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PERIMETER-MINIMIZING CURVES AND SURFACES IN $\mathbf{R}^n$ ENCLOSING PRESCRIBED MULTI-VOLUME*

FRANK MORGAN†

Abstract. Planar curves minimizing length for given area are classically characterized as circular arcs. We give a new generalization to $\mathbf{R}^n$ of such area constraints and characterize the minimizing curves. We also consider surfaces satisfying new generalized volume constraints.

1. Introduction. A smooth oriented hypersurface S in $\mathbf{R}^n$ which minimizes area for given boundary and enclosed volume has constant mean curvature. (Given volume means that for any competitor S', $S' - S$ encloses net algebraic volume 0. Alternatively, $\int_S x_1 dx_2 \wedge \cdots \wedge dx_n$ is prescribed.) Conversely, any small portion of a smooth, constant-mean-curvature hypersurface minimizes area for given boundary and volumes [M4, Rmk. p. 76].

This paper considers m-dimensional surfaces S in $\mathbf{R}^n$ for arbitrary $m < n$ which minimize area for given boundary and "multi-volume": volumes enclosed by orthogonal projections onto all $(m+1)$-planes or just onto axis $(m+1)$-planes, which can be expressed as a multivector in the exterior algebra $\Lambda_{m+1}\mathbf{R}^n$. Of course the volumes of projections onto axis planes may be arbitrarily adjusted by adding spheres in those planes. Theorems 2.1 and 2.3 show that a minimizer exists among the generalized surfaces (rectifiable currents) of geometric measure theory (cf. [M4]) and that it is a real-analytic submanifold on an open dense set.

For the case $m = 1$, Theorem 3.1 classifies all such minimizing curves as simultaneous tracings of circles and straight lines in copies of $\mathbf{R}^2$. Corollary 3.2 infers the general solution to the problem of finding the shortest closed curve of prescribed multi-area. Here Caratheodory's celebrated moment curve ([Ca], [Ga]) makes a surprising appearance in the calculus of variations.

For the case $m = 2$, Proposition 4.2 provides analogous examples but no classification theorem. For $m \geq 3$, the only known examples are unions of constant-mean-curvature hypersurfaces lying in largely orthogonal subspaces (Proposition 4.3).

1.1. The standard thread problem. In contrast to our approach, the standard generalization from $\mathbf{R}^2$ to $\mathbf{R}^n$ of length-minimizing curves C for fixed area and boundary requires a fixed reference curve (or "wire") C_0 with the same boundary and minimizes the length of C given the area of the area-minimizing surface R bounded by $C - C_0$. Actually, in the usual, more physical version of the problem, one fixes the length of C (the "thread") and minimizes the area of R (perhaps a soap film). This problem also generalizes from curves to m-dimensional surfaces. Minimizers generally have constant mean curvature. See [E], [DHKW, Chapt. 10], [N].

1.2. The variational condition. The variational condition 2.2 on our new m-dimensional surfaces S in $\mathbf{R}^n$ minimizing area for given multi-volume and boundary provides an $(m+1)$-vector $\xi \in \Lambda_{m+1}\mathbf{R}^n$ such that the mean curvature vector H satisfies

$$\mathrm{H} = \xi \llcorner \vec{S},$$

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†Department of Mathematics, Williams College, Williamstown, MA 01267, USA (Frank.Morgan@williams.edu).

where $\vec{S}$ is the unit m-plane tangent to S. For hypersurfaces, this condition reduces to constant scalar mean curvature. Already in $\mathbf{R}^4$, it would be interesting to find any nontrivial surfaces satisfying $H = e_1 \wedge e_2 \wedge e_3 \lfloor \vec{S}$ (see §4.2).

1.3. The proof that minimizing curves are simultaneous tracings of circles and straight lines in copies of $\mathbf{R}^2$. An argument using projections shows that all such curves are length minimizing. Conversely, the unit tangent T and curvature vector κ of a curve C minimizing length for given area constraints satisfy the variational condition $\kappa = \xi \lfloor T$. By standard linear algebra, we may assume that ξ is of the form

$$\xi = \omega_1 e_1 \wedge e_2 + \omega_2 e_3 \wedge e_4 + \cdots + \omega_k e_{2k-1} \wedge e_{2k}.$$

Theorem 3.1 deduces that C must be simultaneous tracings of circular arcs in the corresponding 2-planes and possibly a straight line.

