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THE MONODROMY PAIRING*

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For an Abelian variety over a complete local field K, let A' denote its dual and $\Gamma(A)$ its "fundamental group" (see §1 below). In SGA7I, Exposé IX §9, Grothendieck defined a pairing between $\Gamma(A)$ and $\Gamma(A')$ with values in $\mathbf{Z}$, if A has semi-stable reduction, which he called the monodromy pairing. In fact, he proved a p-adic analogue of the Picard-Lefshetz formula when A is a Jacobian of a curve with semi-stable reduction (see §5 below).

Raynaud [R1] also wrote down a definition of a pairing between $\Gamma(A)$ and $\Gamma(A')$ when A has semi-stable reduction using bi-extensions and Grothendieck asserted that Raynaud's pairing is the same as his in Chapter IX, §14.2.5 of SGA7 but gave no details of a proof. We provide a proof in §2 based on Werner's analysis of Raynaud's pairing which uses an observation of Reversat-Van der Put [R-vP].

This result was needed to show, in [CI], that the monodromy operator on the first de Rham cohomology group of a semi-stable curve defined in [pSI] is the same as that of Hyodo-Kato [HK] and it was also used in §3 of [W] to study local heigfht pairings.

The remainder of the paper is a rigid analytic proof of the p-adic Picard-Lefshetz formula based on the aforementioned expression of Raynaud's formula for the monodromy pairing. A sketch of an alternate proof of this result is contained in [FC], Chapter III, Theorem 8.3.

The main new technical result is an explicit relationship, proven in §4, between the rigid residue maps from the regular differentials on a curve with semi-stable reduction over K to K, defined in Reciprocity Laws on Curves [C2], and rigid homomorphisms from $\mathbf{G}_m$ into the Jacobian of the curve.

We first prove, in §5, the Picard-Lefshetz formula in the case of the Jacobians of Mumford curves. Although this is not essential to our ultimate general proof, it provides motivation and in fact we are able to prove more in this case.

In the following, $K \subset \mathbf{C}_p$ will be a finite extension of the completion of the maximal unramified extension of $\mathbf{Q}_p$, R will be its ring of integers, k will be its residue field and π will be a uniformizing parameter of K. We will let v be the valuation on $\mathbf{C}_p$ such that $v(\pi) = 1$. All objects discussed below will be supposed defined over K, unless otherwise indicated.

Generalizing the notions of [C2]: If F is a complete subfield of $\mathbf{C}_p$, by a **discoid** space over F we mean an affinoid which becomes isomorphic to a finite union of disjoint affinoid disks after a finite base extension and by a **wide open space** over F we mean the complement in a smooth complete curve of a discoid subdomain. If W is a wide open space over F the set of **ends** of W is the inverse limit of $CC(W_{\mathbf{C}_p} - X)$ where X runs over the affinoid subdomains of $W_{\mathbf{C}_p}$. The group $Gal(\mathbf{C}_p/F)$ clearly acts on this set. By a **basic wide open pair**, we mean a pair (W, X), where W is a wide open and X is an affinoid in W with good reduction such that after a finite extension the connected components of $W - X$ are annuli and the map from $CC(W - X)$ to the ends of W is a bijection. A wide open which is a member of a basic wide open pair will be called a **basic wide open**. The results and proofs of [C2] carry over easily to this more general context.

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1. Covering Spaces. The following definition is due to Bosch and Lütkebohmert [BL, §8].

DEFINITION. *Suppose $f: W \to Y$ is a morphism of rigid spaces over K. Then W is said to be a* **covering space** *of Y and f is said to be a* **covering map** *if there exists an admissible open covering $\mathcal{C}$ of Y such that for each $U \in \mathcal{C}$, $f^{-1}(U)$ is a disjoint disconnected union of spaces each mapping isomorphically via f onto U. If Z is a connected rigid space with no non-trivial covering spaces, Z is said to be* **simply connected**.

Generalizing the argument of Example 2.5 of [U] one can show that basic wide opens or one dimensional affinoids with good reduction are simply-connected. It follows, in particular, that curves over K with good reduction are simply connected. (This has been generalized to higher dimensions by van der Put in [vP].)

If Y is connected and is admissibly covered by simply connected rigid spaces (this is a special case of what Ullrich [U] calls "semi-locally simply connected") there is connected covering space $\tilde{Y}$ which maps onto any other connected covering space W of Y such that

$$\begin{array}{ccc} \tilde{Y} & & \\ \downarrow & \searrow & \\ Y & \leftarrow & W \end{array}$$

commutes. The space $\tilde{Y}$ is unique up to isomorphism and up to unique isomorphism if a base point of $\tilde{Y}$ is chosen. It is called the **rigid universal cover** of Y. The group of rigid automorphisms of $\tilde{Y}$ over Y is called the fundamental group of Y.

Suppose X is a curve over K with a regular semi-stable model $\mathcal{X}$ over R. Then, if all the irreducible components of $\bar{\mathcal{X}}$ are smooth, the collection of rigid subspaces of X, $\{red^{-1}(A)\}$, where A ranges over the irreducible components of $\bar{\mathcal{X}}$, is an admissible covering of X by simply connected rigid spaces since $red^{-1}(A)$ is a basic wide open or equals X if $\bar{\mathcal{X}}$ is smooth. Thus X has a universal covering $\tilde{X}$ over K. (This is true even if the irreducible components are not smooth. One way to see this is to pass to a ramified quadratic extension K' of K and blow up the singular points of $\mathcal{X}_{R_{K'}}$. This new model will then be regular and we can apply the above analysis to get a universal covering which one can show is actually defined over K.)

We follow Grothendieck [G, §12.3] (except that the edges of our graphs are oriented). Suppose Y is a semi-stable curve over a field L, as in Definition 9.2/6 of [BLR]. In this case, this means that Y is proper and has only ordinary double points as singularities. Let $Gr(Y)$ be the oriented graph whose vertices $ver(Y)$ correspond to the irreducible components of Y, and whose oriented edges, $\mathrm{edg}(Y)$, connecting a vertex a to a vertex b correspond to ordered pairs (x, y) of distinct points on the normalization of Y, Y^n which map to the same point of Y such that $x \in a$, $y \in b$. If $e = (x, y)$ is an edge between a and b, we let $a(e) = a$ and $b(e) = b$.. If X is as above, we set $Gr(X) = Gr(\bar{\mathcal{X}})$, $\mathrm{ver}(X) = \mathrm{ver}(\bar{\mathcal{X}})$ and $\mathrm{edg}(X) = \mathrm{edg}(\bar{\mathcal{X}})$.

If v is a vertex of $Gr(X)$, we let X_v equal the $\mathrm{red}^{-1}(v)$ (this is the tube $]v[$ in the sense of Berthelot) and if $e = (x, y)$ is an edge, $A_e = \mathrm{red}^{-1}(x') =]x'[$, where x' is the image of x on $\bar{\mathcal{X}}$. Then, if $\bar{\mathcal{X}}$ has more than one component, $\mathcal{C} = \{X_v : v \in \mathrm{ver}(X)\}$ is an admissible cover of X by wide opens and if v is smooth,

$$(X_v, X_v - \bigcup_{u \neq v} X_u)$$

is a basic wide open pair. We note that, since $\mathcal{X}$ is regular, any point in $X(K)$ is a point on a unique element of $\mathcal{C}$.

