ASIAN J. MATH.
Vol. 4, No. 2, pp. 383–390, June 2000

LJUSTERNIK-SCHNIRELMANN THEORY AND CONLEY INDEX A NONCOMPACT VERSION*

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Abstract. In this paper we use a noncompact version of Conley index theory to obtain a Ljusternik-Schnirelmann type result in critical point theory: Let X be a complete Finsler manifold, $f \in C^1(X, \mathbb{R})$ which satisfies Palais-Smale condition and φ^t be the flow relative to a pseudo-gradient vector field for f. If $I \subset X$ is a (c)-invariant set with $f(I)$ bounded, then f has at least $\nu_H(h(I)) - 1$ critical points in I where ν_H is the (homotopy) Ljusternik-Schnirelmann category and $h(I)$ is the Conley index of I.

1. Introduction. Conley's homotopy index was first constructed for isolated invariant sets of continuous flows on locally compact metric spaces [2]. The compactness assumption is crucial both in the existence and uniqueness of the Conley index. An important result in Conley index theory is the generalized Morse inequalities which introduced Conley's work as a generalization of Morse theory. The compactness assumption is also crucial in basic results of Morse theory. Indeed every noncompact boundaryless manifold admits a smooth function without critical points. In order to generalize these theories to the noncompact case, we must assume some compactness property of the flow. This has been done in [1] and [7] in two different ways both leading to the generalized Morse inequalities. In classical critical point theory, the compactness assumption is replaced by the Palais-Smale condition:
(P-S) Let X be a Banach manifold and $f \in C^1(X)$. We say that f satisfies Palais-Smale condition if any sequence $\{x_n\}$ such that $f(x_n)$ is bounded and $\|Df(x_n)\| \longrightarrow 0$ possesses a convergent subsequence.

DEFINITION. *A Finsler structure on the tangent bundle of a Banach manifold X is a continuous function $\|.\| : T(X) \longrightarrow [0, +\infty)$ such that*

- *(a) For every $x \in X$, the restriction $\|.\|_x = \|.\||_{T_x(X)}$ is an equivalent norm on the tangent space $T_x(X)$,*
- *(b) For each $x_0 \in X$ and $k > 1$, there is a trivializing neighborhood U of x_0 in which*

$$\frac{1}{k}\|\cdot\|_x \leq \|\cdot\|_{x_0} \leq k\|\cdot\|_x.$$

A C^1-Banach manifold X together with a Finsler structure on its tangent bundle $T(X)$ is called a Finsler manifold.

Now suppose that X is a complete Finsler manifold and $f \in C^1(X)$. In [4], Palais proved that f admits a pseudo-gradient vector field i.e. a map $F : X \longrightarrow T(X)$ such that

- (i) The equation $\dot{x} = F(x)$ has a unique solution for every initial point $x_0 \in X$,
- (ii) $\langle Df(x), F(x)\rangle \geq \alpha(\|Df(x)\|)$ where α is a strictly increasing continous function with $\alpha(0) = 0$,

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(iii) $||F||$ is bounded.

Therefore we can consider the flow relative to a pseudo-gradient vector field for f. The Palais-Smale condition makes this flow to satisfy a compactness property which is essential in constructing variantional systems in [1].

In this paper we follow Benci's approach to the Conley index theory to obtain the following Ljusternik-Schnirelmann type result in critical point theory:

THEOREM 1.1. *Let X be a complete Finsler manifold, $f \in C^1(X, \mathbb{R})$ which satisfies Palais-Smale condition and φ^t be the flow relative to a pseudo-gradient vector field for f. If $I \subset X$ is a (c)-invariant set with $f(I)$ bounded, then f has at least $\nu_H(h(I)) - 1$ critical points in I.*

In the above theorem, $h(I)$ is the Conley index of the (c)-invariant set I which is defined in the next section and ν_H is the (Homotopy) Ljusternik-Schnirelmann category which is discussed in section 3. This theorem will be gradually proven through the examples of section 3.

2. Conley index. Let φ^t be a continuous flow on a metric space X. A subset $I \subset X$ is called an isolated invariant set if it is the maximal invariant set in a closed neighborhood of itself. Such a neighborhood is called an isolating neighborhood. In classical Conley index theory [2], the isolating neighborhood is assumed to be compact. We replace this kind of compactness by Benci's compactness property of the flow [1]. First of all, for every $T \in \mathbb{R}^+$ and $V \subset X$, we define

$$G^T(V) = \{x \in X | \varphi^{[-T,T]}(x) \in V\} = \bigcap_{-T \leq t \leq T} \varphi^t(V).$$

Thus the maximal invariant set in V is $I(V) = \bigcap_{T>0} G^T(V)$.

