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ON THE NEGATIVITY OF KERNELS OF KODAIRA-SPENCER MAPS ON HODGE BUNDLES AND APPLICATIONS*

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Let X denote a smooth complex projective variety, and $S = \sum_i S_i \subset X$ a normal crossing divisor with smooth components. We consider a polarized variation of Hodge structure $\mathbb{V}_0$ on $X_0 = X \setminus S$ and the period map

$$\phi : X_0 \to \mathcal{D}/\Gamma$$

corresponding to $\mathbb{V}_0$.

THEOREM 0.1. *Suppose that ϕ is injective at some points. Let $D \subset X_0$ denote the degeneration locus of ϕ. Then*

i) *$\Omega^1_X(logS)$ is weakly positive on $X_0 \setminus D$ in the sense of Viehweg [V1], [V3]. i.e. for any ample line bundle L on X and for any positive number α there exists some positive number β such that the symmetric tensor*

$$S^{\alpha\beta}(\Omega^1_X(logS)) \otimes L^{\beta}|_{X_0 \setminus D}$$

is generated by global sections on X.

ii) *the Kodaira dimension of* $\det \Omega^1_X(logS)$ *is equal to* $\dim X$. *i.e.* (X, S) *is of logarithmic general type.*

Viehweg recently asked that if the cotangent bundles of moduli varieties of polarized varieties with log-pole along infinity are positive in some sense. It is well known that the moduli space of curves of genus $g \geq 1$ has the semi-positivity property. Theorem 0.1 shall give a positive answer to this question in the case when some Torelli map on the universal family is locally injective. It is extremely very interesting to know if that will still be true without existence of locally injective Torelli maps at all!

In particular, Theorem 0.1 recovers the following two special cases, which were known previously. Let $X_0 =$ an arithmetic variety $\mathcal{D}/\Gamma$, and X is a smooth compactification of X_0. Then Mumford [M] proved that (X, S) is of logarithmic general type by using automorphic forms, which was studied in [BB]. Recently, Luo [L] showed that the moduli spaces of polarized Calabi-Yau manifolds are of log general type. Luo used the method in [SY] and the general Schwarz lemma of Yau [Y]. Our method here is essentially semi-positivity of $\Omega^1_X(\log S)$, and Kawamata-Viehweg vanishing theorem. The author believes that one can also use the methods of [SY] and [Y] to prove ii).

Inspired by a recent paper of Bedulev-Viehweg [BV] Jost and the author [JZ] recently proved similar inequalities, the so called Arakelov type inequalities for Hodge bundles over curves (also see [P2]). The inequalities states that the degree of Hodge bundles are bounded above by the degree of the log-canonical bundles of the curve C multiplies some constants, which depends on the Hodge type of E only, i.e.

$$\deg E^{p,q} \leq r^{p,q}(2g(C) - 2) + \#S).$$

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Those inequalities recover the original Arakelov inequality for families of semi-stable algebraic curves over curves [A]. And they also improve the Faltings inequality for families of semi-stable abelian varieties over curves [F]. Here we have Arakelov-type inequalities for Hodge bundles over any dimensional varieties.

THEOREM 0.2. *Under the same assumption as in Theorem 0.1, and suppose that the monodromy around the components of S are unipotent. Let (E,θ) denote the canonical extension of (E_0,θ_0). Then after a blowing up $\tau:(\hat{X},\hat{S})\to(X,S)$, there exist a positive rational number $r^{p,q}$, which is bounded above by the Hodge type of (E,θ), an effective $\mathbb{Q}-$divisor $P^{p,q}$ and a semi-negative $\mathbb{Q}-$divisor $N^{p,q}$ on $\hat{X}$, such that*

$$\det\tau^*E^{p,q}=r^{p,q}\det\tau^*\Omega^1_X(\log S)-P^{p,q}+N^{p,q}.$$

DEFINITION 0.1. A divisor N on X is called semi-negative, if there exists a Hermitian metric h on the line bundle $\mathcal{O}_X(N)$ possibly degenerates along a subvariety of X, such that the Chern form $c_1(N,h)$ of h is negative semi definite, which represents the Chern class of $\mathcal{O}_X(N)$, and for any morphism $f:Z\to X$ one has

$$\int_Z\wedge^{\dim Z}(-f^*c_1(N,h))\geq 0.$$

Set $L^{p,q'}:=P^{p,q}-N^{p,q}$. Then $L^{p,q'}=\tau^*L^{p,q}$, and the line bundle $L^{p,q}$ is weakly positive on some Zariski open subset of X.

In his paper [G1] Griffiths defined the so-called canonical bundle $K(E_0)$ of the variation of Hodge structure $\mathbb{V}_0$ by:

$$K(E_0)=(\det E_0^{k,0})^k\otimes(\det E_0^{1,k-1})^{k-1}\otimes\ldots\otimes(\det E_0^{k-1,1}),$$

where $\bigoplus E_0^{p,q}$ is the system of Hodge bundles corresponding to $\mathbb{V}_0$ (see below). Griffiths showed that the curvature form of $K(E)$ is positive semi definite, and is positive definite at those points, where the period map is injective. If the monodromies around the components of S are unipotents. Then, there is a canonical extension $\bigoplus E^{p,q}$ on X (see below). Hence, a canonical extension $K(E)$, such that $K(E)$ is nef and big. Applying Theorem 0.2 we obtain the following inequality, which shall be consider as an algebraic geometry version of Yau's Schwarz inequality

COROLLARY 0.3. *After a blowing up $\tau:(\hat{X},\hat{S})\to(X,S)$ there exists a positive rational number r, which is bounded above by the Hodge type of $\mathbb{V}_0$, an effective $\mathbb{Q}-$divisor P and a semi-negative $\mathbb{Q}-$divisor N on $\hat{X}$ such that*

$$K(\tau^*E)=r\det\tau^*\Omega^1_X(\log S)-P+N.$$

In particular, $\frac{1}{r}(K(\tau^*E)-N)$ is a nef and big invertible sub-sheaf of $\det\tau^*\Omega^1_X(\log S)$.

REMARK. The inequality in Cor. 0.3 still holds on the curvature level. So, we may regard this inequality as an effective and geometric version of Griffiths-Schmid's theorem [GS]. Namely, the holomorphic sectional curvature of the horizontal sub-bundle is strictly positive.

QUESTION (Viehweg) Does the difference

$$r \det \tau^* \Omega_X^1(\log S) - K(\tau^* E)$$

have any geometric meanings?

The main idea in the proof of Theorem 0.1 and 0.2 is the negativity of kernels Kodaira-Spencer maps on Hodge bundles, which we are going to describe. A polarized variation of Hodge structure $\mathbb{V}_0$ on X_0 gives rise the Hodge bundles

$$E_0 = \mathrm{gr}_{F_0} \mathcal{V} = \bigoplus_{p+q=k} E_0^{p,q}$$

and the linearlizations of the Gauss-Manin flat connection

$$\theta_0^{p,q} : E_0^{p,q} \to E_0^{p-1,q+1} \otimes \Omega_X^1, \quad \theta_0 = \bigoplus_{p+q=k} \theta_0^{p,q}$$

According Simpson's terminology [S1] we shall call the pair (E_0, θ_0) a system of Hodge bundles. It satisfies $\theta_0 \wedge \theta_0 = 0$. In the case when (E_0, θ_0) comes from a geometry situation, a family of smooth projective varieties $f : Y_0 \to X_0$, then $\mathbb{V}_0 = R^k f_*(\mathbb{C}_{Y_0})$, $E_0^{p,q} = R^q f_*(\Omega^p_{Y_0/X_0})$ and $\theta_0^{p,q}$ is just given by the cup product with the Kodaira-Spencer class.

By works due to Deligne, Griffiths and Schmid the vector bundle $\mathcal{V}_0 = \mathbb{V}_0 \otimes \mathcal{O}_{X_0}$ can be extended over X as vector bundles, the Gauss-Manin flat connection has regular singularities along S and the Hodge filtration F_0 can be extended as a filtration of coherent sheaves over X. In the case when the monodromy around the components of S are unipotent, then there is a canonical extension $\mathcal{V}$ of $\mathcal{V}_0$, F_0 is extended as a filtration F of vector bundles over X [Sch] and θ has at most logarithmic pole along S. The canonically extended system of Hodge bundles will be denoted by

$$(E, \theta) = (\bigoplus_{p+q=k} E^{p,q}, \bigoplus_{p+q=k} \theta^{p,q}), \quad \theta^{p,q}(E^{p,q}) \subset E^{p-1,q+1} \otimes \Omega_X^1(\log(S)).$$

DEFINITION 0.2. We set

$$K(\theta^{p,q}) := \mathrm{Ker}(\theta^{p,q} : E^{p,q} \to E^{p-1,q+1} \otimes \Omega_X^1(\log(S)).$$

From the general fact, $K(\theta^{p,q}) \subset E^{p,q}$ are sub-bundles in the complement of some subvariety of X of codimension ≥ 2 except some obvious cases. For example, we have trivially $K(\theta^{0,k}) = E^{0,k}$.

A prototype of the negativity of $K(\theta^{p,q})$ is a theorem of Griffiths [G1], who showed that the curvature form of $E_0^{0,k}$ is negative semi definite. In the case when the monodromies around the components of S are unipotent, by using this curvature property and variations of mixed Hodge structures Fujita [Fu] for $\dim X = 1$ case, and Kawamata [Ka1] for $\dim X > 1$ case showed that $E^{0,k}$ is semi negative in the algebro-geometric sense. In fact, they showed that $E^{k,0}$ ($\simeq E^{0,k\vee}$) is semi positive. Namely, let $f : C \to X$ be a morphism from a smooth projective curve into X. Then the degrees of any invertible quotient sheaves of $f^*(E^{k,0})$ are non negative. In the case when the monodromy around S are not unipotent, one has to be careful on which extension will be taken.

