ASIAN J. MATH.
Vol. 4, No. 1, pp. 221–226, March 2000

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ON A CONJECTURE OF DEMAILLY AND KOLLÁR*

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1. Introduction. If f is holomorphic in a neighborhood of a compact set K in a complex manifold, define $c_K(f)$ to be the supremum of all real numbers c such that $|f|^{-2c}$ is integrable on some neighborhood of K.

In their recent paper [DK], Demailly and Kollár made the following remarkable conjectures.

CONJECTURE A: Fix the compact set K. For every non-zero holomorphic function f and for every compact set L containing K in its interior, there is a number $\alpha = \alpha(f, K, L) > 0$ such that

$$\sup_L |g - f| < \alpha \quad \Longrightarrow \quad c_K(g) \geq c_K(f)$$

CONJECTURE B: Let

$$\mathcal{C}(n) = \{c_0(f) \ : f \text{ is holomorphic in a neighborhood of the origin of } \mathbf{C}^n\}$$

Then $\mathcal{C}(n)$ satisfies the ascending chain condition: every convergent increasing sequence in $\mathcal{C}(n)$ is stationary.

A version of Conjecture B in algebraic geometry has been formulated earlier, starting with the 1992 work of Shokurov [Sh][K1-2]. This algebraic geometric version of Conjecture B has been established in dimension $n = 2$ by Shokurov in [Sh], and in dimension $n = 3$ by Alexeev in [A]. Related conjectures and results in algebraic geometry can be found in [K1-2]. The exponents $c_0(f)$ also play an important role in the study of the existence of Kähler-Einstein metrics [CY][TY1-2][Si][T1-2] [Y1-2][DK].

In [DK], Demailly and Kollár had proved the following weaker version of Conjecture A: under the same conditions, for any $\epsilon > 0$, there exists a number $\alpha(f, K, L, \epsilon)$ such that

$$(A_\epsilon) \qquad \sup_L |g - f| < \alpha(f, K, L, \epsilon) \quad \Longrightarrow \quad c_K(g) \geq c_K(f) - \epsilon$$

In dimension $n = 2$, this last statement had also been obtained in [T2] and in [PS]. The purpose of this short note is to show that the methods of [PS] can also give the following theorem:

THEOREM. *Conjectures A and B hold when* $n = 2$.

It was already observed in [DK] that Conjecture A follows from Conjecture B combined with the weaker statement (A_ϵ). We shall nevertheless give direct separate proofs for both conjectures, since in our approach, the method of proof of Conjecture A is no different from that of (A_ϵ). It will also emerge from our proof that x is a limit

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point of $\mathcal{C}(2)$ if and only if $x = 0$ or x is a number of the form $2/a$ where a is a positive integer.

2. Proof of Conjecture A when $n = 2$. In [PS], an approach was developed for the study of integrals of $|f|^{-2c}$ in n variables, by iterating sharp estimates for one-dimensional integrals of the form

$$\int_{B_\Lambda} \frac{\sum_{i=1}^{I} |P_i(z)|^\epsilon}{\sum_{j=1}^{J} |Q_j(z)|^\delta} dV \tag{2.0}$$

Here $P_i(z)$ and $Q_j(z)$ are polynomials in the variable $z \in \mathbf{C}$, ϵ and δ are non-negative real numbers, $dV = dxdy$ is the Euclidian measure on $\mathbf{C}$, and B_Λ is the open disk of radius Λ. A key result was that the finite-dimensional space of polynomials $P_i(z)$, $Q_j(z)$ admits a stratification into algebraic varieties, on each of which the *size* of the above integral is given by expressions of the form

$$\frac{(\sum_{i=1}^{\hat{I}} |\hat{P}_i(B_1, \cdots, B_M)|^2)^{\hat{\epsilon}}}{(\sum_{j=1}^{\hat{J}} |\hat{Q}_j(B_1, \cdots, B_M)|^2)^{\hat{\delta}}}$$

where $\hat{P}_i(B_1, \cdots, B_M)$ and $\hat{Q}_j(B_1, \cdots, B_M)$ are polynomials in the coefficients B_1, $\cdots, B_M$ of the original polynomials $P_i(z)$ and $Q_j(z)$ [PS, Theorem 4].

