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A REMARK ON DIVISORS OF CALABI-YAU HYPERSURFACES*

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Abstract. We prove that a non-singular hypersurface of degree $\geq n+1$ in $\mathbb{P}^n$ for $n \geq 4$ does not contain a reduced irreducible divisor which admits a desingularization having nef anticanonical bundle.

1. Introduction. In this paper we shall generalize a theorem [1] of Chang and Ran to the higher dimensional case. They proved that a generic hypersurface of degree ≥ 5 in $\mathbb{P}^3$ or $\mathbb{P}^4$ does not contain a reduced irreducible divisor which admits a desingularization having numerically effective anticanonical bundle. The $\mathbb{P}^3$ case is a conjecture of Harris which is first proven by G. Xu [5] with a different method. The natural generalization of their theorem is the nonexistence of a divisor with numerically effective (nef) anticanonical bundle on a generic hypersurface of degree $\geq n+1$ in $\mathbb{P}^n$ for $n \geq 5$ (See Corollary 3.3). However, the interesting case is the case of Calabi-Yau hypersurfaces (degree equal to $n+1$) since G. Xu gave a geometric genus bound for divisors on generic hypersurfaces of general type. In fact, our setup in this paper is a little more general. We prove that a non-singular complete intersection in Grassmannian with a similar degree assumption does not contain a reduced irreducible divisor which admits a desingularization having numerically effective anticanonical bundle.

Let us fix notations in this paper. Thus let X be a non-singular complete intersection of type $(m_1, m_2, \cdots, m_k)$ in Grassmann variety $G(r, n+1)$ such that $\dim X \geq 3$ and $m = m_1 + m_2 + \cdots + m_k \geq n+1$, and suppose $\bar{D} \subset X$ is an irreducible and reduced divisor. Let $f : D \longrightarrow \bar{D} \subset X$ be a desingularization, l denote the dimension of D and L denote $f^*\mathcal{O}_G(1)$. Obviously, L is nef and big. Let K_D be the canonical bundle of D. Let S and Q be the universal subbundle and universal quotient bundle on $G(r, n+1)$. $Q^\vee$ denotes the dual of Q.

The main technical statement we are going to prove is the following.

PROPOSITION 1.1. *A non-singular complete intersection X of type $(m_1, m_2, \cdots,$ $m_k)$ in Grassmann variety $G(r, n+1)$ such that $m = m_1 + m_2 + \cdots + m_k \geq n+1$ does not contain a reduced irreducible divisor which admits a desingularization having $H^0(K_D \otimes f^*Q^\vee) = 0$ and $H^1(K_D - m_iL) = 0$ for $i = 1, \cdots, k$.*

Here we review the definition and some basic properties of reflexive sheaves (See [3]). Let $\mathcal{F}^{\vee\vee}$ be the double dual of $\mathcal{F}$. A coherent sheaf $\mathcal{F}$ is reflexive if the natural map $\mathcal{F} \longrightarrow \mathcal{F}^{\vee\vee}$ is an isomorphism. Define the singularity set of $\mathcal{F}$ to be the locus where the $\mathcal{F}$ is not free over the local ring.

It is well-known that the singularity set of a torsion-free sheaf on D is at least 2-codimensional. Moreover, the singularity set of a reflexive sheaf on D is at least 3-codimensional. It is also well-known that, in general, any reflexive rank 1 sheaf on an integral and locally factorial scheme is invertible.

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2. Proof of Proposition 1.1. The proof is by contradiction. Assume such divisors $\bar{D}$ exist.