DEFINITION. *If r and s are two points of $X(K)$ we say a* **path** *from r to s is a sequence $(v, e_1, \ldots, e_n, w)$, $n \geq 0$, where $v, w \in ver(X)$, $e_i \in edg(X)$ such that $r \in X_v(K)$, $s \in X_w(K)$, $b(e_i) = a(e_{i+1})$ for $1 \leq i < n$ and $v = w$ if $n = 0$ while $a(e_1) = r$ and $v(e_n) = s$ if $n > 0$.*

We say two paths are **homotopic**, if the paths obtained by successively removing all "doubling backs" (i.e. subsequences of the form $((x,y),(y,x))$) are the same. (Another way to think about this is that we have a natural map of paths on X to paths on $Gr(X)$ and then two paths between the same pair of points in $X(K)$ are homotopic if and only if their images on $Gr(X)$ are in the usual sense.) Now the basic theorem which is an elementary translation of the results in [U] is,

THEOREM 1.1. *Suppose $O \in X(K)$. The points of $\tilde{X}(K)$ are in one-to-one correspondence with pairs (γ, b) where $b \in X(K)$ and γ is the homotopy class of a path from O to b.*

Raynaud [R1] introduced the covering of a semi-stable Abelian variety described in the following theorem and van der Put proved it is the universal cover in [vP]. (See also the comments after the statement of the theorem.)

THEOREM 1.3. *If A is an Abelian variety over K with semi-stable reduction, its universal covering is isomorphic to an extension G of an Abelian variety B with good reduction by a torus T, the covering map from G to A is a homomorphism and its kernel is a free Abelian group Γ of finite rank.*

As Berkovich pointed out, one can also deduce the simply connectedness of Raynaud's covering from Theorem 6.5.1 of [B] and its proof.

While A and $T \to G$, in the above theorem, are algebraic varieties and morphisms, the map $G \to A$ is rigid analytic. The scheme G has a model over R_K whose reduction $\bar{G}$ is an extension of the reduction of B by a torus. Let G^0 be the rigid space associated to the formal completion of this model along $\bar{G}$.

The information in this theorem may be summarized by the following diagram,

$$\begin{array}{ccccc} & & \Gamma & & \\ & & \downarrow & & \\ T & \to & G & \to & B \\ & & \downarrow & & \\ & & A & & \end{array} \tag{1}$$

which we call the **uniformization cross** of A. We also point out that Γ is the fundamental group of A which we denote by $\Gamma(A)$ elsewhere.

Moreover, the free group in the uniformiation cross of A' is $\Gamma' = Hom(T, \mathbf{G}_m)$, the torus is $Hom(\Gamma, \mathbf{G}_m)$ and the Abelian variety with good reduction is the dual of B. The morphisms $\Gamma' \to G'$ and $T' \to G'$ are the one-motive duals of the morphisms $T \to G$ and $\Gamma \to G$.

2. Grothendieck's and Raynaud's Pairings. In this section we prove the equality of the pairing defined by Grothendieck and the one defined by Raynaud.

Let notation be as in te preceding section, so that, in particular, A is an Abelian variety over K with uniformization cross (1.1). Fix a prime ℓ. Let $\Gamma_\ell = \mathbf{Z}_\ell \otimes \Gamma$. We obtain from (1.1) a diagram with exact rows and columns,

$$\begin{array}{ccccccccc} & & 0 & & & & & & \\ & & \downarrow & & & & & & \\ & & \mathcal{T}_\ell(T) & & & & & & \\ & & \downarrow & & & & & & \\ 0 & \to & \mathcal{T}_\ell(G) & \to & \mathcal{T}_\ell(A) & \xrightarrow{\phi_A} & \Gamma_\ell & \to & 0 \\ & & \downarrow & & & & & & \\ & & \mathcal{T}_\ell(B) & & & & & & \\ & & \downarrow & & & & & & \\ & & 0 & & & & & & \end{array}$$

Here, if $\alpha = (\alpha_n)_n \in \mathcal{T}_\ell(A)$ and $(\tilde{\alpha}_n \in G(\bar{K})) \mapsto \alpha_n$, then $\ell^n \tilde{\alpha}_n \in \Gamma$ and

$$\phi_A(\alpha) = \lim_{n \to \infty} \ell^n \tilde{\alpha}_n.$$

Suppose now $\ell \neq p$. There is a natural homomorphism,

$$\rho: I := \operatorname{Gal}(\bar{K}/K) \to \mathcal{T}_\ell(\mathbf{G}_m), \quad \sigma \in I \mapsto ((\pi^{1/\ell^n})^{\sigma - 1})_n.$$

We will henceforth think of $\mathcal{T}_\ell(\mathbf{G}_m)$ as $\mathbf{Z}_\ell(1)$ and use addtive notation when possible. Claim: If $\sigma \in I$ and $\alpha \in \mathcal{T}_\ell(A)$,

$$(\sigma - 1)\alpha \in \mathcal{T}_\ell(T).$$

Indeed, $G'[\ell^n](\overline{K}) \subseteq G'(K)$ and so $\mathcal{T}_\ell(G')^I = \mathcal{T}_\ell(G')$. Suppose $\beta \in \mathcal{T}_\ell(G') \subseteq \mathcal{T}_\ell(A')$. Then, since the points of $\mathbf{G}_m[\ell^n]$ and $G'[\ell^n]$ are defined over K

$$((\sigma - 1)\alpha, \beta)_{Weil} = \sigma(\alpha, \sigma^{-1}(\beta))_{Weil} - (\alpha, \beta)_{Weil} = 0.$$

Thus $(\sigma - 1)\alpha \in \mathcal{T}_\ell(G')^\perp = \mathcal{T}_\ell(T)$.

If

$$(\ ,\ )_\ell: \Gamma_\ell \times \Gamma'_\ell \to \mathbf{Z}_\ell$$

is the extension of scalars of the monodromy pairing, $(\ ,\ )_{Mon}$, defined by Grothendieck then [G, §9]

$$(*) \qquad (\alpha^{\sigma - 1}, \beta)_{Weil} = \rho(\sigma)^{(\phi_A(\alpha), \phi_{A'}(\beta))_\ell}.$$

Now let $\gamma \in \Gamma(A)$ and $\gamma' \in \Gamma(A')$. Then γ' may be thought of as a morphism of T into $\mathbf{G}_m$. Now,

$$T(K)/T^0(K) \cong G(K)/G^0(K)$$

where T^0 and G^0 are the formal completions of T and G along their special fibers and $\gamma'(G^0(K)) \subset R^*$. Thus if $g \in T(K)$ maps to the image of γ in $G(K)/G^0(K)$

$$(\gamma, \gamma')_{unif} := v(\gamma'(g))$$

is a well defined $\mathbf{Z}$-valued pairing. Werner [W] has shown this is the same as the one defined by Raynaud using bi-extensions. We will now show, THEOREM 2.1. $(,)_{unif} = (,)_{Mon}$.

