ASIAN J. MATH.
Vol. 4, No. 2, pp. 351–368, June 2000

006

DENSITY OF RATIONAL POINTS ON ELLIPTIC K3 SURFACES*

F. A. BOGOMOLOV† AND YU. TSCHINKEL‡

1. Introduction. Let X be a smooth projective algebraic variety defined over a number field K. We will say that rational points on X are potentially dense if there exists a finite extension K'/K such that the set $X(K')$ of K'-rational points is Zariski dense. What are possible strategies to propagate rational points on an algebraic variety? We thought of two: using the group of automorphisms $\mathrm{Aut}(X)$ and using additional geometric structures - like elliptic fibrations. The class of K3 surfaces is an ideal test case for both methods.
One of our main results is:

THEOREM 1.1. *Let X be a K3 surface defined over a number field K. Assume that X has a structure of an elliptic fibration or an infinite group of automorphisms. Then rational points on X are potentially dense.*

Here is a more detailed list of what we learned: We don't know if rational points are potentially dense on a general K3 surface with Picard group $\mathrm{Pic}(X_{\mathbb{C}}) = \mathbb{Z}$. In particular, we don't know if rational points are dense on a double cover of $\mathbb{P}^2$ ramified in a general curve of degree 6. However, we can prove potential density for a *divisor* in the space of all such K3 surfaces, corresponding to the case when, for example, the ramification curve is singular (cf. [6]). The overall picture is similar. In any moduli family of algebraic K3 surfaces we can find some union of algebraic subsets, including a divisor, such that rational points are potentially dense on the K3 surfaces corresponding to the points of this subset. More precisely,

THEOREM 1.2. *Let X be a K3 surface, defined over a number field K. Assume that* $\mathrm{rk}\,\mathrm{Pic}(X_{\mathbb{C}}) = 2$ *and that X does not contain a (-2)-curve. Then rational points on X are potentially dense.*

REMARK 1.3. If $\mathrm{rk}\,\mathrm{Pic}(X_{\mathbb{C}}) = 2$ and if X does not contain a (-2)-curve then either it has an elliptic fibration or it has an infinite automorphism group (but not both!). For example, a quartic surface in $\mathbb{P}^3$ containing a smooth curve of genus 2 and degree 6 doesn't admit any elliptic fibrations, but the group $\mathrm{Aut}(X_{\mathbb{C}})$ is infinite (cf. [24] p. 583, [27]).

THEOREM 1.4. *Let X be a K3 surface over K with* $\mathrm{rk}\,\mathrm{Pic}(X_{\mathbb{C}}) \geq 3$. *Then rational points on X are potentially dense, with a possible exception of 8 isomorphy classes of lattices* $\mathrm{Pic}(X_{\mathbb{C}})$.

REMARK 1.5. If $\mathrm{rk}\,\mathrm{Pic}(X_{\mathbb{C}}) = 3$ then there are only 6 types of lattices where we can't prove potential density. There are only 2 types when $\mathrm{rkPic}(X_{\mathbb{C}}) = 4$. Potential density holds for all K3 with $\mathrm{rkPic}(X_{\mathbb{C}}) \geq 5$. All K3 surfaces with $\mathrm{rk}\,\mathrm{Pic}(X_{\mathbb{C}}) = 20$ have infinite groups of automorphisms. We use Nikulin's classification of lattices of algebraic K3 surfaces (cf. [23], [22]).

*Received February 12, 1999; accepted for publication August 31, 1999.
†Courant Institute of Mathematical Sciences, N.Y.U., 251 Mercer str., New York, (NY) 10012, U.S.A. (bogomolo@cims.nyu.edu)
‡Dept. of Mathematics, U.I.C., 851 South Morgan str., Chicago, (IL) 60607-7045, U.S.A. (yuri@math.uic.edu).

First we consider the problem of density for general elliptic fibrations $\mathcal{E} \to \mathbb{P}^1$. Suppose that $\mathcal{E}$ has a zero section (i.e. $\mathcal{E}$ is Jacobian) and that there exists a section of infinite order in the Mordell-Weil group of $\mathcal{E}$. Then a specialization argument shows that rational points are dense in $\mathcal{E}_b$ for a Zariski dense set of fibers $b \in \mathbb{P}^1$ (cf. [28]). It turns out that even in absence of global sections one can sometimes arrive at the same conclusion.

DEFINITION 1.6. Let $\mathcal{E} \to B$ be an elliptic fibration and $\mathcal{M} \subset \mathcal{E}$ an irreducible multisection (defined over $\mathbb{C}$) with the following property: for a general point $b \in B(\mathbb{C})$ there exist two distinct points $p_b, p_b' \in (\mathcal{M} \cap \mathcal{E}_b)(\mathbb{C})$ such that $p_b - p_b'$ is non-torsion in the Jacobian $\mathcal{J}(\mathcal{E}_b)(\mathbb{C})$ of $\mathcal{E}_b$. We will call such a multisection an nt-multisection (non-torsion).

For example, if $\mathcal{M}$ is ramified in a smooth fiber of $\mathcal{E}$ then it is an nt-multisection (cf. 4.4). We will say that $\mathcal{M}$ is *torsion of order* m if for all $b \in B$ and all $p_b, p_b' \in \mathcal{M} \cap \mathcal{E}_b$ the zero-cycle $p_b - p_b'$ is torsion of order m in $\mathcal{J}(\mathcal{E}_b)$. An easy lemma (but not a tautology!) says that if $\mathcal{M}$ is not torsion of order m for any $m \in \mathbb{N}$ then $\mathcal{M}$ is an nt-multisection (cf. 3.8). (There are analogous notions for abelian schemes and torsors under abelian schemes.)

PROPOSITION 1.7. *Assume that $\mathcal{E} \to \mathbb{P}^1$ has an nt-multisection which is a rational or elliptic curve. Then rational points on $\mathcal{E}$ are potentially dense.*

We want to study situations when rational or elliptic multisections occur and to analyze constrains which they impose on the elliptic fibration (possible monodromy, structure of singular fibers etc). We shall call fibrations with finitely many (resp. none) rational or elliptic multisections *hyperbolic* (resp. *strongly hyperbolic*). Unfortunately, we don't know examples of hyperbolic elliptic fibrations (without multiple fibers). The aim of Section 2 is to prove the existence of a least *one* rational multisection on algebraic elliptic K3 surfaces. From this we will deduce in Section 3 the following theorem:

THEOREM 1.8. *Let X be an algebraic K3 surface with $\operatorname{rk}\operatorname{Pic}(X_{\mathbb{C}}) \leq 19$ admitting a structure of an elliptic fibration. Then this fibration has infinitely many rational nt-multisections.*

The proof goes roughly as follows: We find elliptic K3 surfaces $\mathcal{E}' \to \mathbb{P}^1$ admitting a dominant map $\mathcal{E}' \to X$ such that the genus of every irreducible m-torsion multisection $\mathcal{M}' \subset \mathcal{E}'$ is ≥ 2. On the other hand, the deformation theory argument in Section 2 implies that $\mathcal{E}'$ contains a *rational* multisection which must be an nt-multisection. Its image in X is a rational nt-multisection.

ACKNOWLEDGEMENTS. Part of this work was done while both authors were visiting the Max-Planck-Institute in Bonn. We are grateful to the MPI for the hospitality. The first author was partially supported by the NSF. The second author was partially supported by the NSA. We would like to thank Joe Harris and Barry Mazur for their ideas, suggestions and encouragement.

2. K3 surfaces. In this section we prove that every elliptic fibration on an algebraic K3 surface has at least one rational multisection.

2.1. Generalities. There are several approaches to the theory of K3 surfaces. Algebraically, a K3 surface S (defined over some field of characterstic zero) is a smooth projective surface with trivial canonical class $\mathcal{K}_S = 0$ and $\mathrm{H}^1(S, \mathcal{O}_S) = 0$. They are parametrized by an infinite countable set of 19-dimensional algebraic spaces. The main invariant is the Picard group $\mathrm{Pic}(S)$ which is isomorphic to a torsion free primitive lattice of finite rank (≤ 20) equipped with a hyperbolic even integral bilinear form.

Another approach is via Kähler geometry. A K3 surface S is a compact simply connected Kähler surface equipped with a non-degenerate nowhere vanishing holomorphic $(2,0)$-form ω_S. To obtain a natural parametrization we have to consider *marked* K3 surfaces, which are pairs (S,σ) consisting of a K3 surface S and an isometry of lattices

$$\sigma : \mathrm{H}^2(S,\mathbb{Z}) \tilde{\to} \mathcal{L} \simeq 3 \cdot \mathcal{H} \oplus 2 \cdot (-E_8),$$

where $\mathcal{H}$ is the standard lattice with form xy and E_8 is an 8-dimensional even unimodular positive definite lattice. We will denote by $\langle , \rangle$ the intersection form on $\mathrm{H}^2(S,\mathbb{Z})$. Marked K3 surfaces are parametrized by the conformal class $\sigma_{\mathbb{C}}(\mathrm{H}^{2,0}(S,\mathbb{Z}))$ of their non-degenerate holomorphic forms - the *period.* The latter lies in the quadric given by $\langle \omega_S, \omega_S \rangle = 0$ (inside $\mathbb{P}^{21} = \mathbb{P}(\mathrm{H}^2(S,\mathbb{Z})_{\mathbb{C}})$). The period (still denoted by) ω_S satisfies the inequality $\langle \omega_s, \overline{\omega}_S \rangle > 0$. Therefore, marked Kähler K3 surfaces are parametrized by points of a complex homogeneous domain $\Omega = SO_{(3,19)}(\mathbb{R})/SO_{(2,18)}(\mathbb{R})$. (with the standard equivariant complex structure). Unmarked K3 surfaces correspond to orbits of the group $SO_{(3,19)}(\mathbb{Z})$ on this space.

