nabla_{a,A} \circ \beta(\Psi), \Psi)| \leq c' \left(|\nabla_C(\beta)||\Psi|^2 + |\beta||\nabla_{A,a}\Psi||\Psi| \right)$$
$$\leq c' \left[|\nabla_C(\beta)||\Psi|^2 + |\beta| \left(\varepsilon \, |\nabla_{A,a}\Psi|^2 + \frac{1}{\varepsilon} \, |\Psi|^2 \right) \right] ,$$

where c' is a universal constant and ε is any positive number. Choose now $\varepsilon := \frac{1}{2(c' \sup |\beta| + 1)}$, so that the total coefficient of $|\nabla_{A,a}\Psi|^2$ in the expression obtained by replacing $|(\gamma \cdot \nabla_{a,A} \circ \beta(\Psi), \Psi)|$ in (7) with the right hand term of (8) becomes negative. Then we get an inequality of the form

$$\frac{1}{2}\Delta|\Psi|^2 \leq -C|\Psi|^4 + \sup \left(c|F_a^+| + c'|\nabla_C(\beta)| - \frac{s}{4} \right) |\Psi|^2 + \frac{c' \sup |\beta|}{\varepsilon} |\Psi|^2 ,$$

and the assertion follows easily by the maximum principle. □

COROLLARY 4.12. *If Ω is compact, the condition "$\int_{D_x} |F_{A_n}|^2_{g_\gamma} vol_{g_\gamma} < \varepsilon^2$ for all sufficiently large n" in Corollary 4.10 can be replaced by the condition*

$$\text{"} \int_{D_x} |F^{-g_\gamma}_{A_n}|^2_{g_\gamma} vol_{g_\gamma} < \frac{\varepsilon^2}{2} \text{ for all sufficiently large } n \text{"}.$$

Proof. By Proposition 4.11 and the inequality (3), the pointwise norm $|F^{+g_\gamma}_{A_n}|$ of the g_γ-self-dual component of the curvature is apriori bounded by a constant (depending on s_{g_γ} and $\mathfrak{p}$) hence $\int_{D_x} |F^{+g_\gamma}_{A_n}|^2_{g_\gamma}$ can be made arbitrarily small, by replacing eventually D_x with a smaller ball. □

4.2. Regularity. We begin with the following simple

REMARK 4.13. *Let X be a 4-manifold and g, g' two metrics on X. Then the operator $d^*_g + d^{+g'} : A^1 \longrightarrow A^0 \oplus A^2_{+g'}$ is elliptic. If X is compact then the kernel of this operator is the harmonic space $\mathbb{H}^1_g$. The image of its extension $L^2_{k+1} \longrightarrow L^2_k$ is $(A^0)^\perp_k \oplus (A^2_{+g'})^\perp_k$, where $(A^0)^\perp_k$ is the L^2_g-orthogonal complement of $\mathbb{R} \subset (A^0)_k$, and $(A^2_{+g'})^\perp_k$ is the $L^2_{g'}$-orthogonal complement of $\mathbb{H}^2_{+g} \subset (A^2_{+g'})_k$.*

Indeed, one checks easily that the symbol σ of $d^*_g + d^{+g'}$ is injective for non-vanishing cotangent vectors ξ. Indeed, if $\sigma_\xi(\alpha) = 0$, then $(\xi \wedge \alpha)_{+g'} = 0$, hence $\xi \wedge \alpha = 0$. Therefore α has the form $\alpha = c\,\xi$, $c \in \mathbb{R}$. Using now the Λ^0-component of the equation $\sigma_\xi(\alpha) = 0$, we get $c\,|\xi|^2_g = 0$, i. e. $c = 0$. But Λ^1, $\Lambda^0 \oplus \Lambda^2_{+g'}$ have both rang 4, so σ_ξ must be isomorphism.

On compact 4-manifolds one has $\ker d^{+g'} = \ker d$. Therefore $\ker(d^*_g + d^{+g'}) = \ker(d^*_g + d) = \mathbb{H}^1_g(X)$. The image of the $L^2_{k+1} \longrightarrow L^2_k$ extension of $d^*_g + d^{+g'}$ is obviously contained in $(A^0)^\perp_k \oplus (A^2_{+,g'})^\perp_k$. Therefore it must coincide with this space, because $\operatorname{index}(d^*_g + d^{+g'}) = \operatorname{index}(d^*_g + d^{+g}) = b_1 - b_+ - 1$. □

As in the section above we fix $SU(2)$-bundles $S^\pm_s$ on the 4-sphere S such that $\Lambda^1_S \simeq \mathbb{R}SU(S^+_s, S^-_s) = \operatorname{Hom}_{\mathbb{H}}(S^+_s, S^-_s)$ as oriented 4-bundles. The pairs consisting of a metric on the sphere and a $Spin(4)$-structure for that metric are parameterized by linear isomorphic Clifford maps

$$\gamma : \Lambda^1_S \longrightarrow \operatorname{Hom}_{\mathbb{H}}(S^+_s, S^-_s) .$$

We denote by $Clif(S)$ the space of Clifford maps on the sphere. Let again E_s be the trivial $U(2)$ bundle on S.

We fix a Clifford map $\gamma_s : \Lambda^1_S \longrightarrow \mathrm{Hom}_{\mathbb{H}}(S_s^+, S_s^-)$ such that $g_s := g_{\gamma_s}$ has non-negative scalar curvature, strictly positive in the south pole ∞. Therefore the associated selfadjoint Dirac operator $\not{D}_{\gamma_s}$ is injective, by the Weitzenböck formula. Denote by C_s the Levi-Civita connection induced by γ_s in the $SO(4)$-bundle $P_s^u \times_\pi \mathbb{R}^4 = \mathrm{Hom}_{\mathbb{H}}(S_s^+, S_s^-)$ and denote by $\mathfrak{q}_s$ the system of data

$$\mathfrak{q}_s := (\gamma_s, C_s, 0, 0) \in Clif(S) \times \mathcal{A}(P_s^u \times_\pi \mathbb{R}^4) \times \mathcal{A}(\det(P_s^u)) \times A^0(P_s^u \times_\pi \mathbb{R}^4) ,$$

where we used as usually the identification $\mathcal{A}(\det(P_0^u)) = A^1(u(1))$.

Denote by

$$sw_{\mathfrak{p}} : \mathcal{A}(\bar{\delta}(P_s^u)) \times A^0(S_s^+ \otimes E_s) \longrightarrow A^0(S_s^- \otimes E_s) \times A^0(su(S_s^+) \otimes su(2))$$

the Seiberg-Witten map associated with a system of data $\mathfrak{p}$ for the pair (S, P_s^u).

PROPOSITION 4.14. *(Regularity of L^4-small L^2_1-almost solutions with connection component in Coulomb gauge) Let g be an arbitrary fixed metric on the sphere. There are positive constants α, μ, c (depending on g and γ_s) such that for any system of data $\mathfrak{p} = (\mathfrak{q}, K)$ with $\mathfrak{q}$ sufficiently close to $\mathfrak{q}_s$ and $|K - \mathrm{id}_{su(S_s^+)}| < \epsilon$ the following holds.*

Any pair $u = (A, \Psi) \in L^2_1(\Lambda^1(su(2))) \times L^2_1(S_S^+ \otimes E_s)$ satisfying:

(i) $d_g^*(A) = 0$,

(ii) $\| u \|_{L^4} < \alpha$, satisfies the inequality

$$\| u \|_{L^2_1} \leq c \ \| sw_{\mathfrak{p}}(u) \|_{L^2} \ .$$

If, moreover

(iii) $\| sw_{\mathfrak{p}}(u) \|_{L^2} < \mu$,

(iv) $sw_{\mathfrak{p}}(u)$ is smooth,

then u is also smooth.