DEFINITION. We say φ^t satisfies property (c) in V if for every neighborhood U of $I(V)$ there exists $T > 0$ such that $G^T(V) \subset U$. An isolated invariant set $I \subset X$ is called (c)-invariant if it has an isolating neighborhood V in which φ^t satisfies property (c).

In order to define the index pair, we follow Robbin and Salamon [6]. Given a closed pair (N, L) in X, we define the induced semi-flow on N/L by $\varphi^t_\# : N/L \longrightarrow N/L$,

$$\varphi^t_\#(x) = \begin{cases} \varphi^t(x) & \text{if } \varphi^{[0,t]}(x) \subset N - L \\ [L] & \text{otherwise.} \end{cases}$$

NOTE. We shall not distinguish between $N - L$ and $N/L - \{[L]\}$. The following theorem is a generalization of Theorem 4.2 in [6].

THEOREM 2.1. *The induced semi-flow $\varphi_\# : N/L \times \mathbb{R}^+ \longrightarrow N/L$ is continuous if and only if:*

(i) L is positively invariant relative to N, i.e.

$$x \in L, \varphi^{[0,t]}(x) \subset N \Rightarrow \varphi^{[0,t]}(x) \subset L.$$

(ii) Every orbit which exits N goes through L first:

$$x \in N, \varphi^{[0,\infty)}(x) \not\subset N \Rightarrow \exists_{t \geq 0} \text{ with } \varphi^{[0,t]}(x) \subset N \text{ and } \varphi^t(x) \in L$$

or equivalently if $x \in N - L$*, then* $\exists_{t>0}$ *such that* $\varphi^{[0,t]}(x) \subset N - L$.

It is not hard to see that $\varphi_{\#}$ fails to be continuous if (i) or (ii) does not hold [6]. The reverse conclusion is a special case of the following theorem [8].

THEOREM 2.2. *Let* (N, L) *and* (N', L') *be two closed pairs in* X *which satisfy (i) and (ii) and suppose that* $G^T(N - L) \subset N' - L'$ *and* $G^T(N' - L') \subset N - L$ *for some* $T \geq 0$. *Then the map* $\psi : N/L \times [T, \infty) \longrightarrow N'/L'$ *defined by:*

$$\psi([x], t) = \begin{cases} \varphi^{3t}(x) & \textit{if } \varphi^{[0,2t]}(x) \subset N - L \textit{ and } \varphi^{[t,3t]}(x) \subset N' - L' \\ [L'] & \textit{otherwise} \end{cases}$$

is continuous and $\psi^t : N/L \longrightarrow N'/L'$ *is a homotopy equivalence for every* $t \geq T$.

Proof. We first consider the case $\psi(x, t) \neq [L']$, i.e. $\varphi^{[0,2t]}(x) \subset N - L$ and $\varphi^{[t,3t]}(x) \subset N' - L'$. By continuity, there exists $\theta \in \mathbb{R}^+$ such that $\varphi^{[0,2t+2\theta]}(x) \cap L = \varnothing$ and $\varphi^{[t,3t+3\theta]}(x) \cap L' = \varnothing$. Since $[0, 2t + 2\theta]$ and $[t, 3t + 3\theta]$ are compact, there is a neighborhood U of x in $N - L$ such that $\varphi^{[0,2t+2\theta]}(U) \cap L = \varnothing$ and $\varphi^{[t,3t+3\theta]}(U) \cap L' = \varnothing$. Now $U \subset N - L$ and L is the exit set in N, hence by (ii), $\varphi^{[0,2t+2\theta]}(U) \subset N - L$. It follows that $\varphi^{[T,t]}(U) \subset G^T(N - L) \subset N' - L'$. Now L' is the exit set of N', thus $\varphi^{[T,3t+3\theta]}(U) \subset N' - L'$. Therefore ψ^t is the same as φ^{3t} on $U \times [T, t + \theta]$ and hence ψ is continuous at (x, t).