The same principle will be exploited in the proof for the semi-negativity of sub-bundles $N \subset K(\theta^{p,q})$. By using the Griffiths formula for the curvature form of the Hodge metric on E_0 [G2] and the extra property that $\theta^{p,q}(N) = 0$, one shows that the curvature form of $N|_{X_0}$ of the Hodge metric is negative semi-definite. On the boundary S one has the similar situation as on X_0. The intersection of different components gives rise a stratification $S = \cup_I S_I$. The residue of θ on S_I induces a filtration on $N|_{S_I}$, such that each component in the grading again lies in the kernel of the Kodaira-Spencer map on the system of Hodge bundles induced by variation of mixed Hodge structures. In the case when the local monodromies around the components of S are unipotent, a Theorem of Kollár ([K], Th. 5.20) allows the Chern forms of the singular Hodge metric of N to represent the Chern classes of N. We obtain

THEOREM 1.2. *Suppose that $N \subset E^{p,q}$ is a sub-bundle, such that $\theta^{p,q}(N) = 0$. Then the curvature form of the Hodge metric restricted on N_0 is negative semi-definite. If in addition the local monodromies of $\mathbb{V}_0$ around the components of S are unipotent, then the Chern classes of N can be represented by the Chern forms of this curvature form. The Chern classes $c_i(N^\vee)$ is semi-positive in following sense. Let $f : Z \to X$ be a morphism, then the integration*

$$\int_Z f^*(c_1^{i_1}(N^\vee)\dots c_n^{i_n}(N^\vee)) \geq 0.$$

More over, if $f^*c_1(N)L^{\dim Z-1} = 0$ for some ample line bundle L on Z, then the following holds:

i) If $f(Z) \not\subset S$, then the Gauss-Manin connection on f^*E restricts to a flat connection on f^*N, which is compatible with the Hodge metric. Hence, the map $f^*N \to f^*E$ induces a map between the underlying local systems of $f^*N|_0$ and $f^*E|_0$.

ii) If $f(Z) \subset \cap_{i_1,\dots,i_{\#I}} S_{i_k} =: S_I$ and $\not\subset S_J$ for $J \supset I$. Let

$$\bigoplus Gr(f^*N) \subset \bigoplus Grf^*(E)$$

denote the gradings for f^*N and f^*E of the pulled back weight filtration of the residue of θ on S_I. Then the Gauss-Manin connection on $\bigoplus Gr(f^*E)$ (it comes from VHS on the grading of the weight filtration of the monodromy around S_I) restricts to a flat metric connection on $\bigoplus Gr(f^*N)$. Hence the map $f^*N \to f^*E$ induces a map between the underlying local systems of $\bigoplus Gr(N_0)$ and $\bigoplus Gr(f^*E|_0)$.

In order to apply Theorem 1.2 to the semi-positivity of $\Omega^1_X(\log S)$ we consider the induced VHS on the endomorphism $\mathrm{End}(\mathbb{V}_0)$ and the corresponding system of Hodge bundles of weight $k = 0$.

$$(\mathrm{End}(E), \theta^{\mathrm{end}}) = (E, \theta) \otimes (E^\vee, -\theta^\vee).$$

The Kodaira-Spencer map $\theta : E \to E \otimes \Omega^1_X(\log S)$ induces a sheaves map

$$d\phi : T_X(-\log S) \to \mathrm{End}(E),$$

with $d\phi(T_X(-\log S)) \subset \mathrm{End}(E)^{-1,1}$, and such that $d\phi|_{X_0}$ coincides with the differential of the period map $\phi : X_0 \to \mathcal{D}/\Gamma$. From the integrable condition $\theta \wedge \theta = 0$ one gets

PROPOSITION 2.1.

$$\theta^{\mathrm{end}} d\phi(T_X(-\log S)) = 0.$$

Theorem 0.1 and 0.2 follow from Theorem 1.2, Prop.2.1 and Kawamata-Viehweg's vanishing theorem. Here we have further applications of Theorem 1.2 and Prop. 2.1.

Consider $X \setminus S$ as a moduli variety of polarized varieties, carries a universal family, and such that some Torelli map on the family is locally injective. Let $f : (Z, S') \to (X, S)$ be a morphism. A deformation f_t of f, such that $f_t(S') \subset S$ can be thought as a deformation of the family of polarized varieties over Z with the fixed discriminant S'. This gives rise a section $s : \mathcal{O}_Z \to f^*(T_X(-\log S))$. By Theorem 1.2 and Prop. 2.1 we recover Faltings theorem on deformations of families of polarized abelian varieties [F], and the theorem due to Jost-Yau and Peters on deformations of period maps [JY], [P1].

COROLLARY 0.4. *Suppose that $f : Y \to Z$ is a family of polarized varieties, such that the Torelli map for some $R^k f_*(\mathbb{C}_Y)$ is locally injectives. Then a deformation of this family with the fixed Z and the fixed discriminant S induces a non-zero endomorphism on $R^k f_*(\mathbb{C}_Y)$ of $(-1,1)-$Hodge type.*

Corollary 0.4 should have a far reaching generalization. Namely, Cor.0.4 just means that the image of the 0-te cohomology

$$\mathrm{Im}(H^0(Z, f^*T_X(-\log S)) \to H^0(Z, f^*\mathrm{End}(E)))$$

of coherent sheaves can be realized as the 0-te topology cohomology classes in $H^0_{Betti}(Z, f^*\mathrm{End}\mathbb{V}_0)$ of the local system $f^*\mathrm{End}(\mathbb{V}_0)$. We may ask a further question, if for any q the image

$$\mathrm{Im}(H^q(Z, f^*T_X(-\log S)) \to H^q(Z, f^*\mathrm{End}(E)))$$

can also be interpreted as a sub-space of $H^q_{Betti}(Z, f^*\mathrm{End}\mathbb{V}_0)$? Here we have an answer for the projective case.

THEOREM 0.5. *Suppose $f : Z \to X$ is a morphism. If $f(Z) \subset X_0$, then the sheaves morphism*

$$f^*T_X(-\log S) \to f^*End(E)$$

induces an injective map

$$Im(H^q(Z, f^*T_X(-\log S)) \to H^q(Z, f^*End(E))) \to H^q_{Betti}(Z, f^*End(\mathbb{V}_0)).$$

The proof of Theorem 0.5 is just to use Prop. 2.1 and Simpson's three descriptions for the cohomology of a local system. Namely,

$$H^i_{Betti}(Z, f^*\mathbb{V}_0) \simeq H^i_{DR}(Z, f^*\mathcal{V}) \simeq \mathbb{H}^i_{Dol}(Z, f^*(E, \theta))$$

via harmonic forms.

Conjecturaly, Theorem 0.5 shall till be true for the quasi-projective case. If $\dim X = 1$, then Theorem 0.5 should follow from Zucker's theorem [Z]. For higher dimensional quasi-projective varieties one needs to work out Simpson's three descriptions for cohomology of local systems (or a better formulation: the intersection cohomology of local systems) via L^2–harmonic forms. Of course, the theorems in [CKS] for degeneration of Hodge structures will certainly come into play.

Theorem 0.5 and its general form has the following potential application for deformations of subvarieties in moduli spaces of polarized varieties. In order to explain the idea we just consider the moduli space $f : \mathcal{A} \to M_0$ of polarized abelian varieties with a universal family. Let $g : (Z, S') \to (M, S)$ be a morphism and $f : \mathcal{A}|_Z \to Z$ the pulled back family. In their paper [SZ] Saito and Zucker asked the following question. To understand those infinitesimal log-deformations of (Z, S'), which underly projective deformations of $(\mathcal{A}|_Z, f^{-1}(S'))$ as a fibre space. It is the same thing to ask that when an abstract log-deformation of (Z, S') will underly a deformation of the triple $g : (Z, S') \to (M, S)$. Consider the VHS $S^2R^1f_*(\mathbb{C}_\mathcal{A})$. Then the tangent bundle of M with log-zero along S is identified with the last Hodge bundle $S^2R^1f_*(\mathcal{O}_\mathcal{A})$ of $S^2R^1f_*(\mathbb{C}_\mathcal{A})$. The morphism between tangent sheaves reads

$$T_Z(\log S') \to S^2R^1f_*(\mathcal{O}_{\mathcal{A}|_Z}).$$

A calculation in [SZ] shows that the obstruction for an abstract log-deformation of (Z, S') underlying deformation of $(\mathcal{A}|_Z, f^{-1}(S'))$ as a fibre space lies in

$$\mathrm{Im}(H^1(Z, T_Z(\log S') \to H^1(Z, S^2R^1f_*(\mathcal{O}_{\mathcal{A}|_Z})).$$

Assuming Theorem 0.5 is true for the quasi projective case. Then this obstruction lies further in

$$H^1_{Betti}(Z, S^2R^1f_*(\mathbb{C}_{\mathcal{A}|Z})) \subset H^1_{Betti}(Z, \mathrm{End}_0(R^1f_*\mathbb{C}_{\mathcal{A}|Z})).$$

If the induced map $g : Z_0 \to M_0$ induces an isomorphism $g_* : \pi_1(Z_0) \to \pi_1(M_0)$, then it induces an isomorphism

$$H^1_{Betti}(Z, \mathrm{End}_0(R^1f_*\mathbb{C}_{\mathcal{A}|Z})) \simeq H^1_{Betti}(M_0, \mathrm{End}_0(R^1f_*\mathbb{C}_\mathcal{A})).$$

By Weil's locally rigidity theorem the cohomology group on the right site vanishes, we expect the following statement:

CONJECTURE 0.6. *Let $f : \mathcal{A} \to Z$ be a semi-stable family a polarized abelian varieties over a smooth projective variety Z with the discriminant divisor S'. If the induced map $g : Z_0 \to M_0$ to the moduli space is an isomorphims $g_* : \pi_1(Z_0) \to \pi_1(M_0)$ then any infinitesimal log-deformation of (Z, S') underlies a projective deformation of $(\mathcal{A}, f^{-1}(S'))$ as a fibre space.*

The same statement for global deformations was recently proved in [JLZ]. It is, in fact, a statement about variations of Hodge conjecture in the non-abelian case.