We will identify cycles and forms on S with their (co)homology classes. We will call a homology class h *primitive*, if $h \neq mZ$ for some $m > 1$ and some effective cycle Z. We denote by $\Lambda_{\mathrm{eff}}(S)$ the monoid of all classes in $\mathrm{Pic}(S)$ represented by effective divisors. (This differs slightly from the standard definition of the effective cone as a cone in $\mathrm{Pic}(S)_{\mathbb{R}}$. In particular, the smallest closed *cone* in $\mathrm{Pic}(S)_{\mathbb{R}}$ containing $\Lambda_{\mathrm{eff}}(S)$ could be finitely generated with $\Lambda_{\mathrm{eff}}(S)$ being infinitely generated.)

We want to describe, in this setting, the subset of algebraic and elliptic K3 surfaces. A Kähler K3 surface S is algebraic if there is a primitive element $x \in \mathrm{H}^2(S,\mathbb{Z})$ such that $\langle x,x \rangle > 0$ and $\langle \omega_S, x \rangle = 0$. Conversely, every primitive $x \in \mathrm{H}^2(S,\mathbb{Z})$ determines a hyperplane $\{\langle \omega_S, x \rangle = 0\}$. The intersection of this hyperplane with Ω will be denoted by $\Omega(x)$. For a generic point of $\Omega(x)$ with $\langle x,x \rangle > 0$ one of the classes $\pm x$ defines a polarization of the corresponding marked K3 surface.

Every element h which is a generator of $\Lambda_{\mathrm{eff}}(S)$ with $\langle h,h \rangle = -2$ is represented by a smooth rational curve. Similarly, every generator of $\Lambda_{\mathrm{eff}}(S)$ with $\langle h,h \rangle = 0$ is represented by a smooth elliptic curve (which defines an elliptic fibration without multiple fibers $S \to \mathbb{P}^1$). In particular, this class is also represented by a (singular) rational curve, contained in the singular fibers of the fibration. Therefore, (marked) elliptic K3 surfaces constitute a set of hyperplanes $\Omega(h)$ with $\langle h,h \rangle = 0$ (and primitive h). For a generic member of $\Omega(h)$ the element h defines the class of a fiber of the corresponding elliptic fibration.

2.2. Deformation theory. In this section we work over $\mathbb{C}$. An *immersion* of a smooth curve $f : C \to X$ into a smooth variety X is a regular map of degree 1 onto its image such that the differential $\mathrm{d}f$ is non-zero everywhere. An *embedding* is an immersion with smooth image.

REMARK 2.1. If $f : C \to S$ is an immersion of a smooth curve into a smooth surface then there exists a local neighborhood U of C (abstractly) to which the map f extends as a local isomorphism $f : U \to S$. The normal bundle $\mathcal{N}_C(U)$ of C in U is defined by restriction of the canonical bundle $\mathcal{K}_S$ to $f(C)$. In particular, if S is a K3 surface then the normal bundle $\mathcal{N}_C(U) = \mathcal{K}_C$.

PROPOSITION 2.2. *Let C_0 be a smooth rational curve, S_0 a K3 surface and $f_0 : C_0 \to S_0$ an immersion. Let $\mathcal{S} \to \mathcal{T}$ be a smooth scheme over a complex ball $\mathcal{T}$ of dimension 20 with fibers smooth K3 surfaces S_t (local deformations of S_0). Consider the smooth subfamily $\mathcal{S}'_{\mathcal{T}'} = \mathcal{S}' \to \mathcal{T}'$ corresponding to deformations such that the class of $[f_0(C_0)] \in \mathrm{H}^2(S_0, \mathbb{Z}) \simeq \mathrm{H}^2(S_t, \mathbb{Z})$ remains algebraic for all $t \in \mathcal{T}'$ (dimension of $\mathcal{T}'$ equals 19). Then for all $t \in \mathcal{T}'$ (close enough to t_0) there exists a smooth family of smooth curves $\mathcal{C}_{\mathcal{T}'} = \mathcal{C} \to \mathcal{T}'$ and a holomorphic map $f' : \mathcal{C}_{\mathcal{T}'} \to \mathcal{S}_{\mathcal{T}'}$ such that $f'|_{t_0} = f_0$*

Proof. Construct a complex 2-dimensional neighborhood U_0 of C_0 with the property that f_0 extends to a holomorphic map $g_0 : U_0 \to S_0$ such that g_0 is a local isomorphism. This is possible since $\mathrm{d}f_0 \neq 0$. There is a non-degenerate $(2,0)$-form on U_0 induced from S_0. The curve C_0 is smooth in U_0 and its normal bundle in U_0 is isomorphic to $\mathcal{O}_{C_0}(-2)$. It is well known that in this situation there exists a local neighborhood of C_0 which is isomorphic to a small neighborhood of the zero section in the bundle $\mathcal{O}_{C_0}(-2)$.

The deformation of the complex structure on S_0 induces (by means of g_0) a deformation of the complex structure on U_0. We obtain a smooth family $g : \mathcal{U}_{\mathcal{T}} \to \mathcal{S}_{\mathcal{T}}$ (with $g|_{t_0} = g_0$) of deformations of complex structures on U_0. The base of the space of versal deformations for U_0 is a 1-dimensional disc. In the neighborhood of $t_0 \in \mathcal{T}$ every deformation of U_0 is induced from the versal deformation space by a holomorphic map. As a preimage of zero we obtain a local divisor $\mathcal{D}_0 \subset \mathcal{T}$. It follows that $C_0 \times \mathcal{D}_0$ is contained in the restriction of the family $\mathcal{U}_{\mathcal{T}}$ to $\mathcal{D}_0$.

On the other hand, outside the divisor $\mathcal{T}' \subset \mathcal{T}$ the class $[C_0] \in \mathrm{H}^2(U_t, \mathbb{Z}) = \mathbb{Z}$ is not algebraic. This is equivalent to the property that the integral of the holomorphic form ω_t over the class $[f_0(C_0)]$ is not zero. Then the integral of the induced form $g^*(\omega_t)$ over $[C_0]$ is not equal to zero as well (where ω_t is the non-degenerate holomorphic form on S_t induced by deformation). Therefore, the class $[C_0] \in \mathrm{H}^2(U_t, \mathbb{Z})$ cannot be realized by a holomorphic curve if $t \notin \mathcal{T}'$. Since we have obtained a realization of this class over $\mathcal{D}_0$ we can conclude that the local divisor $\mathcal{D}_0$ is contained in $\mathcal{T}'$. Since $\mathcal{T}'$ is irreducible (it is a smooth disc), both divisors coincide. Therefore, the map f' is obtained by restriction of g to $\mathcal{C}_{\mathcal{D}_0} = \mathcal{C}_{\mathcal{T}'}$.

REMARK 2.3. This proof imitates the approach of S. Bloch who introduced the notion of semi-regularity for *embedded* varieties ([5]). Here we use a similar technique for immersed varieties. This deformation technique was extended to the case of general maps by Z. Ran (cf. [25] and [26]).

2.3. Effective divisors.

THEOREM 2.4. (Bogomolov-Mumford) *Every class in $\Lambda_{\mathrm{eff}}(S)$ can be represented by a sum of (classes of) rational curves.*

Proof. The monoid of effective divisors $\Lambda_{\mathrm{eff}}(S)$ of a K3 surface S is generated by classes of (-2)-curves (represented by smooth rational curves), classes x with $\langle x, x\rangle =$ 0 (represented by smooth elliptic curves, cuspidal elliptic curves or nodal elliptic curves) and by primitive classes x with $\langle x, x\rangle > 0$. Any smooth elliptic curve defines an elliptic fibration. This fibration always has singular fibers (Euler characteristic) and they consist of rational curves. It remains to show the following

PROPOSITION 2.5. *Let S be a K3 surface. Every primitive effective class in* $\mathrm{Pic}(S)$ *with* $\langle x, x\rangle > 0$ *can be represented by a sum of (classes of) rational curves (with multiplicities).*

The rest of this section is devoted to a proof of this fact. An alternative proof is contained in [21].

We first show that every primitive class is uniquely determined by its square. Next we give a direct construction of a K3 surface containing a rational curve which represents a primitive class, with a given square. Finally we apply a deformation argument.

EXERCISE 2.6. Let $\mathcal{L}$ be an indefinite unimodular lattice containing $3 \cdot \mathcal{H}$, where $\mathcal{H}$ is the standard form given by xy. The orbit of any primitive element under the group $\mathrm{SO}(\mathcal{L})$ is uniquely determined by the square of this element.

Proof. First we show it for elements with x with $\langle x, x\rangle = 0$. Indeed, since x is primitive, there exists a y with $\langle x, y\rangle = 1$. Then x, y generate a sublattice $\mathcal{H} \subset \mathcal{L}$. Since any sublattice $\mathcal{H}$ is a direct summand, we have the result. Similarly, if z is any element such that $\langle z, x\rangle = 1$ for some $x \in \mathcal{L}$ with $\langle x, x\rangle = 0$ then z is equivalent to the element with coordinates $(\langle z, z\rangle, 1)$ in the sublattice $\mathcal{H}$ (with standard coordinates). This concludes the exercise (see also [12], p. 224).

COROLLARY 2.7. *Let S be a K3 surface. Every primitive class in $\Lambda_{\mathrm{eff}}(S)$ is uniquely determined by its self-intersection.*

Proof. Identify $\mathrm{Pic}(S)$ with a sub-lattice in $\mathcal{L} = 3 \cdot \mathcal{H} \oplus 2 \cdot (-E_8)$.

PROPOSITION 2.8. *For any even $n \in \mathbb{N}$ there exists a pair $f : C \hookrightarrow S$ consisting of a smooth rational curve C immersed in a K3 surface S and having self-intersection equal to n.*

Proof. Let R be a curve of genus 2 and $\mathcal{J}(R)$ its Jacobian. Let $\mathbb{Z}/\ell\mathbb{Z} \subset \mathcal{J}(R)$ be a cyclic subgroup of odd order ℓ. Consider the map $\pi : R \to \mathcal{J}(R)/(\mathbb{Z}/\ell\mathbb{Z})$.