Now suppose that $(x, t) \in (N - L) \times [T, \infty)$ such that $\psi(x, t) = [L']$ and ψ is not continuous at (x, t). Then there is an open set U in N'/L' with $[L'] \in U$ and a sequence $(x_i, t_i) \in (N - L) \times [T, \infty)$ such that $(x_i, t_i) \longrightarrow (x, t)$ and $\psi(x_i, t_i) \notin U$. It follows that $\psi(x_i, t_i) \neq [L']$ which means that $\varphi^{[0,2t_i]}(x_i) \subset N - L$ and $\varphi^{[t_i,3t_i]}(x) \subset N' - L'$. Since $(x_i, t_i) \longrightarrow (x, t)$, by continuity $\varphi^{[0,2t]}(x) \subset N, \varphi^{[t,3t]}(x) \subset N'$ and $\varphi^{3t}(x) \notin L'$. Thus $\varphi^{[t,3t]}(x) \subset N' - L'$ by (i) and hence $\varphi^{2t}(x) \in G^T(N' - L') \subset N - L$ and by (i) $\varphi^{[0,2t]}(x) \subset N - L$ which contradicts $\psi(x, t) = [L']$.

It remains to prove the continuity of ψ in $[L] \times [T, \infty)$. Let U be an open set in N'/L' with $[L'] \in U$ and $t \in [T, \infty)$. We shall find an open set V in N/L such that $[L] \in V$ and $\psi(V \times [T, t]) \subset U$. Observe that there is an open set U' in X such that $L' \subset U'$ and U is obtained by $U' \cap N'$. Now for $x \in L$ and $s \in [T, t]$, suppose that $\varphi^{[0,2s]}(x) \subset N$ and $\varphi^{[s,3s]}(x) \subset N'$, then we claim that $\varphi^{3s}(x) \in L'$. Otherwise $\varphi^{[s,3s]}(x) \subset N' - L'$ by (i), hence $\varphi^{2s}(x) \in G^T(N' - L') \subset N - L$ and again by (i), $\varphi^{[0,2s]}(x) \subset N - L$ which contradicts $x \in L$. Now we construct V as follows. If $x \in L$ and $\varphi^{[3T,3t]} \subset U'$, then by continuity, there is an open set V_x in X such that $x \in V_x$ and $\varphi^{[3T,3t]}(V_x) \subset U'$. If $x \in L$ and $\varphi^{[3T,3t]}(x) \not\subset U'$, there exists $s \in [T, t]$ such that $\varphi^{[3T,3s]}(x) \subset U'$ and either $\varphi^{[0,2s]}(x) \not\subset N - L$ or $\varphi^{[s,3s]}(x) \not\subset N' - L'$. Now by continuity, there exists an open set V_x such that $\varphi^{[3T,3s]}(V_x) \subset U'$ and for every $y \in V_x$, either $\varphi^{[0,2s]}(y) \not\subset N - L$ or $\varphi^{[s,3s]}(x) \not\subset N' - L'$. It is not hard to see that $\psi([y], s) \in U$ for every $x \in L, s \in [T, t]$ and $y \in N \cap V_x$. Now $V' = \bigcup_{x \in L} V_x$ defines an open set V in N/L such that $[L] \subset V$ and $\psi(V \times [T, t]) \subset U$. The continuity of ψ in $[L] \times [T, \infty)$ is easily concluded now.

As a result the maps $f : N/L \longrightarrow N'/L'$ defined by

$$f([x]) = \begin{cases} \varphi^{3t}(x) & \text{if } \varphi^{[0,2t]}(x) \subset N - L \text{ and } \varphi^{[t,3t]}(x) \subset N' - L' \\ [L'] & \text{otherwise,} \end{cases}$$

and $g : N'/L' \longrightarrow N/L$ defined by

$$g([x]) = \begin{cases} \varphi^{3t}(x) & \text{if } \varphi^{[0,2t]}(x) \subset N' - L' \text{ and } \varphi^{[t,3t]}(x) \subset N - L \\ [L] & \text{otherwise,} \end{cases}$$

are continuous for any $t \geq T$. The composition $g \circ f : N/L \longrightarrow N/L$ is given by

$$g \circ f([x]) = \begin{cases} \varphi^{6t}(x) & \text{if } \varphi^{[0,2t]}(x) \subset N-L, \varphi^{[t,5t]}(x) \subset N'-L' \text{ and } \varphi^{[4t,6t]}(x) \subset N-L \\ [L] & \text{otherwise.} \end{cases}$$