Proof. We use the method of continuity as in the proof of 4.4.13 [DK]. The essential fact used in the proof of that theorem is that the map

$$B \longmapsto (d^*B, F_B^+)$$

can be written as the sum of an injective elliptic first order operator and a quadratic map. By Remark 4.13, the map $(d_g^*, sw_{\mathfrak{p}})$ has the same property. Note that we do not require the metric g to be close to g_{γ_s}.

As in the proof of Theorem 4.7, the system $\mathfrak{q} = (\gamma, C, a, \beta)$ defines an elliptic first order operator on the sphere

$$D_{\mathfrak{q}} := \begin{array}{ccccc} \not{D}_{\mathfrak{q}} & : & A^0(S_s^+ \otimes E_s) & \longrightarrow & A^0(S_s^- \otimes E_s) \\ \oplus & & \oplus & & \oplus \\ d_g^* + \Gamma_\gamma \circ d & : & A^1(su(2)) & \longrightarrow & A^0(su(2))^\perp \oplus A^0(su(S_s^+) \otimes su(E_s)) \end{array}$$

Here $\not{D}_{\mathfrak{q}}$ stands for the Dirac operator $\not{D}^C_\gamma + \beta + \gamma(\frac{a}{2})$, and $A^0(su(2))^\perp$ for the L^2_g-orthogonal complement of the 3-dimensional space of constant $su(2)$-valued functions.

By Remark 4.13 and elliptic semicontinuity, it follows that $D_{\mathfrak{q}}$ is injective if $\mathfrak{q}$ is sufficiently C^0-close to $\mathfrak{q}_s$. Moreover, the $L^2_{k+1} \longrightarrow L^2_k$ extension of $D_{\mathfrak{q}}$ is an isomorphisms depending continuously on $\mathfrak{q}$ with respect to the C^k-topology.

We extend the operator d_g^* on pairs by putting $d_g^*(B,\Phi) := d_g^*(B)$. With this convention note that the map $d_g^* + sw_{\mathfrak{p}}$ can be written as

$$(d_g^* + sw_{\mathfrak{p}})(B,\Phi) = D_{\mathfrak{q}}(B,\Phi) + \begin{bmatrix} \gamma(B)\Phi \\ \Gamma_\gamma(B \wedge B) - K(\Phi\bar{\Phi})_0 \end{bmatrix} = D_{\mathfrak{q}}(B,\Phi) + B_{\gamma,K}(B,\Phi) ,$$

where $B_{\gamma,K}$ is the quadratic map defined by the square bracket.

Claim 1: If α is sufficiently small, there exists a constant $c = c(g,\gamma_s)$ such that for any L_1^2-pair v with $d_g^* v = 0$, $\| v \|_{L^4} < \alpha$, one has the estimate

$$\| v \|_{L_1^2} < c \| sw(v) \|_{L^2} . \tag{1}$$

Indeed, the Coulomb gauge condition $d_g^*(v) = 0$ implies

$$D_{\mathfrak{q}}(v) = -B_{\gamma,K}(v) + sw(v) . \tag{2}$$

This gives an estimate of the form

$$\| v \|_{L_1^2} \leq C_{\mathfrak{q}} \| D_{\mathfrak{q}}(v) \|_{L^2} \leq C_{\mathfrak{q}} C_{\gamma,K} \| v \|_{L^4}^2 + \| sw(v) \|_{L^2} \leq$$

$$\leq C C_{\mathfrak{q}} C_{\gamma,K} \| v \|_{L^4} \| v \|_{L_1^2} + \| sw(v) \|_{L^2} ,$$

Since $\mathfrak{q}$ is assumed to be close to $\mathfrak{q}_s$ and K belongs to a bounded family, it follows that the constants $C_{\mathfrak{q}}$, $C_{\gamma,K}$ can be chosen independently of $\mathfrak{p}$. The claim follows by the same rearrangement argument used in the proof of Theorem 4.7, taking $\alpha \leq \frac{1}{2CC_{\mathfrak{q}}C_{\gamma,K}}$. This proves the claim and the first part of the theorem.

Claim 2: If α is sufficiently small, then for any two L_1^2-sections v_1, v_2 with $d_g^*(v_i) = 0$, $\| v_1 \|_{L^4} < \alpha$, $\| v_2 \|_{L^4} < \alpha$ and $sw(v_1) = sw(v_2)$ it follows $v_1 = v_2$.

Indeed, let $b_{\gamma,K}$ be the $\mathbb{R}$-bilinear map associated with $B_{\gamma,K}$. One has

$$D_{\mathfrak{q}}(v_1 - v_2) = b_{\gamma,K}((v_2 - v_1), v_1) + b_{\gamma,K}(v_1, (v_2 - v_1)) ,$$

hence, by the injectivity of $D_{\mathfrak{q}}$, we get an estimate of the form

$$\| v_1 - v_2 \|_{L^4} \leq C \| v_1 - v_2 \|_{L_1^2} \leq C C_{\mathfrak{q}} \| b_{\gamma,K}((v_2 - v_1), v_1) + b_{\gamma,K}(v_1, (v_2 - v_1)) \|_{L^2}$$

$$\leq C C_{\mathfrak{p}} (\| v_1 \|_{L^4} + \| v_2 \|_{L^4}) \| v_2 - v_1 \|_{L^4}$$

where $C_{\mathfrak{p}}$ is a constant depending continuously of $\mathfrak{p}$ with respect to the C^0-topology. Therefore, we may suppose as above that $C_{\mathfrak{p}} = C_1$ is independent of $\mathfrak{p}$. Take $\alpha \leq \frac{1}{4CC_1}$.

Claim 3: If α is sufficiently small, then for any smooth pair v with $d_g^*(v) = 0$, $\| v \|_{L^4} < \alpha$ one has estimates of the form

$$\| v \|_{L_{k+1}^2} \leq C_{\mathfrak{p},k} \| sw(v) \|_{L_k^2} + P_{\mathfrak{p},k}(\| sw(v) \|_{L_{k-1}^2}) ,$$

where $C_{\mathfrak{p},k}$ is a positive constant and $P_{\mathfrak{p},k}$ is a polynomial with positive coefficients and without constant term.

To see this use again the rearrangement argument above to estimate the L_2^2 and the L_3^2 norms of v (compare with the proof of Theorem 3.7). For the higher Sobolev norms apply the usual bootstrapping procedure to the elliptic equation (2).

Claim 4: If α is sufficiently small, there exists a positive number μ such that for every smooth section $f \in A^0(S_s^- \otimes E_s \oplus su(S_s^+) \otimes su(2))$ with $\| f \|_{L^2} < \mu$, the equation

$$sw(v) = f \ , \ d_g^*(v) = 0$$

has a smooth solution v satisfying $\| v \|_{L^4} < \alpha$.