THEOREM 0.7. *Let $\pi : \mathcal{Z}_0 \to B_0$ be a family of smooth quasi-projective varieties over a smooth quasi-projective variety B_0. Suppose that over one fibre Z_0 there is* *a family of smooth polarized abelian varieties* $f : \mathcal{A} \to Z_0$ *and the induced map* $g : Z_0 \to M_0$ *to the moduli space is an isomorphims* $g_* : \pi_1(Z_0) \to \pi_1(M_0)$. *Then the family* $f : \mathcal{A} \to Z_0$ *can be extended as a family of smooth polarized abelian varieties* $f : \mathcal{A} \to \mathcal{Z}_0$ *after a base changing of* B_0.

Before we finish this section, we would like ask:

QUESTION. Let $\phi : X_0 \to \mathcal{D}/\Gamma$ be a locally injective period map corresponding to a VHS $\mathbb{V}_0$. Suppose that for any morphism $f : C_0 \to X_0$, such that the composition $\phi f : C_0 \to \mathcal{D}/\Gamma$ is non-constant, there exists no endomorphism of $(-1,1)-$type twisted by a unitary local system on the underlying local system of $f^*\mathbb{V}_0$. Is $\Omega^1_X(\log S)$ big on some non-empty Zariski open subset $U \subset X_0$?

Here the bigness of $\Omega^1_X(\log S)$ on U (see [V3]) means that for any line bundle M over X there exists some $\gamma >> 0$, such that

$$S^{\gamma}(\Omega^1_X(\log S) \otimes M^{-1}$$

weakly positive on U

It is easy to see the converse direction of this question is true. Suppose $\mathbb{V}_0$ arises from the geometry situation, Viehweg recently asked if some extra geometry properties on the fibres shall force $\Omega^1_X(\log S)$ to be big. For example, the contangent bundle along the fibres is big. On the other hand, the existence of a $(-1,1)-$type endomorphism on $R^k f_*(\mathbb{C}_{Y_0})$ implies that this VHS splits over $\mathbb{C}$ of some very special types. It would be interesting to see if this property has any consequence to the geometry on the fibres.

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1. Negativity of kernels of Kodaira-Spencer maps. Let X denote a smooth complex projective variety, and $S \subset X$ a normal crossing divisor with smooth components. For the convenience of the readers we give a short review on variation of Hodge structure (see [Sch])

A variation of polarized Hodge structure $\mathbb{V}_0$ of weight k on $X_0 = X \setminus S$ consists of:

i) a finite dimensional real vector space V, and a homomorphism

$$\rho : \pi_1(X_0, x_0) \to \mathrm{GL}(V)$$

Let $\mathbb{V}_0$ denote the local system corresponding to ρ and $\mathcal{V}_0 = \mathbb{V}_0 \otimes \mathcal{O}_{X_0}$ the holomorphic vector bundle with the integrable connection (the Gauss-Manin connection)

$$\nabla : \mathcal{V}_0 \to \mathcal{V}_0 \otimes \Omega^1_{X_0}, \quad \nabla^2 = 0.$$

ii) A flat, real and non-degenerated bilinear form

$$Q : \mathbb{V}_0 \otimes \mathbb{V}_0 \to \mathbb{R}$$

iii) A filtration of $\mathcal{V}_0$ by holomorphic sub-bundles

$$\mathcal{V}_0 = F_0^0 \supset F_0^1 \supset \dots \supset F_0^k$$

This filtration is called the Hodge filtration and satisfies the infinitesimal period relation

$$\nabla(F_0^i) \subset F_0^{i-1} \otimes \Omega^1_{X_0}.$$

A VHS induces a so-called system of Hodge bundles as follows. Let

$$E_0^{p,k-p} := F_0^p / F_0^{p+1}.$$

Because of the infinitesimal relation, the integral connection induces a sheaves homomorphism

$$\theta_0^{p,q} : E_0^{p,q} \to E_0^{p-1,q+1} \otimes \Omega^1_{X_0}.$$

We set

$$(E_0, \theta_0) = \Big(\bigoplus_{p+q=k} E_0^{p,q}, \bigoplus_{p+q=k} \theta_0^{p,q} \Big).$$

The Hodge bundles satisfies the following Riemman-Hodge bilinear relations

$$Q(E_0^{p,q}, E_0^{r,s}) = 0 \quad \text{unless} \quad p = s,\ q = r,$$
$$i^{p-q} Q(v, \bar{v}) > 0, \quad \text{if} \quad v \in E_0^{p,q},\ v \neq 0.$$

The so-called Weil operator $C : E_0 \to E_0$ is defined by

$$Cv = i^{p-q} v, \quad \text{if} \quad v \in E_0^{p,q}.$$

The Hermitian form $h(u, \bar{v}) := Q(Cu, \bar{v})$ is positive definite and defines the so-called Hodge metric h on E_0.

Suppose that the local monodromy around each component of S is unipotent. By works of Deligne and Schmid there is a canonical extension $\mathcal{V}$ of $\mathcal{V}_0$, F_0 is extended as a filtration F of vector bundles over X, and θ has at most logarithmic pole along S. This extension is locally described as follows: Let $s = \sum_i f_i s_i$ be a single-valued section of of $\mathcal{V}_0$, where f_i are multi-valued holomorphic functions and s_i are multi-valued flat sections of $\mathbb{V}_0$. Then s can be extended over S to a local section of $\mathcal{V}$ if and only if every f_i has at most logarithmic singularities along S. This canonically extended system of Hodge bundles will be denoted by

$$(E, \theta) = \Big(\bigoplus_{p+q=k} E^{p,q}, \bigoplus_{p+q=k} \theta^{p,q} \Big), \quad \theta^{p,q} : E^{p,q} \to E^{p-1,q+1} \otimes \Omega^1_X(\log(S)).$$

PROPOSITION 1.1. *Let $N \subset E^{p,q}$ be a sub-bundle and $\theta^{p,q}(N) = 0$. Then N is semi-negative in the following sense:*

i) *The curvature form $\Theta(N|_{X_0}, h)$ of the induced Hodge metric h is negative semi definite. More over, if $\Theta(N|_{X_0}, h) = 0$. Then the metric connection on N is flat, and coincides with the Gauss-Manin connection.*

ii) *In general, we consider $N|_{S_{I,0}}$. There exists a filtration of sub-bundles of $E|_{S_I}$, called the weight filtration of $res_{S_I}\theta$*

$$W_{0,\dots,0,\dots,0} \subset \dots \subset W_{l_1,\dots,l_m,\dots,l_{\#I}} \subset W_{l_1,\dots,l_m+1,\dots,l_{\#I}} \subset \dots \subset W_{2k,\dots,2k,\dots,2k} = E|_{S_I},$$

which is preserved by θ, and such that the quotient $Gr_{l_1,\dots,l_m,\dots,l_{\#I}}(E|_{S_I})$ together with the descent θ forms a direct sum of systems of Hodge bundles $loc_{S_I}(E, \theta)$ corresponding to a direct sum of polarized VHS on S_I. Furthermore, this filtration restricts to a filtration on $N|_{S_I}$

$$W_{0,\dots,0,\dots,0}(N|_{S_I}) \subset \dots \subset W_{l_1,\dots,l_m,\dots,l_{\#I}}(N|_{S_I}) \subset \dots \subset W_{2k,\dots,2k,\dots,2k}(N|_{S_I}) = N|_{S_I},$$

such that

$$Gr_{l_1,\dots,l_m,\dots,l_{\#I}}(N|_{\bar{S}_I})) \subset Ker(loc_S\theta : loc_S E^{p,q} \to loc_S E^{p-1,q+1} \otimes \Omega^1_{\bar{S}_I}(log(S_I \cap \sum_{J \supset I} S_J)).$$

Hence, by i) the curvature form of the induced Hodge metric on $Gr_{l_1,\dots,l_m,\dots,l_{\#I}}(N|_{S_{I,0}})$ is again negative semi definite. And, if the curvature form vanishes. Then the metric connection on $Gr_{l_1,\dots,l_m,\dots,l_{\#I}}(N|_{S_{I,0}})$ is flat, and coincides with the Gauss-Manin connection on $loc_{S_I}(E, \theta)$.

Proof of i). Let h denote the Hodge metric on the system of Hodge bundle (E_0, θ), and $\Theta(E_0, h)$ the curvature form of (E_0, h). By [G] we have

$$\Theta(E_0, h) + \theta \wedge \bar{\theta}_h + \bar{\theta}_h \wedge \theta = 0,$$

where $\bar{\theta}_h$ is the complex conjugation of θ with respect to h.

Suppose $N \subset E^{p,q}$ is a sub-bundle and $\theta(N) = 0$. We shall use the last equation above to deduce that the curvature form of $N|_{X_0}$ is negative semi definite.