If $\varphi^{[t,5t]}(x) \subset N'-L'$, then $\varphi^{[2t,4t]}(x) \subset G^T(N'-L') \subset N-L$. Moreover if $\varphi^{[0,6t]}(x) \subset N-L$, then $\varphi^{[t,5t]}(x) \subset G^T(N-L) \subset N'-L'$. Therefore $g \circ f = \varphi_{\#}^{6t}$ which is homotopic to the identity through the continuous semi-flow $\varphi_{\#}$ on N/L. Similarly $f \circ g$ is homotopic to the identity and hence ψ^t is a homotopy equivalence for every $t \geq T$. □

DEFINITION. *An index pair for an isolated invariant set $I \subset X$ is a closed pair (N, L) such that $\overline{N-L}$ is an isolating neighborhood for I in which φ^t satisfies property (c) and the induced semi-flow $\varphi_{\#}^t$ on N/L is continuous.*

THEOREM 2.3. *Every (c)-invariant subset $I \subset X$ admits an index pair and if (N, L) and (N', L') are two index pairs for I, then there is a flow-defined homotopy equivalence between $(N/L, [L])$ and $(N'/L', [L'])$.*

Proof. Let V be an isolating neighborhood for I in which φ^t satisfies property (c). Then there is a $T \in \mathbb{R}^+$ such that $G^T(V) \subset \overset{\circ}{V}$. If we define

$$\Gamma^T(V) = \{x \in G^T(V) | \varphi^{[0,T]}(x) \cap \partial V \neq \varnothing\},$$

then $(G^T(V), \Gamma^T(V))$ is an index pair for I. (cf. [1])

Now suppose that (N, L) and (N', L') are two index pair for I. Since φ^t satisfies property (c) in $\overline{N-L}$ and $\overline{N'-L'}$, there is a $T \in \mathbb{R}^+$ such that $G^T(\overline{N-L}) \subset N'-L'$ and $G^T(\overline{N'-L'}) \subset N-L$. By Theorem 3.2, ψ^t is a homotopy equivalence between $(N/L, [L])$ and $(N'/L', [L'])$ for every $t \geq T$. □

DEFINITION. *Let I be a (c)-invariant subset of X and (N, L) be an index pair for I. Conley index of I is the homotopy type of $(N/L, [L])$. The Conley index of I is denoted by $h(I)$ and it is well-defined by the above theorem.*

In Ljusternik-Schnirelmann theory, it is important to know whether a point admits a contractible neighborhood or not. We want to prove that $[L]$ has a contractible neighborhood in N/L for every index pair (N, L).

LEMMA 2.4. *If (N, L) is an index pair, then there is a $T \in \mathbb{R}^+$ such that $\varphi^{[0,T]}(x) \not\subset \overline{N-L}$ for every $x \in L$.*

Proof. Suppose the contrary, then there are sequences $t_i \longrightarrow +\infty$ and $x_i \in L$ with $\varphi^{[0,t_i]}(x_i) \subset \overline{N-L}$. Since L is positively invariant relative to N, we obtain $\varphi^{[0,t_i]}(x_i) \subset L$. It follows that $\varphi^{t_i/2}(x_i) \in G^{t_i/2}(\overline{N-L})$ and $\varphi^{t_i/2} \in L$ which contradicts property (c) in $\overline{N-L}$. □

PROPOSITION 2.5. *If (N, L) is an index pair, then $[L]$ admits a closed contractible neighborhood in N/L.*

Proof. By the above lemma, there exists $T \in \mathbb{R}^+$ such that $\varphi^{[0,T]}(x) \not\subset \overline{N-L}$ for every $x \in L$. Thus $\varphi^t(x) \notin \overline{N-L}$ for some $t \in [0, T]$ and by continuity of φ^t there is an open set U_x such that $x \in U_x$ and $\varphi^t(U_x) \cap \overline{N-L} = \varnothing$. Now $U = \bigcup_{x \in L} U_x$ is an open set with $L \subset U$ and $\varphi^{[0,T]}(x) \not\subset \overline{N-L}$ for every $x \in U$. Thus $\varphi_{\#}^T([x]) = [L]$

for every $x \in U \cap N$ and hence $(\varphi_{\#}^{T})^{-1}([L])$ is a neighborhood of $[L]$, it is obviously closed and contractible to $[L]$ through $\varphi_{\#}^{t}$. □