Proof. It is enough to verify this after extending scalars to $\mathbf{Z}_\ell$.
Identify Γ' with $Hom(T, \mathbf{G}_m)$ and so

$$\Gamma'_\ell = Hom_{\mathbf{Z}_\ell}(\mathcal{T}_\ell(T), \mathbf{Z}_\ell(1)).$$

If $\delta \in \mathcal{T}_\ell(T)$ and $\beta \in \mathcal{T}_\ell(A')$, then

$$(\delta, \beta)_{Weil} = \phi_{A'}(\beta)(\delta).$$

Then,

$$((\sigma - 1)\alpha, \beta)_{Weil} = \phi_{A'}(\beta)((\sigma - 1)\alpha).$$

Suppose $\phi_{A'}(\beta) = \gamma' \in \Gamma'$ and $\phi_A(\alpha) = \gamma \in \Gamma$. Write $\gamma = t + g$ where $t \in T(K)$ and $g \in G^0(K)$. Claim: We can pick $\tilde{\alpha}_n \in G(\overline{K})$, $t_n \in T(\bar{K})$ and $g_n \in G^0(K)$ so that $\tilde{\alpha} \mapsto \alpha_n$, $\tilde{\alpha}_n = t_n + g_n$, $\ell t_{n+1} = t_n$ and $\ell g_{n+1} = g_n$. Indeed, we know $\ell^n \tilde{\alpha}_n \equiv \gamma \bmod \ell^\Gamma$. This means we can assume $\ell^n \tilde{\alpha}_n = \gamma$. Let $t_n \in T(\overline{K})$ such that $\ell t_{n+1} = t_n$, $n \geq 0$, and $t_0 = t$. Then,

$$\ell^n(\tilde{\alpha}_n - t_n) = g$$

and so $\tilde{\alpha}_n - t_n \in G^0(K)$. Then

$$(\sigma - 1)\alpha = ((\sigma - 1)\tilde{\alpha}_n)_n = ((\sigma - 1)t_n)_n$$

and

$$\begin{aligned} ((\sigma - 1)\alpha, \beta)_{Weil} &= \gamma'\Big(\big((\sigma - 1)t_n\big)_n \Big) \\ &= \big(\gamma'(t_n)^{\sigma - 1})\big)_n \\ &= v(\gamma'(t_0))\rho(\sigma) \\ &= (\gamma, \gamma')_{unif}\rho(\sigma). \end{aligned}$$

This completes the proof.

3. Statement of p-adic Picard-Lefshetz. Let notation be as in Section 1. In particular, X is a curve with semi-stable reduction. Let $E(X)$ be the free Abelian group on the oriented edges of $Gr(X)$ and $V(X)$ be the free Abelian group on the vertices of $Gr(X)$. Let τ be the automorphism of $E(X)$ which sends $(x, y) \in$ edg(X) to (y, x), let $E^-(X)$ be the quotient group of $E(X)$ by the relation $\tau(s) = -s$. If f is a function on edg(X) such that $f(\tau(e)) = -f(e)$, $\sum'_e f(e)e$ will denote the element in $E^-(X)$ represented by a sum of elements $f(e)e$ where one e is chosen from each pair $\{c, \tau(c)\}$, $c \in$ edg(X). Let $\partial: E(X) \to V(X)$ be the map such that

$$\partial(e) = b(e) - a(e).$$

Then ∂ induces a map from $E^-(X)$ to $V(X)$ and its kernel in this group is naturally isomorphic to $H_1^{Bet}(Gr(X), \mathbf{Z})$. Now we define a pairing on $E(X)$ by setting

$$(e, f) = \begin{cases} 1 & if \;\; \text{e=f} \\ -1 & if \;\; f = \tau(e) \\ 0 & otherwise \end{cases}$$

for edges e and f. This induces a pairing $(,)_{PL}$ on $E^-(X)$ and hence by restriction on $H_1^{Bet}(Gr(X), \mathbf{Z})$. Now $H_1^{Bet}(Gr(X), \mathbf{Z})$ is canonically isomorphic to $\Gamma(J(X))$ where $J(X)$ is the Jacobian of X.

If we base change to a finite extension K' of K of ramification index e, we obtain $Gr(X_{K'})$ by replacing each edge of $Gr(X)$ by a chain of e edges and $e-1$ vertices. Using, this we get a natural isomorphism $\iota: H_1^{Bet}(Gr(X), \mathbf{Z}) \to H_1^{Bet}(Gr(X_{K'}), \mathbf{Z})$ and we have

$$(1) \qquad (\iota(h), \iota(g))_{PL,K'} = e(h,g)_{PL,K}.$$

The p-adic Picard-Lefshetz Theorem, proven by Grothendieck in Chapter IX of SGA7I [G] is,

THEOREM 3.1.

$$(\, , \,)_{J(X),Mon} = (\, , \,)_{X,PL}.$$

The remainder of this paper is devoted to giving a rigid analytic proof of this theorem.

4. Residues. Let X be a curve over K as in §3 whose minimal model $\mathcal{X}$ has semi-stable reduction with smooth irreducible components. Let J be the Jacobian of X and $\alpha: X \to J$ an Albanese morphism. Then in the uniformization cross of J, T may be identified with $\mathrm{Hom}(H_1^{Bet}(Gr(X), \mathbf{Z}), \mathbf{G}_m)$ and so we may identify $\mathrm{Hom}(T, \mathbf{G}_m)$ with $H_1^{Bet}(Gr(X), \mathbf{Z})$. Let f be the rigid morphism from T to J. For an edge e of $Gr(X)$, let Res_e denote the map $Res_{a(e),A_e}: \Omega^1_{X_e/K} \to K$ as described in [C2, page 221].

THEOREM 4.1. *Suppose* $\omega \in H^0(J, \Omega^1_{J/K})$ *and* $h \in H_1^{Bet}(Gr(X), \mathbf{Z})$ *are such that* $f^*\omega = h^* dz/z$. *Then,*

$$h = \sum\nolimits_e' Res_e(\alpha^*\omega)e.$$

Since, using Proposition 4.3 of [C2], we see that if ω is a holomorphic differential on X, over K, $\sum_e' Res_e(\omega)e$ represents a one-cycle on $Gr(X)$ with coefficients in K, it follows that,

COROLLARY 4.1.1. *Let* Λ *be the group of holomorphic differentials* ω *on* X *such that* $Res_e\omega \in \mathbf{Z}$ *for all* $e \in edg(X)$ *and* Λ^0 *be the subgroup of* $\omega \in \Lambda$ *such that* $Res_e\omega = 0$. *Then the map*

$$\omega \mapsto \sum\nolimits_e' Res_e(\omega)e,$$

induces an isomorphism of Λ/Λ^0 *with* $H_1^{Bet}(Gr, \mathbf{Z})$.