LEMMA 2.9. *For a generic R the curve $\pi(R)$ contains exactly 6 points of order 2 of the quotient abelian variety $\mathcal{J}(R)/(\mathbb{Z}/\ell\mathbb{Z})$. These points are non-singular points of $\pi(R)$.*

Proof. It suffices to show that the only torsion points of $\mathcal{J}(R)$ contained in R (for a generic R) are the standard 6 points of order 2. (Indeed, a point $\pi(Q)$, where $Q \in R$ is a point of order 2 in $\mathcal{J}(R)$, is a singular point of $\pi(R)$ if and only if there exists a point $P \neq Q$ in R such that $\ell \cdot P = Q$ in $\mathcal{J}(R)$. Thus P has to be a torsion point of order 2ℓ.)

Consider the universal family $\mathcal{C} \to \mathbf{M}(2,2)$ of smooth curves of genus 2 with 2 level structure. This family is imbedded as a subvariety into the universal family of principally polarized abelian varieties $\mathcal{J} \to \mathbf{M}_J(2,2)$ (Jacobians) of dimension 2 and level 2. The family $\mathcal{C} \to \mathbf{M}(2,2)$ has 6 natural sections (points of order 2 in the Jacobian). The family $\mathcal{J} \to \mathbf{M}_J(2,2)$ has 16 natural sections and 6 of them are contained in $\mathcal{C}$.

The monodromy of the family $\mathcal{C} \to \mathbf{M}(2,2)$ is a congruence subgroup of the group $\mathrm{Sp}_4(\mathbb{Z})$ (which we denote by $\Gamma_{\mathcal{C}}$). The torsion multisections of $\mathcal{J} \to \mathbf{M}_J(2,2)$ split into a countable union of irreducible varieties T_m corresponding to the orbits of monodromy $\Gamma_{\mathcal{C}}$ on $(\mathbb{Q}/\mathbb{Z})^4$.

Thus if a generic element R_t contains a torsion point of order ℓ then it also contains its $\Gamma_{\mathcal{C}}$ orbit. Remark that for odd ℓ this orbit consists of all primitive elements of order ℓ in the torsion group of the fiber. If $\ell = 2n$ then the corresponding orbit contains all primitive torsion points x, y of order ℓ with $nx = ny$. Thus the intersection cycle $R_t + a \cap R_t$ (where a is an element of order ℓ or $n = \ell/2$ in the even case) consists of primitive points x such that $x+a$ is also primitive. The degree of this cycle is $\geq \phi(\ell)\ell^2$ (where $\phi(\ell)$ is the Euler function.) (In fact, for any primitive a the degree is greater than the number of points x which are primitive modulo the subgroup generated by a - hence the number of primitive points in $(\mathbb{Z}/\mathbb{Z}\ell)^3$ estimates the corresponding number from below.) On the other hand $\langle R_t + a, R_t\rangle = 2$. Hence we obtain a contradiction if $m > 2$.

It shows that the only torsion points which can lie on a generic curve of genus 2 are the points of order 2. Since any point of order 2 which lies on R_t has to be invariant under the standard involution there are exactly six points of this kind on any R.

LEMMA 2.10. *The self-intersection* $\langle \pi(R), \pi(R)\rangle = 2\ell$.

Proof. Indeed, the preimage $\pi^{-1}(\pi(R))$ consists of translations of R by $\mathbb{Z}/\ell\mathbb{Z}$. Since $\langle R, R\rangle = 2$ we have $\langle \pi(R), \pi(R)\rangle = \frac{1}{\ell} \cdot 2\ell^2$.

LEMMA 2.11. *For every even $n > 0$ there exists a K3 surface S_n containing a rational curve which represents a primitive class c_n such that $\langle c_n, c_n\rangle = n$.*

Proof. After dividing $\mathcal{J}(R)/(\mathbb{Z}/\ell\mathbb{Z})$ by $\mathbb{Z}/2\mathbb{Z}$ we obtain a rational curve $\pi(R)/(\mathbb{Z}/2\mathbb{Z})$ on the singular Kummer surface $\mathcal{J}(R)/\mathcal{D}_{2\ell}$ (where $\mathcal{D}_{2\ell}$ is the dihedral group). After blowing up $\mathcal{J}(R)/\mathcal{D}_{2\ell}$ at the images of the 16 points of order 2 on $\mathcal{J}(R)/(\mathbb{Z}/\ell\mathbb{Z})$ we obtain an immersed rational curve with square $\ell - 3$. This curve represents a primitive class because its intersection with each of the 6 blown up (-2)-curves equals to one.

LEMMA 2.12. *Let S be any K3 surface with an effective primitive class x with square equal to n. Then there exists a 1-dimensional smooth family of K3 surfaces $f : \mathcal{S} \to \mathcal{T}_1$ such that x is an effective class in* $\mathrm{Pic}(S_t)$ *for all $t \in \mathcal{T}$ and such that $S_{t_0} = S$, $S_{t_1} = S_n$ (for some $t_0, t_1 \in \mathcal{T}$) and the class $x \in$* $\mathrm{Pic}(S_{t_1})$ *is represented by an immersed rational curve.*

Proof. Consider a subvariety in the moduli space of marked K3 surfaces where a given class x is algebraic. It is given by a hyperplane section with the equation $\langle \omega, x\rangle = 0$ in the intersection of the open domain $\langle \omega, \overline{\omega}\rangle > 0$ with the quadric $\langle \omega, \omega\rangle = 0$. This is a connected smooth domain, which we denote by $\Omega(x)$. This domain is invariant under the action of a subgroup of $SO_{(2,19)}(\mathbb{R})$. The arithmetic subgroup $\Gamma(x) \subset SO_{(3,19)}(\mathbb{Z})$ stabilizing x acts discretely on $\Omega(x)$ (since it stabilizes a 3-dimensional subspace generated by $x, \omega, \overline{\omega}$ which has a positive definite intersection form) and the quotient is a possibly singular algebraic variety with at most quotient singularities. It is a (coarse) moduli space of K3 surfaces with a fixed class x. There exists a subgroup $\Gamma(x)'$ of finite index in $\Gamma(x)$ which acts freely on $\Omega(x)$. The quotient $\tilde{\Omega}(x)$ is the fine moduli space of K3 surfaces with a fixed class x such that for a generic point of $\tilde{\Omega}(x)$ the corresponding K3 surface carries a polarization with class x. There is a point in $\Omega(x)$ which corresponds to a Kummer surface S_n with a class $c_n = x$ represented by an immersed rational curve. (Indeed, the classes c_n, x lie in the same orbit under the action of $SO_{(3,19)}(\mathbb{Z})$.) We have a smooth algebraic curve $\tilde{\mathcal{T}}(x) \subset \tilde{\Omega}(x)$ which connects the projections of points corresponding to S_n and S. Observe that we can choose the curve $\tilde{\mathcal{T}}(x)$ such that it contains only a finite number of points $\tilde{t}$ where x is not a polarization of the corresponding K3 surface $S_{\tilde{t}}$. The family of effective cycles $C_{\tilde{t}}(x)$ (represented by sums of rational curves) which represent the class x in the group $\mathrm{Pic}(S_{\tilde{t}})$ is an algebraic ruled surface which projects surjectively onto the generic point of $\tilde{\mathcal{T}}(x)$ (this follows from the surjectivity in the neighborhood of S_n). Hence, there is a smooth relative compactification of this ruled surface with a proper (fiberwise) map to the corresponding family of K3 surfaces. The class of the image of any fiber $C_{\tilde{t}}(x)$ is x.

REMARK 2.13. Let x be a primitive class which is one of the generators of $\Lambda_{\mathrm{eff}}(S)$. Then it is represented by an irreducible rational curve.

REMARK 2.14. There are similar results about immersions of stable curves (not necessarily rational curves) and substantially more general theorems on the existence of curves and families of curves. For example, Mori and Mukai proved that a generic K3 surface can be covered by a family of elliptic curves (cf. [20]). Yau, Zaslow and Beauville found a formula for the number of (singular) rational curves in a given class on generic K3 surfaces ([31], [4]). Xi Chen constructs such curves deforming them from combinations of rational curves on degenerations of K3 surfaces (cf. [8], [9]). (However, their results don't imply the existence of infinitely many rational multisections on elliptic K3 surfaces.) Let us also mention the work of C. Voisin on Lagrangian immersions of algebraic varieties into hyperkähler varieties. We decided to include the initial argument of the first author since it is direct, transparent and sufficient for our purposes.

PROPOSITION 2.15. *The set $\mathbf{M}(S,h)$ of elliptic K3 surfaces $\mathcal{E} \to \mathbb{P}^1$ with a fixed Jacobian $\mathcal{J}(\mathcal{E}) = S$ is given by $\mathbf{M}(S,h) = \{\omega_t = \omega_S + th\}_{t\in\mathbb{C}} \subset \Omega(h)$, where t is a complex parameter and h is a representative of the class of the elliptic fiber S_b ($b \in \mathbb{P}^1$).*

Proof. Let h be any $(1,1)$-form induced from the base $\mathbb{P}^1$. Then the form $\omega_h := \omega_S + th$ defines a complex structure on S. Indeed, ω_h is non-zero everywhere, its square is identically zero, it is a closed form and it is non-degenerate on the real sub-bundle of the tangent bundle. If its class is homologous to zero then the variation is trivial. Otherwise, we obtain a line $\mathbf{M}(S,h)$ in the space $\Omega(h)$.

Assume now that $\mathcal{E}' \to \mathbb{P}^1$ is an elliptic K3 surface with the same given Jacobian S. Then there is a smooth (fiberwise) isomorphism $\iota : \mathcal{E} \to \mathcal{E}'$ which is holomorphic along the fibers. The holomorphic forms $\omega_{\mathcal{E}}$ and $\omega_{\mathcal{E}'}$ correspond to the sections s, s' of $\mathrm{H}^0(\mathbb{P}^1, \mathcal{O})$. Therefore, the difference $\omega_{\mathcal{E}} - \iota^*(\omega_{\mathcal{E}'})$ is a closed form which has a non-trivial kernel on the tangent sub-bundle to elliptic fibers. Therefore, this difference is a form of rank at most 2 induced from the base of the elliptic fibration.

3. Elliptic fibrations.

3.1. Generalities. In this section we continue to work over $\mathbb{C}$. We have to use parallel theories of elliptic fibrations in the analytic and in the algebraic categories. All algebraic constructions carry over to the analytic category. As in the case of K3 surfaces there are some differences which we explain along the way.