1.4. Prescribed mean curvature vector. Earlier work of R. Gulliver and F. Duzaar and M. Fuchs (especially [DF2, Thm. 3.2]), in seeking surfaces with "prescribed mean curvature vector" as a function of tangent plane, considers a similar problem with just a single volume constraint. Then in the resulting variational condition $H = \xi \lfloor \vec{S}$, ξ comes from the constraint. Gulliver [Gu1, p. 118] gives one interpretation of our helical minimizer as the path of "a charged particle moving in a magnetic field." Gulliver and Duzaar and Fuchs also consider variable curvature associated with a generalized volume constraint.

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2. Existence and regularity. Theorems 2.1 and 2.3 show that our minimizers exist and are real-analytic submanifolds on an open dense set. The development includes the variational curvature condition 2.2.

THEOREM 2.1. *There exists an area-minimizing m-dimensional rectifiable current S in $\mathbf{R}^n$ with given boundary and multi-volume. (The only hypothesis on ∂S is that it bound some rectifiable current.)*

Proof. There is some rectifiable current satisfying the prescriptions because volumes may be arbitrarily adjusted by adding spheres in axis $(m+1)$-planes. The only problem in applying the Compactness Theorem [M4, Thm. 5.5] to obtain a minimizer is that a minimizing sequence may not stay bounded. This problem may be solved by judicious truncation and the restoration of the prescribed volume conditions by adding spheres as in [DF2, Thm. 3.2] or by repeating the process with translations of the discarded material as in [M1, §4] (summarized in [M4, §§13.3-13.7]). The proof of [M1] generalizes to clusters. □

The following result gives the variation curvature condition for a minimizing surface S: the mean curvature H and unit tangent plane $\vec{S}$ satisfy $H = \xi \lfloor \vec{S}$ for some fixed $(m+1)$-vector ξ. (By definition, $\xi \lfloor \vec{S}$ is characterized by

$$\left(\xi \lfloor \vec{S}\right) \bullet v = \xi \bullet \left(\vec{S} \wedge v\right)$$

for all vectors v. For example, $(e_1 \wedge e_2 + e_3 \wedge e_4) \lfloor e_1 = e_2$. Because there is a fixed inner product, we need not distinguish between vectors and covectors.)

The derivation of the Lagrange multiplier condition 2.2(2) or the equivalent curvature condition 2.2(3) is complicated by the possible degeneracy of the constraints. Our proof is simpler than the proof of a similar result in [DF2, Thm. 4.2], which does not apply to curves ($m = 1$). We avoid degenerate constraints and higher order correction terms by considering only especially simple variations.

THEOREM 2.2 (Variational curvature condition). *Let S be an m-dimensional oriented surface (rectifiable current) in $\mathbf{R}^n$. Let $\{e_I\}$ be a set of oriented axis $(m+1)$-planes in $\mathbf{R}^n$ (an orthonormal basis for $\Lambda_{m+1}\mathbf{R}^n$). Then the following conditions are equivalent:*

(1) S is stationary for area for prescribed multi-volume.

(2) For some λ_I, S is stationary for $A + \Sigma\lambda_I V_I$, where A denotes area and V_I denotes projected volume in e_I.

(3) For some $\xi \in \Lambda_{m+1}\mathbf{R}^n$, the mean curvature of S weakly satisfies

$$H = \xi \lfloor \vec{S},$$

where $\vec{S}(x)$ *is the unit tangent plane to S at x.*

The natural correspondence between (2) and (3) is given by $\xi = \Sigma\lambda_I e_I$. We may take ξ in span$\left\{\vec{S}(x) \wedge v : v \in \mathbf{R}^n\right\}$.