We may identify $\mathrm{Hom}(\mathbf{G}_m, J)$ with $H^1_{Bet}(Gr(X), \mathbf{Z})$. If $e \in edg(X)$, let h_e be the class of the one-cycle $f \in edg(X) \mapsto (e, f)$, where $(\, , \,)$ is the pairing defined in section 3. Then an equivalent version of Theorem 4.1 is,

THEOREM 4.2. *Suppose* $e \in edg(X)$ *and* $\omega \in H^0(J, \Omega^1_{J/K})$, *then*

$$Res_e(\alpha^*\omega) = Res_{z=0}(h_e^*\omega).$$

The key result needed for the proof of these theorems is contained in §20 (*partie polaire*), *Chapitre VII of Groupes algebriques et corps de classes* [S].

Let C be a smooth complete curve over an algebraically closed field F and D be a reduced effective divisor on C over F (a *module* in the language of [S]). We'll also abuse notation and also use D to denote the set of points in its support. Let $\mathcal{E}(D)$ be the group of divisors supported on D and $\mathcal{E}^0(D)$ the subgroup of divisors of degree 0. Then, the generalized Jacobian J_D of C with respect to D is an extension of the Jacobian of C by the torus $T_D = \mathrm{Hom}(\mathcal{E}^0(D), \mathbf{G}_m)$. Let $\alpha_D: C - D \to J_D$ be a generalized Albanese morphism.

For each point $P \in D$, let t_P be a rational function on C over F which is a uniformizing parameter at P and is such that $t_P(P') = 1$ if $P' \in D$, $P' \neq P$. Let ψ be the function on the complement U of the support of the union of divisors of the t_P, $P \in D$, into T_D defined by

$$\psi(Q) = (\sum_{P \in D} a_P P \mapsto \prod_{P \in D} t_P(Q)^{a_P}).$$

Then the result Serre proves in [20] may be rephrased, in this case, as,

PROPOSITION 4.3. *The morphism $\alpha_D - \psi$ from U to J_D extends to $U \cup D$.*

Let Y be a semi-stable curve over F. Then, as explained in Example 9.2/8 of [BLR], we have a natural exact sequence,

$$0 \to T_Y \to Pic^0(Y) \to \prod_{A \in ver(Y)} Pic^0(A) \to 0,$$

where $T_Y = \mathrm{Hom}\big(H_1^{Bet}(Gr(Y), \mathbf{Z}), \mathbf{G}_m\big)$. If $A \in ver(Y)$, let m_A denote the inverse image on A of the singular locus of Y and G_A the generalized Jacobian of A with respect to m_A. Then there are natural maps from $H_1^{Bet}(Gr(Y), \mathbf{Z})$ to $\mathcal{E}^0(m_A)$ and $\iota_A: G_A \to Pic^0(Y)$ such that the following diagram commutes,

$$\begin{array}{ccc} \mathrm{Hom}(\mathcal{E}^0(m_A), \mathbf{G}_m) & \longrightarrow & T_Y \\ \downarrow & & \downarrow \\ G_A & \xrightarrow{\iota_A} & Pic^0(Y) \\ \downarrow & & \downarrow \\ Pic^0(A) & \longrightarrow & \prod_{B \in C} Pic^0(B) \end{array}$$

REMARK. *If C is a smooth curve over F and $m_1, m_2, \ldots, m_n$ are disjoint effective divisors on C over F, let $G_{m_1,\ldots,m_n}$ denote the generalized Jacobian of C which classifies $n+1$-tuples $(\mathcal{L}, \iota_1, \ldots, \iota_n)$ where $\mathcal{L}$ is a degree zero invertible sheaf on C and ι_i is a trivialization of $\mathcal{L}$ at m_i, i.e., an isomorphism from $\mathcal{L}/\mathfrak{m}_{m_i}\mathcal{L}$ onto $\mathcal{O}_C/\mathfrak{m}_{m_i i}\mathcal{O}_C$. Here, $\mathfrak{m}_{m_i}$ is the sheaf of ideals whose sections are the sections of $\mathcal{O}_C$ which vanish at m_i. Now, with notation as above, put the following equivalence relation $\sim_A$ on m_A: Say for $e, d \in m_A$, $e \sim_A d$ if and only if the interiors of the edges which correspond to e and d lie on the same connected component of $Gr(X)\backslash\{A\}$. Let $m_1, \ldots, m_r$ be the equivalence classes in m_A with respect to $\sim_A$. Then one can show that the image of G_A in $Pic^0(Y)$ is naturally isomorphic to $G_{m_1,\ldots,m_r}$.*

Let X and $\mathcal{X}$ be as above. Let $v \in ver(X)$, $P \in X(K)$ such that $\bar{P}$ is a smooth point on the image of v in $\bar{\mathcal{X}}$ and suppose $a: X \to J(X)$ is the Albanese morphism from X into its Jacobian such that $a(P) = 0$. Then if $\mathcal{X}^{ns}$ is the non-singular locus of $\mathcal{X}$ and $\mathcal{N}$ is the Néron model of $J(X)$, a extends to a morphism $\alpha: \mathcal{X}^{ns} \to \mathcal{N}$.

Let v' denote the image of $v - m_v$ in $\bar{\mathcal{X}}$. Then, $\bar{\alpha}(v')$ is contained in the connected component of the origin of $\bar{\mathcal{N}}$ which by Theorem 9.5/4 of [BLR] is naturally isomorphic to $Pic^0(\bar{\mathcal{X}})$. Moreover, making this identification, $\bar{\alpha}|_{v'}$ is the composition of the generalized Albanese morphism from v' to $G_{v'}$ sending $\bar{P}$ to 0 and the homomorphism ι_v discussed above.

REMARK. *From this and the previous remark it follows that $\bar{\alpha}$ restricted to v' is a closed immersion if the genus of X is positive and all the sets m_i have cardinality at least 2, (or equivalently, if all the singular points on the image of v are* *non-disconnecting double points). We make this more explicit in the special case of Mumford curves below (see Theorem 5.1). In §9 of [E], Edixhoven studied to what extent α is a closed immersion. In particular, his results imply that α is a closed immersion if X has genus at least one and all the singularities of $\bar{X}$ are non-disconnecting double points.*

End of proof of Theorem 4.1.