DEFINITION 3.1. Let $\mathcal{E}$ be a smooth projective algebraic surface. An elliptic fibration is a morphism $\varphi : \mathcal{E} \to B$ onto a smooth projective irreducible curve B with connected fibers and with generic fiber a smooth curve of genus 1. A Jacobian elliptic fibration is an elliptic fibration with a section $e : B \to \mathcal{E}$.

To every elliptic fibration $\varphi : \mathcal{E} \to B$ one can associate a Jacobian elliptic fibration $\varphi_{\mathcal{J}} : \mathcal{J} = \mathcal{J}(\mathcal{E}) \to B$ (cf. [1]). Over the generic point $\mathcal{J}_b$ is given by classes of divisors of degree zero in the fiber $\mathcal{E}_b$. The zero section $e_{\mathcal{J}}$ corresponds to the trivial class. A Jacobian elliptic fibration $\mathcal{J}$ can be viewed simultaneously as a group scheme over B (defining a sheaf over B) and as a surface (the total space). In order to distinguish, we will sometimes use the notation $\mathcal{J}$ and $S(\mathcal{J})$, respectively. Most of the time we will work with $B = \mathbb{P}^1$.

We will only consider elliptic fibrations without multiple fibers. They are locally isomorphic to the associated Jacobian elliptic fibration $\mathcal{J} = \mathcal{J}(\mathcal{E})$ (for every point in the base $b \in B$ there exists a neighborhood U_b such that the fibration $\mathcal{E}$ restricted to $U_b \subset B$ is Jacobian). The fibration $\mathcal{E}$ is a principal homogeneous space (torsor) under $\mathcal{J}$ and the set of all (isomorphism classes of) $\mathcal{E}$ with fixed Jacobian is identified with $\mathrm{H}^1_{\mathrm{et}}(B, \mathcal{J})$ (where $\mathcal{J}$ is considered as a sheaf of sections in the Jacobian elliptic fibration). In the analytic category we have a similar description of elliptic fibrations $\mathcal{E}$ with given Jacobian $\mathcal{J}$ (where $\mathcal{J}$ is always algebraic). The group of isomorphism classes of $\mathcal{E}$ with a given Jacobian $\mathcal{J}$ is identified with $\mathrm{H}^1_{\mathrm{an}}(B, \mathcal{J})$.

In the presence of singular fibers we have

$$\mathrm{H}^1(B, \mathcal{J}) = \mathrm{H}^2(S(\mathcal{J}), \mathcal{O})/\mathrm{Image}(\mathrm{H}^2(S(\mathcal{J}), \mathbb{Z})).$$

The subgroup of algebraic elliptic fibrations $\mathrm{H}^1_{\mathrm{et}}(B, \mathcal{J})$ coincides with the torsion subgroup in this quotient ([10], Section 1.5). It can also be described as the union of the images of $\mathrm{H}^1_{\mathrm{an}}(B, \mathcal{J}_m)$, noting the exact sequence

$$\mathrm{H}^0_{\mathrm{an}}(B, \mathcal{J}) \to \mathrm{H}^1_{\mathrm{an}}(B, \mathcal{J}_m) \to \mathrm{H}^1_{\mathrm{an}}(B, \mathcal{J})$$

where $\mathcal{J}_m$ is the sheaf of elements of order m in $\mathcal{J}$ (the elements of order m lie in the image of $\mathrm{H}^1_{\mathrm{an}}(B, \mathcal{J}_m)$).

3.2. Multisections. DEFINITION 3.2. Let $\varphi : \mathcal{E} \to B$ be an elliptic fibration (analytic or algebraic). We say that a subvariety (analytic or algebraic) $\mathcal{M} \subset \mathcal{E}$ is a *multisection* of degree $d_{\mathcal{E}}(\mathcal{M})$ if $\mathcal{M}$ is irreducible and if the degree $d_{\mathcal{E}}(\mathcal{M})$ of the projection $\varphi : \mathcal{M} \to B$ is non-zero. The definition of degree extends to formal linear combinations of multisections.

REMARK 3.3. If an analytic fibration $\mathcal{E} \to B$ has an analytic multisection then both the fibration and the multisection are algebraic.

There is a natural map

$$\mathrm{Rest} : \mathrm{Pic}(\mathcal{E}) \to \mathrm{Pic}(\mathcal{E}_b)/\mathrm{Pic}^0(\mathcal{E}_b) = \mathbb{Z}.$$

DEFINITION 3.4. The *degree* $d_{\mathcal{E}}$ of an algebraic elliptic fibration $\mathcal{E} \to B$ is the index of the image of $\mathrm{Pic}(\mathcal{E})$ under the map Rest.

Clearly, the degree of any multisection $\mathcal{M}$ of $\mathcal{E}$ is divisible by $d_{\mathcal{E}}$.

LEMMA 3.5. *There exists a multisection $\mathcal{M} \subset \mathcal{E}$ with $d_{\mathcal{E}}(\mathcal{M}) = d_{\mathcal{E}}$.*

Proof. Let $\mathcal{D}$ be a divisor in $\mathcal{E}$ representing the class having intersection $d_{\mathcal{E}}$ with the class of the generic fiber of $\mathcal{E}$. Then there is an effective divisor in the class of $\mathcal{D}' = \mathcal{D} + n \cdot \mathcal{E}_b$ for some $n \geq 0$. Indeed, consider $\langle \mathcal{D}', \mathcal{D}' \rangle = \langle \mathcal{D}, \mathcal{D} \rangle + 2nd_{\mathcal{E}}(\mathcal{D})$. By Riemann-Roch, the Euler characteristic is

$$\frac{1}{2}\langle \mathcal{D}', \mathcal{D}' - \mathcal{K}_{\mathcal{E}} \rangle + c_1^2 + \frac{c_2}{12} = \frac{1}{2}\langle \mathcal{D}, \mathcal{D} \rangle + nd_{\mathcal{E}}(\mathcal{D}) - \langle \mathcal{K}_{\mathcal{E}}, \mathcal{D} \rangle + c_1^2 + \frac{c_2}{12}$$

Hence, for n big enough, it is positive. By Serre-duality, we know that

$$\mathrm{h}^2(\mathcal{E}, \mathcal{D}') = \mathrm{h}^0(\mathcal{E}, \mathcal{K}_{\mathcal{E}} - \mathcal{D}') = 0,$$

since the latter has a negative intersection with the generic fiber $\mathcal{E}_b$. Thus, the class of $\mathcal{D}'$ contains an effective divisor and $\mathcal{D}' \cap \mathcal{E}_b = d_{\mathcal{E}}(\mathcal{D}) = d_{\mathcal{E}}$. Then the divisor $\mathcal{M}$ is obtained from this effective divisor by removing the vertical components (clearly, $\mathcal{M}$ is irreducible).

COROLLARY 3.6. *The order of $[\mathcal{E}] \in \mathrm{H}^1(B, \mathcal{J})$ is equal to $d_{\mathcal{E}}$.*

DEFINITION 3.7. A multisection $\mathcal{M}$ is said to be torsion *of order m* if m is the smallest positive integer such that for any $b \in B$ and any pair of points $p_b, p'_b \in \mathcal{M} \cap \mathcal{E}_b$ the image of the zero-cycle $p_b - p'_b$ in $\mathcal{J}_b$ is torsion of order m. We call a multisection an *nt*-multisection (non-torsion), if for a general point $b \in B$ there exist two points $p_b, p'_b \in \mathcal{M} \cap \mathcal{E}_b$ such that the zero-cycle $p_b - p'_b \in \mathcal{J}_b$ is non-torsion.

LEMMA 3.8. *If an irreducible multisection $\mathcal{M} \subset \mathcal{E}$ is not a torsion multisection of order n for any $n \in \mathbb{N}$ then $\mathcal{M}$ is an nt-multisection.*

Proof. We work over $\mathbb{C}$. The union of all torsion multisections of $\mathcal{E}$ is a countable union of divisors. So it can't cover all of $\mathcal{M}$ unless $\mathcal{M}$ is contained in some torsion multisection.

Let $\mathcal{E} \to B$ be an elliptic fibration (without multiple fibers). There is a natural set of elliptic fibrations $\mathcal{J}^m = \mathcal{J}^m(\mathcal{E})$ over B parametrizing classes of cycles of degree m on the generic fiber of $\mathcal{E}$. The Jacobian of each $\mathcal{J}^m$ is isomorphic to $\mathcal{J}^0 = \mathcal{J}$. The class $[\mathcal{J}^m] \in \mathrm{H}^1_{\mathrm{an}}(B, \mathcal{J})$ equals $m \cdot [\mathcal{E}]$. If β is a cocycle defining $\mathcal{J}^\beta$ then the cocycle $m \cdot \beta$ is defined by pointwise multiplication by m in the Jacobian fibration. Thus, we obtain the fibration $\mathcal{J}^{m\beta}$.

We have natural rational maps of algebraic varieties $\mathcal{J}^m \times_B \mathcal{J}^k \to \mathcal{J}^{m+k}$ which fiberwise is the addition of cycles. These maps provide the set of isomorphism classes of $\mathcal{J}^m$ with the structure of a cyclic group. The order of this group for algebraic $\mathcal{E}$ coincides with $d_{\mathcal{E}}$. The identification $\mathcal{J}^{d_{\mathcal{E}}} = \mathcal{J}^0$ is not canonical. It is defined modulo the action of $\mathrm{H}^0(B, \mathcal{J})$ on $\mathcal{J}^{d_{\mathcal{E}}}$ (for example, choosing a multisection of degree $d_{\mathcal{E}}$ in $\mathcal{J}^1 = \mathcal{E}$ will fix the identification).

The construction provides maps $\eta^m : \mathcal{J}^1 \to \mathcal{J}^m$ for any $m \in \mathbb{N}$, since $\mathcal{J}^1$ imbeds diagonally into the fiber product of m copies of $\mathcal{J}^1$. All the above maps are well defined on the open subvarieties obtained by deleting the singular fibers of the fibrations. They are algebraic and they extend to meromorphic maps.