Proof. To see the equivalence of (2) and (3) with $\xi = \Sigma\lambda_I e_I$, note that for any smooth variation v, the first variation $\delta^1(V_I) = \int \left(e_I \lfloor \vec{S}\right) \bullet v$. Hence

$$\delta^1(A + \Sigma\lambda_I V_I) = \delta^1(A) + \Sigma\lambda_I \int \left(e_I \lfloor \vec{S}\right) \bullet v = \delta^1(A) + \int \left(\xi \lfloor \vec{S}\right) \bullet v,$$

which vanishes for all v if and only if $H = \xi \lfloor \vec{S}$. Note that $\xi \lfloor \vec{S}$ depends only on the projection of ξ onto span$\left\{\vec{S}(x) \wedge v : v \in \mathbf{R}^n\right\}$.

Since condition (2) immediately implies (1), it remains to be shown that (1) implies (3). We consider variations of the form $S_t = S + tv$ with v of the special form $v = \varphi \cdot v_0$ for some scalar function φ vanishing at the boundary and constant vector v_0. (A variation of this simple form respects the volume constraints if and only if $\int_S \vec{S} \wedge v = 0$.)

Suppose that for all $\xi \in \Lambda_{m+1}\mathbf{R}^n$, $\int_S \left(\xi \lfloor \vec{S}\right) \bullet v = 0$. Then $\int_S \vec{S} \wedge v = 0$, the variation respects the volume constraints, the first variation must vanish, and $\int_S H \cdot v = 0$. It follows that as a linear functional, H lies in the vectorspace $\left\{\xi \lfloor \vec{S} : \xi \in \Lambda_{m+1}\mathbf{R}^n\right\}$; i.e., for some $\xi \in \Lambda_{m+1}\mathbf{R}^n$,

$$\int_S H \bullet v = \int_S \left(\xi \lfloor \vec{S}\right) \bullet v$$

for all v. It follows that $H = \xi \lfloor \vec{S}$, as desired. □

THEOREM 2.3 (Regularity Theorem). *Let S be an m-dimensional rectifiable current in $\mathbf{R}^n$ stationary for fixed boundary and multi-volume. Then S is a real-analytic submanifold on an open dense set away from its boundary. If $m = 1$ and S is* minimizing, *then S is a real-analytic embedded curve (possibly with multiplicity).*

Proof. By 2.2(3), S has weakly bounded mean curvature. By Allard's regularity theorem [A, §8], on an open dense set S is a $C^{1,\alpha}$ submanifold. By Morrey's higher regularity for C^1 weak solutions to elliptic variational problems [My3, Thm. 9.2, p. 158], S is $C^{2,\alpha}$ (actually C^∞), and hence real-analytic by [My1] or [My2, Thm. 6.7.6, p. 271] on that open dense set.

If $m = 1$, weakly bounded curvature already implies that S is a $C^{1,1}$ immersed curve, hence real-analytic by 2.2(3) and the theory of differential equations. If two strands are tangent at a point, they coincide. Suppose two strands intersect transversally. Then reattaching them to each other contradicts bounded curvature. □

REMARK. Minimizers for fixed multi-volume may well enjoy the same regularity as the subclass of minimizers (without volume constraints); cf. [M4, Chapt. 8], [DF1, §5]. Such regularity is not known even for minimizers of $A + \Sigma\lambda_I V_I$ (cf. [DS, Intro. and 5.5(iii)]). For minimizers of fixed multi-volume, it is not known even whether a tangent cone is minimizing, because the cost of small volume adjustments is not known to be linear. (Note e.g. the extra hypothesis required in [DF2, Thm. 5.1].)

3. Length-minimizing curves for given multi-area. Theorem 3.1 gives a complete characterization of our minimizing curves. Corollary 3.2 gives the general solution to the problem of finding the shortest closed curve with prescribed multi-area.

3.1. Linear algebra. Standard linear algebra shows that for any 2-vector $\xi \in \Lambda_2\mathbf{R}^n$ there is an orthonormal basis such that

$$\xi = \omega_1 e_1 \wedge e_2 + \omega_2 e_3 \wedge e_4 + \cdots + \omega_k e_{2k-1} \wedge e_{2k} \tag{3.1}$$

with $0 < \omega_1 \leq \omega_2 \leq \cdots \leq \omega_k$. (This is essentially just normal form for skew-symmetric matrices.) Consider the complex structure $ie_{2j-1} = e_{2j}$. If the ω_j are distinct, then the choice of basis is unique up to rotation in the complex lines $e_{2j-1} \wedge e_{2j}$. More generally, if for example $\omega_1 = \omega_2 = \cdots = \omega_p$, then $e_1, e_3, \ldots, e_{2p-1}$ may be any unitary basis for their complex span (fixed complex structure).