Indeed, choose first α such that the conclusions of Claims 1-3 hold. We use the continuity method to find a smooth solution of the equations $sw(v) = f$, $d_g^*(v) = 0$. Let (SW^t) be the equation

$$(SW^t) \qquad (d_g^* + sw_{\mathfrak{p}})(v) = t\, f \ .$$

We have to find a smooth solution of (SW^1) whose L^4-norm is bounded by α. Let N be the set

$$N := \{t \in [0,1] |\ (SW^t) \text{ has a smooth solution } v \text{ with } \| v \|_{L^4} < \alpha\}$$

The set N contains 0. We assert that, taking a smaller bound α if necessary, N becomes an open set. We use the implicit function theorem. Let v_0 be a solution of (SW^{t_0}) satisfying $d_g^*(v_0) = 0$, $\| v_0 \|_{L^4} < \alpha$. We have

$$\frac{\partial}{\partial v}(d_g^* + sw_{\mathfrak{p}})(\dot{v}) = D_{\mathfrak{q}}(\dot{v}) + b_{\gamma,K}(\dot{v}, v) + b_{\gamma,K}(v, \dot{v})$$

This shows that, for $v = 0$, the operator $\frac{\partial}{\partial v}|_0(d_g^* + sw_{\mathfrak{p}})$ defines an isomorphism:

$$\begin{array}{ccc} L_1^2(S_s^+ \otimes E_s) & \longrightarrow & L^2(S_s^- \otimes E_s) \\ \oplus & & \oplus \\ L_1^2(\Lambda^1(su(2))) & \longrightarrow & L^2(su(2))^{\perp} \oplus L^2(su(S_s^+) \otimes su(E_s)) \end{array}$$

If $\| v \|_{L^4}$ is sufficiently small, then the $L_1^2 \longrightarrow L^2$ extension of $\frac{\partial}{\partial v}(d_g^* + sw_{\mathfrak{p}})$ is still an isomorphism. By the Fredholm alternative it follows that the $L_3^2 \longrightarrow L_2^2$ extension is an isomorphism, too. Therefore, there exists $\varepsilon > 0$ and an L_3^2 solution v_t of (SW^t) for any $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ such that $v_{t_0} = v_0$. Using the usual bootstrapping procedure, it follows that v_t must be smooth.

We claim that N is closed, if the bound μ of $\| f \|_{L^2}$ is sufficiently small. Indeed, if $t_n \to t_0$, and if v_n is a smooth solution of (SW^{t_n}) with $\| v_n \|_{L^4} < \alpha$, then Claim 3. shows that there is a subsequence $(v_{n_m})_{m \in \mathbb{N}}$ converging in the $\mathcal{C}^\infty$-topology to a smooth section v_0, which must solve the equation (SW^{t_0}). Of course, it is not clear that the strict inequality $\| v_{n_m} \|_{L^4} < \alpha$ is preserved at the limit. On the other hand, using the estimate (1) proved in Claim 1. and the boundedness of the inclusion $L_1^2 \subset L^4$, we see that, choosing μ sufficiently small, we can assure that

$$\| v_n \|_{L^4} \leq \frac{\alpha}{2} \ .$$

Therefore v_0 satisfies the stronger inequality $\| v_0 \|_{L^4} \leq \frac{\alpha}{2}$. Now the second assertion in the theorem follows immediately: If $\| u \|_{L^4} < \alpha$, $d_g^*(u) = 0$, $\| sw_{\mathfrak{p}}(u) \|_{L^2} < \mu$, and $sw_{\mathfrak{p}}(u)$ is smooth, we can find a smooth solution v of the equations $d_g^* v = 0$, $sw_{\mathfrak{p}}(v) = sw_{\mathfrak{p}}(u)$ with $\| v \|_{L^4} < \alpha$. But, by Claim 2., this solution must coincide with u. □

COROLLARY 4.15. *With the notations and assumptions of the theorem, the following holds: There exists a positive constant α_1 (depending on (g, γ_s)) such that any L_1^2-pair $u = (A, \Psi)$ with $d_g^*(A) = 0$, $\| u \|_{L_1^2} \leq \alpha_1$ and $sw_{\mathfrak{p}}(u)$ smooth, is also smooth.*

4.3. Removable singularities. We notice first that Corollary 4.8 (Estimates in terms of the curvature) can be easily generalized to an *arbitrary* system of data $\mathfrak{p}' = (\mathfrak{q}', K')$ for the pair $(\bar{B}, P_0^u)$, not necessarily close to a standard system. The only difference is that the constant ε in the conclusion of the theorem will depend on $\mathfrak{p}'$. To see this it is enough to notice that the operator $D_{\mathfrak{q}}$ constructed in the proof of Theorem 4.7 is always elliptic by Remark 4.13 (even if the metric g_γ is not close to the metric g_s). We can use in fact the standard constant curvature metric on the sphere for the Coulomb condition, as in [DK]. $D_{\mathfrak{q}}$ will be in general non-injective, but the injectivity of this operator is not essential in the proof of 4.7: the corresponding elliptic estimates (el), $(el)_k$ will contain on the right the additional term $\| u \|_{L^2}$, which can be estimated in terms of $\| u \|_{L^4}$ using the volume of the sphere endowed with the metric g_γ.

An alternative argument uses a division of the unit ball in small balls, the scale invariance of the equations (Remark 4.5), the original Theorem 4.7, and the patching arguments explained on p. 162 [DK] in the instanton case.

Using this generalization of Corollary 4.8, we get the following analogon of Proposition 4.4.10 [DK]:

LEMMA 4.16. *Let Ω be a strongly simply connected 4-manifold endowed with a $Spin^{U(2)}(4)$-bundle P^u with $P^u \times_\pi \mathbb{R}^4 \simeq \Lambda^1_\Omega$, $\bar{\delta}(P^u) \simeq \Omega \times PU(2)$. Fix a trivialization of the $PU(2)$-bundle $\bar{\delta}(P^u)$. Let $\mathfrak{p} = (\gamma, C, a, \beta, K)$ be a system of data for the bundle P^u such that pointwise $|K - \mathrm{id}| < \mathfrak{e}$.*

There exists a positive constant $\varepsilon_{\mathfrak{p}}$, and for every precompact interior domain $\Omega' \Subset \Omega$ there exists a positive constant $M_{\mathfrak{p},\Omega'}$ such that any solution (A, Ψ) of the $PU(2)$-monopole equations for $\mathfrak{p}$ with $\| F_A \|_{L^2_{g_\gamma}} < \varepsilon_{\mathfrak{p}}$ is gauge equivalent over Ω' to a pair (A', Ψ') satisfying

$$\| A' \|_{L^4_{g_\gamma}(\Omega')} < M_{\mathfrak{p},\Omega'} \| F_A \|_{L^2_{g_\gamma}} .$$

REMARK 4.17. *Given a fixed system of data $\mathfrak{p}_0$, we can find constants ε_0, $M_{0,\Omega'}$ (independent of $\mathfrak{p}$) such that the conclusion of the theorem holds with these constants, for every $\mathfrak{p}$ sufficiently close to $\mathfrak{p}_0$. Moreover, the statement is true if we use the fixed metric g_{γ_0} to compute the Sobolev norms.*

We will need these results in the following particular case:

Let $\mathcal{N}$, $\mathcal{N}'$ be the annuli

$$\mathcal{N} := \{x \in B | \ \frac{1}{2} < |x| < 1\} , \quad \mathcal{N}' := \{x \in B | \ \frac{4}{6} < |x| < \frac{5}{6}\} .$$

Denote by $\mathcal{N}_r$, $\mathcal{N}'_r$ the images of $\mathcal{N}$, $\mathcal{N}'$ under the homothety h_r. We recall that we denoted by P_0^u the trivial $Spin^{U(2)}(4)$-bundle on $\bar{B}$, which is associated with the triple of $SU(2)$-bundles $S_0^\pm := \bar{B} \times \mathbb{H}_\pm$, $E_0 := \bar{B} \times \mathbb{C}^2$.