In this note we require the special case of these formulas, when the integrand in (2.0) reduces to $|Q(z)|^{-\delta}$ with $Q(z)$ a polynomial of degree N, and δ is a real number in the range $2/N < \delta < 4/N$ such that $2/\delta$ in non-integral. In this case the formulas simplify substantially, and we have [PS, Theorems 2 and 3]

PROPOSITION 1. *Fix a positive integer N. For every r in the range $1 \leq r \leq (N/2)$, there exist polynomials $D_{r,j} \in \mathbf{Z}[A_1, ..., A_N]$, with $1 \leq j \leq h(r) = N!/(2r)!$ with the following property. If we let*

$$\Delta_r = \Delta_r(a_1, ..., a_N) = \sum_{q=1}^{h(r)} |D_{r,q}(a_1, ..., a_N)|^{1/h(r)!}$$

then for all real numbers $\delta \in (2/(N-r+1), 2/(N-r))$ and every positive real number Λ we have

$$\int_{B_\Lambda} \frac{1}{|Q(z)|^\delta} \, dV \sim \frac{1}{\Delta_{r+1}^{(N-r+1)\delta-2} \Delta_r^{2-(N-r)\delta}} \tag{2.1}$$

for all monic polynomials $Q(z) = \sum_{i=0}^{N} a_i z^{N-i}$ whose roots lie in $B_{\Lambda/2}$. If $\delta < 2/N$ then

$$\int_{B_\Lambda} \frac{1}{|Q(z)|^\delta} \, dV \sim \Lambda^{2-N\delta} \tag{2.2}$$

The implied constants in (2.1) and (2.2) depend on δ and Λ, but they are independent of the coefficients of Q.

We can establish now Conjecture A in 2 dimensions. It suffices to establish it when K is a single point 0. Let $f(z, w)$ be a holomorphic function in a neighborhood of the origin in $\mathbf{C}$ and assume $f(0) = 0$, $f(z, w)$ is not identically zero. Let M be the order of vanishing of $f(z, w)$ at the origin, i.e., the lowest degree with a non-vanishing monomial in the Taylor expansion of $f(z, w)$ at 0. Then, after a suitable rotation of coordinates, the Weierstrass Preparation Theorem says that on some polydisk $U \times V$ centered at the origin, f can be factored as $f = u_f p_f$ where u_f is nowhere vanishing holomorphic function and

$$p_f(z,w) \quad = \quad w^M + a_1(z)w^{M-1} + \cdots a_M(z).$$

Here $a_i(z) = a_{i,f}(z)$ are holomorphic functions satisfying the condition

$$a_i(0) = 0. \tag{2.3}$$

Moreover, there is a polydisk $U' \times V' \subseteq U \times V$, centered at the origin, and an $\alpha > 0$ such that if g is holomorphic on $U \times V$ and if $\sup_{U\times V} |g - f| < \alpha$, then g can be factored as $g = u_g p_g$ on $U' \times V'$ where u_g is a holomorphic function such that $|u_g|$ is bounded below by a positive constant, and

$$p_g(z,w) \quad = \quad w^M + a_{1,g}(z)w^{M-1} + \cdots a_{M,g}(z)$$

Here the $a_{i,g}$ are holomorphic on V', although they may no longer satisfy (2.3). The map $g \to a_{i,g}$ (*resp.* $g \to u_g$) is continuous with respect to the sup norm metrics on $U \times V$ and V' (*resp.* $U \times V$ and $U' \times V'$). Thus we can choose α and U' sufficiently small so that $||g - f|| = \sup_{U\times V} |g - f| < \alpha$ implies that $u_g u_f^{-1} \sim 1$ and for every $z \in U'$, all the roots of $p_g(z,w) = 0$ are in $\frac{1}{2}V'$ (see e.g. Lemma 5.2 in [PS]).

Let $N = 2M$ and define $b_{i,g}(z)$ by the formula

$$p_g(z,w)^2 = \sum_{i=0}^{N} b_{i,g}(z)w^{N-i}$$

For every r in the range $1 \le r \le M$, and for every holomorphic function g satisfying $||g - f|| < \alpha$, let $F_{r,g}(z) = \Delta_r(b_{1,g}, ..., b_{N,g})$, and let $n(r,g)$ be the unique real number such that

$$|z|^{-n(r,g)} F_{r,g}(z)$$

is continuous and non-vanishing at the origin. Proposition 1 implies that

$$n(r,g)(N!)! \in \mathbf{Z}. \tag{2.4}$$

By shrinking α and U' even further, we may assume (using, for example, the winding number principle) that for all g satisfying $||g - f|| < \alpha$, and for all r, we have $n(r,g) \le n(r,f)$.