We obtain an action of $\mathcal{J} = \mathcal{J}^0$ on $\mathcal{E} = \mathcal{J}^1$ which is regular in non-singular points of the fibers of $\mathcal{J}$ and $\mathcal{E}$ and which induces a transitive action of the fibers $\mathcal{J}_b$ on $\mathcal{E}_b$ (for smooth fibers).

The maps η^m allow to transfer irreducible multisections between the elliptic fibrations $\mathcal{J}^m$ (modulo $d_{\mathcal{E}}$). More precisely, we have

LEMMA 3.9. *Let $\mathcal{M} \subset \mathcal{J}^k$ be a torsion multisection of order t (with $d_{\mathcal{J}^k}|t$). Consider the map $\eta^m : \mathcal{J}^k \to \mathcal{J}^{km}$. Then $\eta^m(\mathcal{M}) \subset \mathcal{J}^{mk}$ is a torsion multisection of order exactly $t/\gcd(t,m)$. Moreover, if $\mathcal{M}$ is non-torsion or torsion of order coprime to m then the restriction $\eta^m : \mathcal{M} \to \eta^m(\mathcal{M})$ is a birational map and $d_{\mathcal{J}^k}(\mathcal{M}) = d_{\mathcal{J}^{km}}(\eta^m(\mathcal{M}))$.*

Proof. Locally, we have a Jacobian elliptic fibration and the map η^m is multiplication by m. Therefore, if $\eta^m(x) = \eta^m(y)$ for some $x, y \in \mathcal{J}^k$ then $m \cdot (x - y) = 0$ (in $\mathcal{J}$). Since $\mathcal{M}$ is irreducible, either $(x - y)$ is torsion of order $\gcd(t, m)$ for any pair of points $x, y \in \mathcal{J}^k_b$ for a general fiber b or these pairs constitute a divisor in $\mathcal{M}$. In the latter case, it follows that the restriction of η^m to $\mathcal{M}$ is a birational map and hence $\eta^m(\mathcal{M})$ is a multisection of $\mathcal{J}^{mk}$ of the same degree.

COROLLARY 3.10. *Let p be a prime number and $\mathcal{E}$ an elliptic fibration with $d_{\mathcal{E}} = p$. Let $\mathcal{M}$ be a torsion multisection of $\mathcal{E}$. Then $\mathcal{M}$ admits a surjective map onto one of the p-torsion multisections of $\mathcal{E}$ or onto one of the non-zero p-torsion multisections of $\mathcal{J}(\mathcal{E})$.*

Proof. Suppose that $\mathcal{M}$ is a torsion multisection of order $p^k t$ where $(t, p) = 1$ and $k \geq 1$. If $k = 1$ choose an α such that $\alpha t = 1 \bmod p$. We have a map

$$\eta^{\alpha t} : \mathcal{J}^1(\mathcal{E}) \to \mathcal{J}^{\alpha t}(\mathcal{E}) \simeq \mathcal{J}^1(\mathcal{E})$$

and $\eta^{\alpha t}(\mathcal{M})$ is a torsion multisection of order p.

If $k > 1$, then $\eta^{p^{k-1}t}(\mathcal{M})$ is a non-trivial p-torsion multisection (but not a section) in $\mathcal{J}(\mathcal{E})$.

DEFINITION 3.11. Let $\mathcal{Z}$ be any cycle of degree $d_{\mathcal{E}}(\mathcal{Z})$ on $\mathcal{E}$ which is given by a combination of multisections with integral coefficients. We define a class map

$$\tau_{\mathcal{Z}} : \mathcal{E} \to \mathcal{J}$$

by the following rule:

$$\tau_{\mathcal{Z}}(p) = [d_{\mathcal{E}}(\mathcal{Z}) \cdot p - \mathrm{Tr}_{\mathcal{Z}}(\varphi(p))]$$

for $p \in \mathcal{E}$. Here we denote by $\mathrm{Tr}_{\mathcal{Z}}(b)$ the zero-cycle $\mathcal{Z} \cap \mathcal{E}_b$.

3.3. Monodromy. Denote by $b_1, ..., b_n$ the set of points in B corresponding to singular fibers of $\mathcal{E}$. Consider the analytic fibration $\mathcal{E}^* \to B^*$, where $B^* = B \backslash \{b_1, ..., b_n\}$, obtained by removing all singular fibers from $\mathcal{E}$. We have a natural action of the free group $\pi_1(B^*)$ on the integral homology of the fibers. The group of automorphisms of the integral homology of a generic fiber $\mathcal{E}_b$ which preserve orientation is the group $\mathrm{SL}_2(\mathbb{Z})$. One of the main characteristics of an elliptic fibration $\mathcal{E} \to B$ is its global monodromy group Γ.

DEFINITION 3.12. The global monodromy group $\Gamma = \Gamma(\mathcal{E})$ of $\mathcal{E} \to B$ is the image of $\pi_1(B^*)$ in $\mathrm{SL}_2(\mathbb{Z})$. Denote by $\mathrm{ind}(\Gamma) = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$ the index of the global monodromy. A cycle around a point b_i (for any $i = 1, ..., n$) defines a conjugacy class in $\pi_1(B^*)$. The corresponding conjugacy class in $\mathrm{SL}_2(\mathbb{Z})$ is called local monodromy around b_i. So we obtain (a class of) cyclic subgroups $T_i \subset \mathrm{SL}_2(\mathbb{Z})$ (up to conjugation).

REMARK 3.13. The monodromy group $\Gamma = \Gamma(\mathcal{E})$ of an elliptic fibration $\mathcal{E}$ coincides with $\Gamma(\mathcal{J})$ of the corresponding Jacobian elliptic fibration. In particular, for locally isotrivial elliptic fibrations the monodromy group $\Gamma(\mathcal{E})$ is a finite subgroup of $\mathrm{SL}_2(\mathbb{Z})$. For non-isotrivial elliptic fibrations $\mathcal{E} \to \mathbb{P}^1$ we have $\mathrm{ind}(\Gamma(\mathcal{E})) < \infty$.

The group $\mathrm{SL}_2(\mathbb{Z})$ has a center $\mathbb{Z}/2\mathbb{Z}$ and we shall denote by Γ_c the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ obtained by adjoining the center to Γ.

REMARK 3.14. A generic elliptic fibration $\mathcal{E} \to \mathbb{P}^1$ has monodromy group $\mathrm{SL}_2(\mathbb{Z})$. More precisely, $\mathrm{SL}_2(\mathbb{Z})$ has two standard nilpotent generators a, b. Assume that all singular fibers of $\mathcal{E}$ are *nodal* (rational curves with one self-intersection). In this case, we can select a system of vanishing arcs from some points in B^* so that all T_i split into two clusters I_a and I_b (of equal cardinality) such that $T_i = \langle a \rangle$ for $i \in I_a$ and $T_i = \langle b \rangle$ for $i \in I_b$ (cf. [10], p. 171). In particular, any two local monodromies corresponding to different classes generate $\mathrm{SL}_2(\mathbb{Z})$.

Jacobian elliptic fibrations over $\mathbb{P}^1$ arise in families $\mathcal{F}_r$ parametrized by an integer r which is defined through the standard Weierstrass form

$$y^2 = x^3 + p(t)x + q(t)$$

where p (resp. q) is a polynomial of degree $4r$ (resp. $6r$), satisfying some genericity conditions. There are lists of possible singular fibers, possible local monodromy groups and actions of these groups on the torsion sections of the nearby fibers as well as a list of possible torsion groups of the singular fibers (cf. [13] or [1]).

For any non-isotrivial elliptic fibration $\mathcal{E} \to B$ we have a map $j_B : B \to \mathbb{P}^1$ defined by $j_B(b) := j(\mathcal{E}_b)$ (where j is the standard j-invariant of an elliptic curve with values in $\mathbb{P}^1 = \overline{PSL_2(\mathbb{Z})\backslash H}$).

REMARK 3.15. If $B = \mathbb{P}^1$ and $\mathcal{E} \in \mathcal{F}_r$ then $j_B = \frac{4p^3}{4p^3+27q^2}$. Hence, the degree of the map j_B in this case is bounded by $12r$.

PROPOSITION 3.16. *Let $\mathcal{E} \to B$ be a non-isotrivial elliptic fibration. Then* $\mathrm{ind}(\Gamma) \leq 2\deg(j_B)$.

Proof. The map j_B is the same for an elliptic fibration $\mathcal{E}$ and for the Jacobian of $\mathcal{E}$. Thus we reduce to the case of Jacobian elliptic fibrations. Consider the Γ-covering $\tilde{\mathcal{E}}^* \to B^*$. It is a Jacobian elliptic fibration over an open analytic curve $\tilde{B}^*$. Since it is topologically trivial the map j_B lifts to a holomorphic map $\tilde{j}_B : \tilde{B}^* \to H$ (where H is the upper-half plane). This map is Γ-equivariant and it defines a map $j_\Gamma : B^* \to \Gamma\backslash H$. Therefore, the map j_B on B^* is a composition $j_B = r_\Gamma \circ j_\Gamma$, where r_Γ is the map $r_\Gamma : \Gamma\backslash H \to \mathrm{SL}_2(\mathbb{Z})\backslash H$. The group Γ acts on H through its homomorphism to $\mathrm{PSL}_2(\mathbb{Z})$. Therefore, the degree of r_Γ is equal to the index $\mathrm{ind}(\Gamma_c)$ if Γ contains the center $\mathbb{Z}/2$ and equal to $\frac{1}{2}\mathrm{ind}(\Gamma)$ otherwise.

COROLLARY 3.17. *The number of possible monodromies in any family of elliptic fibrations with bounded degree of j_B is finite. In particular, for the families $\mathcal{F}_r$ with a given r all monodromy groups have index $\leq 24r$.*

REMARK 3.18. If we have an algebraic variety which parametrizes elliptic fibrations then global monodromy changes only on algebraic subvarieties, where the topological type of the projection $\varphi : \mathcal{E} \to B$ changes. This variation normally occurs in big codimension. Indeed, the monodromy is completely determined by its action outside of small neighborhoods of singular fibers. Hence, it doesn't vary under small smooth variations of $\mathcal{E}$.