We may and will suppose the reduction of $\mathcal{X}$ has more than one component. Let v be a vertex of $Gr(X)$ and Y_v the affinoid in X_v equal to $X_v - \bigcup_{w \in ver(X), w \neq v} X_w$. Then, as we remarked above (X_v, Y_v) is a basic wide open pair and $X_v - Y_v$ is a disjoint union of annuli which we call ends. In fact, it equals

$$\bigcup_{\substack{e \in edg(X) \\ a(e)=v}} A_e.$$

Moreover, the reduction of Y_v is naturally isomorphic to v'. Let $\alpha: X \to J$ be the Albanese morphism sending P to 0. There exists a unique lifting $\tilde{\alpha}: X_v \to G$ which takes P to $0 \in G$. We can complete X_v to a complete smooth curve C with good reduction isomorphic to v by gluing in affinoid disks as in the proof of Proposition 3.3 (ii) of [C2]. For each $e \in m_v$, let t_e be a rational function on C regular on the residue classes corresponding to the the elements of m_v (which we shall label $\bar{e}$, for $e \in m_v$) and invertible on the annulus A_e such that $\bar{t}_e$ is a uniformizing parameter at $\bar{e}$ and $\bar{t}_e(\bar{f}) = 1$, if $f \in m_v$ and $f \neq e$. Let U be the rigid subspace of X_v where the t_e are defined and invertible and for an edge c of $Gr(X)$ define the function on U,

$$z_c = \begin{cases} t_c & \textit{if } \text{a(c)=v} \\ t_{\tau(c)}^{-1} & \textit{if } \text{b(c)=v} \\ 1 & \textit{otherwise} \end{cases}$$

and let $\psi: U \to \mathrm{Hom}(H_1(Gr(X), \mathbf{Z}), \mathbf{G}_m)$ be defined by

$$\psi(Q) = \Big(\sum_{e \in edg(X)} a_e e \mapsto \prod_{e \in edg(X)} z_e(Q)^{a_e} \Big).$$

Let $\bar{Z}$ be the affine open of v' where $\bar{\psi}$ is regular (it is non-empty) and Z the affinoid subdomain of Y_v above it. Let $g: U \to G$ be the morphism $\tilde{\alpha} - \psi$. We know $g(Z) \subset G^0$ (the rigid space associated to the completion $\hat{G}$ of G along its reduction. In fact, if $\pi: G \to J$ is the natural map $\pi|_{G^0}$ is an isomorphism of G^0 with the connected component of the origin in the rigid space associated with the formal completion of $\mathcal{N}$ along its reduction. In fact,

PROPOSITION 4.4. *There exists a wide open neighborhood W of Z in U such that $g(E)$ is contained in a residue class of G^0 for each end E of W contained in an end of X_v.*

We will prove this below. (We can show that, in fact, $g^{-1}(G^0)$ contains A_e for all $e \in m_v$). Thus if V_e is the annulus $A_e \cap W$ and if ω is any invariant differential on G, since the residue classes of $\hat{G}$ are open disks, $g^*\omega$ is exact on V_e. Thus for any orientation on V_e,

$$Res_{V_e} g^*\omega = 0,$$

for all e such that $a(e) = v$. As this residue also equals $Res_e g^*\omega$, we see that, with notation as in the statment of theorem,

$$Res_c a^*\omega = Res_c(\psi^*(h^*\frac{dz}{z}))$$
$$= Res_c(\sum_{e\in edg(X)} a_e\frac{dz_e}{z_e}) = a_c - a_{\tau(c)},$$

if h is represented by $\sum a_e e$, for all $c \in m_v$, (see §II of [C2]). This completes the proof.

Proof of Proposition 4.4.
To prove the proposition, for each $e \in m_v$, choose rigid functions, $f_{e,1},\dots,f_{e,d}$, on the open affinoid in G^0 above an affine neighborhood of $\bar{g}(e)$ in $Pic^0(\mathcal{X})$ which reduce to a system of local parameters at $\bar{g}(e)$. Let γ_e be a lifting of $\bar{g}(e)$ to a point $G^0(K)$. Now use Proposition 4.3 and the following lemma applied successively to each of $f_{e,i}\circ g - f_{e,i}(\gamma_e)$.

LEMMA 4.5. *Suppose (B,Y) is a basic wide open pair over K and h is a rigid function on B and $|h|_Y \le 1$. Let C be the completion of $\bar{Y}$ to a smooth curve over k. Suppose $e \in C - \bar{Y}$ and $\overline{h|_Y}$ extends to $\bar{Y}\cup\{e\}$ so that $\overline{h|_Y}(e) = 0$. Then, there exists a wide open neighborhood W of Y in B such that if E is the end of B corresponding to e, $h(E\cap W)\subseteq B(0,1)$.*

Proof. Let $\tilde{C}$ be a smooth curve over K with reduction C obtained from B by gluing in affinoid disks and B_e the residue disk of C above e. Then, there is a rational function z on C (a function which reduces to a uniformizing parameter at e) and $r \in |\mathbf{C}_p|$, $1 > r > 0$ such that

$$E(\mathbf{C}_p) = \{x \in B_e(\mathbf{C}_p): 1 > |z(x)| > r\}.$$

If we expand h in z,

$$h(z) = \sum_{-\infty}^{\infty} a_n z^n,$$

then the hypotheses that h is a rigid function on B over K and $|h|_Y \le 1$ imply $a_n \in K$, $|a_n| \le 1$ and

$$\lim_{n\to-\infty} |a_n|s^n = 0, \qquad \text{if } 1 > s > r.$$

The hypothesis that $\bar{h}(e) = 0$ implies $|a_n| < 1$ for $n \le 0$. Now suppose $1 > t > r$, then we know there exists an $N > 0$ such that $|a_{-n}|t^{-n} < 1$, for all $n \ge N$. Let,

$$u = \max_{1<n<N} |a_{-n}|^{1/n}.$$

Then $u < 1$. It follows that on the annulus A in E, $1 > |z| > \max\{t,u\}$, $|h| < 1$. Thus we may take $W = (W - E)\cup A$. □

5. Toric Jacobians. Let X be a regular Mumford curve of genus at least one over K (which is a finite extension of $\mathbf{Q}_p^{nr}$) such that all the components of its reduction are smooth, and let J be its Jacobian. Then we have

$$0 \to \Gamma \to T \to J \to 0,$$

where $T = \mathrm{Hom}(H_1^{Bet}(Gr,\mathbf{Z}),\mathbf{G}_m)$. We regard X as embedded in J via an Albanese map so that $0 \in X$. Let $\tilde{X}$ be the universal covering of X. If $\tilde{0}$ is a point of $\tilde{X}$ above

0 we have a unique map ι from $\tilde{X}$ to T so that $\tilde{0} \mapsto 0_T$ and the following diagram commutes

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\iota} & T \\ \downarrow & & \downarrow \\ X & \to & J \end{array}$$

We know that $\mathrm{Hom}(T, \mathbf{G}_m)$ is naturally isomorphic to a lattice inside $H^0(J, \Omega^1_J)$ ($f \mapsto f^*(dt/t)$, which is a form invariant under the action of Γ so comes from J) and we know, by Corollary 4.1.1, that the pullback of this lattice to X via ι^* is the lattice Λ of differentials inside $H^0(\Omega^1_X)$ such that $Res_e(\omega) \in \mathbf{Z}$, for $e \in$ edg(X) . We also have, by Corollary 4.1.1 an isomorphism of Λ with $H_1^{Bet}(Gr, \mathbf{Z})$,

$$\omega \mapsto f_\omega := \sum\nolimits_e' Res_e(\omega) e.$$

We will now make explicit the map from $\tilde{X}$ to T.