LEMMA 4.18. *Let $\mathfrak{p} = (\gamma, C, a, \beta, K)$ be a system of data for the trivial $Spin^{U(2)}(4)$-bundle P_0^u on the ball $\bar{B}$, such that pointwise $|K - \mathrm{id}| < \mathfrak{e}$, and such that $\gamma|_{\Lambda^1_0} : \mathbb{R}^4 \longrightarrow \mathbb{H} = (P_0^u \times_\pi \mathbb{R}^4)_0$ is the standard identification. Then there exists constants $\varepsilon(K_0) > 0$, $M(K_0)$ such that for any sufficiently small $r > 0$ the following holds:*

Any solution (A, Ψ) of the $PU(2)$-monopole equations for $\mathfrak{p}|_{\mathcal{N}_r}$ with $\| F_A \|_{L^2(\mathcal{N}_r)} < \varepsilon(K_0)$ is gauge equivalent over $\mathcal{N}'_r$ to a pair (A', Ψ') satisfying

$$\| A' \|_{L^4(\mathcal{N}'_r)} < M(K_0) \| F_A \|_{L^2(\mathcal{N}_r)} .$$

The constants $\varepsilon(K_0) > 0$, $M(K_0)$ are independent of r, and the Sobolev norms are computed with respect to the standard euclidean metric.

Proof. We use the same argument as in Remark 4.5. Let

$$h_r : (\mathcal{N}', \mathcal{N}) \longrightarrow (\mathcal{N}'_r, \mathcal{N}_r)$$

the homothety of slope r.

The pair $(h_r^*(A), rh_r^*(\Psi))$ solves the monopole equations associated with the system of data $(\gamma_r := rh_r^*(\gamma), h_r^*(C), h_r^*(a), rh_r^*(\beta), h_r^*(K))$, which converges to the standard system $\mathfrak{p}_{K_0}$ restricted to $\mathcal{N}$, as $r \to 0$.

The result follows now from 4.16, 4.17 and the conformal invariance of the L^4-norm on 1-forms and of the L^2-norm on 2-forms. □

We shall use the following notations

$$\Omega_r := B \setminus \bar{B}(r) \ , \quad B^\bullet = B \setminus \{0\} \ , \quad B^\bullet(R) = B(R) \setminus \{0\} \ , S^\bullet = S \setminus \{0\} \ .$$

LEMMA 4.19. *Let $\mathfrak{p} = (\gamma, C, a, \beta, K) = (\mathfrak{q}, K)$ be a system of data for the trivial bundle P_0^u on the ball B, and let (A, Ψ) be a pair on $B^\bullet$ solving the monopole equations for $\mathfrak{p}|_{B^\bullet}$ such that*

$$\int_{B^\bullet} |F_A|^2 < \infty$$

Then for any sufficiently small $r > 0$, there exist an $SU(2)$-bundle E_r over B, a pair $(A_r, \Psi_r) \in \mathcal{A}(E_r) \times A^0(S_0^+ \times E_r)$ and an $SU(2)$-isomorphism

$$\rho_r : E_r|_{\Omega(r)} \longrightarrow E|_{\Omega(r)}$$

such that:

i) $\rho_r^*(A, \Psi) = (A_r, \Psi_r)$,

ii) $\| sw_{\mathfrak{p}}(A_r, \Psi_r) \|_{L^2(B)} \to 0$ *as* $r \to 0$.

Proof. Let φ be a cut-off map $\varphi : B \longrightarrow [0, 1]$ which is identically 1 on $B \setminus B(\frac{5}{6} - \varepsilon)$ and identically 0 on $B(\frac{4}{6} + \varepsilon)$.

Put $\varphi_r := \varphi \circ h_r^{-1}$. Note first that, by the conformal invariance of the L^4-norm on 1-forms, the norm $\| d\varphi_r \|_{L^4}$ (computed with the euclidean metric) does not depend on r.

Consider now the restriction of the pair (A, Ψ) to $\mathcal{N}_r$. Since the total integral of $|F_A|^2$ on the ball is finite, it follows that for any sufficiently small $r > 0$ we have

$$\| F_A \|_{L^2(\mathcal{N}_r)} < \varepsilon(K_0) \ ,$$

so that Lemma 4.18 applies. The conclusion of this Lemma can be reformulated as follows: There exists an $SU(2)$-trivialization $\mathcal{N}'_r \times \mathbb{C}^2 \xrightarrow{\tau_r} E_0|_{\mathcal{N}'_r}$ such that the connection matrix of $\tau_r^*(A)$ (which we also denote by $\tau_r^*(A)$) satisfies the estimate

$$\| \tau_r^*(A) \|_{L^4(\mathcal{N}'_r)} \leq M(K_0) \| F_A \|_{L^2(\mathcal{N}_r)} \tag{1}$$

We define the $SU(2)$-bundle E_r by gluing (over the annulus $\mathcal{N}'_r$) the trivial bundles $B(0, \frac{5r}{6}) \times \mathbb{C}^2$, $E_0|_{\Omega(\frac{4r}{6})}$ via the isomorphism τ_r.

Let P_r^u be the $Spin^{U(2)}(4)$-bundle associated with the triple $(S_0^\pm, E_r)$. The system $\mathfrak{p}$ can be also regarded as a system of data for the bundle P_r^u.

Now denote by u the initial pair $u := (A, \Psi)$, and by u_r the pair

$$u_r \in \mathcal{A}(E_r) \times A^0(S_0^+ \otimes E_r) ,$$

which coincides with u on $\Omega((\frac{5}{6} - \varepsilon)r)$ and with the cut-off $\varphi_r \tau_r^*(u)$ of $\tau_r^*(u)$ on $B(0, \frac{5r}{6})$.

The section $sw_{\mathfrak{p}}(u_r)$ vanishes identically on $\Omega(\frac{5r}{6})$, where u_r coincides with u. Therefore, in order to prove *ii)* we only have to estimate the L^2 norm of $sw_{\mathfrak{p}}(\varphi_r \tau_r^*(u))$ on $B(0, \frac{5r}{6})$, where E_r coincides with the trivial bundle $B(0, \frac{5r}{6}) \times \mathbb{C}^2$.

On $B(0, \frac{5r}{6})$ the Seiberg-Witten map $sw_{\mathfrak{p}}$ can be written as a sum between a first order differential operator and a quadratic map:

$$sw_{\mathfrak{p}}(B, \Phi) = \begin{bmatrix} \not{D}_{\mathfrak{q}} \Phi \\ \Gamma_\gamma(dB) \end{bmatrix} + \begin{bmatrix} \gamma(B)(\Phi) \\ \Gamma_\gamma(B \wedge B) - K(\Phi\bar{\Phi})_0 \end{bmatrix} = T_{\mathfrak{q}}(B, \Phi) + B_{\gamma,K}(B, \Phi)$$

Since $T_{\mathfrak{q}}$ is a first order operator, we have an identity of the form

$$T_{\mathfrak{q}}(fv) = A_{\mathfrak{q}}(df)(v) + f T_{\mathfrak{q}}(v) ,$$

where $A_{\mathfrak{q}}(df)$ is a 0-order operator whose coefficients depend linearly on the first order derivatives of the real function f.