First, consider the dual tautological sequence.

$$0 \longrightarrow Q^\vee \longrightarrow \bigoplus_{(n+1)\text{ copies}} \mathcal{O}_G \longrightarrow S^\vee \longrightarrow 0. \tag{2.1}$$

We pull back the dual tautological sequence tensoring with f^*Q.

$$0 \longrightarrow f^*Q \otimes f^*Q^\vee \longrightarrow \bigoplus_{(n+1)\text{ copies}} f^*Q \longrightarrow f^*T_G \longrightarrow 0. \tag{2.2}$$

The top cohomology group $h^l(f^*Q) = h^0(K_D \otimes f^*Q^\vee) = 0$ makes

$$H^l(f^*T_G) = 0.$$

Second, we pull back the defining sequence of normal bundle of X.

$$0 \longrightarrow f^*T_X \longrightarrow f^*T_G \longrightarrow \bigoplus m_i L \longrightarrow 0. \tag{2.3}$$

Note that we need the smoothness of X to get the above sequence. Then we have $h^{l-1}(m_iL) = h^1(K_D - m_iL) = 0$ which implies

$$H^l(f^*T_X) = 0.$$

Third, consider the defining sequence of normal sheaf N_f.

$$0 \longrightarrow T_D \longrightarrow f^*T_X \longrightarrow N_f \longrightarrow 0. \tag{2.4}$$

With the above three sequences, we obtain

$$H^l(N_f) = 0$$

and

$$c_1(N_f) = K_D + (n+1-m)L.$$

Let $N_f^{\vee\vee}$ be the double dual of N_f. $N_f^{\vee\vee}$ is a reflexive sheaf of rank 1 so it is invertible. The image of N_f in $N_f^{\vee\vee}$ under the canonical map is torsion-free since $N_f^{\vee\vee}$ is torsion-free. The singularity set of a torsion-free sheaf is at least 2-codimensional. Therefore, we have an exact sequence

$$0 \longrightarrow \tau \longrightarrow N_f \longrightarrow N_f^{\vee\vee} \longrightarrow \phi \longrightarrow 0 \tag{2.5}$$

with support of ϕ at least 2-codimensional. Divide the above sequence into two short exact sequences.

$$\begin{aligned} &0 \longrightarrow \tau \longrightarrow N_f \longrightarrow \psi \longrightarrow 0, \\ &0 \longrightarrow \psi \longrightarrow N_f^{\vee\vee} \longrightarrow \phi \longrightarrow 0. \end{aligned} \tag{2.6}$$

Then $H^l(N_f) = 0$ implies

$$H^l(N_f^{\vee\vee}) = 0.$$

On the other hand, we have $c_1(N_f^{\vee\vee}) = K_D + (n+1-m)L - c_1(\tau)$. Note that the first chern class of a torsion sheaf is always effective.([4] V.6.14) Therefore,

$$h^l(N_f^{\vee\vee}) = h^0(K_D - N_f^{\vee\vee}) = h^0((m-n-1)L + c_1(\tau)) > 0$$

gives a contradiction.

3. Main Theorems. For $r = 1$, we identify $G(1, n+1)$ with $\mathbb{P}^n$.

PROPOSITION 3.1. *A non-singular complete intersection X of type $(m_1, m_2, \cdots, m_k)$ in $\mathbb{P}^n$ for $n \geq 4$ such that $m = m_1 + m_2 + \cdots + m_k \geq n+1$ does not contain a reduced irreducible divisor which admits a desingularization having $H^0(K_D - L) = 0$ and $H^1(K_D - m_i L) = 0$ for $i = 1, \cdots, k$.*

Proof. Replace the dual tautological sequence in the proof of Proposition 1.1 with the Euler sequence.

$$0 \longrightarrow \mathcal{O}_D \longrightarrow \bigoplus_{(n+1)\text{ copies}} L \longrightarrow f^*T_{\mathbb{P}^n} \longrightarrow 0. \tag{3.1}$$

$h^l(L) = h^0(K_D - L) = 0$ concludes

$$H^l(f^*T_{\mathbb{P}^n}) = 0.$$

and the remaining proof is the same as the proof in Proposition 1.1. □

Note that we can get the above proposition immediately from Proposition 1.1 if we identify $\mathbb{P}^n$ with $G(n, n+1)$.