Let

$$\Sigma(f) = \{c \in \mathbf{R} : |f|^{-2c} \text{ is integrable on some neighborhood of the origin}\}.$$

It is well-known that $\Sigma(f)$ is an open set (using for example Hironaka's theorem on resolution of singularities), and thus $\Sigma(f) = (-\infty, c_0(f))$. We must prove that if $||g - f|| < \alpha$, then $\delta \in \Sigma(f)$ implies $\delta < c_0(g)$. In fact, if T is at most a countably infinite set of real numbers, it suffices to prove that $\delta \in \Sigma(f)\backslash T$ implies $\delta < c_0(g)$.

Lemma 5.1 of [PS] guarantees that $\Sigma(f) \subseteq (0, 4/N)$. Choose $\delta \in \Sigma(f)\backslash T$ where T is the set of δ in $\Sigma(f)$ such that $2/\delta$ is an integer.

Since u_f is nowhere vanishing, we have

$$\int_{V'} \frac{1}{|g(z,w)|^{2\delta}}\, dV(w) \sim \int_{V'} \frac{1}{|p_g^2|^{\delta}}\, dV(w) \tag{2.5}$$

where the implied constant depends on f and δ, but not on g.

If $\delta < 2/N$, then (2.2) implies that (2.5) is finite, so that $\delta < c_0(g)$.

If $\delta > 2/N$, choose r, $1 \leq r \leq M$, such that $\delta \in (2/(N-r+1), 2/(N-r))$. Then applying Proposition 1 to the right hand side of (2.5), we obtain, for every $z \in U'$,

$$\int_{V'} \frac{1}{|g(z,w)|^{2\delta}}\, dV(w) \sim \frac{1}{F_{r+1,g}(z)^{(N-r+1)\delta-2} F_{r,g}(z)^{2-(N-r)\delta}}$$

Thus we see that $\delta \in \Sigma(g)$ if and only if

$$n(r+1,g)((N-r+1)\delta-2) + n(r,g)(2-(N-r)\delta) < 2. \tag{2.6}$$

Since $\delta \in \Sigma(f)$, (2.6) holds when $g = f$. But $n(k,g) \leq n(k,f)$ for all k. Thus $\delta \in \Sigma(g)$.

3. Proof of Conjecture B when $n = 2$. Let $\mathcal{C} = \mathcal{C}(2)$. It suffices to show that for every $r > 0$, the set $\{c \in \mathcal{C} : c \geq r\}$ satisfies the ascending chain condition.

If f is holomorphic in a neighborhood of the origin, and if $f(0) = 0$, then $\Sigma(f) \subseteq (0,4)$. Moreover, if f is a Weierstrass polynomial of degree M, i.e., an expression of the form $w^M + a_1(z)w^{M-1} + \cdots + a_M(z)$, then $c_0(f) < 2/M$. Since $2/M < r$ for M sufficiently large, it suffices to prove that for each $M > 0$,

$$\mathcal{C}(M) = \{c_0(f) : f \text{ is a Weierstrass polynomial of degree } M\} \tag{3.1}$$

satisfies the ascending chain condition. Thus we fix $M > 0$ and let $N = 2M$. We have shown that $\mathcal{C}(M) \subseteq [1/M, 2/M]$. It therefore suffices to show that for every integer r such that $1 \leq r \leq M$, the set $\mathcal{C}(M) \cap [2/(N-r+1), 2/(N-r)]$ satisfies the ascending chain condition (since $[2/N, 4/N]$ is a union of such intervals).