Therefore

$$\begin{aligned} sw_{\mathfrak{p}}(\varphi_r \tau_r^*(u)) &= A_{\mathfrak{q}}(d\varphi_r)(\tau_r^*(u)) + \varphi_r T_{\mathfrak{q}}(\tau_r^*(u)) + \varphi_r^2 B_{\gamma,K}(\tau_r^*(u)) \\ &= \varphi_r sw_{\mathfrak{p}}(\tau_r^*(u)) + A_{\mathfrak{q}}(d\varphi_r)(\tau_r^*(u)) + (\varphi_r^2 - \varphi_r) B_{\gamma,K}(\tau_r^*(u)) \\ &= A_{\mathfrak{q}}(d\varphi_r)(\tau_r^*(u)) + (\varphi_r^2 - \varphi_r) B_{\gamma,K}(\tau_r^*(u)) . \end{aligned}$$

Therefore, taking into account that $d\varphi_r$ and $(\varphi_r^2 - \varphi_r)$ vanish outside $\mathcal{N}_r'$, we get

$$\begin{aligned} \| sw_{\mathfrak{p}}(u_r) \|_{L^2(B)} &= \| sw_{\mathfrak{p}}(\tau_r^*(u)) \|_{L^2(B(\frac{5r}{6}))} \\ &\leq C_{\mathfrak{q}} \| d\varphi_r \|_{L^4} \| \tau_r^*(u) \|_{L^4(\mathcal{N}_r')} + C_{\mathfrak{p}}' \| \tau_r^*(u) \|_{L^4(\mathcal{N}_r')}^2 \end{aligned}$$

Since $\| d\varphi_r \|_{L^4}$ does not depend on r we have only to prove that $\| \tau_r^*(u) \|_{L^4(\mathcal{N}_r')}$ converges to 0 as $r \to 0$. But the estimate (1) shows that the L^4-norm of the connection component of $\tau_r^*(u)$ converges to 0 as $r \to 0$.

On the other hand, by the inequality (3) Section 4.1 and the second monopole equation, one has pointwise in $\mathcal{N}_r'$.

$$|\tau_r^*(\Psi)|^4 = |\Psi|^4 \leq \left[C_1^{-1} |\Gamma_\gamma(F_A)|\right]^2 .$$

This gives an estimate of $\| \tau_r^*(\Psi) \|_{L^4(\mathcal{N}_r')}$ in terms of $\| F_A \|_{L^2(\mathcal{N}_r')}^{\frac{1}{2}}$, which obviously converges to 0 as $r \to 0$. □

We recall from [DK] the following important

THEOREM 4.20. *(Gauge fixing on the sphere* [5]*) Let g_c be the standard constant curvature metric on the sphere S^4. Then there are constants ε_c, M_c such that any*

[5]Note that in [DK] it is stated a slightly weaker form of this theorem (Proposition 2.3.13 p. 63): The hypothesis requires that A can be joined to the flat connection by a path of connections with L^2-small curvature . However, the second proof of this result, which is given in section 2.3.10, *does not* use this additional assumption. I am grateful to Peter Kronheimer for pointing me out this important detail. On the other hand, note that this second proof works only for the standard constant curvature metric, and can be generalized to conformally flat metrics with non-negative sectional curvature. Since our regularity theorem works for solutions whose connection component is in Coulomb gauge with respect to *any* metric, not necessary close to the metric defined by the $Spin^{U(2)}$-structure, we don't need this generalization

connection A in the trivial $SU(2)$-bundle E_s with $\| F_A \|_{L^2} < \varepsilon_c$ is gauge equivalent to a connection $\tilde{A}$ satisfying

$$d^*_{g_c}(\tilde{A}) = 0 \ , \quad \| \tilde{A} \|_{L^2_1} < M_c \| F_A \|_{L^2} \ . \qquad \square$$

We can prove now

THEOREM 4.21. *(Removable singularities) Let $\mathfrak{p} = (\mathfrak{q}, K)$ a system of data for the trivial $Spin^{U(2)}(4)$-bundle P^u_0 on $\bar{B}$ and let $u = (A, \Psi)$ be a pair on the punctured ball solving the monopole equations for $\mathfrak{p}|_{B^\bullet}$ such that*

$$\int_{B^\bullet} |F_A|^2 < \infty \ .$$

There exists an $SU(2)$-bundle F on the ball, and an $SU(2)$-isomorphism $\rho : F|_{B^\bullet} \longrightarrow E_0|_{B^\bullet}$ such that $\rho^*(A, \Psi)$ extends to a global smooth solution of the monopole equations associated with $\mathfrak{p}$ and the $Spin^{U(2)}(4)$-bundle defined by $(S^{\pm}_0, F)$.

Proof. We use similar arguments as in the proof of the "Removable singularities" theorem for the instanton equation (Theorem 4.4.12 [DK]). The only difference is that the L^2_1-bound of the approximate solutions we construct, does not follow directly from Theorem 4.20 (Gauge fixing on the sphere).

Identify $\bar{B}$ with the upper hemisphere of the 4-sphere S, and extend the system $\mathfrak{p}$ to a system for the $Spin^{U(2)}(4)$-bundle P^u_s. The extended system will be denoted by the same symbol $\mathfrak{p}$, and we can assume that $\mathfrak{p}$ has the form $(\mathfrak{q}, K)$ with $\mathfrak{q}$ close to the system $\mathfrak{q}_s$ constructed in the proof of Theorem 4.7, so that Theorem 4.14 and Corollary 4.15 applies. We shall use these results in the particular case $g = g_c$; with respect to this metric connections with L^2-small curvature can be brought in the Coulomb gauge, by 4.20.

Step 1. For a sufficiently small positive number $R < 1$ we use Lemma 4.18 to get a trivialization of $E_0|_{\mathcal{N}'_R}$, such that the L^4-norm of the corresponding connection matrix is controlled by $\| F_A \|_{L^2(\mathcal{N}_R)}$. By the same gluing procedure we get a bundle E^R on the punctured sphere $S^\bullet$, trivialized on $S \setminus \bar{B}(\frac{4R}{6})$. We cut off the pair u this time towards the outer boundary of the ball, and we get a pair $u^R = (A^R, \Psi^R)$. It holds

$$\lim_{R \to 0} \| sw_{\mathfrak{p}}(u^R) \|_{L^2} = \lim_{R \to 0} \| F_{A^R} \|_{L^2} = \lim_{R \to 0} \| \Psi^R \|_{L^4} = 0 \ , \tag{2}$$

The first two relations follow as in the proof of Lemma 4.19, since both maps $sw_{\mathfrak{p}}(\cdot)$, $F.$ can be written as the sum of a first order operator and a quadratic map, hence the perturbations produced by of the two cut-off operations can be estimated in terms of the L^2-norm of the curvature restricted to the corresponding annuli.

To get the third formula, it is enough to notice that the pointwise norm of the spinor is invariant under bundle isomorphisms, and that the L^4-norm of $\Psi|_{B^\bullet(R)}$ can be estimated in terms of $\| F_{A|_{B^\bullet(R)}} \|^{\frac{1}{2}}_{L^2}$.

Suppose now that $r < R < 1$ and use the same procedure (to modify the bundle and cut off the solution), but this time in both directions.