The metric h induces a metric h on $N|_{X_0} =: N_0$, and C^∞−decomposition

$$E_0 = N_0 \oplus N_0^\perp.$$

This gives

$$\Theta(N_0, h) = \Theta(E_0, h)|_{N_0} + \bar{A} \wedge A = -\theta \wedge \bar{\theta}_h|_{N_0} - \bar{\theta}_h \wedge \theta|_{N_0} + \bar{A} \wedge A,$$

where $A \in A^{1,0}(\mathrm{Hom}(N_0, {N_0}^\perp)$ is the second fundamental form of the sub-bundle $N_0 \subset E_0$. Note that $\theta(N) = 0$, we have $\bar{\theta}_h \wedge \theta|_{N_0} = 0$. Hence

$$\Theta(N_0, h) = -\theta \wedge \bar{\theta}_h|_{N_0} + \bar{A} \wedge A \leq 0.$$

If $\Theta(N_0, h) = 0$, then $\theta \wedge \bar{\theta}_h|_{N_0} = 0$ and $\bar{A} \wedge A = 0$. This implies that $A = 0$ and $\bar{\theta}_h|_{N_0} = 0$. This means that the above C^∞−decomposition is holomorphic, and the decomposition of the Gauss-Manin connection on N_0 becomes $D = D_h$. i) is proved.

Proof of ii). The main idea here is similar to Kawamata's proof for $E^{k,0}$ [Ka1]. There he used the weight filtration of the monodromy around S, and showed that $E^{k,0}$ is semi positive. Here we shall prove the semi negativity of the kernels of Kodaira-Spencer map on the Hodge bundles. Since they are algebraic subsheaves of Hodge bundles, we have to use the following weight filtration defined by $\mathrm{res}_S\theta$. The underlying spaces of those two filtrations are the same. But, they have different holomorphic structures in general.

It is a θ−invariant filtration

$$W_{0,\dots,0,\dots,0} \subset \dots \subset W_{l_1,\dots,l_m,\dots,l_{\#I}} \subset W_{l_1,\dots,l_m+1,\dots,l_{\#I}} \subset \dots \subset W_{2k,\dots,2k,\dots,2k} = E|_{S_I},$$

such that the grading together with descent θ forms a direct sum of systems of Hodge bundles corresponding to the polarized VHS on S_I induced by variation of mixed Hodge structure along S_I.

First of all we shall review the notion of variation of mixed Hodge structure (VMHS) and the systems of mixed Hodge bundles corresponding to them. The following discussion on mixed Hodge structure can be found in [Sch], [CKS] and [Ka1].

We write $I = \{i_1, i_2, \dots, i_{\#I}\}$ with the fix chosen order $i_l < i_{l+1}$ The VHS $H_{\mathbb{C}}$ on $\bar{X} \setminus \bar{S}$ induces a VMHS on S_I inductively (it depends on the order that we have chosen).

Start with $I = \{i\}$. Let γ_i denote a short loop around S_i. One has the unipotent matrix

$$\rho(\gamma_i) = I + N_i \in \mathrm{GL}(V), \quad N_i^{k+1} = 0.$$

There exists a unique filtration [Sch]

$$W_0^i \subset W_1^i \subset \dots \subset W_l^i \subset W_{l+1}^i \subset \dots \subset W_{2k}^i = V$$

such that $N_i(W_l) \subset W_{l-2}$. Set $\mathrm{Gr}_l^i(V) := W_l^i/\dot{W}_{l-1}^i$. Then

$$N_i^l : \mathrm{Gr}_{k+l}^i(V) \to \mathrm{Gr}_{k-l}^i(V)$$

is an isomorphism, for each $l \geq 0$.

If $l \geq k$, let $\mathcal{P}_l^i \subset \mathrm{Gr}_l^i(V)$ denote the kernel of

$$N_i^{l-k+1} : \mathrm{Gr}_l^i(V) \to \mathrm{Gr}_{2k-l-2}^i(V)$$

and set $\mathcal{P}_l = 0$ if $l < 0$. Then one has the so-called primitive decomposition

$$\mathrm{Gr}_l^i(V) = \bigoplus_j N_i^j(\mathcal{P}_{l+2j}^i), \quad j \geq \max(k-l, 0).$$

Q defines a nondegenerated bilinear form Q_l^i on $\mathrm{Gr}_l(W_*^i)$ by

$$Q_l^i(\tilde{u}, \tilde{v}) = Q(u, N^{l-k}v), \quad \text{if } u,\, v \in W_l^i \text{represent } \tilde{u}, \tilde{v}.$$

The isomorphism above is then an isometry, and the decomposition above is then orthogonal with respect to this bilinear form.

Let U_i denote a tube neighborhood of S_i in X. Then this weight filtration gives rise a filtration of flat sub-bundles of $\mathcal{V}_0|_{U_i \setminus S_i}$, since N_i commutes with $\rho|_{U_i \setminus S_i}$. $\mathcal{V}_0$ has a canonical extension $\mathcal{V}$ over X, locally described as follows.

Let $s = \sum_i f_i s_i$ be a single-valued section of of $\mathcal{V}_0$, where f_i are multi-valued holomorphic functions and s_i are multi-valued flat sections of $\mathcal{V}_0$ with respect to the Gauss-Manin connection. s can be extended over S to a local section of $\mathcal{V}$ if and only if every f_i has at most logarithmic singularities along S.

The Gauss-Manin connection is then extended as meromorphic integrable connection with log pole along S

$$\nabla' : \mathcal{V} \to \mathcal{V} \otimes \Omega_X(\log S).$$

The weight filtration $\{W^i_l\}$ can be extended as a filtration of sub-bundles of $\mathcal{V}|_{U_i}$

$$W^i_0 \subset \dots \subset W^i_l \subset \dots \subset W^i_{2k} = \mathcal{V}|_{U_i},$$

such that the induced Gauss-Manin connection on $\mathrm{Gr}^i_l(\mathcal{V}|_{U_i \setminus S_i})$ extends to a holomorphic integrable connection on $\mathrm{Gr}^i_l(\mathcal{V}|_{U_i})$. There is a way to describe local flat sections of $\mathrm{Gr}^i_l(\mathcal{V}|_{U_i})$.

Let $U_x \subset X$ be a small neighborhood of $x \in S_i$, and $\tilde{u}$ be a multi-valued flat section of $W^i_l|_{U_x \setminus S_i}$. The multi-valued property of $\tilde{u}$ is exactly caused by the nilpotent part N_i of the unipotency matrix $\rho(\gamma_i) = I + N_i$. Since $N_i(W^i_l) \subset N_i(W^i_{l-2})$, the action of $\rho(\gamma_i)$ on $\mathrm{Gr}^i_l(\mathcal{V}|_{U_i \setminus S_i})$ is identity. Hence $\tilde{u}$ projects to a single valued flat section u of $\mathrm{Gr}^i_l(\mathcal{V}|_{U_x \setminus S_i})$, which extends to a flat section of $\mathrm{Gr}^i_l(\mathcal{V}|_{U_x})$. Q^i_l also extends to a flat bilinear form on $\mathrm{Gr}^i_l(\mathcal{V}|_{U_i})$, which will be again denoted by Q^i_l.

Moreover, W. Schmid showed that the Hodge filtration $\{F^p_0\}$ has an extension

$$\mathcal{V} = F'^0 \supset F^1 \supset \dots \supset F^p \supset \dots \supset F^k,$$

such that $\{F^p\}$, $\{W^i_l\}$ and together with Q^i_l define a variation of polarized mixed Hodge structures on $\mathcal{V}|_{S_i}$. i.e. the projection of $\{F^p\}$ to $\mathrm{Gr}^i_l(\mathcal{V}|_{S_i})$ defines a direct sum of VHS on the components of the primitive decomposition of $\mathrm{Gr}^i_l(\mathcal{V}|_{S_i})$, and those VHS are polarized by the bilinear form Q^i_l.

Now let $I = \{i_1, i_2\}$, we consider two components of S. Let $\gamma_{i_1}, \gamma_{i_2}$ be the two short loops around S_{i_1}, S_{i_2} and N_{i_1}, N_{i_2} the nilpotent parts in $\rho(\gamma_{i_1}, \rho(\gamma_{i_2}$ respectively. Since N_{i_1} and N_{i_2} commute, the property of the weight filtration of N_{i_1} in i) is preserved by N_{i_2}. The weight filtration of N_{i_1}

$$W^{i_1}_0 \subset W^{i_1}_1 \subset \dots \subset W^{i_1}_l \subset W^{i_1}_{l+1} \subset \dots \subset W^{i_1}_{2k} = V$$

is also preserved by N_{i_2}. Hence, N_{i_2} descends to a nilpotent endomorphism

$$\hat{N}_{i_2} : \mathrm{Gr}_{l_1}(W^{i_1}_*) \to \mathrm{Gr}_{l_1}(W^{i_1}_*),$$

and defines then a weight filtration on $\mathrm{Gr}_{l_1}(W^{i_1}_*)$

$$W^{i_1,i_2}_{l_1,0} \subset W^{i_1,i_2}_{l_1,1} \subset \dots \subset W^{i_1,i_2}_{l_1,l} \subset W^{i_1,i_2}_{l_1,l+1} \subset \dots \subset W^{i_1,i_2}_{l_1,2k} = \mathrm{Gr}^{i_1}_{l_1}(\mathcal{V}).$$

Let U_{i_1,i_2} denote a tube neighborhood of S_{i_1,i_2} in S_{i_1}. Then we just repeat the above discussion again here. The weight filtration $\{W^{i_1,i_2}_{l_1,l}\}$ defines a filtration of flat sub-bundles of $\mathrm{Gr}^{i_1}_{l_1}(\mathcal{V})|_{U_{i_1,i_2} \setminus S_{i_1,i_2}}$, and extends to a filtration of sub-bundles on $\mathrm{Gr}^{i_1}_{l_1}(\mathcal{V})|_{U_{i_1,i_2}}$ such that the Gauss-Manin connection ∇' induces a holomorphic integrable connections on the quotient

$$\mathrm{Gr}^{i_1,i_2}_{l_1,l_2}(\mathcal{V}) := W^{i_1,i_2}_{l_1,l} / W^{i_1,i_2}_{l_1,l-1}.$$

Further more, the Hodge filtration on $\mathrm{Gr}^{i_1}_{l_1}(\mathcal{V})$ projects to the primitive parts of $\mathrm{Gr}^{i_1,i_2}_{l_1,l_2}$ is a VHS, and polarized by

In general, for any S_I, we construct inductively a grading

$$V = \bigoplus_{l_1,l_2,\dots,l_{\#I}}^{i_1,i_2,\dots,i_{\#I}} \mathrm{Gr}^{i_1,i_2,\dots,i_{\#I}}_{l_1,l_2,\dots,l_{\#I}},$$

and show that each component of this grading carries a direct sum of polarized VHS on $S_{i_1,\dots,i_{\#I}}$.