A general multi-area prescription of area A_{ij} in the axis plane $e_i \wedge e_j$ may be represented by the 2-vector $\Sigma A_{ij} e_i \wedge e_j$ and hence for appropriate basis takes the simpler form (3.1) as described above.

THEOREM 3.1 (Characterization of minimizing curves). *Let C be a simple smooth curve in $\mathbf{R}^n$ with unit tangent vector T and curvature vector κ satisfying the variational curvature condition $\kappa = \xi\lfloor T$ for some 2-vector $\xi \in \Lambda_2\mathbf{R}^n$ (2.2(3)). Choose an orthonormal basis $e_1, e_2 = ie_1, \ldots, e_{2k-1}, e_{2k} = ie_{2k-1}, \ldots, e_n$ for $\mathbf{R}^n$ so that such ξ takes the form (3.1)*

$$\xi = \omega_1 e_1 \wedge e_2 + \omega_2 e_3 \wedge e_4 + \cdots + \omega_k e_{2k-1} \wedge e_{2k} \tag{3.2}$$

with $0 < \omega_1 \leq \omega_2 \leq \cdots \leq \omega_k$. Then C has an arclength parameterization of the form

$$C(s) = a_0 + a_1 e^{i\omega_1 s} e_1 + \cdots + a_k e^{i\omega_k s} e_{2k-1} + a_{k+1} s \tag{3.3}$$

with $a_0 \in \mathbf{R}^n$; $a_1, \ldots, a_k \in \mathbf{R}$; $a_{k+1} \in \operatorname{span}\{e_{2k+1}, \ldots, e_n\}$. The 2-vector ξ may be chosen so that $a_j \neq 0$ for $1 \leq j \leq k$ and $0 < \omega_1 < \omega_2 < \cdots < \omega_k$.

Conversely, every such C in which each $\omega_j \Delta s$ is at most 2π radians is up to translation uniquely minimizing for given multi-area.

Proof. Consider an arclength parameterization

$$C(s) = (C_1(s), C_2(s), \ldots, C_k(s), C_{k+1}(s))$$

with $C_j(s)$ in the span of $\{e_{2j-1}, e_{2j}\}$, except that $C_{k+1}(s)$ lies in the span of $\{e_{2k+1}, \dots, e_n\}$. Then

$$T(s) = \left(C_1'(s), C_2'(s), \dots, C_k'(s), C_{k+1}'(s)\right),$$

$$\kappa(s) = \left(C_1''(s), C_2''(s), \dots, C_k''(s), C_{k+1}''(s)\right).$$

Since $\kappa = \xi \lfloor T$, $C_1''(s)$ is perpendicular to $C_1'(s)$ and hence $C_1(s)$ is parametrized by a multiple of arclength. Therefore its curvature is a multiple of $C_1''(s)$ and has constant magnitude, so $C_1(s)$ is a circle. The same argument holds for $C_2, \dots, C_k$. Finally $C_{k+1}''(s) = 0$ and C_{k+1} is constant or a straight line parametrized by a constant multiple of arclength, as desired. The condition $\kappa = \xi \lfloor T$ implies that the ω_j occurring in C equal the coefficients of ξ.

ξ may be chosen to minimize k. If say $a_k = 0$, then $\kappa = \xi \lfloor T = \xi' \lfloor T$, where $\xi' = \xi - \omega_k e_{2k-1} \wedge e_{2k}$, and k can be reduced. If say $\omega_1 = \omega_2$, then for the new orthonormal basis with

$$e_1' = \frac{a_1 e_1 + a_2 e_3}{\sqrt{a_1^2 + a_2^2}}, e_2' = i e_1', e_3' = \frac{-a_2 e_1 + a_1 e_3}{\sqrt{a_1^2 + a_2^2}}, e_4' = i e_3',$$

the expression for ξ still begins $\omega_1 e_1 \wedge e_2 + \omega_2 e_3 \wedge e_4$, but the expression for $C(s)$ begins $a_0 + \sqrt{a_1^2 + a_2^2} e^{i\omega_1 s} e_1' + 0 e^{i\omega_1 s} e_2'$, and then k can be reduced as before, the desired contradiction.