Thus we fix r such that $1 \leq r \leq M$, and we let $\delta \in (2/(N-r+1), 2/(N-r))$. Let f be a Weierstrass polynomial of degree M . It follows from (2.6) that if c is a positive real number, then $c \in \mathcal{C} \cap (2/(N-r+1), 2/(N-r))$ if and only if

$$n(r+1)((N-r+1)c-2) + n(r)(2-(N-r)c) = 2$$

$$\text{and}\quad c \in \left(\frac{2}{(N-r+1)}, \frac{2}{(N-r)}\right) \tag{3.2}$$

where we have denoted $n(r,f)$ by $n(r)$ for simplicity. Now (3.2) is equivalent to, via simple algebraic manipulations,

$$c \;=\; 2\frac{n(r+1)-n(r)+1}{n(r+1)(N-r+1)-n(r)(N-r)} \;\textit{and}$$

$$n(r+1) > N-r \;\textit{and}\; n(r) < N-r+1 \tag{3.3}$$

The fact that $n(r) < N-r+1$, together with (2.4) tells us that there are only finitely many possibilities for $n(r)$. For each such choice of $n(r)$, there may be infinitely many possibilities for the value of $n(r+1)$, but these possibilities must also satisfy (2.4). In particular, the possible values of $n(r+1)$ form a subset of

$$X = \{n \in \mathbf{Q} : n(N!)! \in \mathbf{Z}, n > N-r\}.$$

Now fix $n(r)$ and let $n(r+1)$ range over X in equation (3.3). The key observation is that as $m \in X$ tend towards infinity, the values of c form a *decreasing* sequence, converging to $2/(N-r+1)$. This proves Conjecture B, in the case $n = 2$. It also shows that the limit points of $\mathcal{C}(2)$ are either 0, or rational numbers of the form $2/a$

with a a positive integer. In other words, the limit set of $\mathcal{C}(2)$ is contained in $\mathcal{C}(1)$. In fact, it is equal to $\mathcal{C}(1)$, as we shall see in the next section*.

4. The limit set of $\mathcal{C}(n)$. It follows immediately from the definitions that $\mathcal{C}(n) \subseteq \mathcal{C}(n+1)$ for $n \geq 1$.

PROPOSITION 2. If $c \in \mathcal{C}(n)$ then $c = \lim x_m$, where $x_1, x_2, \ldots \in \mathcal{C}(n+1)$ forms a strictly decreasing sequence.

Proof. Let f be holomorphic in a neighborhood of the origin of $\mathbf{C}^n$ such that $c_0(f) = c$. Let m be a sufficiently large positive integer and let $g_m(z_1, \ldots, z_n, w) = w^m - f$. Then

$$\int_B \frac{1}{|w^m - f|^{2\delta}} dV(w) \sim \frac{1}{|f|^{2(\delta - 1/m)}}$$

Thus $c_0(g_m) = c + \frac{1}{m}$.

REMARKS. 1. The same proofs apply in dimension two to give the analogues of Conjectures A and B in the real-analytic setting, using the results in [PSS]. In dimensions 3 and higher, it is known that even the weaker conjecture A_ϵ is not true, due to a counterexample of Varchenko.

2. It is tempting to speculate that $\mathcal{C}(n)$ is exactly the limit set of $\mathcal{C}(n+1)$. This was also suggested earlier by Kollár, who also formulated Proposition 2 in [K2].

3. More specifically, it is likely that the structure of $\mathcal{C}(n+1)$ is determined inductively by that of $\mathcal{C}(n)$ in the manner suggested above:

$$\mathcal{C}(n+1) = \mathcal{C}(n) \cup \bigcup_{c \in \mathcal{C}(n), 1 \leq m < \infty} x(c, m)$$

where for each c, the $x(c, m)$ forms a strictly decreasing sequence such that

$$\lim_{m \to \infty} x(c, m) = c.$$

ACKNOWLEDGEMENTS. The authors would like to thank Professor J. Kollár for informing them about the works of Shokurov, Alexeev, and Igusa, and for providing them with the references [S], [A], [I], and [K1]. They would also like to thank Professor S.T. Yau for his encouragement and for providing them with references on Kähler-Einstein metrics.

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* In dimension $n = 2$, the exponents $c_0(f)$ were determined by Igusa [I] to be of the form $c_0(f) = \frac{1}{m} + \frac{1}{n}$, $m, n \in \mathbf{N}^*$, for irreducible power series $f \in \mathbf{C}[[x, y]]$. But for a general holomorphic function f in two variables, they can be more complicated, as we just saw. For example, for $f(x, y) = y(y^2 + x^{89})$, the exponent $c_0(f)$ is given by $c_0(f) = \frac{91}{267}$, which cannot be expressed as $\frac{1}{m} + \frac{1}{n}$ for any integers m, n.

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