Now we shall write down the system of mixed Hodge bundles corresponding to VMHS constructed above. Consider the canonical extension

$$\theta : E \to E \otimes \Omega^1_{\bar{X}}(\log S)$$

of the system of Hodge bundle corresponding to a VHS. θ has the expression

$$\theta = \sum_{i=1}^{m} \theta_i \frac{dz_i}{z_i} + \sum_{i\geq m+1} \theta_i,$$

near $S = \{z|z_1...z_m = 0\}$, where all θ_i are holomorphic, and $\theta_i\theta_j = \theta_j\theta_i$.

The so-called residue map around S_i of S is an endomorphism defined by

$$\mathrm{res}_{S_i}\theta : E|_{S_I} \xrightarrow{\theta_i} E|_{S_i}.$$

In our situation $\mathrm{res}_{S_i}\theta$ is automatically nilpotent, and $(\mathrm{res}_{S_i}\theta)^{k+1} = 0$. So, it defines a weight filtration on $E|_{S_i}$

$$W_0(E|_{\bar{S}_i}) \subset W_1(E|_{S_i}) \subset ... \subset W_l(E|_{S_i}) \subset W_{l+1}(E|_{S_i}) \subset ... \subset W_{2k}(E|_{S_i}) = E|_{S_i},$$

which is preserved by the action of $\theta|_{S_i}$. Simpson [S2] showed that this weight filtration is the same as the weight filtration corresponding to the nilpotent part of the local monodromy around $\bar{S}_i$. So, this filtration is a filtration of sub-bundles. Denote $\mathrm{Gr}^i_l(E) = W_l(E|_{S_i})/W_{l-1}(E|_{S_i})$, one also has a primitive decomposition

$$\mathrm{Gr}^i_l(E) = \bigoplus_j (\mathrm{res}_{S_i}\theta)^j(\mathcal{P}^i_{l+2j}).$$

We consider now the following two exact sequences

$$0 \to \mathrm{End}(E) \otimes \Omega^1_{S_i}(\log(S_i \sum_{j\neq i} S_j) \to \mathrm{End}(E) \otimes \Omega^1_X(\log(S)|_{S_i} \overset{\mathrm{res}_{S_i}}{\to} \mathrm{End}(E)|_{S_i} \to 0$$

and

$$0 \to \mathrm{End}(\mathrm{Gr}^i_l(E)) \otimes \Omega^1_{S_i}(\log(S_i \cap \sum_{j\neq i} S_j) \to \mathrm{End}(\mathrm{Gr}^i_l(E)) \otimes \Omega^1_X(\log(S)|_{S_i}$$

$$\overset{\mathrm{res}_{S_i}}{\to} \mathrm{End}(\mathrm{Gr}^i_l(E)) \to 0.$$

Since

$$\theta \in H^0(S_i, \mathrm{End}(E) \otimes \Omega^1_X(\log(S)|_{S_i})$$

preserves the weight filtration, it descends to a section

$$\theta \in H^0(S_i, \mathrm{End}(\mathrm{Gr}^i_l)|_{S_i}) \otimes \Omega^1_X(\log(S)|_{S_i}).$$

The image

$$\mathrm{res}_{S_i}(\theta) \in H^0(S_i, \mathrm{End}(\mathrm{Gr}^i_l(E))$$

is zero, since it comes from

$$\mathrm{res}_{S_i}(\theta) \in H^0(S_i, \mathrm{End}(E)|_{S_i}),$$

and it has the trivial action on the quotient Gr^i_l of its weight filtration. Hence θ factors to a section

$$\theta^i_l \in H^0(S, \mathrm{End}(\mathrm{Gr}^i_l(E) \otimes \Omega^1_{\bar{S}_i}(\log(S_i \cap \sum_{j \neq i} S_j).$$

This section together with the projection of the Hodge sub-bundles of $E|_{S_i}$ to $\mathrm{Gr}^i_l(E)$ is the direct sum of systems of Hodge bundles

$$(\mathrm{Gr}^i_l(E), \theta^i_l) = \bigoplus_j ((\mathrm{res}_{S_i}\theta)^j(\mathcal{P}^i_{l+2j}), \theta^i_{l,j})$$

corresponding to the direct sum of polarized VHS on the primitive decomposition of $\mathrm{Gr}^i_l(H)$. We call the following direct sum of systems of Hodge bundles

$$\bigoplus_{l,j} ((\mathrm{res}_{S_i}\theta)^j(\mathcal{P}^i_{l+2j}), \theta^i_{l,j}) =: \mathrm{loc}_{\bar{S}_i}(E, \theta)$$

the localization of (E, θ) on S_i.

Now we consider two components S_{i_1}, S_{i_2} of S. The residue

$$\mathrm{res}_{S_{i_1,i_2}} \mathrm{loc}_{S_{i_1}} \theta =: \mathrm{res}_{S_{i_1,i_2}} \theta$$

gives a weight filtration

$$W^{i_1,i_2}_{l_1,0} \subset W^{i_1,i_2}_{l_1,1} \subset \ldots \subset W^{i_1,i_2}_{l_1,l} \subset W^{i_1,i_2}_{l_1,l+1} \subset \ldots \subset W^{i_1,i_2}_{l_1,2k} = \mathrm{Gr}^{i_1}_{l_1}(E).$$

Let $\mathrm{Gr}^{i_1,i_2}_{l_1,l}(E) = W^{i_1,i_2}_{l_1,l} / W^{i_1,i_2}_{l_1,l-1}$. Then by the same argument as above $\mathrm{loc}_{S_{i_1}}\theta$ descends to an endomorphism

$$\theta^{i_1,i_2}_{l_1,l} : \mathrm{Gr}^{i_1,i_2}_{l_1,l}(E) \to \mathrm{Gr}^{i_1,i_2}_{l_1,l}(E) \otimes \Omega^1_{\bar{S}_{i_1,i_2}}(\log(S_{i_1,i_2} \cap \sum_{j \neq i_1,i_2} S_j).$$

This endomorphism together with the projection of Hodge sub-bundles of $\mathrm{Gr}^{i_1}_{l_1}(E)$ to $\mathrm{Gr}^{i_1,i_2}_{l_1,l}(E)$ is a direct sum

$$(\mathrm{Gr}^{i_1,i_2}_{l_1,l}(E), \theta^{i_1,i_2}_{l_1,l}) = \bigoplus_j ((\mathrm{res}_{\bar{S}_{i_1,i_2}}\theta)^j(\mathcal{P}^{i_1,i_2}_{l_1,l+2j}), \theta^{i_1,i_2}_{l_i,l,j})$$

of systems of Hodge bundles corresponding to the polarized VHS on the primitive decomposition of $\mathrm{Gr}^{i_1,i_2}_{l_1,l_2}(V)$.

We call the direct sum of systems of Hodge bundles

$$\bigoplus_{l_1,l_2,j} ((\mathrm{res}_{\bar{S}_{i_1,i_2}}\theta)^j(\mathcal{P}^{i_1,i_2}_{l_1,l_2+2j}), \theta^{i_1,i_2}_{l_i,l_2,j}) =: \mathrm{loc}_{S_{i_1,i_2}}(E,\theta)$$

the localization of (E,θ) on S_{i_1,i_2}. In general, for $I = \{i_1, i_2, ..., i_{\#I}\}$ we just construct the residue $\mathrm{res}_{S_I}(\theta)$ and its weight filtration inductively. The pulled back the weight filtration on each individual step to $E|_{S_I}$ gives rise a filtration with a multi index set

$$W_{0,...,0,...,0} \subset ... \subset W_{l_1,...,l_m,...,l_{\#I}} \subset W_{l_1,...,l_m+1,...,l_{\#I}} \subset ... \subset W_{2k,...,2k,...,2k} = E|_{S_I},$$

such that the grading together with the descent θ

$$\bigoplus_{l_1,...,l_m,...,l_{\#I}} (\mathrm{Gr}_{l_1,...,l_m,...,l_{\#I}}(E|_{S_I}, \theta_{l_1,...,l_m,...,l_{\#I}})$$

is a direct sum of systems of Hodge bundles corresponding to the polarized VHS on the primitive decompositions of the components of the grading

$$V = \bigoplus_{l_1,l_2,...,l_{\#I}}^{i_1,i_2,...,i_{\#I}} \mathrm{Gr}^{i_1,i_2,...,i_{\#I}}_{l_1,l_2,...,l_{\#I}}(V).$$

Now let $N \subset E^{p,q}$ be a sub-bundle and $\theta(N) = 0$. We consider the grading $\mathrm{Gr}_{l_1,...,l_m,...,l_{\#I}}(N|_{S_I})$ of the filtration restricted to N. It is contained in the kernel of $\mathrm{loc}_{S_I}\theta$. Applying the same argument in i) to this grading we prove ii).