LEMMA 2.6. *Let V be an isolating neighborhood for I in which φ^t satisfies property (c). Then for every neighborhood U of I, there exists $T \in \mathbb{R}^+$ such that for every $x \in V$, either $\varphi^{[0,2T]}(x) \not\subset V$ or $\varphi^T(x) \in U$.*

Proof. Suppose the contrary, then there exist a sequence $t_i \in \mathbb{R}^+$ with $t_i \longrightarrow \infty$ and a sequence $x_i \in V$ such that $\varphi^{t_i}(x_i) \notin U$ and $\varphi^{[0,2t_i]}(x_i) \subset V$. Thus $\varphi^{t_i}(x_i) \in G^{t_i}(V)$ which contradicts property (c). □

3. Category. Let X be a topological space. A category is a map $\nu : 2^X \longrightarrow Z \cup \{+\infty\}$ which satisfies the following axioms:

i) If $A \subset B$, then $\nu(A) \leq \nu(B)$.
ii) $\nu(A \cup B) \leq \nu(A) + \nu(B)$.
iii) For every subset $A \subset X$ there exists an open set $U \subset X$ such $A \subset U$ and $\nu(A) = \nu(U)$.
iv) If $f : X \longrightarrow X$ is a continuous map homotopic to the identity id_X, then $\nu(A) \leq \nu(f(A))$ for every subset $A \subset X$.

EXAMPLE. (Homotopy Ljusternik-Schnirelmann category) The HLS-category $\nu_H(A) = \nu_H(A; X)$ is defined to be the minimum number of open sets contractible in X required to cover A. If such a cover does not exist, we set $\nu_H(A) = +\infty$ and if it exists, A is called H-categorizable (in X). It is not hard to show that ν_H is a category and an invariant of homotpy type [3]. Moreover a subset $A \subset X$ is H-categorizable if and only if $\nu_H(\{x\}) = 1$ for every $x \in A$. The following useful lemma gives a generalization of axiom (iv) for the HLS-category.

LEMMA 3.1. *If Y dominates X, (i.e. there are continuous maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ with $g \circ f \sim id_X$), then for every H-categorizable subset $A \subset Y, f^{-1}(A)$ is H-categorizable and $\nu_H(f^{-1}(A)) \leq \nu_H(A)$. In particular if Y is H-categorizable, then so is X and $\nu_H(X) \leq \nu_H(Y)$.*

Proof. It is enough to prove that for every open set $U \subset Y$ contractible in Y, $f^{-1}(U)$ is contractible in X. Consider the following commutative diagram in which i and j are inclusion maps.

$$\begin{array}{ccc} X & \underset{g}{\overset{f}{\rightleftarrows}} & Y \\ {\scriptstyle i}\big\uparrow & & \big\uparrow{\scriptstyle j} \\ V = f^{-1}(U) & \xrightarrow{f|_V} & U \end{array}$$

$f \circ i = j \circ f|_V \Rightarrow g \circ f \circ i = g \circ j \circ f|_V \Rightarrow i \sim g \circ j \circ f\ |_V \sim$ constant. □

REMARK 3.2. Suppose that I is a (c)-invariant subset of X with two index pairs (N, L) and (N', L'). Then by Theorem 2.2, there is a flow-defined homotopy equivalence between N/L and N'/L' which leaves I invariant. Thus by the above lemma, $\nu_H(N/L)$ and $\nu_H(I; N/L)$ is independent of (N, L) and hence $\nu_H(h(I))$ and $\nu_H(I; h(I))$ make sense. Moreover if I is H-categorizable in N/L, then it is also H-categorizable in N'/L'.

THEOREM 3.3. *Let φ^t be a continuous flow on a metric space X and I be a (c)-invariant subset of X with an index pair (N, L). Then for every category ν on N/L, $\nu(N/L) \le \nu(I) + \nu([L])$. Moreover if $L = \varnothing$, then $\nu(N) = \nu(I)$.*

Proof. Choose an open set $U \subset N - L$ such that $I \subset U$ and $\nu(I) = \nu(U)$. By Lemma 2.6, there is a $T \in \mathbb{R}^+$ such that $N/L = (\varphi^T_\#)^{-1}(U) \cup (\varphi^{2T}_\#)^{-1}([L])$. Thus by axioms (ii) and (iv), $\nu(N/L) \le \nu((\varphi^T_\#)^{-1}(U)) + \nu((\varphi^{2T}_\#)^{-1}([L])) \le \nu(U) + \nu([L]) = \nu(I) + \nu([L])$. If $L = \varnothing$, then N is a positively invariant subset of X in which φ^t satisfies property (c). Thus $G^T(N) = \varphi^T(N) \subset U$ for some $T \in \mathbb{R}^+$ and hence $\nu(N) \le \nu(\varphi^T(N)) \le \nu(U) = \nu(I)$. since $I \subset N$, by axiom (i), $\nu(I) \le \nu(N)$ and hence $\nu(N) = \nu(I)$. □