THEOREM 3.1. *A non-singular complete intersection X of type $(m_1, m_2, \cdots, m_k)$ in $\mathbb{P}^n$ such that $\dim X \geq 3$ and $m = m_1 + m_2 + \cdots + m_k \geq n+1$ does not contain a reduced irreducible divisor which admits a desingularization having nef anticanonical bundle.*

Proof. If $-K_D$ is nef, $-K_D + L$ and $-K_D + m_i L$ are nef and big. With the Kawamata-Viehweg Vanishing Theorem, we obtain that $H^0(K_D - L) = 0$ and $H^1(K_D - m_i L) = 0$ for $i = 1, \cdots, k$ (Note that $\dim D = \dim X - 1 \geq 2$). Hence the theorem follows. □

COROLLARY 3.2. *A non-singular hypersurface of degree $\geq n+1$ in $\mathbb{P}^n$ for $n \geq 4$ does not contain a reduced irreducible divisor which admits a desingularization having nef anticanonical bundle.*

For $n = 3$, a hypersurface X of degree $d = 4$ in $\mathbb{P}^3$ is a K3 surface. The divisor $\bar{D}$ becomes a curve. Therefore, $h^1(K_D - dL) = h^0(dL)$ is never zero. Hence, our proof doesn't work for this case. By the way, it is well-known that K3 surfaces have rational curves.

Now assume that $r \geq 2$. We can get a similar result.

THEOREM 3.2. *A non-singular complete intersection X of type $(m_1, m_2, \cdots, m_k)$ in Grassmann variety $G(r, n+1)$ such that $m = m_1 + m_2 + \cdots + m_k \geq n+1$ and $(k+1) + (n+1-r) \leq \dim G(r, n+1)$ does not contain a reduced irreducible divisor which admits a desingularization having nef anticanonical bundle.*

Proof. If $-K_D$ is nef, $-K_D + m_i L$ is nef and big. With the Kawamata-Viehweg Vanishing Theorem, we obtain that $H^1(K_D - m_i L) = 0$ for $i = 1, \cdots, k$. In order to get $H^0(K_D \otimes f^*Q^\vee) = 0$, we need to prove that $H^0(f^*Q^\vee) = 0$.

If $H^0(f^*Q^\vee)$ is non-trivial, from the the pull back of the dual tautological sequence, the non-trivial section of $H^0(f^*Q^\vee)$ gives a linear form F and $\bar{D}$ is contained in $(F)_0$, the zero locus of F. We may identify $(F)_0$ with the Schubert cycle $\sigma_{1,\cdots,1}$. Since X is a complete intersection, we also can identify X with the intersection of Schubert cycles $(\Pi m_i)\sigma^k_{1,0,\cdots,0}$. $\bar{D}$ is a divisor of X so it is also a complete intersection. Hence we may identify $\bar{D}$ with a multiple of $\sigma^{k+1}_{1,0,\cdots,0}$. Now consider a Schubert cycle

$\sigma_{(n+1)-r,0,\cdots,0}$, which does not intersect $\sigma_{1,\cdots,1}$. On the other hand, $\sigma_{(n+1)-r,0,\cdots,0}$ does intersect $\sigma^{k+1}_{1,0,\cdots,0}$ if $(k+1)+(n+1-r) \leq \dim G(r, n+1)$. We get a contradiction. Therefore $H^0(f^*Q^\vee) = 0$.

If K_D is trivial, then we get $H^0(K_D \otimes f^*Q^\vee) = H^0(f^*Q^\vee) = 0$. If K_D is not trivial, $H^0(K_D) = 0$. By the injectivity of

$$0 \longrightarrow K_D \otimes Q^\vee \longrightarrow \bigoplus_{(n+1) \text{ copies}} K_D, \tag{3.2}$$

we also get $H^0(K_D \otimes f^*Q^\vee) = 0$. □

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