Suppose $c \in \tilde{X}(K)$ corresponds to a path

$$\gamma = (v, e_1, \ldots, e_n, w)$$

from 0 to b. We will produce a homomorphism from Λ into $\mathbf{G}_m$. Let $X_i = X_{b(e_i)}$ for $1 \le i \le n$ and $X_0 = X_v$. Suppose $\omega \in \Lambda$, then we know that on X_j there is an invertible function h_j such that

$$\omega = dh_j/h_j.$$

We can choose h_j so that $h_0(0) = 1$ and $h_i = h_{i+1}$ on A_{e_i} (the annulus corresponding to e_i) for $1 \le i \le n$. Then the image of c in $T(K)$ is the homomorphism which takes ω to $h_n(b)$.

It follows that the image of the fundamental group (homotopy classes of paths which begin and end at 0 and which is isomorphic to $\pi_1(Gr(X))$) is contained in Γ. Since this map is a homomorphism (which one sees after a moment's reflection) it is surjective because X generates J and it factors through the Abelianization of the fundamental group which is $H_1^{Bet}(Gr, \mathbf{Z})$.

We can make this last statement more explicit.

Suppose $g \in H_1^{Bet}(Gr, \mathbf{Z})$ is represented by $g = \sum_e a_e e \in E(X)$ and $\omega \in \Lambda$. Suppose, for a vertex v, k_v is an invertible function on X_v such that $dk_v/k_v = \omega$ on X_v. Then for an edge e, $c_e =: (k_{a(e)}/k_{b(e)})\big|_{A_e}$ is in K. Set

$$Exp \int_g \omega =: \prod_{e \in edg(X)} c_e^{a_e}.$$

This is well defined because $c_e = c_{\tau(e)}^{-1}$. Then $\omega \mapsto Exp \int_g \omega$ is the element of $\mathrm{Hom}(\Lambda, \mathbf{G}_m)$ to which g maps. (I am trying to avoid any confusion caused by the fact that Λ and Γ are canonically isomorphic.) In fact, if λ is the homomorphism from T to $\mathbf{G}_m$ such that $d\lambda/\lambda$ (considered as a form on J) pulls back to ω, and γ is the image of g in T,

$$\lambda(\gamma) = Exp \int_g \omega.$$

One can see this because the function h_j above is just the restriction of λ to the appropriate image of $X_{i(x_j)}$.

The Picard-Lefswhetz formula, in this case, comes down to the computation

$$v(\lambda(\gamma)) = v(Exp \int_g \omega) = \sum_e a_e v(c_e)$$
$$= \sum_e a_e Res_e(\omega),$$

by the Lemma 7.1, since we may suppose $k_i \in A^0(X_i)$. But this latter is just $(g, f_\omega)_{PL}$ by Theorem 4.7.

Using the above considerations we can translate the arguments in [U],

THEOREM 5.1. *If X is a Mumford curve, every edge of whose graph lies in a cycle, and Y is a component of the non-singular locus of the reduction of its minimal model $\mathcal{X}$, the morphism from Y to the reduction of the Néron model of the Jacobian of X_K is a closed immersion.*

Proof. Let X^0 be the formal completion of the non-singular locus of $\mathcal{X}$ along its reduction, Y^0 the connected component of X^0 corresponding to Y and J^0 be the formal completion of the Néron model of J along its reduction. Let Z be the connected component of J^0 to which Y^0 maps. We must show the map from Y to Z is a closed immersion. It is enough to show that the morphism from Y^0 to J^0 is a closed immersion (in fact, this is precisely the result needed in the proof of Theorem 3.17 in [CKR]).

Now, Y^0 is isomorphic to the formal completion of $\mathbf{P}^1_R - S$ along its reduction where S is a finite subset of $\mathbf{P}^1_k(k)$ which corresponds to the set of singular points of $\bar{\mathcal{X}}$ which lie on $\bar{Y}$. After a translation, we may suppose Z is the connected component of the origin in J^0 and so is isomorphic to the formal completion of T along its reduction.

It follows from the results in §4 that if f: edg$(X) \to \mathbf{Z}$ is a one-cycle, i.e., if $f(e) = -f(\tau(e))$ and for all $x \in ver(X)$, $\sum_{a(e)=x} f(e) = 0$, then there exists a homomorphism h from T to $\mathbf{G}_m$ such that

$$Res_e \iota^*(\mathrm{d}h/h) = f(e).$$

Since every edge e of $Gr(X)$ lies in a cycle, there exists an $h \in \mathrm{Hom}(T, \mathbf{G}_m)$ such that

$$Res_e(\mathrm{d}(\iota^* h)/(\iota^* h)) = 1. \tag{2}$$

Now these h induce homomorphisms h^0 from T^0 into $\mathbf{G}^0_m$ and so $\iota^* h^0$ is a formal function on Y and because of (2), $R\langle\{\iota^* h^0 \colon h \in \mathrm{Hom}(T, \mathbf{G}_m)\}\rangle$ is the ring of formal functions on Y. This completes the proof. □

6. The Key Lemma.

LEMMA 6.1. *Suppose $h \colon Y \to B$ is a rigid analytic morphism over K from an absolutely irreducible affinoid subdomain of $\mathbf{B}^1_K$ into an Abelian variety over K with good reduction. Then $h(Y)$ is contained in a residue class of $\bar{B}$ corresponding to a point of $B(k)$ (i.e. h factors through the inclusion of a residue class).*

Proof. Let $\bar{B}$ be the reduction of B and suppose $\bar{U}$ is an affine open in $\bar{B}$. Let U be the affinoid in B which is the reduction inverse of $\bar{U}$. If $h(Y)$ is contained in U, then since $h \colon Y \to U$ is a morphism of affinoids, it has a reduction $\bar{Y} \to \bar{U}$ and hence h reduces to a morphism from $\bar{Y}$ to $\bar{B}$. Since the reduction of Y is isomorphic to several affine opens of $\mathbf{A}^1_k$ crossing at a point, as we can see using Theorem 9.7.2/2 of [BGR], and there are no non-constant rational maps from $\mathbf{P}^1$ into an Abelian variety by the corollary of Lemma 3.9/7 in [S], we conclude $h(Y)$ is contained in a residue class.