To prove the converse, first note that the length of any curve and the lengths L_i of its projections C_i satisfy

$$\text{length}\, C = \int \sqrt{dL_1^2 + \cdots + dL_{k+1}^2} \geq \sqrt{L_1^2 + \cdots + L_{k+1}^2} \tag{3.4}$$

because of the algebraic inequality

$$\sqrt{A_1^2 + B_1^2} + \sqrt{A_2^2 + B_2^2} \geq \sqrt{(A_1 + A_2)^2 + (B_1 + B_2)^2},$$

with equality if and only if the projections have proportional parameterizations. Now let C be as in (3.3) and let $\tilde{C}$ be any minimizer. The projection lengths $\tilde{L}_i \geq L_i$, because planar arcs of at most 2π radians are uniquely minimizing for prescribed boundary and area (up to translation for the case of a full circle without boundary points). Now by (3.4), $\text{length}\, \tilde{C} \geq \text{length}\, C$, and equality holds if and only if the projections agree (possibly up to translation) and have proportional parameterizations, i.e., $\tilde{C}$ is a translation of C. □

COROLLARY 3.2. *For prescribed multi-area in $\mathbf{R}^n$, the length-minimizing curves without boundary are as follows. For some orthonormal basis $e_1, e_2 = ie_1, \dots, e_{2k-1}, e_{2k} = ie_{2k-1}, e_{2k+1}, \dots, e_n$ for $\mathbf{R}^n$, the prescription becomes area A_j in the $e_{2j-1} \wedge e_{2j}$ plane ($1 \leq j \leq k$) and 0 in the other axis planes, with $A_1 \geq \cdots \geq A_k > 0$. If the A_j are distinct, then the basis is unique up to rotations in each span$\{e_{2j-1}, e_{2j}\}$. More generally, if for example $A_1 = \cdots = A_p$, then $e_1, e_3, \dots, e_{2p-1}$ may be any unitary basis for their complex span (fixed complex structure). Corresponding to each such choice of basis there is a shortest curve*

$$C(s) = a_0 + a_1 e^{is} e_1 + a_2 e^{2is} e_3 + \cdots + a_k e^{kis} e_{2k-1}$$

parametrized by a multiple of arclength $s \in [0, 2\pi]$ *with* $j\pi a_j^2 = A_j$ $(1 \leq j \leq k)$.

REMARK. Thus Caratheodory's famous "moment curve" ([Ca], [Ga]) makes an appearance in the calculus of variations.

Proof. By Theorem 2.3, a minimizer is a real-analytic embedded curve. Since it has no boundary, it must have just one component, because translating two components to cross transversally would contradict regularity.

By Theorem 3.1, every candidate must be of the form

$$C(s) = a_0 + a_1 e^{i\omega_1 s} e_1 + \cdots + a_k e^{i\omega_k s} e_{2k-1},$$

$0 \leq s \leq 2\pi$, $0 < \omega_1 < \omega_2 < \cdots < \omega_k$. Moreover, for C to be a closed curve, each ω_j must be an integer. The projected areas are $\omega_j \pi a_j^2$ in the $e_{2j-1} \wedge e_{2j}$ plane and 0 in the other axis planes (because the ω_j are distinct). The length L satisfies

$$\frac{L^2}{4\pi} = \omega_1^2 \pi a_1^2 + \cdots + \omega_k^2 \pi a_k^2 = \omega_1 A_1 + \cdots + \omega_k A_k,$$

which is uniquely minimized when $\omega_1 = 1, \ldots \omega_k = k$ and $\omega_j \pi a_j^2 = A_j$ (i.e., the larger projected areas correspond to the smaller coefficients). □

NONUNIQUENESS. In addition to the (sometimes nonunique) minimizers of Corollary 3.2, there are infinitely many other curves stationary for length, obtained by other integral choices of the ω_j. Such curves are locally minimizing by Theorem 3.1. Furthermore, there is the union of disjoint circles in the $e_{2j-1} \wedge e_{2j}$; when $A_1 = \cdots = A_k$, it has the permutation symmetry so conspicuously lacking for the minimizer.