COROLLARY 3.4. *Let I be a (c)-invariant subset of X. If I is H-categorizable in $h(I)$, then $h(I)$ is H-categorizabel and $\nu_H(h(I)) \le \nu_H(I; h(I)) + 1$.*

Proof. It is enough to observe that for every index pair (N, L) for I, $\nu_H([L]\,; N/L) = 1$ by Proposition 2.5. □

Now we introduce a class of metric space in which the Conley index of every (c)-invariant subset is H-categorizable.

DEFINITION. *A topological space X is called semi-locally contratible if for every $x \in X$ and open set $U \subset X$ with $x \in U$, there exists a neighborhood V of x such that $x \in V \subset U$ and V is contractible in U.*

COROLLARY 3.5. *If I is a (c)-invariant subset of a semi-locally contractible metric space X, then I is H-categorizable in $h(I)$ and hence $h(I)$ is H-categorizable.*

Proof. Let (N, L) be an index pair for I. Since X is semi-locally contractible, every $x \in I$ admits an open neighborhood contractible in $N - L$. Thus I is H-categorizable in N/L and by the above corollary, $h(I)$ is H-categorizable. □

In order to apply these results to critical point theory, we need the concept of Morse decomposition. When the space is compact, the Morse decomposition is defined by means of the ω-limit and α-limit sets [8]. This definition is not efficient in the noncompact case because the ω-limit and the α-limit sets may be empty. We also need something like Palais-Smale condition to obtain the desired results in critical point theory. Now we define the Morse-Palais-Smale decomposition as follows:

DEFINITION. *Let φ^t be a continuous flow on a metric space X and $I \subset X$ be an invariant set. A collection $\{M_i\}_{i=1}^n$ of invariant subsets of I is called an MPS-decomposition for I if for every set of neighborhoods $U_1, \cdots, U_n$ of $M_1, \cdots, M_n$ respectively, there exist $t_1, \cdots, t_n \in \mathbb{R}$ such that $I \subset \bigcup_{i=1}^n \varphi^{t_i}(U_i)$.*
When I is compact, every Morse decomposition for I is an MPS-decomposition [5]. *The following example shows the relation between this definition and Palais-Samle condition.*

EXAMPLE 1. Let X be a complete Finsler manifold and $f \in C^1(X, \mathbb{R})$ which satisfies Palais-Smale condition. Let φ^t be the flow relative to a pseudo-gradient vector field for f introduced in [4]. Suppose that f has a finite number of critical values $c_1 < c_2 < \cdots < c_n$ in $[a, b]$ and I is the maximal invariant set in $f^{-1}[a, b]$. If we set $M_i = \{x \in X | f(x) = c_i, f'(x) = 0\}$, then $\{M_i\}_{i=1}^n$ is an MPS-decomposition for I. (See the proof of Deformation Theorem [4]). This is also valid for every variational system [1].

EXAMPLE 2. Let I be an invariant set with an MPS-decomposition $\{M_i\}_{i=1}^n$. If S is a closed invariant subset of I, then $\{M_i \cap S\}_{i=1}^n$ is an MPS-decomposition for S. Moreover if $\{M_{i_1}, \cdots, M_{i_k}\}$ is an MPS decomposition for M_i, then $\{M_1, \cdots, M_{i-1}, M_{i_1}, \cdots, M_{i_k}, M_{i+1}, \cdots, M_n\}$ is an MPS-decomposition for I. In the case of the above example, each M_i may be decomposed into its connected components. In particular, if $f^{-1}[a,b]$ contains a finite number of critical points $x_1, \cdots, x_n$ then $\{x_i\}_{i=1}^n$ is an MPS-decomposition for I. Thus for every closed invariant subset $S \subset I$, the set of all critical points of f in S gives an MPS-decomposition for S.