Proof of Theorem 1.2. We first consider a special case, where $f : Z \to X$ with $f(Z) \not\subset S$. By Kollár's theorem ([K] Th. 5.20) the Chern classes $c_i(f^*N^\vee)$ of the dual bundle can be represented by the corresponding Chern forms $c_i(f^*N_0^\vee, h)$ of the induced Hodge metric, and they are semi positive by i), Prop. 1.1. Hence

$$\int_Z f^*(c_1^{i_1}(N^\vee)...c_n^{i_n}(N^\vee)) = \int_{Z_0} f^*(c_1^{i_1}(N_0^\vee, h)...c_n^{i_n}(N_0^\vee, h)) \geq 0.$$

We consider now the general case. Suppose that $f(Z) \subset S_I$ and $f(Z) \not\subset \cap_{J\supset I}S_J$. We pull back the weight filtration of $\mathrm{res}_{S_I}\theta$ to Z, and enlarge it to a filtration

$$W_{0,...,0,...,0}(f^*N) \subset ... \subset W'_{l_1,...,l_m,...,l_{\#I}}(f^*N) \subset ... \subset f^*N,$$

such that it is a filtration of vector bundles on the complement of a subvariety of codimension two. We then take a blowing up $\pi : \hat{X} \to Z$, such that the pulled back filtration can be enlarged to a filtration of vector bundles

$$W_{0,...,0,...,0}(\pi^*f^*N) \subset ... \subset W_{l_1,...,l_m,...,l_{\#I}}(\pi^*f^*N) \subset ... \subset \pi^*f^*N.$$

Now, all the quotient $\mathrm{Gr}_{l_1,...,l_m,...,l_{\#I}}(\pi^*f^*N)$ are sub-bundles of $\pi^*f^*(\mathrm{loc}_{S_I}E^{p,q})$, and lies in the kernel of $\pi^*f^*\mathrm{loc}_{S_I}\theta$ by ii, Prop. 1.1. Applying the same argument as before, the Chern classes of $\mathrm{Gr}_{l_1,\ldots,l_m,\ldots,l_{\#I}}(pi^*f^*N^\vee)$ are semi-positive. Hence the Chern classes of $\pi^*f^*N^\vee$ are semi positive. This implies that

$$\int_Z f^*(c_1^{i_1}(N^\vee)\ldots c_n^{i_n}(N^\vee)) = \int_{\hat{Z}} c_1^{i_1}(\pi^*f^*N^\vee)\ldots c_n^{i_n}(\pi^*f^*N^\vee) \geq 0.$$

Suppose that $c_1(N)L^{\dim Z-1} = 0$ for some ample line bundle L. We consider first a special case, where $f(Z) \not\subset S$. we shall show that the Curvature form on f^*N_0 vanishes. Let C be a smooth projective curve of the complete intersection of sections of the linear system $|mL|$. Since the first Chern form of $f^*N|_{C_0}$ is negative semi definite and represents the first Chern class of $f^*N|_C$, this Chern form vanishes. By choosing different curve C passing through any point in Z and along any direction, we show that the first Chern form of f^*N_0 vanishes. This implies that the curvature form on f^*N_0 of the Hodge metric h vanishes. Hence, the Gauss-Manin connection restricts a flat $h-$connection on on f^*N_0.

Suppose now $f(Z) \subset S_I$ for some I, and $\not\subset \cap_{J\supset I}S_J$, then by the same argument as above we show that the same property holds for the grading $\mathrm{Gr}(N)$ of the pulled back weight filtration of the residue of θ on S_I. Theorem 1.2 is proved.

2. Semi positivity of parameter spaces of Hodge structures. Let $\tilde{\phi}$: $\tilde{X}_0 \to \mathcal{D}$ denote the equivariant holomorphic map from the universal cover X_0 into the classifying space of polarized Hodge structures corresponding to a VHS $\mathbb{V}_0$ with the system of Hodge bundles (E_0, θ_0). The differential of ϕ induces a sheaves homomorphism

$$d\tilde{\phi} : T_{\tilde{X}_0} \to \phi^*T_{\mathcal{D}}.$$

We have the following Hodge-theoretical description for $d\tilde{\phi}$ [G1] and [Sch]. Let $V_{\mathbb{R}}$ denote the real vector space of the fibre of the local system at x_0 with the Hodge filtration

$$V_{\mathbb{C}} = F_0^0 \supset F_0^1 \supset \ldots \supset F_0^k.$$

The orthogonal group of the bilinear form Q is a linear algebraic group defined over $\mathbb{R}$

$$G_{\mathbb{R}} = \{g \in \mathrm{GL}(V_{\mathbb{R}}) | Q(g(u), g(v) = Q(u,v) \quad \text{for all} \quad u,\, v \in V_{\mathbb{R}}\}.$$

The group of its $\mathbb{C}-$points

$$G_{\mathbb{C}} = \{g \in \mathrm{GL}(V_{\mathbb{R}}) | Q(g(u), g(v) = Q(u,v) \quad \text{for all} \quad u,\, v \in V_{\mathbb{C}}\}.$$

The Lie algebra $g_{\mathbb{C}}$ of $G_{\mathbb{C}}$ is then

$$g_{\mathbb{C}} = \{X \in \mathrm{End}(V_{\mathbb{C}}) | Q(X(u), v) + Q(u, X(v)) = 0 \quad \text{for all} \quad u,\, v \in V_{\mathbb{C}}\}.$$

The quotient space

$$\mathcal{D} = G_{\mathbb{R}}/G_{\mathbb{R}} \cap B, \quad \text{where} \quad B = \{g \in G_{\mathbb{C}} | gF_0^p = F^p\, \forall p\}$$

is the so called classifying space of Hodge structures, which parameterizes $Q-$polarized Hodge structures with fixed Hodge numbers $h^{p,q}$, $\sum_{p+q=k} h^{p,q} = \dim V_{\mathbb{C}}$, and carries the universal $Q-$polarized Hodge filtration, and the universal Hodge bundles $\mathcal{E}$.

Since the Hodge structure is functorial, the dual space $V_{\mathbb{R}}^{\vee}$ is equipped naturally a Hodge structure of weight $-k$ by

$$E_0^{\vee} := \bigoplus_{p'+q'=-k} E^{\vee}{}_0^{p',q'},$$

where $E^{\vee}{}_0^{p',q'} = E^{-p',-q'}{}_0^{\vee}$.

If the Hodge structure on $V_{\mathbb{R}}$ is polarized by a bilinear form. Then this bilinear form induces naturally a bilinear form on $V_{\mathbb{R}}^{\vee}$ which polarizes the Hodge structure on $V_{\mathbb{R}}^{\vee}$.

Further more, these two Hodge structures on $V_{\mathbb{C}}$ and on $V_{\mathbb{C}}^{\vee}$ induces a Hodge structure on $\mathrm{End}(V_{\mathbb{C}})$

$$\bigoplus_{r+s=0} \mathrm{End}(E)_0^{r,s} = \bigoplus_{(p+p')+(q+q')=0} E_0^{p,q} \otimes E^{\vee}{}_0^{p',q'}.$$

The product of the polarizations on $V_{\mathbb{R}}$ and on $V_{\mathbb{R}}^{\vee}$ gives the polarization on $V_{\mathbb{R}} \otimes V_{\mathbb{R}}^{\vee}$. The Lie algebra $\mathrm{g}_{\mathbb{R}}$ of $G_{\mathbb{R}}$ is then

$$\mathrm{g}_{\mathbb{R}} = \{X \in \mathrm{End}(V_{\mathbb{R}}) | Q(X(u), v) + Q(u, X(v)) = 0 \quad \text{for all} u,\, v \in V_{\mathbb{R}}\}.$$

This sub-space $\mathrm{g}_{\mathbb{R}} \subset \mathrm{End}(\mathrm{V}_{\mathbb{R}})$ carries a sub-Hodge structure with

$$\mathrm{g}_{\mathbb{C}} = \bigoplus_{\mathrm{r}} \mathrm{g}^{\mathrm{r},-\mathrm{r}},$$

where

$$\mathrm{g}^{\mathrm{r},-\mathrm{r}} = \{\mathrm{X} \in \mathrm{g}_{\mathbb{C}} | \mathrm{X}\mathcal{E}^{\mathrm{p},\mathrm{q}} \subset \mathcal{E}^{\mathrm{p}+\mathrm{r},\mathrm{q}-\mathrm{r}}, \quad \forall\, \mathrm{p},\, \mathrm{q}\}.$$

The Hodge bundle $\mathrm{g}^{-1,1}$ is identified with the so-called horizontal sub-bundle of holomorphic tangent bundle $T_{\mathcal{D}}$. We have then

$$T_{\tilde{X}_0} \overset{d\tilde{\phi}}{\to} \phi^*\mathrm{g}^{-1,1} \overset{\mathrm{i}}{\hookrightarrow} \phi^*\mathrm{End}(\mathcal{E}),$$

which descends to

$$T_{X_0} \overset{d\phi}{\to} \mathrm{g}^{-1,1} \overset{\mathrm{i}}{\hookrightarrow} \mathrm{End}(\mathrm{E}_0),$$

where $\mathrm{g}^{-1,1}$ is the $(-1,1)-$type Hodge bundle of the descent sub-system of Hodge bundles of (E_0, θ_0), and the composition map $id\phi$ coincides with the map

$$T_{X_0} \to \mathrm{End}(E_0)$$

induced by the Kodaira-Spencer map $\theta_0 : E_0 \to E_0 \otimes \Omega^1_{X_0}$.