EXAMPLE 3.19. For the family $\mathcal{F}_r$ the monodromy is $\mathrm{SL}_2(\mathbb{Z})$ provided that at least two nodal fibers from different clusters I_a, I_b remain unchanged. The dimension of the subvariety in $\mathcal{F}_r$ with monodromy different from $\mathrm{SL}_2(\mathbb{Z})$ is $\leq \frac{1}{2}\dim\mathcal{F}_r + 1$.

3.4. Torsion multisections. In this section we will work over $\mathbb{C}$. Let $\varphi : \mathcal{E} \to B$ be an elliptic fibration and $\mathcal{M}$ an irreducible multisection of $\mathcal{E}$.

PROPOSITION 3.20. *Let $\mathcal{J} \to \mathbb{P}^1$ be a non-isotrivial Jacobian elliptic fibration with global monodromy $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$. Then there exists a constant c (for example, $c = \frac{6}{\pi^2}$) such that for all torsion multisections $\mathcal{M} \subset \mathcal{J}$ of degree $d_{\mathcal{J}}(\mathcal{M})$ and order m we have*

$$d_{\mathcal{J}}(\mathcal{M}) > \frac{c \cdot m^2}{\mathrm{ind}(\Gamma)}.$$

Proof. For each $b \in \mathbb{P}^1$ we have an action of Γ on the cycle $\mathcal{M} \cap \mathcal{J}_b$ and also an action of Γ on the set of points of order m of this fiber. It follows that $\mathcal{M} \cap \mathcal{J}_b$ must coincide with an orbit of Γ on the m-torsion points. The size of the corresponding orbit for the full group $\mathrm{SL}(2,\mathbb{Z})$ acting on *primitive* torsion points of order m (e.g., points of order exactly m) is equal to the product $m^2 \cdot \prod_{p|m}(1 - 1/p^2)$. Hence, the size of any orbit of Γ on the primitive m-torsion points of a general fiber is

$$> \frac{m^2}{\mathrm{ind}(\Gamma)} \cdot \prod_{p|m}(1 - 1/p^2).$$

PROPOSITION 3.21. *Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a subgroup of finite index. There exists a constant $m_0(\Gamma)$ such that for all non-isotrivial Jacobian elliptic fibrations $\mathcal{J} \to B$, with at least 4 singular fibers, with global monodromy Γ and for all torsion multisections $\mathcal{M} \subset \mathcal{J}$ of order m with $m > m_0(\Gamma)$ we have $g(\mathcal{M}) \geq 2$.*

Proof. Although this fact is probably well known we decided to give an argument.

Every orbit of the (linear) action of Γ on m-torsion points defines an irreducible m-torsion multisection in $\mathcal{J}$ (and vice versa). Thus we can identify the orbit for a given multisection with the quotient Γ/Γ', where Γ' is a subgroup of finite index in Γ. The corresponding orbit for a singular fiber $\mathcal{J}_{b_i}$ is equal to the quotient Γ/Γ_i where Γ_i is a subgroup of Γ generated by Γ' and the local subgroup T_i (even though T_i are, in principle, defined only up to conjugation in Γ, but specifying the multisection we also specify the pair T_i, Γ' modulo common conjugation). Therefore, the Euler characteristic of the normalization $\tilde{\mathcal{M}}$ of $\mathcal{M}$ will be equal to

$$\chi(\tilde{\mathcal{M}}) = |\Gamma/\Gamma'| \cdot \left(\chi(B) - \sum_i (1 - a_i)\right), \tag{3.1}$$

where the sum is over the set of singular fibers and the contribution a_i for a singular fiber is computed as follows: denote by $p_{ii'}$ the reduced irreducible components of the zero cycle $\mathcal{M} \cap \mathcal{E}_{b_i}$. The number a_i is computed as a sum of local contributions from $p_{ii'}$, via monodromy.

The above formula calculates the Euler characteristic of the topological normalization of $\mathcal{M}$ - the latter amounts to the separation of different local branches of $\mathcal{M}$ over the base. Therefore, it is equal to the Euler characteristic of the algebraic normalization of $\mathcal{M}$. In order to prove our theorem it suffices to observe that this formula implies the growth of the absolute value of $\chi(\mathcal{M})$ (and consequently the genus of the normalization of $\mathcal{M}$) as $m \to \infty$.

The fibration $\mathcal{J} \to B$ contains at least one fiber of potentially multiplicative reduction (pullback of ∞ of the j-map). The local monodromy around any fiber of this type is an infinite cyclic group which includes a subgroup of small index (2,3,4,6) generated by the unipotent transformation $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ where k is the number of components in the fiber. The number of m-torsion elements in this singular fiber is at most $m \cdot k$. But the degree of the torsion multisection grows like m^2 (cf. 3.21). Hence the contribution a_i for such fiber tends to zero when $m \to \infty$. Indeed,

$$a_i = \frac{\sum_d b_d}{\sum_d b_d d},$$

where b_d is the number of branches of $\mathcal{M}$ (around $\mathcal{E}_b$) of local degree d. The sum $\sum_d b_d d$ is equal to the global degree of $\mathcal{M}$ over $\mathbb{P}^1$ which for a torsion multisection of order m grows like m^2. On the other hand the sum $\sum_d b_d$ can be estimated by $m^{1+\epsilon}$ (this follows from local computations).

Similarly, for singular fibers with potentially good reduction the corresponding local monodromy groups are among the standard finite subgroups of $\mathrm{SL}_2(\mathbb{Z})$: $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}$. They all act effectively on the points of order m for $m \geq 5$. Hence, every fiber of this type contributes at least 5/6, 3/4, 2/3 or 1/2, respectively (again this follows from local computations).

Asymptotically, for $m \gg 0$, the contribution from every singular fiber of potentially multiplicative reduction will tend to 1, the contribution from other fibers is $\geq 1/2$. Since we have at least 4 singular fibers, the theorem follows.

An alternative argument would be to observe that $\mathcal{M}$ admits a map onto a modular curve. Choosing a point on $\mathcal{M}$ amounts to choosing an elliptic curve and an m-torsion point on it. As m increases the genus of $\mathcal{M}$ has to go up.

We have a similar result for non-Jacobian elliptic fibrations. Before stating it we recall some generalities: Let $\mathcal{E} \to \mathbb{P}^1$ be a (non-Jacobian) elliptic fibration and $\mathcal{J}(\mathcal{E})$ the associated Jacobian elliptic fibration. The fibration $\mathcal{J}(\mathcal{E})$ contains the grouplike part $\mathcal{G}$, obtained by removing multiple components and singular points of singular fibers. All the sections of $\mathcal{J}(\mathcal{E})$ are contained within $\mathcal{G}$. The fibers $\mathcal{G}_b$, (for $b \in B$) are abelian algebraic groups - sometimes non-connected.

The fibration $\mathcal{E}$ contains an open subvariety which is a principal homogeneous space under $\mathcal{G}$. It is defined by a cocycle $c_{\mathcal{E}} \in H^1(\mathbb{P}^1, \mathcal{G}_l)$ if l is the order of $\mathcal{E}$ in the Tate-Shafarevich group. Here $\mathcal{G}_l$ is the l-torsion group subscheme of $\mathcal{G}$, whose generic fiber $\mathcal{G}_l$ is isomorphic to $T_l := \mathbb{Z}/l\mathbb{Z} + \mathbb{Z}/l\mathbb{Z}$. We have:

1. The minimum degree of a multisection in $\mathcal{E}$ is l;
2. The fibration of relative zero cycles of degree $l, \mathcal{E}_l \simeq \mathcal{J}(\mathcal{E})$.

The latter isomorphism is unique up to the action of the group of global sections $H^0(\mathbb{P}^1, \mathcal{J}(\mathcal{E}))$.

Consider the restriction of $\mathcal{E}$ onto the open part $B^* \subset \mathbb{P}^1$ where the fibers of $\mathcal{E}$ are smooth. Now we define a cocycle $c_{\mathcal{M}} \in H^1(B^*, \mathcal{G}_l)$ as follows: we have a variety $\mathcal{T}_l$ which fiberwise (in smooth fibers) is defined as the set of points of $\mathcal{E}$ which differ from $\mathcal{M}$ by torsion of order l (in the Jacobian). Thus we have an accociated cocycle $c_{\mathcal{M}}$ on the open part B^* and a locally constant sheaf $\mathcal{T}_l$ with fiber T_l together with an affine action of $\pi_1(B^*)$ on T_l. Here we consider T_l as an affine plane over $\mathbb{Z}/l\mathbb{Z}$. Thus we get a homomorphism $A_c : \pi_1(B^*) \to \mathrm{ASL}_2(\mathbb{Z}/l\mathbb{Z})$ to the affine group $\mathrm{ASL}_2(\mathbb{Z}/l\mathbb{Z})$. The linearization of this homomorphism is the composition of the monodromy homomorphism $\pi_1(B^*) \to \Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ with the reduction mod l: $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/l\mathbb{Z})$. Denote by H_c the image of $\pi_1(B^*)$ in $\mathrm{ASL}_2(\mathbb{Z}/l\mathbb{Z})$. Thus the fiber of $\mathcal{T}_l$ is a finite affine module $(\mathbb{Z}/l\mathbb{Z} + \mathbb{Z}/l\mathbb{Z})$.

It also defines $\mathcal{E}$ as a (compactification of a) principal homogeneous space under $\mathcal{J}$ (under the natural embedding of sheaves $\mathcal{T}_l \to \mathcal{J}$). It follows that the order of the cocycle $c_{\mathcal{E}}$ divides the order of the cocycle $c_{\mathcal{M}}$.

The total space $\mathcal{T}_l \to B^*$ is a union of connected components. One of these components is the open part of $\mathcal{M}^*$ of $\mathcal{M}$ lying over B^*. Now it is a simple topological fact that connected components of $\mathcal{T}_l$ correspond to the orbits of H_c on T_l (under the affine action $H_c \subset \mathrm{ASL}_2(\mathbb{Z})$. Thus $\mathcal{M}$ defines in fact several torsion multisections (which are components of the subset of points in $\mathcal{E}$ which fiberwise differ from $\mathcal{M}$ by torsion elements).