We get $SU(2)$-bundles, E^R_r on the sphere, which come with trivializations over $B(\frac{5r}{6})$, $S \setminus \bar{B}(\frac{4R}{6})$, and with an isomorphism

$$E^R_r|_{B(\frac{5R}{6}) \setminus \bar{B}(\frac{4r}{6})} \xrightarrow{\simeq \rho_{r,R}} E_s|_{B(\frac{5R}{6}) \setminus \bar{B}(\frac{4r}{6})} \ ,$$

as well as cut-off pairs

$$u_r^R = (A_r^R, \Psi_r^R) \in \mathcal{A}(E_r^R) \times A^0(S_s^+ \otimes E_r^R) \ .$$

With this construction, it holds

$$\lim_{r\to 0} \| sw_{\mathfrak{p}}(u_r^R) \|_{L^2} = \| sw_{\mathfrak{p}}(u^R) \|_{L^2} \ , \quad \lim_{r\to 0} \| F_{A_r^R} \|_{L^2} = \| F_{A^R} \|_{L^2} \ , \tag{3}$$

$$\lim_{r\to 0} \| \Psi_r^R \|_{L^4} = \| \Psi^R \|_{L^4} \ .$$

Note that the double gluing-procedure we used could apriori give rise to a non-trivial $SU(2)$-bundle $E_{r,R}$ on the sphere. But since the curvature $F_{A_r^R}$ can be made as small as we please, it follows that all the bundles $E_{r,R}$ become trivial, if R is small.

Step 2. Using (2), (3) and Theorem 4.20 it follows that, once R is small, there exists an $SU(2)$-isomorphism $\theta_r^R : E_s \longrightarrow E_r^R$ such that $B_r^R := \theta_{r,R}^*(A_r^R)$ satisfies

$$d_{g_c}^*(B_r^R) = 0 \ , \quad \| B_r^R \|_{L_1^2} \leq M_c \| F_{A_r^R} \|_{L^2} \tag{4}$$

Put $\Phi_r^R := (\theta_r^R)^*(\Psi_r^R)$, $v_r^R := (B_r^R, \Phi_r^R)$.

Step 3. Using (2), (3), (4) and the boundedness of the embedding $L_1^2 \subset L^4$, it follows that, if R is small enough, the L^4-norm of the pair v_r^R can be made smaller as the constant α in the Regularity Theorem 4.14, so that we get an estimate of the form

$$\| v_r^R \|_{L_1^2} \leq c \| sw_{\mathfrak{p}}(v_r^R) \|_{L^2} = \| sw_{\mathfrak{p}}(u_r^R) \|_{L^2} \ . \tag{5}$$

The relations (2), (3) imply now that , choosing R small, we can assure that

$$\| v_r^R \|_{L_1^2} \leq \alpha_1 \ , \tag{6}$$

where α_1 is the constant in Corollary 4.15. From this point the proof goes further like in the instanton case: We choose R sufficiently small such that all the mentioned properties are fulfilled, and we let r tend to 0. Using the L_1^2-boundedness obtained in (6) it follows that we can find a sequence $r_i \to 0$ such that $v_i = (B_i, \Phi_i) := v_{r_i}^R$ converges weakly in L_1^2 to an L_1^2-pair $v = (B, \Phi)$.

Step 4. We want to prove that v is smooth. The weak limit v must also satisfies $\| v \|_{L_1^2} \leq \alpha_1$ by the weak-semicontinuity of the norm in reflexive Banach spaces. Therefore, by Corollary 4.15, we only have to prove that the L^2-section $sw(v)$ is smooth.

But on any small ball D , $\bar{D} \subset S^\bullet$, the pairs $v_i = (B_i, \Phi_i)$ remain in the same gauge equivalence class. Recall now from [DK] that the Sobolev norms of any connection H in Coulomb gauge can be estimated in terms of the gauge invariant expressions

$$\| F_H \|_{L^\infty} \ , \ \| \nabla_H^{(i)} F_H \|_{L^2} \ ,$$

as soon as its L^4-norm is sufficiently small. Using the estimate (4) and the scale invariance of the L^4-norm on 1-forms, this condition will be also fulfilled (for all small balls D), if R is sufficiently small.

On the other hand one can easily bound the Sobolev norms of a spinor Ξ in terms of the gauge invariant expressions $\| \nabla_H^{(i)} \Xi \|_{L^2}$ and the Sobolev norms of the connection H.

Therefore, taking a subsequence if necessary, we can assume that v_i converges in the Fréchet $\mathcal{C}^\infty$-topology on $S^\bullet$, so that $sw(v)$ is smooth on the punctured sphere.

But, by Lemma 4.19, $\lim_i \| sw_{\mathfrak{p}}(v_i|_{B(\frac{4R}{6})}) \|_{L^2} \to 0$, so $sw(v)$, which is the limit of $sw(v_i)$ in the distribution sense, vanishes in a neighbourhood of 0.

On the other hand, for any ball D, $\bar{D} \subset B^\bullet(\frac{4R}{6})$, the isomorphism $\theta^R_{r_i}$ intertwines the connection matrices A, B_i, and B_i converges in the $\mathcal{C}^\infty$ topology on such a ball. Therefore a subsequence $\theta^R_{r_{i_n}}$ converges in the $\mathcal{C}^\infty$ topology on $B^\bullet(\frac{5R}{6})$ to a smooth bundle isomorphism θ, such that

$$\theta^*(A|_{B^\bullet(\frac{5R}{6})}) = B|_{B^\bullet(\frac{5R}{6})} \ .$$

Taking the limit of $[\theta^R_{r_{i_n}}]^*(\Psi|_{B(\frac{5R}{6})\setminus\bar{B}(\frac{4r_{i_n}}{6}))}) = \Phi^R_{r_{i_n}}|_{B(\frac{5R}{6})\setminus\bar{B}(\frac{4r_{i_n}}{6}))}$ for $n \to \infty$, we also get

$$\theta^*(\Psi|_{B^\bullet(\frac{5R}{6})}) = \Phi|_{B^\bullet(\frac{5R}{6})}) \qquad \square$$

4.4. Compactified moduli spaces. Let X be a closed oriented 4-manifold. For a $Spin^{U(2)}(4)$-bundle P^u with $P^u \times_\pi \mathbb{R}^4 \simeq \Lambda^1$ and a system of data $\mathfrak{p} = (\gamma, C, a, \beta, K)$ for P^u denote by $\mathcal{M}_{\mathfrak{p}}(P^u)$ the moduli space of pairs $(A, \Psi) \in \mathcal{A}(\bar{\delta}(P^u)) \times A^0(\Sigma^+(P^G))$ solving the $PU(2)$-monopole equations associated with $\mathfrak{p}$.

By Proposition 2.1, the data of a $Spin^{U(2)}(4)$-bundle P'^u on X with $\det(P'^u) \simeq \det(P^u)$, $P'^u \times_\pi \mathbb{R}^4 \simeq P^u \times_\pi \mathbb{R}^4$ is equivalent via the map $\bar{\delta}$ to the data of $PU(2)$-bundle $\bar{P}'$ whose Pontrjagin class satisfies

$$p_1(\bar{P}') \equiv (w_2(X) + \bar{c}_1(\det(P^u)))^2 \bmod 4 \ .$$

For every number $l \in \mathbb{N}$ we fix:

1. A $Spin^{U(2)}(4)$-bundle P^u_l with

$$l = \frac{1}{4}\left(p_1(\bar{\delta}(P^u_l)) - (p_1(\bar{\delta}(P^u))\right)$$

2. Identifications

$$(id) \qquad P^u_l \times_\pi \mathbb{R}^4 \xrightarrow{\sim} P^u \times_\pi \mathbb{R}^4 \ , \quad \det(P^u_l) \xrightarrow{\sim} \det(P^u).$$

These bundle isomorphisms allow us to identify the spaces of perturbations-data associated with the bundles P^u, P^u_l.