Suppose that the monodromies of $\mathbb{V}_0$ around components of S are unipotent. Then by works of Deligne and Schmid we have a natural extension

$$E \to E \otimes \Omega^1_X(\log S),$$

which induces an extension

$$T_X(-\log S) \overset{d\phi}{\to} \mathrm{g}^{-1,1} \overset{\mathrm{i}}{\hookrightarrow} \mathrm{End(E)}.$$

In the general, by [Sch] the monodromies around the components of S are unimodular. i.e. the norms of the eigenvalues of the monodromy matrix are equal to 1. In this case, one still extensions

$$E_\alpha \to E_\alpha \otimes \Omega^1_X(\log S),$$

which induces an extension

$$T_X(-\log S) \stackrel{d\phi}{\to} \mathrm{g}_\alpha^{-1,1} \stackrel{\mathrm{i}}{\hookrightarrow} \mathrm{End}(\mathrm{E}_\alpha).$$

PROP. 2.1. *Let* $(End(E), \theta^{end})$ *denote the system of Hodge bundles corresponding to the polarized VHS on* $End(H_{\mathbb{R}})$*, induced by the polarized VHS on* $V_{\mathbb{R}}$*. Then*

$$\theta^{end}(d\phi(T_X)(-\log S))) = 0.$$

Proof. Let (E, θ) be the system of Hodge bundles corresponding to the VHS on V. Then the system of Hodge bundles corresponding to the induced VHS on $V^\vee$ is

$$(E^\vee = \bigoplus_{p'+q'=-k} E^{\vee p',q'}, -\theta^\vee),$$

where $E^{\vee p',q'} = E^{-p',-q'\vee}$, and $\theta^\vee = \theta$ under the natural isomorphism

$$\mathrm{End}(E^\vee) = \mathrm{End}(E).$$

The system of Hodge bundles corresponding to the induced VHS on $\mathrm{End}(V)$ is

$$(\mathrm{End}(E), \theta^{\mathrm{end}} = (\bigoplus_{r+s=0} \mathrm{End}(E)^{r,s}, \theta^{\mathrm{end}}) = (\bigoplus_{(p+p')+(q+q')=0} E^{p,q} \otimes E^{\vee p',q'}, \theta^{\mathrm{end}})$$

with

$$\theta^{\mathrm{end}}(u \otimes v^\vee) = \theta(u) \otimes v^\vee - u \otimes \theta^\vee(v^\vee).$$

Let $U \subset X$ be a Zariski open subset, such that $E|_U$ and Ω^1_X are trivial with the base $\{e_1, ..., e_m\}$ and the base $\{dz_1, ..., dz_n\}$ respectively. Then

$$\theta|_U = \sum_{i=1}^n A_i dz_i,$$

with

$$A_i = (\sum_{r,s} a_i^{r,s} e_r \otimes e_s^\vee) \in H^0(U, \mathrm{End}(E))$$

and

$$\theta^\vee|_U = \sum_{i=1}^n A_i^\vee dz_i,$$

with

$$A_i^\vee = (\sum_{r,s} a_i^{\vee r,s} e_r^\vee \otimes e_s) \in H^0(U, \mathrm{End}(E^\vee)), \quad \text{with} \quad a_i^{\vee r,s} = a_i^{s,r}, \quad \forall i,\ r,\ s.$$

$T_X|_U$ has the base $\{1/\partial z_1, ..., 1/\partial z_n\}$, and the map

$$d\phi : T_X \to \mathrm{End}(E)$$

is defined by sending $1/\partial z_i$ to A_i. A simple calculation gives

$$\theta^{\mathrm{end}}(A_i) = \sum_{j=1}^{n} (A_i A_j - A_j A_i) dz_j.$$

Since $A_i A_j - A_j A_i = 0$ for all i, j we get $\theta^{\mathrm{end}}(A_i) = 0$. Prop. 2.1 is proved.

Proof of Theorem 0.1. i) We first show i) for the case when the monodromy around the components of S are nilpotent. We shall prove that $\Omega_X^1(\log S)$ is weakly positive on the complement of the degeneration locus D of $d\phi$.

By blowing up $\pi : \hat{X} \to X$ on some sub variety of codimension ≥ 2 and contained in the degeneration locus D of $d\phi$, we may enlarge the sub-sheaf

$$\pi^* d\phi(T_X)(-\log S)) \subset T \subset \pi^* \mathrm{g}^{-1,1}$$

to a sub-bundle T with the torsion quotient $T/\pi^* d\phi(T_X)(-\log S))$. By Prop. 2.1 $\theta^{\mathrm{end}}(T) = 0$. Since the monodromies of $\mathrm{End}(\mathbb{V}_0)$ around the components of S also are unipotent, by Theorem 1.2 the Chern classes $T^\vee$ are semi positive. This implies that $T^\vee$ is weakly positive. Now Let L be an ample line bundle over X, and $E \subset \hat{X}$ the exceptional locus of the blowing up π. Then $\pi^* L^r(-E)$ is ample for some $r >> 1$. Hence, for any positive α there exists a positive number β such that

$$S^{\alpha\beta}(T^\vee) \otimes (\pi^* L^r(-E))^\beta$$

is generated by the global sections on $\hat{X}$.

Since

$$T^\vee|_{\hat{X}\setminus\pi^* D} \simeq \pi^* \Omega_X^1(\log S)|_{\hat{X}\setminus\pi^* D},$$

by take the direct image of π we get that

$$S^{\alpha\beta}(\Omega_X^1(\log S))|_{X\setminus D} \otimes L^{r\beta}$$

is generated by global sections on X. We proved i) in Theorem 0.1 for the unipotent case.

We shall now prove i) in Theorem 1 for the general case. By [Sch] the system of Hodge bundles

$$\theta_0 : E_0 \to E_0 \otimes \Omega_{X_0}^1$$

has an extension as a system of Hodge coherent sheaves

$$\theta : E \to E \otimes \Omega_X^1(\log S).$$

After a blowing up at a subvariety of S, we get an extension of system of Hodge bundles. This gives rise a map

$$T_X(-\log S) \to \mathrm{End}(E).$$

We claim that the sub-sheaf $T_X(-\log S) \to \mathrm{End}(E)$ is generated by those local sections, whose Hodge norms have at most logarithmic singularities on S.

Given some real numbers $\alpha_1, \alpha_2, \ldots$. Let $U \ni x \in S$ be a small open subset, such that $U \cap S = \{z \in U | z_1 z_2 \ldots z_k = 0\}$. One defines an extension by taking local sections s in E with the growth w.r.t. the Hodge norm h

$$|s|_h \leq |z_1|^{\alpha_1 - \epsilon_1} \ldots |z_k|^{\alpha_k - \epsilon_k}$$

for any positive numbers $\epsilon_1, \ldots, \epsilon_k$. By [S2] for the $\dim X = 1$ case and [JLZ] for the $\dim X > 1$ case, one gets an extension for the Kodaira-Spencer map

$$\theta : E_\alpha \to E_\alpha \otimes \Omega^1_X(\log S).$$

This means that the extended Kodaira-Spencer map preserves the polynomial growth condition of local sections in E. Hence, the sub-sheaf $T_X(-\log S) \to \mathrm{End}(E)$ must be generated by those local sections of $\mathrm{End}(E)$, whose Hodge norms have at most logarithmic singularities on S.

Now the argument in the proof of Theorem 5.20 in [K] can be still applied to this sub-sheaf $T_X(-\log S) \to \mathrm{End}(E)$. So, we also prove i) in Theorem 0.1 for the general case by the same argument as in the unipotent case.

Proof of ii) in Theorem 0.1. We need the following lemma

LEMMA 2.2. *Under the same notion as in the proof of i). Suppose that $d\phi$ is injective at some point. Then the self-intersection* $(\det T^\vee)^{\dim X} > 0$.

Proof. Since by Prop. 2.1 and i) in Theorem 1 the first Chern form of $T|_X$ with the restricted Hodge metric is negative semi definite, we only need to show that this first Chern form of T is strictly negative at some point in X_0. The main idea here is to apply Griffith-Schmid's theorem that the holomorphic sectional curvature in the horizontal directions is strictly negative [GS].

Suppose that $d\phi$ is injective at some point $x_0 \in X_0$. Then there exists an analytic open neighborhood U of x_0, such that $\phi : U \to \mathcal{D}$ is an embedding.

Let $W = \phi(U)$. We have $T_X|_U = T|_U$, and $T_X|_U$ together with the Hodge metric is the pulled back of the tangent bundle T_W together with the universal Hodge metric via ϕ. Let $\Theta(T_X|_U)$ denote the curvature form of $T_X|_U$ of the Hodge metric and $\Theta(T_W)$ denote the curvature form of T_W with the universal Hodge metric. Then we have $\Theta(T_X|_U) = \phi^*(\Theta(T_W))$.

By Prop.1.1 the Hermitian form $(\Theta(T_X|_U), v, \bar{v})$ is negative semi definite for any holomorphic tangent vector $v \in T_X|_{x_0}$. In order to show it is strictly negative definite, we only need to show that at least one eigenvalue of $(\Theta(T_X|_U), v, \bar{v})$ is strictly negative for $v \neq 0$.

We shall show at least one eigenvalue of $(\Theta(T_W), u, \bar{u})$ is strictly negative for any $u \neq 0 \in T_W|_{y_0}$. By a theorem due to Griffith-Schmid [GS] the holomorphic sectional curvature of $\Theta(\mathrm{g}^{-1,1})$ of the universal Hodge metric is strictly negative. i.e.the product

$$u(\Theta(\mathrm{g}^{-1,1}),\mathrm{u},\bar{\mathrm{u}})\bar{\mathrm{u}}<0,\quad \forall \mathrm{u}\neq 0\in \mathrm{g}^{-1,1}.$$

Consider now the holomorphic sub-bundle $T_W\subset \mathrm{g}^{-1,1}|_{\mathrm{V}}$, the Hodge metric induces an orthogonal C^∞−decomposition

$$\mathrm{g}^{-1,1}|_{\mathrm{W}}=\mathrm{T_W}\oplus \mathrm{T_W^\perp}.$$

Let $A\in A^{1,0}(\mathrm{Hom}(T_W,T_W^\perp)$ denote the second fundamental form. Then

$$\Theta(T_W)=\Theta(\mathrm{g}^{-1,1})|_{\mathrm{T_W}}+\bar{\mathrm{A}}\wedge \mathrm{A}.$$

Since the Hermitian form $(\bar{A}\wedge A,u,\bar{u})$ is negative semi definite, the product

$$u(\Theta(T_W),u,\bar{u})\bar{u}=u(\Theta(\mathrm{g}^{-1,1}),\mathrm{u},\bar{\mathrm{u}})\bar{\mathrm{u}}+\mathrm{u}(\bar{\mathrm{A}}\wedge\mathrm{A},\mathrm{u},\bar{\mathrm{u}})\bar{\mathrm{u}}<0$$

for any non zero tangent vector $u\in T_W|_{y_0}$. Hence, we obtain

$$v(\Theta(T_X|_U),v,\bar{v})\bar{v}=\phi_*v(\Theta(T_W),\phi_*v,\bar{\phi_*v})\bar{\phi_*v}<0$$

for any non zero vector $v\in T_X|_{x_0}$. This implies that the Hermitian form $(\Theta(T_X|_U),v,\bar{v})$ has at least one negative eigenvalue for any non zero holomorphic tangent vector $v\in T_X|_{x_0}$. Hence, the first Chern form of the dual bundle $c_1(T^\vee,h)$ is strictly positive on U. Applying Kollár's theorem we get

$$(\det T^\vee)^{\dim X}=\int_{\hat{X}}\wedge^{\dim X}c_1(T^\vee,h)>0.$$

Lemma 2.2 is proved.