PROPOSITION 3.22. *Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a subgroup of finite index. There exists a $p_0 > 0$ (which depends only on Γ) such that for every elliptic surface $\mathcal{E} \to \mathbb{P}^1$ with at least 4 singular fibers, global monodromy Γ and for any torsion multisection $\mathcal{M} \subset \mathcal{E}$ of order $p > p_0$ (where p is a prime number) the genus of the normalization of $\mathcal{M}$ is ≥ 2.*

Proof. The minimal index of a proper subgroup of $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ grows with p. This implies that for any subgroup Γ of finite index in $\mathrm{SL}_2(\mathbb{Z})$ its projection onto $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ is surjective for all $p > p_0$.

LEMMA 3.23. *Suppose that Γ surjects onto $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$, that the elliptic fibration $\mathcal{E}$ is non-Jacobian and that $\mathcal{M} \in \mathcal{E}$ has torsion order p. Then $\mathcal{M} \cap \mathcal{E}_b$ has cardinality p^2 (for almost all b).*

Proof. The fiber of $\mathcal{M}$ over the generic point is an orbit of H_c in T_p where $H_c \subset \mathrm{ASL}_2(\mathbb{Z}/p\mathbb{Z})$ surjects on $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ and H_c is not contained in $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})_x \subset \mathrm{ASL}_2(\mathbb{Z}/p\mathbb{Z})_x$ for all x in the affine space T_p (where $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})_x$ is the subgroup stabilizing x). In other words, H_c can not be linearized - otherwise we would have a global section of $\mathcal{E} \to \mathbb{P}^1$. Then T_p is the only orbit of H_c.

If H_c is not isomorphic to $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ (under the linear projection) then it contains the group of translations $(\mathbb{Z}/p\mathbb{Z})^2$ and we are done. Otherwise, we have two conjugate semisimple elements of order $p-1$ with different fixed points and different invariant directions. The orbit under the group generated by these elements is the whole $(\mathbb{Z}/p\mathbb{Z})^2$.

We return to the proof of Proposition 3.22. Now we use Formula (3.1) and we obtain that in this case the contribution from each singular fiber is at least $1/2$ and that there is at least one singular fiber with contribution asymptotically (for $p \to \infty$) 1. Thus the absolute value of Euler characteristic of the normalizaton of $\mathcal{M}$ grows as $p^2/2$.

PROPOSITION 3.24. *For every finite index subgroup* $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ *there exists a* p_0 *such that for all primes* $p > p_0$ *and all (non-isotrivial) non-Jacobian elliptic fibrations* $\mathcal{E} \to \mathbb{P}^1$ *of degree* $d_{\mathcal{E}} = p$, *with at least 4 singular fibers and with global monodromy* Γ *every torsion multisection* $\mathcal{M}$ *of* $\mathcal{E}$ *has genus* $g(\mathcal{M}) \geq 2$.

Proof. First observe that the class of order p in the Shafarevich-Tate group corresponds to a cocycle with coefficients in the p-torsion sub-sheaf of $\mathcal{J}(\mathcal{E})$. Therefore, the elliptic fibration $\mathcal{E}$ corresponding to a cocycle of order p contains a p-torsion multisection $\mathcal{M}$.

By 3.10, we know that every torsion multisection $\mathcal{M}' \subset \mathcal{E}$ admits a map onto the p-torsion multisection in $\mathcal{M} \subset \mathcal{E}$ or a p-torsion multisection in the corresponding Jacobian elliptic fibration $\mathcal{J}(\mathcal{E})$. Now we apply 3.22.

PROPOSITION 3.25. *Let* $\mathcal{E} \to \mathbb{P}^1$ *be an elliptic fibration (with at least 4 singular fibers and fixed monodromy group* Γ *as above). Let* $p > p_0$ *a prime number not dividing the degree* $d_{\mathcal{E}}$. *Let* $\mathcal{E}' \to \mathbb{P}^1$ *be an elliptic fibration of degree* $p \cdot d_{\mathcal{E}}$, *obtained by dividing the cocycle corresponding to* $\mathcal{E}$ *by* p. *Then* $\mathcal{E}'$ *has no rational or elliptic torsion multisections.*

Proof. Let $\mathcal{E}'' = \mathcal{E}'^{d_{\mathcal{E}}}$. It is a fibration of order p (with the same monodromy group Γ). Any torsion multisection of $\mathcal{E}'$ is mapped to a torsion multisection of $\mathcal{E}''$. By 3.24, the genus of any torsion multisection in $\mathcal{E}''$, and therefore in $\mathcal{E}'$ is ≥ 2.

LEMMA 3.26. *Any elliptic K3 surface* $S \to \mathbb{P}^1$ *with* $\mathrm{Pic}(S) \leq 19$ *has at least 4 singular fibers, including at least one potentially multiplicative fiber.*

Proof. The proof is topological and works for Jacobian and non-Jacobian elliptic fibrations. Denote by $\chi(\mathcal{E}_b)$ the Euler characteristic and by $r(\mathcal{E}_b)$ the rank of the lattice spanned by classes of the irreducible components of the singular fiber $\mathcal{E}_b$. Then $\chi(\mathcal{E}_b) - r(\mathcal{E}_b) = 1$ if the fiber has multiplicative reduction (Type I_n), or $\chi(\mathcal{E}_b) - r(\mathcal{E}_b) = 2$ otherwise. We have $\sum \chi(\mathcal{E}_b) = 24$ and $\sum r(\mathcal{E}_b) \leq 18$ (for more details see, for example [32], pp. 7–9).

REMARK 3.27. In [3] Beauville proves that every *semi-stable* non-isotrivial elliptic fibration has at least 4 singular fibers and classifies those which have exactly 4. (These are 6 modular families, cf. [2], p. 658.) There is a complete classification of elliptic K3 surfaces with 3 singular fibers in [32]. For recent work concerning the minimal number of singular fibers in fibrations with generic fiber a curve of genus ≥ 1 see [29], [32].

As a corollary we obtain Theorem 1.8 stated in the introduction:

COROLLARY 3.28. *Every algebraic elliptic K3 surface* $S \to \mathbb{P}^1$ *with* $\mathrm{rk}\,\mathrm{Pic}(S) \leq 19$ *has infinitely many rational nt-multisections.*

Proof. If S is Jacobian we denote by S' some algebraic non-Jacobian elliptic K3 surface with Jacobian $\mathcal{J}(S') = S$. Otherwise, we put $S' = S$. Dividing (the cocycle defining) S' by different primes $p > p_0$ we obtain elliptic K3 surfaces $\mathcal{E}_p$ (of different degrees). By proposition 3.25, $\mathcal{E}_p$ don't contain rational or elliptic torsion multisections. At the same time, by deformation theory, they contain rational multisections of degree divisible by $d_{\mathcal{E}_p}$. Therefore, we can produce a sequence of rational nt-multisections in S' (and consequently, in S) of increasing degrees.

4. Density of rational points.

4.1. Multisections. From now on we will work over a number field K and we restict to the case of the base $B = \mathbb{P}^1$.

PROPOSITION 4.1. *[7] Let $\varphi_{\mathcal{E}} : \mathcal{E} \to \mathbb{P}^1$ be an elliptic fibration defined over K with a nt-multisection $\mathcal{M}$. Then for all but finitely many $b \in \varphi_{\mathcal{J}}(\mathcal{M}(K)) \subset \mathbb{P}^1(K)$ the fibers $\mathcal{E}_b$ have infinitely many rational points.*

Proof. Since $\mathcal{M}$ is an nt-multisection, we have a birational map

$$\tau : \mathcal{M} \to \tau(\mathcal{M}) \subset \mathcal{J}(\mathcal{E}).$$

An argument using Merel's theorem (or simply base change to $\tau(\mathcal{M})$) implies that rational points are dense in the fibers $\mathcal{J}_b$ for almost all $b \in \varphi_{\mathcal{J}}(\tau(\mathcal{M})(K))$ (for a sufficiently large finite extension $K/\mathbb{Q}$). Then one can translate points in $(\mathcal{E}_b \cap \mathcal{M})(K)$ (for $b \in \varphi(\mathcal{M}(K))$) to obtain a Zariski dense set of rational points in the fibers $\mathcal{E}_b$ and consequently in $\mathcal{E}$.

COROLLARY 4.2. *Let $S \to \mathbb{P}^1$ be an elliptic K3 surface defined over a number field K. Then rational points on S are potentially dense.*

Proof. By 3.28, every algebraic elliptic K3 surface with $\mathrm{rk}\,\mathrm{Pic}(S) \leq 19$ has infinitely many rational nt-multisections. If $\mathrm{rk}\,\mathrm{Pic}(S) = 20$ we use 4.10.

DEFINITION 4.3. Let $\varphi : \mathcal{E} \to B$ be an elliptic fibration. A *saliently ramified multisection* of $\mathcal{E}$ is a multisection $\mathcal{M}$ which intersects a fiber $\mathcal{E}_b$ at some smooth point p_b with local intersection multiplicity ≥ 2.

PROPOSITION 4.4. *[6] Suppose that $\mathcal{M} \subset \mathcal{E}$ is a saliently ramified rational or elliptic multisection. Then it is an nt-multisection. Consequently, rational points on $\mathcal{E}$ are potentially dense.*

COROLLARY 4.5. *Let S be an algebraic surface admitting two elliptic fibrations over $\mathbb{P}^1$. Then rational points on S are potentially dense.*

REMARK 4.6. An alternative approach to potential density of rational points on elliptic K3 surfaces $\mathcal{E} \to \mathbb{P}^1$ would be to show that there exists an family of elliptic curves "transversal" to the given elliptic fibration. Then a generic elliptic curve in the transversal elliptic fibration is a saliently ramified multisection of $\mathcal{E} \to \mathbb{P}^1$. It remains to apply 4.4.