Now suppose $\{\bar{U}_i\}$ is a finite affine open cover of $\bar{B}$ and $\{U_i\}$ is the corresponding affinoid cover of B. Then $X_i = h^{-1}(U_i)$ is an affinoid subdomain of Y. After a finite extension of K we may suppose that X_i is a union of finitely many absolutely irreducible affinoids. By the above argument h takes each of these affinoids into a residue class. Since residue classes are pairwise disconnected and Y is connected, h must take all of these affinoids and hence Y into a single residue class. It must correspond to a point of B defined over k since it is clearly Galois stable. □

Since annuli (wide open, half wide open or affinoid) may be admissibly covered by an increasing union of affinoids satisfying the hypotheses of Lemma 6.1, we obtain,

COROLLARY 6.1.1. *If $h: A \to B$ is a rigid morphism from an annulus to an Abelian variety B with good reduction, then the image of h is contained in a residue class.*

Suppose (X, Z) is a basic wide open pair over K and G is an extension of an Abelian variety B with good reduction by a torus T and we have the rigid picture

$$\begin{array}{ccc} & & T \\ & & \downarrow f \\ X & \xrightarrow{g} & G \\ & & \downarrow \pi \\ & & B \end{array}$$

Let $\mathcal{E}^0(X)$ denote the subgroup of the free Abelian group on the ends of X consisting of elements of degree 0. If η is a differential on X, let $Res_X(\eta) =: \sum_e Res_e(\eta)e$ where e runs over the ends of X. It follows that,

LEMMA 6.2. *Suppose we have an invariant differential ω on G such that $f^*\omega$ is a logarithmic differential on T, then $Res_X(g^*\omega) \in \mathcal{E}^0(X)$ and is 0 if $f^*\omega = 0$.*

Proof. We may assume X is a wide open annulus. In this case, by Corollary 6.1.1, $\pi(g(X))$ is contained a residue class U of B which after translation we may suppose to be the kernel of reduction. Now $G_U \cong U \times T$ and the hypotheses imply that the restriction of ω to G_U is of the form $\alpha + d\lambda/\lambda$ where α is an exact differential and λ is an invertible function on G_U. The first assertion follows. If $f^*\omega = 0$, then ω is the pullback of an invariant differential η on B and so is exact on G_U. The second assertion follows. □

COROLLARY 6.2.1. *There is a positive integer N which only depends on X so that if ω is an invariant differential on G whose pullback to T is logarithmic there is a unit h in $A(X)$ such that*

$$Res_X(Ng^*\omega - \frac{dh}{h}) = 0. \tag{1}$$

Proof. Let M be the exponent of the group of divisor classes represented by divisors of degree 0 supported at ∞ on $\bar{X}$. Suppose C is a complete curve with good reduction with the property that X is isomorphic to the complement in C of a discoid subdomain. For each end e of X, let D_e be the corresponding disk in this subdomain and choose a point $P_e \in D_e(K)$. Choose an end e_0. Embed C in its Jacobian J so that P_{e_0} goes to the origin. The image of D_{e_0} in the Jacobian J of C generates an open subgroup U of the kernel of reduction J_0. Then, since K is discretely valued there exists a positive integer r so that $p^r J_0(K) \subseteq U$. Let $\sum_e n_e e \in \mathcal{E}^0(X)$. It follows that the class of $M \sum_e n_e P_e$ lies in $J_0(K)$ and hence the class of $N \sum_e n_e P_e$ lies in U

where $N = p^r M$ and so there is a divisor E of degree 0 supported in D so that

$$p^r M \sum_e n_e P_e = (h)$$

for some function h on C. Hence,

$$Res_X(\frac{dh}{h}) = N \sum_e n_e e. \ \square$$

KEY LEMMA 6.3. *Suppose* $\omega \in Inv_K(G)$ *and* $f^*\omega = d\lambda/\lambda$ *where* $\lambda \in Hom_K(T, \mathbf{G}_m)$. *Suppose* $P, Q \in X(K)$ *and* $a, b \in G^0(K)$ *such that* $P' := g(P) - a$ *and* $Q' := g(Q) - b$ *lie in* $T(K)$. *Then if* $h \in A(X)^*$ *satisfies (1),*

$$v\big(h(P)/h(Q)\big) = Nv\big(\lambda(P')/\lambda(Q')\big).$$

(Note that the left hand side of this equation doesn't change if h is replaced by kh where $Res_X(dk/k) = 0$.)

Proof. First we show,

SUBLEMMA 6.4. *Suppose* w *is a morphism from* X *into* G. *Then, there exists a* $t \in T(K)$ *such that* $w(X) \subset t + G^0$ *if and only if* $Res_X(w^*\eta) = 0$ *for all invariant differentials* η *on* G.

Proof. Let W be one of the connected components of $X - Z$. Then W is an annulus. It follows from Corollary 6.1.1 that $\pi(w(W))$ is contained in a residue class D_W of B. Since D_W is a polydisk, the T^0-torseur $G^0_{D_W}$ over D_W splits as does the T-torseur G_{D_W} over D_W. I.e., $G^0_{D_W} \cong D_W \times T^0$ and $G_{D_W} \cong D_W \times T$ compatibly.

Suppose $w(X) \subset G^0$. As $G^0_{D_W} \cong D_W \times T^0$ for every connected component W of $X - Z$ and every function on a wide open annulus into rigid $\mathbf{G}^0_m$ reduces to a constant, it follows that $w(W)$ is contained in a residue class of G^0 and since every invariant differential η of G is exact on any residue class of G^0, it follows that $Res_X(w^*\eta) = 0$.

By Theorem 6.2 of [BS], the formal Néron model (see Definition 1.1 of [BS]) of G is the formal completion along its reduction of its tft Néron model. In particular, its generic fiber is the union of all translates of G^0 by elements in $G(K)$. Thus, by the rigid analytic Néron mapping property, since Z is an affinoid with good reduction, $w(Z) \subseteq t + G^0$ for some $t \in G(K)$. So, after a translation by an element of $T(K)$, we may assume $w(Z) \subset G^0$. We will now assume $Res_X(w^*\eta) = 0$ for all invariant differentials η on G and prove $w(X) \subset G^0$.

Suppose W is a connected component of $X - Z$. The hypotheses imply that every closed differential on $D_W \times T$ pulls back to an exact differential on W. This, in turn, implies that there exists a residue class V in $G^0_{D_W}$ and an $s \in T(K)$ such that $w(W) \subset s + V$. Indeed, suppose $h_1, \ldots, h_d$ are homomorphisms from T onto $\mathbf{G}_m$ such that

$$\tau =: (h_1, \ldots, h_d): T \to \mathbf{G}^d_m$$

is an isomorphism. Identify T with $\mathbf{G}^d_m$ via τ. Then $\log(h_i)$ is analytic on the image of W for all i. This means that there exists $r \in \mathbf{G}^d_m(K)$ such that $w(W) \subset D \times rB(1,1)^d$, since the domains of analyticity of the logarithm are the disks $uB(1,1)$ for $u \in \mathbf{C}^*_p$. Since G^0 and $s + G^0$ are disconnected if $s \in T(K) - T^0(K)$ while W is connected to Z, we must have $w(W) \subset G^0$. Since this is true for all connected components of $X - Z$, we see that $w(X) \subset G^0$ as claimed. $\square$

We can always change any morphism w from X into G without changing the projection to B by adding a morphism r of X into T. Suppose $\eta \in Inv(G)$, then

$$(w + f \circ r)^*\eta = w^*\eta + r^*(f^*\eta).$$

Now we note, that if n is an integer and η is an invariant differential on G, $(ng)^*\eta = n(g^*\eta)$.