We are in the position to prove ii). By the proof for i) in Theorem 1 $\det T^\vee$ is nef, and by Lemma 2.2 $(\det T^\vee)^n>0$. Hence, by Kawamata-Viehweg's vanishing theorem ([Ka2], [V2],3.2) the Kodaira dimension of $\det T^\vee$ is equal to $\dim X$. Since the quotient of $T_{\hat{X}}(-\log S)\subset T$ i s a torsion sheaf, we get a non zero sheaves homomorphism $\det T^\vee\to\det\Omega^1_{\hat{X}}(\log S)$. Hence, the bigness of $\det T^\vee$ implies the bigness of $\det\Omega^1_{\hat{X}}(\log S)$. Note that the Kodira-dimension of (X,S) is a birational invariant, we prove ii) in Theorem 0.1.

3. Arakelov inequalities for Hodge bundles. In this section we shall prove Theorem 0.2. First we review some basic facts from representation theory [FH]. Let V and W be two vector spaces of $\dim V=m$, and $\dim W=n$. Then one has

$$\wedge^d(V\otimes W)=\bigoplus \mathbb{S}_\lambda V\otimes\mathbb{S}_{\lambda'}W,$$

where $\mathbb{S}_\lambda V$ respectively $\mathbb{S}_{\lambda'}W$, is an irreducible representation of $\mathrm{GL}(V)$ respectively an irreducible representation of $\mathrm{GL}(W)$. The sum over partitions λ with at most $\dim V$ rows and at most $\dim W$ columns in the associated Young diagram. And λ' is the conjugate of λ, which is defined by interchanging rows and columns of λ. It is known that the representation $\mathbb{S}_\lambda V$ can be realized as a $\mathrm{GL}(V)$−invariant subspace of a tensor product $V^{\otimes l}$, which depends on λ. One can do the same thing for vector bundles (see [B]).

Now, we do one step further. Let $N \subset E$ be a sub-bundle of a system of Hodge bundles (E, θ) corresponding to a VHS $\mathbb{V}$, and $\theta(N) = 0$. Then by taking the representation $\mathbb{S}_\lambda(\mathbb{V})$, we obtain the system of Hodge bundles corresponding to this representation $(\mathbb{S}_\lambda E, \mathbb{S}_\lambda \theta)$, which has the property $\mathbb{S}_\lambda N \subset \mathbb{S}_\lambda E$ and $\mathbb{S}_\lambda(\theta)\mathbb{S}_\lambda N = 0$.

By Theorem 1.2 we obtain

LEMMA 3.1. *$\mathbb{S}_\lambda N$ is semi negative w.r.t. the induced Hodge metric via the embedding*

$$\mathbb{S}_\lambda N \subset \mathbb{S}_\lambda E.$$

Proof of Theorem 0.2. We consider the differential of the period map

$$d\phi : T_X(-\log S) \to \mathrm{End}(E)$$

and let $T =: d\phi(T_X(-\log S))$. Then the Kodaira-Spencer map takes value in $T^\vee$,

$$E^{p,q} \overset{\theta^{p,q}}{\to} E^{p-1,q+1} \otimes T^\vee \overset{\theta^{p-1,q+1}\otimes i}{\to} E^{p-2,q+2} \otimes T^{\vee\otimes 2} \overset{\theta^{p-2,q+2}\otimes i^2}{\to}$$

$$\ldots \to E^{0,k} \otimes T^{\vee\otimes(k-q)},$$

where $i : T^\vee \to T^\vee$ is the identity map. We set

$$I_l := (\theta_{p-l,q+l} \otimes i^l)\ldots(\theta_{p-1,q+1} \otimes i)(E^{p,q}),$$

and

$$K_l := \mathrm{Ker}(I_l \overset{\theta_{p-l-1,q+l+1}\otimes i^{l+1}}{\to} I_{l+1}).$$

After a successive blowing up $\tau : \hat{X} \to X$ we may assume that all $T^\vee$, I_l, K_l are vector bundles.

By the additivity of divisors of determinant line bundles in short exact sequences, to prove Theorem 0.2 is enough to show that it holds on $\hat{X}$

$$\det I_l = r_l \det \Omega^1_{\hat{X}}(\log \hat{S}) - P_l + N_l$$

and

$$\det K_l = r'_l \det \Omega^1_{\hat{X}}(\log \hat{S}) - P'_l + N'_l$$

for some positivie rational numbers r_l, r'_l, which are bounded above by the Hodge type of E, and some semi negative $\mathbb{Q}$–divisors N_l, N'_l and some effective $\mathbb{Q}$–divisors P_l, P'_l.

Since for some large l, we have eventually $I_l = K_l$ and again by the additivity of divisors of determinente line bundles in short exact sequences we only need to show the second equality.

Since $\theta_{p-l-1,q+l+1} \otimes i^{l+1}(K_l) = 0$, we have

$$K_l \subset K(\theta^{p-l-1,q+l+1}) \otimes T^{\vee\otimes(l+1)}.$$

Let $d := \mathrm{rk}K_l$. By [FH] we have then

$$\det K_l \subset \wedge^d K(\theta^{p-l-1,q+l+1}) \otimes T^{\vee\otimes(l+1)} = \bigoplus \mathbb{S}_\lambda K(\theta^{p-l-1,q+l+1}) \otimes \mathbb{S}_{\lambda'} T^{\vee\otimes(l+1)}.$$

We find then a non-zero projection for some λ

$$\det K_l \to \mathbb{S}_\lambda K(\theta^{p-l-1,q+l+1}) \otimes \mathbb{S}_{\lambda'} T^{\vee\otimes(l+1)}.$$

Hence, a non-zero sheaves map

$$f : \det K_l \otimes \mathbb{S}_{\lambda'} T^{\otimes(l+1)} \to \mathbb{S}_\lambda K(\theta^{p-1-1,q+1+1}).$$

Set

$$G := f(\det K_l \otimes \mathbb{S}_{\lambda'} T^{\otimes(l+1)}) \subset \mathbb{S}_\lambda K(\theta^{p-l-1,q+l+1}).$$

After a further blowing up of $\hat{X}$, one has a short exact sequence of vector bundles

$$0 \to \mathrm{Ker}(f) \to \det K_l \otimes \mathbb{S}_{\lambda'} T^{\otimes(l+1)} \to G \to 0,$$

and $G \subset G'$ such that G'/G is a torsion sheaf and $G' \subset \mathbb{S}_\lambda K(\theta^{p-l-1,q+l+1})$ is a sub-bundle. By Lemma 3.1 G' is semi negative. Hence $\det G = \det G' - P'$ with $\det G'$ is a semi negative divisor and P' is an effective divisor.

A calculation of the determinantes gives

$$\mathrm{rk}(\mathbb{S}_{\lambda'} T^{\otimes(l+1)}) \det K_l + \det \mathbb{S}_{\lambda'} T^{\otimes(l+1)} = \det G + \det \mathrm{Ker}(f) = \det G' - P' + \det \mathrm{Ker}(f).$$

Further more, we consider the sub-bundle

$$F := \mathrm{Ker}(f) \otimes \det K_l^{\vee} \subset \mathbb{S}_{\lambda'} T^{\otimes(l+1)}.$$

By Lemma 3.1 this sub-bundle F is semi negative. Hence,

$$-\mathrm{rkKer}(f) \det K_l + \det \mathrm{Ker}(f) = \det F \leq 0.$$

From those two equalities of divisors we obtain

$$(\mathrm{rk}(\mathbb{S}_{\lambda'} T^{\otimes(l+1)}) - \mathrm{rkKer}(f)) \det K_l = \det \mathbb{S}_{\lambda'} T^{\vee\otimes(l+1)} + \det G - P' + \det F.$$

Since the map f is non-zero, $\mathrm{rk}(\mathbb{S}_{\lambda'} T^{\otimes(l+1)}) - \mathrm{rkKer}(f) =: r' \geq 1$. And it is also clear $\det \mathbb{S}_{\lambda'} T^{\vee\otimes(l+1)} = r'' \det T^{\vee}$, where r'' is a positive rational number and bounded above by the Hodge type of E. Set $N_l := \det G + \det F$, and notice that $\det \tau^* \Omega^1_X(\log S) = \det T^{\vee} + P''$, where P'' is an effective divisor, we get

$$\det K_l = \frac{r''}{r} \det \tau^* \Omega^1_X(\log S) - \frac{r''}{r} P'' - \frac{1}{r'} P' + \frac{1}{r'} N_l.$$

Theorem 0.2 is proved.

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