In $\mathbf{R}^4$ with $A_1 = A_2 = 1$, the problem is U_2 invariant and there is a three-dimensional family of distinct minimizers. No curve could possibly be U_2 invariant because the orbits of U_2 are all three dimensional, except for the origin.

4. Area-minimizing surfaces in $\mathbf{R}^n$ for given multi-volume. Proposition 4.2 gives examples of our area-minimizing 2-dimensional surfaces in $\mathbf{R}^n$, based on the linear algebra Lemma 4.1. Proposition 4.3 gives more general but less interesting examples of m-dimensional minimizers in $\mathbf{R}^n$. Section 4.1 considers some particular examples without boundary. Section 4.2 discusses the problem of characterizing minimizers. Theorem 4.4 applies the theory of calibrations to prove surfaces minimizing.

LEMMA 4.1. *Consider linear maps* $L_1, \ldots, L_k : \mathbf{R}^2 \to \mathbf{R}^n$, $L = \oplus L_i : \mathbf{R}^2 \to \mathbf{R}^{kn}$. *Then the Jacobians satisfy*

$$J_2 L \geq \sum J_2 L_i,$$

with equality if and only if for some i_0, *orthogonal maps* A_i, *and* $\lambda_i \geq 0$,

$$L_i = \lambda_i A_i L_{i_0}. \tag{4.1}$$

Proof. Choose an orthonormal basis e_1, e_2 for $\mathbf{R}^2$ such that $J_2 L = |L(e_1)|\,|L(e_2)|$. Then

$$\begin{aligned} J_2 L = |L(e_1)|\,|L(e_2)| &= \left(\sum |L_i(e_1)|^2\right)^{\frac{1}{2}} \left(\sum |L_i(e_2)|^2\right)^{\frac{1}{2}} \\ &\geq \sum |L_i(e_1)|\,|L_i(e_2)| \geq \sum J_2 L_i, \end{aligned}$$

with equality if and only if each $L_i(e_1) \perp L_i(e_2)$ and the vectors $(|L_i(e_i)|, |L_i(e_2)|)$ are all multiples of one $(|L_{i_0}(e_1)|, |L_{i_0}(e_2)|)$. □

REMARK. The sharp inequality of Lemma 4.1 generalizes to higher dimensions, but equality holds only if all but one L_i is zero. This corresponds to the fact that for a closed three-dimensional surface S in $\mathbf{R}^4$, for example, the diagonal $(id \times id)(S)$ has greater area than $S \times \{0\} \cup \{0\} \times S$ in $\mathbf{R}^8 = \operatorname{span}\{e_1, \ldots, e_8\}$. This is not a higher-dimensional counterexample to Proposition 4.2, because volume in $\operatorname{span}\{e_1, e_2, e_3, e_8\}$ for example changes.

The following Proposition 4.2 constructs, for example from pairs of isometric, two-dimensional, constant-mean-curvature surfaces, immersed surfaces $f(M)$ in $\mathbf{R}^n$ which minimize area for given boundary and multi-volume.

PROPOSITION 4.2. *Let M be a smooth 2-dimensional Riemannian surface (in general with boundary). Consider isometric immersions $f_1, \ldots, f_k : M \to \mathbf{R}^n$ which minimize area for given boundary and multi-volume (as does any small piece of a constant-mean-curvature surface in $\mathbf{R}^3$), for example from a 1-parameter family of associated constant-mean-curvature surfaces in $\mathbf{R}^3$ of Bonnet (see [Ch], [Ls, Thm. 8]). Then $f(M) = (\lambda_i f_i(M)) \subset \mathbf{R}^{kn}$ minimizes area for given boundary and multi-volume.*

REMARK. If the $f_i(M)$ are minimal surfaces, then $f(M)$ is a minimal surface, and a small portion is known to minimize area for given boundary, even without any volume constraint ([F]; see [LM, §2]).