SUBLEMMA 6.5. *Let $\Gamma' = Hom(T, \mathbf{G}_m)$. Consider the subgroup A of* $\mathrm{Hom}(\Gamma', \mathcal{E}^0(X))$ *consisting of elements of the form*

$$(\gamma \in \Gamma') \mapsto Res_X r^*\big(\gamma^* \frac{dz}{z}\big),$$

where r runs over morphisms from X into T. Then, $A \supseteq NHom(\Gamma', \mathcal{E}^0(X))$.

Proof. Suppose $\gamma_1, ..., \gamma_d \in \Gamma'$ such that $\delta =: (\gamma_1, ..., \gamma_d): T \to \mathbf{G}_m^d$ is an isomorphism. Let $\alpha \in \mathrm{Hom}(\Gamma', \mathcal{E}^0(X))$. Then by the proof of Corollary 6.2.1, for each $1 \le i \le d$ there is an invertible function f_i on X such that $Res_X df_i/f_i = N\alpha(\gamma_i)$. Let $r = \delta^{-1}(f_1, \ldots, f_g)$. Then, for $\gamma \in \Gamma'$

$$N\alpha(\gamma) = Res_X(r^*(\gamma^* \frac{dz}{z})). \ \square$$

Since, if η is an invariant differential on G, $Res_X(g^*\eta)$ only depends on the image of η in $Inv(T)$, by Lemma 6.2, and the image of Γ' in $Inv(T)$ (by $\gamma \mapsto \gamma^*(dT/T)$) is a lattice, we deduce from the previous lemma that there exists a morphism t from X into T such that

$$Res_X(t^*(f^*\eta)) = NRes_X(g^*\eta)$$

for all invariant differentials η on G. Thus if $g_0 = Ng - f \circ t$, we conclude from Sublemma 6.4 that $g_0(X) \subset G^0$. Also, $\pi \circ g_0 = \pi \circ Ng$.

Then,

$$\begin{aligned} Ng^*\omega &= (Ng)^*\omega \\ &= t^*(f^*\omega) + g_0^*\omega \\ &= t^*\frac{d\lambda}{\lambda} + g_0^*\omega. \end{aligned}$$

As $Res_X(g_0^*\omega) = 0$, $g_0(X) \subset G^0$ and so

$$Res_X \frac{dt^*\lambda}{t^*\lambda} = Res_X \frac{dh}{h}.$$

Since $t^*\lambda/h$ is a unit on X the value of $v(t^*\lambda/h)$ is constant, say equal to r, on Z. By Lemma 2.2 of [C2], this function is constant on each end and so by Corollary 3.7b of [C2] it follows that it is equal to r on X. As $v(\lambda(t(P)) = v(\lambda(P'))$, this concludes the proof of the key lemma. $\square$

7. The general case. Suppose now that X is a curve with a regular semi-stable model and whose Jacobian J has the uniformization cross:

$$\begin{array}{ccccc} & & \Gamma & & \\ & & \downarrow & & \\ T & \to & G & \to & B \\ & & \downarrow & & \\ & & J & & \end{array}$$

If $a \in G(K)$ and λ is a homomorphism from T to $\mathbf{G}_m$, let $(v\lambda)(a)$ denote $v(\lambda(a-b))$ for any $b \in G^0(K)$ such that $a - b \in T(K)$.

Let Λ be the subgroup of holomorphic differentials ω on X such that $Res_e\omega \in \mathbf{Z}$ for all edges e. By Corollary 4.1.1, this is the pullback of the group of invariant differentials on J whose pullback to T is the logarithmic derivative of a homomorphism from T into $\mathbf{G}_m$.

Suppose $c \in \tilde{X}$ corresponds to a path, $\gamma =: (v, e_1, \ldots, e_n, w)$, from 0 to b. Let $X_i = X_{b(e_i)}$ for $1 \le i \le n$ and $X_0 = X_v$. Let $\omega \in \Lambda$, then we know, by Corollary 6.2.1, that there exists a positive integer N and for each $0 \le j \le n$ an $h_j \in A(X_j)^*$ such that

$$Res_{X_j}(N\omega - dh_j/h_j) = 0.$$

We can choose h_j so that $h_0(0) = 1$ and $v(h_i/h_{i+1}) = 0$ on A_{e_i}, for $0 \le i < n$. Then our Key Lemma implies that if $\lambda : T \to \mathbf{G}_m$ is the homomorphism which corresponds to ω and d is the image of c in G, then

$$v(h_n(b)) = N(v\lambda)(d).$$

Suppose $u \in H_1^{Bet}(Gr, \mathbf{Z})$ is represented by $\sum_e a_e e$ and $\omega \in \Lambda$. Suppose, for $v \in ver(X)$, $k_v \in A(X_v)^*$ and $Res_{X_v}(dk_v/k_v - N\omega) = 0$. It follows that for an edge e, $k_{a(e)}/k_{b(e)}$ is a function on the annulus A_e with constant valuation v_e. In fact,

LEMMA 7.1. *Suppose e is an edge of $Gr(X)$, ω is a differential on $X_{a(e)} \cup X_{b(e)}$, $h_c \in A(X_{c(e)})^* \cap A^0(X_{c(e)})$ and*

$$Res_e\left(N\omega - \frac{dh_c}{h_c}\right) = 0$$

for $c \in \{a, b\}$. Then h_a/h_b has constant valuation on A_e equal to $NRes_e\omega$.

Proof. Let $n = Res_e(\omega) = -Res_{\tau(e)}(\omega)$. Let z be a uniformizing parameter on $X_{[x,y]}$ such that $|z(x)|$ decreases to 1 as x approaches $X_{a(e)}$. Since, X is regular, $|z(a)|$ increases to $|\pi|$ as x approaches $X_{b(e)}$. It follows from Lemma 2.2 of [C1] and the proof of Lemma 2.1 of [C2] that on A_e, $h_a = z^{nN}f$ and $h_b = (\pi/z)^{-nN}g$ where f and g are units on X_e of constant absolute value 1. Hence,

$$v(h_x/h_y) = v(\pi^{nN}f/g) = NRes_e(\omega),$$

as claimed. □

Set

$$S_u\omega := \frac{1}{N}\sum_{e \in edg(X)} a_e v_e.$$

In fact, if λ is the homomorphism from T to $\mathbf{G}_m$ such that $d\lambda/\lambda$ is the pullback of a form on J which pulls back to ω, and γ is the image of u in G,

$$(\gamma, \lambda)_{Mon} = (v\lambda)(\gamma) = S_u\omega.$$

The Picard-Lefshetz formula in the not necessarily toric split case follows from the computation

$$\begin{aligned}(v\lambda)(\gamma) = S_u\omega &= \frac{1}{N}\sum_{e \in edg(X)} a_e v_e \\ &= \sum_{e \in edg(X)} a_e Res_e(\omega),\end{aligned}$$

by Lemma 7.1, since we may suppose $k_i \in A^0(X_i)$. But this latter is just

$$(u, \sum\nolimits_e' Res_e(\omega)e)_{PL},$$

and the right hand side is the element of $H_1^{Bet}(Gr, \mathbf{Z})$ corresponding to λ by Theorem 4.1.

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