4.2. Automorphisms. Let X be a K3 surface defined over a number field K. We have a hyperbolic lattice $\mathrm{Pic}(X) := \mathrm{Pic}(X_{\mathbb{C}}) \subset \mathcal{L}$ where $\mathcal{L} = 3 \cdot \mathcal{H} \oplus 2 \cdot (-E_8)$ and a monoid of effective divisors $\Lambda_{\mathrm{eff}}(X) \subset \mathrm{Pic}(X)$. We denote by $\mathrm{Aut}(X)$ the group of (regular) algebraic automorphisms of X (over $\mathbb{C}$). Observe that $\mathrm{Aut}(X)$ is finitely generated. We can guarantee that $\mathrm{Aut}(X)$ is defined over K', for some finite extension K'/K.

REMARK 4.7. V. Nikulin proved that there are only finitely many isomorphism types of lattices $\mathrm{Pic}(X)$ for K3 surfaces with $\mathrm{rk}\,\mathrm{Pic}(X) \geq 3$ such that the corresponding group $\mathrm{Aut}(X)$ is finite (cf. [23]). We can prove potential density for those surfaces from Nikulin's list which contain (semipositive) elements with square zero. For example, there are 17 lattices that give finite automorphism groups $\mathrm{Aut}(X)$ for

$\mathrm{rk}\,\mathrm{Pic}(X) = 4$ and of those 17 lattices 15 contain elements with square zero (and therefore admit elliptic fibrations) (cf. [30], [23]).

EXAMPLE 4.8. There exists a K3 surface of rank 4 with the following Picard lattice:

$$\begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & -2 & 0 & 0 \\ -1 & 0 & -2 & 0 \\ -1 & 0 & 0 & -2 \end{pmatrix}$$

There are no elements or square zero and the group of automorphisms $\mathrm{Aut}(X)$ is finite. We don't know whether or not rational points on X are potentially dense.

LEMMA 4.9. *Suppose that* $\mathrm{Aut}(X)$ *is infinite. Then* $\Lambda_{\mathrm{eff}}(X)$ *is not finitely generated.*

Proof. If suffices to identify $\mathrm{Aut}(X)$ (up to a finite index) with the subgroup of $\mathrm{Aut}(\mathcal{L})$ which preserves $\Lambda_{\mathrm{eff}}(X)$. The set of generators of $\Lambda_{\mathrm{eff}}(X)$ is preserved under $\mathrm{Aut}(X)$. If this set is finite $\mathrm{Aut}(X)$ must be finite as well.

THEOREM 4.10. *Let X be a K3 surface over a number field K with an infinite group of automorphisms. Then rational points on X are potentially dense.*

Proof. It suffices to find a rational curve $C \subset X$ such that the orbit of C under $\mathrm{Aut}(X)$ is infinite. The monoid $\Lambda_{\mathrm{eff}}(X)$ is generated by classes of (-2)-curves, curves with square zero and primitive classes with positive square. It follows from (2.13) that every generator of $\Lambda_{\mathrm{eff}}(X)$ is represented by a (possibly singular) irreducible rational curve. Suppose that orbits of $\mathrm{Aut}(X)$ on the generators of $\Lambda_{\mathrm{eff}}(X)$ are all finite. Then the group $\mathrm{Aut}(X)$ is finite and the number of elements is bounded by a function depending only on the rank of the lattice. (Indeed, any group acting on a lattice of rank n embedds into $\mathrm{SL}_n(\mathbb{Z}_3)$. The normal subgroup of elements in $\mathrm{SL}_n(\mathbb{Z}_3)$ equal to the identity modulo 3 consists of elements of infinite order. Hence any subgroup of the automorphisms of the lattice has a subgroup of finite index which consists of elements of infinite order.) So there exists an element of infinite order. For this element the orbit of some generator of $\Lambda_{\mathrm{eff}}(X)$ is infinite. This class is represented by a rational curve C. The orbit of C is not contained in any divisor in X. Extending the field, if necessary, we can assume that rational points on C are Zariski dense. This concludes the proof.

REMARK 4.11. Certainly, there are algebraic varieties X such that the orbit under $\mathrm{Aut}(X)$ of any given rational point is always contained in a divisor. For example, consider a generic Jacobian elliptic surface $\mathcal{J}$ with a non-torsion group of sections. Then $\mathrm{Aut}(\mathcal{J})$ is generated by the group of fiberwise involutions with respect to the sections. In particular, inspite of the fact that the group is infinite the fibers are preserved and the orbit of any point is contained in a divisor. However, rational points on X are Zariski dense, as there is a rational section of infinite order (in $\mathrm{Aut}(X)$ and in $\mathcal{J}$).

COROLLARY 4.12. *Let X be a K3 surface such that* $\mathrm{rk}\,\mathrm{Pic}(X) \geq 2$ *and* $\mathrm{Pic}(X)$ *contains no classes with square zero and square* (-2). *Then* $\mathrm{Aut}(X)$ *is infinite and rational points on X are potentially dense.*

Proof. The monoid $\Lambda_{\mathrm{eff}}(X)$ is infinitely generated.

REFERENCES

[1] W. Barth, C. Peters, and A. van de Ven, *Compact Complex Surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete 4 (3), Springer, Berlin-New York, 1984.

[2] A. Beauville, *Les familles stables de courbes elliptiques sur $\mathbb{P}^1$ admettant quatre fibres singulières*, C. R. Acad. Sci. Paris, Ser. I Math., 294:19 (1982), pp. 657–660.

[3] A. Beauville, *Le nombre minimum de fibres singulières d'une courbe stable sur $\mathbb{P}^1$*, in Astérisque 86, 1991, pp. 97–108.

[4] A. Beauville, *Counting rational curves on K3 surfaces*, Duke Math. Journ., 97:1 (1999), pp. 99–108.

[5] S. Bloch, *Semi-regularity and deRham cohomology*, Invent. Math., 17 (1972), pp. 51–66.

[6] F. Bogomolov and Yu. Tschinkel, *Density of rational points on Enriques surfaces*, Math. Res. Letters, 5 (1998), pp. 623–628.

[7] F. Bogomolov and Yu. Tschinkel, *On the density of rational points on elliptic fibrations*, Journ. Reine und Angew. Math., 511 (1999), pp. 87–93.

[8] X. Chen, *Rational curves on K3 surfaces*, alg-geom 9804075, 1998.

[9] X. Chen, *Singularities of rational curves on K3 surfaces*, alg-geom 9812050, 1998.

[10] R. Friedman and J. Morgan, *Smooth Four-Manifolds and Complex Surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete 27 (3), Springer, Berlin, 1994.

[11] J. Harris and Yu. Tschinkel, *Rational points on quartics*, alg-geom 9809015, 1998.

[12] V. Iskovskikh and I. Shafarevich, *Algebraic surfaces*, in Current Problems in Mathematics. Fundamental Directions 35, Moscow, 1989, pp. 131–263.

[13] K. Kodaira, *On compact complex analytic surfaces I*, Ann. of Math., 71:2 (1962), pp. 111–152.

[14] J. Kollár, Y. Miyaoka, and Sh. Mori, *Rational curves on Fano varieties*, in Classification of irregular varieties (Trento, 1990), LNM 1515, Springer, Berlin, 1992, pp. 100–105.

[15] Yu. I. Manin, *Notes on the arithmetic of Fano threefolds*, Compositio Math., 85:1 (1993), pp. 37–55.

[16] B. Mazur, *The topology of rational points*, Experiment. Math., 1:1 (1992), pp. 35–45.

[17] L. Merel, *Bornes pour la torsion des courbes elliptiques sur les corps de nombres*, Invent. Math., 124:1-3 (1996), pp. 437–449.

[18] R. Miranda and U. Persson, *Configurations of I_n fibers on elliptic K3 surfaces*, Math. Z., 201 (1989), pp. 339–361.

[19] Sh. Mori and Sh. Mukai, *Classification of Fano 3-folds with $B_2 \geq 2$*, Manuscripta Math., 36:2 (1981/21), pp. 147–162.

[20] Sh. Mori and Sh. Mukai, *The uniruledness of the moduli space of curves of genus 11*, in Algebraic Geometry (Tokyo/Kyoto, 1982), LNMM 1016, Springer, Berlin-New York, 1983, pp. 334–353.

[21] Sh. Mori and Sh. Mukai, *On Fano 3-folds with $B_2 \geq 2$*, in Algebraic varieties and analytic varieties (Tokyo, 1981), Adv. Stud. Pure Math. 1, North-Holland, Amsterdam, 1983, pp. 101–129.

[22] V. V. Nikulin, *On the classification of arithmetic groups generated by reflections in Lobachevskiĭ spaces*, Izv. Akad. Nauk SSSR Ser. Mat., 45:1 (1984), pp. 113–142.

[23] V. V. Nikulin, *Discrete reflection groups in Lobachevsky spaces and algebraic surfaces*, in Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986), Providence, RI, 1987, pp. 654–671.

[24] I. Piatetski-Shapiro and I. Shafarevich, *A Torelli theorem for algebraic surfaces of type K3*, Math. USSR, Izv., 5:3 (1972), pp. 547–588.

[25] Z. Ran, *Hodge theory and deformations of maps*, Compositio Math., 97:3 (1995), pp. 309–328.

[26] Z. Ran, *Deformations of maps*, in Algebraic curves and projective geometry (Trento), LNM 1389, Springer, Berlin, 1989, pp. 246–253.

[27] F. Severi, *Complementi alla theoria della base per la totalità delle curve di una superficie algebrica*, Rend. Circ. Mat. Palermo, 30 (1910), pp. 265–288.

[28] J. Silverman, *Heights and the specialization map for families of abelian varieties*, J. Reine Angew. Math., 342 (1983), pp. 197–211.

[29] Sh.-Li Tan, *The minimal number of singular fibers of a semistable curve over $\mathbb{P}^1$*, J. Algebraic Geom., 4:3 (1995), pp. 591–596.

[30] E. Vinberg, *Discrete reflection groups in Lobachevsky spaces*, in Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983), 1984, pp. 593–601.

[31] S.-T. Yau and E. Zaslow, *BPS states, string duality, and nodal curves on $K3$*, Nuclear Phys. B, 471:3 (1996), pp. 503–512.

[32] Q. Ye, *On extremal elliptic K3 surfaces*, alg-geom 9901081, 1999.