Proof. Let $\mathbf{R}^n_i$ denote the i^{th} copy of $\mathbf{R}^n$. For any competing surface S, the projection S_i of S into $\mathbf{R}^n_i$ satisfies $\operatorname{area} S_i \geq \operatorname{area} f_i(M)$. By Lemma 4.1,

$$\operatorname{area} S \geq \sum \operatorname{area} S_i \geq \sum \operatorname{area} f_i(M) = \operatorname{area} f(M).$$

For the parenthetical minimizing property of constant-mean-curvature hypersurfaces, see [M4, Rmk. p. 76]. □

The following theorem is an immediate generalization of [M2, Thm. 5]. Consequential examples of surfaces minimizing area for given multi-volume include (a) the union of two orthogonal 2-spheres and (b) the union of two 3-spheres lying in $\operatorname{span}\{e_1, e_2, e_3, e_4\}$ and $\operatorname{span}\{e_1, e_5, e_6, e_7\}$.

THEOREM 4.3 (cf. [M2, Thm. 5]). *For integers $2 \leq m \leq n$ let S_1, S_2 be m-dimensional rectifiable currents minimizing area for given multi-volume. Suppose they lie in subspaces P_1, P_2 of $\mathbf{R}^n$ with*

$$\dim\left(P_1 \cap P_2^{\perp}\right) \geq \dim P_1 - m + 2. \tag{4.2}$$

Then $S_1 + S_2$ minimizes area for given multi-volume. Suppose further that

$$\dim(P_1 \cap P_2) < m - 2 \tag{4.3}$$

and S' minimizes area with the same boundary and multi-volume as $S_1 + S_2$. Then $S' = S_1' + S_2'$ with S_i' in P_i minimizing area with the same boundary and multi-volume as S_i. (If S_i has no boundary, S_i' may lie in a translation of P_i.)

4.1. Examples without boundary. What is an area-minimizing closed two-dimensional surface enclosing multi-volume

$$V_1 = e_1 \wedge e_2 \wedge e_3 + e_1 \wedge e_2 \wedge e_4,$$

or

$$V_2 = e_1 \wedge e_2 \wedge e_5 + e_3 \wedge e_4 \wedge e_5,$$

or

$$V_3 = e_1 \wedge e_2 \wedge e_3 + e_4 \wedge e_5 \wedge e_6?$$

For the first problem, since $V_1 = e_1 \wedge e_2 \wedge (e_3 + e_4)$, the answer is simply a round sphere in the $e_1 \wedge e_2 \wedge (e_3 + e_4)$ plane with area $2\sqrt[3]{9\pi} \approx 6.09$. For the third problem an answer is two round spheres by Theorem 4.3, with area $2\sqrt[3]{36\pi} \approx 9.67$. For the most interesting, second problem, experiments by K. Brakke on his Surface Evolver [B] indicate a solution in the form of a "generalized sphere":

$$\left(f(z)\, e^{it}, g(z)\, e^{2it}, z\right) \subset \mathbf{C} \times \mathbf{C} \times \mathbf{R} \cong \mathbf{R}^5,$$

with area of about 8.78601821587. By Corollary 3.2, the slices $z = z_0$ are length minimizing precisely when $A_1 = \pi f(z_0)^2 \leq A_2 = 2\pi g(z_0)^2$. Since $\int A_1\, dz = \int A_2\, dz = 1$, the inequality holds for some but not all z_0.

4.2. Characterizing minimizers. It would be interesting to generalize Theorem 3.1 and characterize smooth m-dimensional surfaces S in $\mathbf{R}^n$ with nonzero mean curvature vector H satisfying

$$H = \xi \llcorner \vec{S}, \tag{4.4}$$

for some fixed $(m+1)$-vector ξ in $\Lambda_{m+1}\mathbf{R}^n$. The few known examples have constant scalar mean curvature $|H|$. For the trivial case $m = n - 1$ of hypersurfaces, (4.4) just says S has constant scalar mean curvature. For the case $m = n - 2$, one may assume $\xi = e_1 \wedge \cdots \wedge e_{n-1}$. For the case of $m = 2$, $n = 5$, one may assume $\xi = e_1 \wedge (e_2 \wedge e_3 + a e_4 \wedge e_5)$. For the case of $m = 2$, $n = 6$, one may assume

$$\xi = e_1 \wedge e_2 \wedge e_3 + a_1 e_1 \wedge e_5 \wedge e_6 + a_2 e_4 \wedge e_2 \wedge e_6 + a_3 e_4 \wedge e_5 \wedge e_3 + a_4 e_4 \wedge e_5 \wedge e_6$$