DEFINITION 4.22. *An ideal $PU(2)$-monopole of type $(P^u, \mathfrak{p})$ is a pair $([A', \Psi'], \{x_1, \ldots, x_l\})$ consisting of an element $\{x_1, \ldots, x_l\}$ in a symmetric power $S^l(X)$ of X and a monopole $[A', \Psi'] \in \mathcal{M}_{\mathfrak{p}}(P^u_l)$.*

We denote by $I\mathcal{M}_{\mathfrak{p}}(P^u)$ the space of ideal monopoles of type $(P^u, \mathfrak{p})$.

Let δ_x be the Dirac measure associated with a point $x \in X$. If $\mathfrak{p} = (\gamma, A, a, \beta, K)$, we always use the metric g_γ to compute the norms and to define (anti-)self-duality for 2-forms.

LEMMA 4.23. *The map $F : I\mathcal{M}_{\mathfrak{p}}(P^u) \longrightarrow [\mathcal{C}^0(X, \mathbb{R})]^*$, defined by*

$$F([A', \Psi'], \{x_1, \ldots, x_l\}) = |F_{A'}|^2 + 8\pi^2 \sum_{i=1}^{l} \delta_{x_i} \ ,$$

is bounded with respect to the strong topology in the dual space $[\mathcal{C}^0(X,\mathbb{R})]^*$.

Proof. Let $\varphi \in \mathcal{C}^0(X,\mathbb{R})$ with $\sup_X |\varphi| \leq 1$. Then

$$|\langle F([A',\Psi'],\{x_1,\ldots,x_l\}),\varphi\rangle| \leq [\| F_{A'}^- \|_{L^2}^2 - \| F_{A'}^+ \|_{L^2}^2] + 2 \| F_{A'}^+ \|_{L^2}^2 + 8\pi^2 l$$

$$= -2\pi^2 p_1(\bar{\delta}(P^u)) + 2C \| \Psi' \|_{L^4}^4 ,$$

where C is a universal positive constant. The assertion follows from the apriori $\mathcal{C}^0$-boundedness of the spinor component of a solution (Proposition 4.11). □

Let $\mathfrak{m}' = ([A',\Psi'],s')$ be an ideal monopole of type $(P^u,\mathfrak{p})$ with $s' \in S^{l'}(X)$ and $[A',\Psi'] \in \mathcal{M}_{\mathfrak{p}}(P^u_{l'})$. For a positive number ε we define $U(\mathfrak{m}',\varepsilon)$ to be the set of ideal monopoles $\mathfrak{m}'' = ([A'',\Psi''],s'')$ of type $(P^u,\mathfrak{p})$ with $s'' \subset s'$, and which have the following property:

There exists an isomorphism of $Spin^{U(2)}(4)$-bundles

$$\varphi : P^u_{l''}|_{X\setminus s'} \longrightarrow P^u_{l'}|_{X\setminus s'}$$

which is compatible with the identifications (id) such that

$$d_1(\varphi^*(A',\Psi'),(A'',\Psi'')) < \varepsilon ,$$

where d_1 is a metric defining the Fréchet $\mathcal{C}^\infty$-topology in the product

$$\mathcal{A}(\bar{\delta}(P^u_{l''}|_{X\setminus s'})) \times A^0(\Sigma^+(P^u_{l''}|_{X\setminus s'}))) .$$

Let $M > 0$ be a bound for the map F defined above. The weak topology in the ball of radius M in $[\mathcal{C}^0(X,\mathbb{R})]^*$ is metrisable (see [La], Theorem 9.4.2). Let d_2 be a metric defining this topology.

We endow $I\mathcal{M}_{\mathfrak{p}}(P^u)$ with a metric topology by taking as basis of open neighbourhoods for an ideal monopolem' of type $(P^u,\mathfrak{p})$ the sets of the form $U(\mathfrak{m}',\varepsilon) \cap F^{-1}(B_{d_2}(F(\mathfrak{m}'),\varepsilon))$, $\varepsilon > 0$.

THEOREM 4.24. *With respect to the metric topology defined above the moduli space* $\mathcal{M}_{\mathfrak{p}}(P^u) \subset I\mathcal{M}_{\mathfrak{p}}(P^u)$ *is an open subspace with compact closure* $\overline{\mathcal{M}_{\mathfrak{p}}(P^u)}$.

Proof. The first assertion is obvious. For the second, we use the same argument as in the instanton case, but we make use in an essential way of the $\mathcal{C}^0$-boundedness of the spinor:

Let $\mathfrak{m}_n$ a sequence of ideal monopoles. It is easy to see that we can reduce the general case to the case where $\mathfrak{m}_n = [A_n,\Psi_n] \in \mathcal{M}_{\mathfrak{p}}(P^u)$. By Lemma 4.23, the sequence of measures $\mu_n := F(\mathfrak{m}_n)$ is bounded, so after replacing $\mathfrak{m}_n$ by a subsequence, if necessary, it converges weakly to a (positive) measure μ of total volume $\mu(1) \leq M$. The set

$$S_\varepsilon := \{x \in X | \exists n \in \mathbb{N}\ \forall m \geq n\ (\mu_m(D) \geq \varepsilon^2 \text{ for every geodesic ball } D \ni x)\}$$

contains at most $\frac{M}{\varepsilon^2}$ points, so it is finite for every positive number ε. Choosing the constant ε provided by the "Global compactness" theorem (Corollary 4.10), it follows by a standard diagonal procedure that there exists a subsequence $(\mathfrak{m}_{n_m})_m$ and gauge transformations f_m on $X \setminus S_\varepsilon$, such that $f_m^*(\mathfrak{m}_{n_m})$ converges to a solution (A_0,Ψ_0) of the monopole equations $SW_{\mathfrak{p}}$ restricted to $X \setminus S_\varepsilon$. By the "Removable Singularities" theorem, we can extend this solution to a global solution $(\tilde{A}_0,\tilde{\Psi}_0)$ of the monopole

equations associated with $\mathfrak{p}$ and a new $Spin^{U(2)}(4)$-bundle P'^u, which comes with identifications

$$P'^u \times_\pi \mathbb{R}^4 \xrightarrow{\sim} P^u \times_\pi \mathbb{R}^4 \ , \quad \det(P'^u) \xrightarrow{\sim} \det(P^u).$$

We have

$$|F_{\tilde{A}_0}|^2 = |F_{A_0}|^2 = \mu - 8\pi^2 \sum_{x \in S_\varepsilon} \lambda_x \delta_x$$

with positive numbers λ_x. It remains to prove that the λ_x are integers. Since $F(\mathfrak{m}_{n_m}) \to \mu$, we have for small enough $r > 0$

$$\begin{aligned}\lambda_x &= \lim_{m\to\infty} \frac{1}{8\pi^2} \int\limits_{B(x,r)} |F_{A_{n_m}}|^2 - |F_{\tilde{A}_0}|^2 \\ &= \lim_{m\to\infty} \frac{1}{8\pi^2} \int\limits_{B(x,r)} -\mathrm{Tr}(F^2_{A_{n_m}}) + \mathrm{Tr}(F_{\tilde{A}_0})^2 + 2\left(|(\Psi_{n_m}\bar{\Psi}_{n_m})_0|^2 - |(\tilde{\Psi}_0\bar{\tilde{\Psi}}_0)_0|^2\right) .\end{aligned}$$

As in the instanton case we get

$$\begin{aligned}\lim_{m\to\infty} \frac{1}{8\pi^2} \left[\int\limits_{B(x,r)} -\mathrm{Tr}(F^2_{A_{n_m}}) + \mathrm{Tr}(F_{\tilde{A}_0})^2 \right]_{\mathrm{mod}\ \mathbb{Z}} &= \lim_{m\to\infty} (\tau_{S(x,r)}(A_{n_m}) - \tau_{S(x,r)}(\tilde{A}_0)) \\ &= 0 \text{ in } \mathbb{R}/\mathbb{Z}\end{aligned}$$

by the convergence $f_m^*(A_{n_m}|_{X\setminus S_\varepsilon}) \to \tilde{A}_0|_{X\setminus S_\varepsilon}$. Here $\tau_S(B)$ denotes the Chern-Simons invariant of the connection B on a 3-manifold S ([DK]).