THEOREM 3.6. *Let I be an invariant subset of X with an MPS-decomposition $\{M_i\}_{i=1}^n$. Then fort every category ν on X, $\nu(I) \leq \sum_{i=1}^n \nu(M_i)$.*

Proof. We choose a neighborhood U_i of M_i with $\nu(M_i) = \nu(U_i)$. Now there are $t_1, \cdots, t_n \in \mathbb{R}$ such that $I \subset \bigcup_{i=1}^n \varphi^{t_i}(U_i)$ and hence $\nu(I) \leq \sum_{i=1}^n \nu(\varphi^{t_i}(U_i)) \leq \sum_{i=1}^n \nu(U_i) = \sum_{i=1}^n \nu(M_i)$. □

EXAMPLE. Consider the flow relative to a pseudo-gradient vector field for a function $f \in C^1(X, \mathbb{R})$ which satisfies Palais-Smale condition. If I is a closed invariant subset of X such that $f(I)$ is bounded, then f has at least $\nu_H(I)$ critical points in I. To see this, we may assume that f has a finite number of critical points $x_1, \cdots, x_n$ in I and then $\{x_i\}_{i=1}^n$ gives an MPS-decomposition for I. Now by the above theorem $\nu_H(I) \leq \sum_{i=1}^n \nu_H(x_i)$ and in a Banach manifold, every point admits a contractible neighborhood. Thus $\nu_H(x_i) = 1$ and $\nu_H(I) \leq n$ which means that f has at least $\nu_H(I)$ critical points in I.

THEOREM 3.7. *Let I be a (c)-invariant set with an MPS-decomposition $\{M_i\}_{i=1}^n$ and an index pair (N, L). Then for every category ν on N/L, $\nu(N/L) \leq \nu([L]) + \sum_{i=1}^n \nu(M_i)$.*

Proof. Since $\nu(N/L) \leq \nu([L]) + \nu(I)$ by Theorem 3.3, it is enough to show that $\nu(I) \leq \sum_{i=1}^n \nu(M_i)$. We choose neighborhoods $U_1, \cdots, U_n$ of $M_1, \cdots, M_n$ with $\nu(M_i) = \nu(U_i)$ respectively. Now there are $t_1, \cdots, t_n \in \mathbb{R}$ such that $I \subset \bigcup_{i=1}^n \varphi^{t_i}(V_i)$. Thus $I = \varphi^T(I) \subset \bigcup_{i=1}^n \varphi^{(t_i+T)}(U_i)$ and we may assume that $t_i \leq 0$. Now $I = \bigcup_{i=1}^n (\varphi^{-t_i})^{-1}(U_i \cap I) \subset \bigcup_{i=1}^n (\varphi_\#^{-t_i})^{-1}(U_i)$ and hence $\nu(I) \leq \sum_{i=1}^n \nu((\varphi_\#^{-t_i})^{-1}(U_i)) \leq \sum_{i=1}^n \nu(U_i) = \sum_{i=1}^n \nu(M_i)$. □

COROLLARY 3.8. *Let I be a (c)-invariant set with an MPS-decomposition $\{M_i\}_{i=1}^n$. Then $\nu_H(h(I)) \leq 1 + \sum_{i=1}^n \nu_H(M_i; h(I))$.*

Proof. Notice that $\nu_H(M_i, h(I))$ makes sense since M_i is invariant. (See Remark 3.2). For every index pair $(N, L), \nu_H(N/L) \leq \nu_H([L]; N/L) + \sum_{i=1}^n \nu_H(M_i; N/L)$ and $\nu_H([L]; N/L) = 1$, thus $\nu_H(h(I)) \leq 1 + \sum_{i=1}^n \nu_H(M_i; h(I))$. □

Now consider the last example and suppose that I is a (c)-invariant subset of X such that $f(I)$ is bounded, Since every Banach manifold is locally contractible, $h(I)$ is H-categorizable and hence $\nu_H(x, h(I)) = 1$ for every $x \in I$. If f has a finite number of critical points $x_1, \cdots, x_n$ in I, then $\nu_H(h(I)) \leq 1 + \Sigma \nu_H(x_i, h(I)) = 1 + n$ and it follows that f has at least $\nu_H(h(I)) - 1$ critical points in I. This finishes the proof of Theorem 1.1 which is also valid for every variational system [1].

ACKNOWLEDGMENTS. The author would like to thank Professor Hesaaraki who encouraged me to read Benci's approach to Conley index theory. This research was supported in part by Institute for studies in theoretical Physics and Mathematics, IPM.

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