[M3, Thm. 4.1]. The first open case is $m = 2$, $n = 4$, with $H = e_1 \wedge e_2 \wedge e_3 \llcorner \vec{S}$, i.e., mean curvature perpendicular to e_4 and of magnitude $\sin\theta$, where θ is the angle e_4 makes with $\vec{S}$. Each known example either lies in $\mathbf{R}^3$ ($\theta = 90°$) or locally takes the form $\mathbf{R} \times C$ for a space curve C (θ constant). There must be others, because for these examples either $\int x_4\, dx_1 dx_2 = 0$ or $\int x_1\, dx_2 dx_3 = 0$ and there are smooth surfaces satisfying other constraints by Theorem 2.1.

In response to a preprint of this paper, Robert Bryant reports that the theory of exterior differential systems guarantees a local solution to (4.4) for real-analytic initial data (along an $(m-1)$-dimensional submanifold). For the aforementioned case

$$H = e_1 \wedge e_2 \wedge e_3 \llcorner \vec{S}, \tag{4.5}$$

the e_4 coordinate u is harmonic, with $|du|^2 = \cos^2\theta$. If $\theta(u, v)$ is a solution to

$$\cot\theta\,(\theta_{uu} + \theta_{vv}) + \theta_u^2 + \theta_v^2 + 1 = 0 \tag{4.6}$$

(a version of the sinh-Gordon equation in disguise), then there is an immersion $\mathbf{x}(u,v)$ into $\mathbf{R}^3$ with first and second fundamental forms

$$\mathrm{I} = \tan^2\theta\, du^2 + \sec^2\theta\, dv^2,$$
$$\mathrm{II} = \sec\theta\tan\theta\ \left(du^2 + dv^2\right),$$

and the immersion $\mathbf{x}(u,v) + ue_4$ satisfies (4.5). To recognize (4.6) as sinh-Gordon, rewrite it in the form

$$\Delta\left(\log\left(\sec\theta + \tan\theta\right)\right) + \sec\theta\tan\theta = 0$$

(which makes sense even when $\sin\theta = 0$), and set $\sin\theta = -\tanh\left(\frac{f}{2}\right)$ to obtain the sinh-Gordon equation

$$\Delta f + \sinh f = 0,$$

which is known to have many global solutions.

Since there is no Weierstrass formula for solutions of the sinh-Gordon equation, it follows that, unlike the case of minimal surfaces in $\mathbf{R}^3$, there will be no Weierstrass formula for solutions to (4.5).

The following theorem extends the theory of calibrations to our minimizers. The differential form φ is called a *d-constant calibration* in [M5, §1.2].

THEOREM 4.4 (Calibrations theorem). *Let S be an oriented m-dimensional surface (rectifiable current) in $\mathbf{R}^n$. Suppose there is a smooth differential form φ with* $\mathrm{d}\varphi$ *constant such that φ attains its maximum (say 1) on the tangent planes to S. Then S minimizes area for given boundary and multi-volume.*

Proof. Write any competitor in the form $S + \partial R$. By the volume constraint, $\int_R \mathrm{d}\varphi = 0$ and hence $\int_{\partial R}\varphi = 0$. Therefore

$$\operatorname{area}(S) = \int_S \varphi = \int_{S+\partial R}\varphi \leq \operatorname{area}(S + \partial R).$$

□

REMARK. Unfortunately there seem to be no nontrivial easy examples. A small portion of a round circle or sphere (or any constant-mean-curvature hypersurface) is calibrated ([M4, Rmk. p. 76], [M5, §1.2]), but a whole circle or sphere cannot be calibrated, because it is not minimizing among chains with real coefficients: a circle with half the density and twice the area has less weighted perimeter.

Note that while a circle C is minimizing, $\mathbf{R}\times C$ is not. On the other hand, if an m-dimensional surface S in $\mathbf{R}^n$ is calibrated by an m-form φ, then $\mathbf{R}^k \times S \subset \mathbf{R}^{k+n}$ is calibrated by $\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_k\wedge\varphi$. More general products of minimizers are not generally minimizing (or even stationary).

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