On the other hand, by the apriori C^0-bound of the spinor component on the space of monopoles, the term $\int\limits_{B(x,r)} 2\left(|(\Psi_{n_m}\bar{\Psi}_{n_m})_0|^2 - |(\tilde{\Psi}_0\bar{\tilde{\Psi}}_0)_0|^2\right)$ can be made as small as we please by choosing r sufficiently small. This shows that the λ_x are integers, and that

$$\sum_{x\in S_\varepsilon} \lambda_x = \frac{1}{4}\left(p_1(\bar{\delta}(P'^u)) - (p_1(\bar{\delta}(P^u))\right) ,$$

which completes the proof. □

REFERENCES

[Ad] R. Adams, *Sobolev Spaces*, Academic Press, 1975.

[AHS] M. Atiyah, N. J. Hitchin, and I. M. Singer, *Self-duality in four-dimensional Riemannian geometry*, Proc. R. Lond. A., 362 (1978), pp. 425–461.

[B] A. Besse, *Einstein Manifolds*, Springer Verlag, 1987.

[BB] B. Booss and D. D. Bleecker, *Topology and Analysis. The Atiyah-Singer index formula and Gauge-Theoretic Physics*, Springer Verlag, 1985.

[C] B. Chen, *Geometry of Submanifolds*, Pure and Applied Mathematics, Marcel Dekker, INC., New York, 1973.

[D] S. Donaldson, *Anti-self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles*, Proc. London Math. Soc., 3 (1985), pp. 1–26.

[DK] S. Donaldson and P. B. Kronheimer, *The Geometry of Four-Manifolds*, Oxford Science Publications, 1990.

[F] P. FEEHAN, *Generic metrics, irreducible rank-one $PU(2)$-monopoles, and transversality*, preprint, dg-ga/9809001, to appear in Communications in Analysis and Geometry.

[FL] P. FEEHAN AND TH. LENNES, *$PU(2)$-monopoles. I : regularity, Uhlenbeck compactness, and transversality*, Journal of Differential Geometry, 49 (1998), pp. 265–410.

[FU] D. S. FREED AND K. UHLENBECK, *Instantons and Four-Manifolds*, Springer-Verlag, 1984.

[GHL] S. GALLOT, D. HULIN, AND J. LAFONTAINE, *Riemannian Geometry*, Springer Verlag, 1987.

[HH] F. HIRZEBRUCH AND H. HOPF, *Felder von Flächenelementen in 4-dimensionalen 4-Mannigfaltigkeiten*, Math. Ann., 136 (1958), pp. 56–172.

[H] N. HITCHIN, *Harmonic spinors*, Adv. in Math., 14 (1974), pp. 1–55.

[Ke] J. KELLEY, *General Topology*, Springer Verlag, 1955.

[K] S. KOBAYASHI, *Differential Geometry of Complex Vector Bundles*, Princeton University Press, 1987.

[KM] P. KRONHEIMER AND T. MROWKA, *The genus of embedded surfaces in the projective plane*, Math. Res. Letters, 1 (1994), pp. 797–808.

[LL] T. LI AND A. LIU, *General wall crossing formula*, Math. Res. Lett., 2 (1995), pp. 797–810.

[LM] J. M. F. LABASTIDA AND M. MARINO, *Non-abelian monopoles on four manifolds*, Preprint, Departamento de Fisica de Particulas, Santiago de Compostela, April, 1995.

[La] R. LARSEN, *Functional Analysis: An Introduction*, Marcel Dekker, Inc., New York, 1973.

[LMi] H. B. LAWSON AND M. L. MICHELSON, *Spin Geometry*, Princeton University Press, New Jersey, 1989.

[LT] M. LÜBKE AND A. TELEMAN, *The Kobayashi-Hitchin Correspondence*, World Scientific Publishing Co., 1995.

[M] K. MIYAJIMA, *Kuranishi families of vector bundles and algebraic description of the moduli space of Einstein-Hermitian connections*, Publ. R.I.M.S. Kyoto Univ., 25 (1989), pp. 301–320.

[OST] CH. OKONEK, A. SCHMITT, AND A. TELEMAN, *Master spaces for stable pairs*, Topology, 38:1 (1999), pp. 117–139.

[OT1] CH. OKONEK AND A. TELEMAN, *The coupled Seiberg-Witten equations, vortices, and moduli spaces of stable pairs*, Int. J. Math., 6:6 (1995), pp. 893–910.

[OT2] CH. OKONEK AND A. TELEMAN, *Les invariants de Seiberg-Witten et la conjecture de Van De Ven*, Comptes Rendus Acad. Sci. Paris, t., Série I, 321 (1995), pp. 457–461.

[OT3] CH. OKONEK AND A. TELEMAN, *Seiberg-Witten invariants and rationality of complex surfaces*, Math. Z., to appear.

[OT4] CH. OKONEK AND A. TELEMAN, *Quaternionic monopoles*, Comptes Rendus Acad. Sci. Paris, t., Série I, 321 (1995), pp. 601–606.

[OT5] CH. OKONEK AND A. TELEMAN, *Quaternionic monopoles*, Commun. Math. Phys., 180:2 (1996), pp. 363–388.

[OT6] CH. OKONEK AND A. TELEMAN, *Seiberg-Witten invariants for manifolds with $b_+ = 1$, and the universal wall crossing formula*, Int. J. Math., 7:6 (1996), pp. 811–832.

[OT7] CH. OKONEK AND A. TELEMAN, *Recent developments in Seiberg-Witten theory and complex geometry*, in Several complex variables (Berkeley, CA, 1995 - 1996), Math. Sci. Res. Inst. Publ. 37, Cambridge Univ. Press, Cambridge, 1999, pp. 391–428.

[PT1] V. PIDSTRIGACH AND A. TYURIN, *Invariants of the smooth structure of an algebraic surface arising from the Dirac operator*, Russian Acad. Izv. Math., 40:2 (1993), pp. 267–351.

[PT2] V. PIDSTRIGACH AND A. TYURIN, *Localisation of the Donaldson invariants along the Seiberg-Witten classes*, dg-ga/9507004, 1995.

[Sm] S. SMALE, *An infinite dimensionale version of Sard's theorem*, American Journal of Math., 87 (1965), pp. 861–866.

[T1] A. TELEMAN, *Non-abelian Seiberg-Witten theory*, Habilitationsschrift, Universität Zürich, 1996.

[T2] A. TELEMAN, *Non-abelian Seiberg-Witten theory and stable oriented pairs*, Internat. J. Math., 8:4 (1997), pp. 507–535.

[T3] A. TELEMAN, *Fredholm L^p-theory for coupled Dirac operators*, C. R. Acad. Sci. Paris Sr. I Math., 329:1 (1999), pp. 5–10.

[T4] A. TELEMAN, *Almost virtual $PU(2)$-monopoles*, in preparation.

[W] E. WITTEN, *Monopoles and four-manifolds*, Math. Res. Letters, 1 (1994), pp. 769–796.