ASIAN J. MATH.
Vol. 4, No. 1, pp. 165–172, March 2000

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A NOTE ON THE COSEGRE CLASS OF A SUBVARIETY*

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In this short note we introduce the notion of the cosegre class of a subvariety of a nonsingular algebraic variety, which is more geometric than the Segre class of a subvariety, and with it give a proof,which is essentially in the framework of algebraic geometry, to a theorem of Brylinski,Dubson and Kashiwara[1,Corollary 5].

Let Z be a subvariety of an algebraic variety X and denote by $C_Z X$ the normal cone to Z in X. In Fulton[2] the Segre class $\mathrm{s}(Z, X)$ of Z in X is defined to be the Segre class of $C_Z X$. The normal cone $C_Z X$ is defined to be $\mathrm{Spec}(\sum \mathcal{I}^k/\mathcal{I}^{k+1})$, where $\mathcal{I}$ is the ideal sheaf defining Z in X and has the projective completion $q : P(C_Z X \oplus 1) \to Z$ with the canonical line bundle $\mathcal{O}(1)$. The Segre class $\mathrm{s}(Z, X)$ is defined as

$$\mathrm{s}(Z,X) = q_*(\sum_i c_1(\mathcal{O}(1))^i \cap [P(C_Z X \oplus 1)]).$$

Hereafter we assume that X is a nonsingular algebraic variety of dimension n and Z is an irreducible subvariety of X. For simlplicity's sake, we assume that $\dim Z < \dim X$. In this note cycles are always algebraic cycles and the intersection of cycles are the refined intersection of Fulton[2] in Borel-Moore homology groups. Let $T^*_{Z_{sp}} X$ be the conormal bundle to the nonsingular part Z_{sp} of Z in X. Then the closure of $T^*_{Z_{sp}} X$ in the cotangent bundle T^*X is a conic subvariety, which we denote simply by $T^*_Z X$. The closure of $T^*_Z X$ in the projective completion $P(T^*X \oplus 1)$ is denoted by $\overline{T^*_Z X}$. Let $q : P(T^*X \oplus 1) \to X$ be the projection and $\mathcal{O}(1)$ the canonical line bundle on $P(T^*X \oplus 1)$. We call

$$s^*(Z,X) = q_*(\sum_i c_1(\mathcal{O}(1))^i \cap [\overline{T^*_Z X}])$$

the cosegre class of Z in X. We define linearly the cosegre class $s^*(z, X)$ of a cycle z on X, which is an element of $H_.(|z|, \mathbb{Z})$, where $|z|$ is the support of z and $\mathbb{Z}$ is the ring of integers. The following lemma is obtained in Sabbah[7,Lemma 1.2.1].

Lemma. *Let $\check{c}_M(Z)$ be the checked Mather-Chern class of Z and $c(T^*X)$ the Chern class of the cotangent bundle of X. Then we have*

$$\check{c}_M(Z) = c(T^*X) \cap s^*(Z,X).$$

Letting $\mathcal{M}^\cdot$ be a bounded complex of $\mathcal{D}_X$ modules with regular holonomic cohomology. Letting $\mathrm{Ch}(\mathcal{M}^\cdot) = \sum m_j T^*_{V_j} X$ be the characteristic cycle of $\mathcal{M}^\cdot$, where $\{V_j\}$ is a stratification of X, we define the checked Mather-Chern class $\check{c}_M(\mathcal{M}^\cdot)$ to be

$$\check{c}_M(\mathcal{M}^\cdot) = \sum m_j \check{c}_M(\bar{V_j}).$$

*Received December 21, 1999; accepted for publication July 19, 2000.

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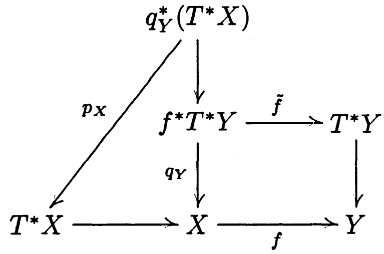
In this note we denote the cycle $\sum m_j \overline{T^*_{V_j} X}$ on T^*X by the same letter $\mathrm{Ch}(\mathcal{M}^\cdot)$. By the above lemma we have

$$\check{c}_M(\mathcal{M}^\cdot) = c(T^*X) \cap s^*(\sum m_j \bar{V}_j, X) = c(T^*X) \cap q_*(\sum c_1(\mathcal{O}(1))^i \cap \mathrm{Ch}(\mathcal{M}^\cdot)).$$

Let Y be a nonsingular algebraic variety of dimension m and $f : X \to Y$ a projective map. Let $q_Y : f^*T^*Y \to X$ be the induced bundle, $\tilde{f} : f^*T^*Y \to T^*Y$ the projection and α_Y the canonical 1-form on T^*Y. Then $\tilde{f}^*(\alpha_Y)$ may be considered to be a section of the induced bundle $q_Y^* T^*X$ over f^*T^*Y.



The following theorem is a corollary of a theorem of Ginsburg[3](cf. Kawai[5])

Theorem 1. *Let $\int_f \mathcal{M}^\cdot$ be the direct image of $\mathcal{M}^\cdot$ by the map f. Then we have*

$$\mathrm{Ch}(\int_f \mathcal{M}^\cdot) = \tilde{f}_*(p_X^*(\mathrm{Ch}(\mathcal{M}^\cdot)) \cdot \tilde{f}^*(\alpha_Y)([f^*T^*Y])).$$

Here the intersection is the intersection in $q_X^*(T^*X)$ and $\tilde{f}^*(\alpha_Y)([f^*T^*Y])$ is the image of the the cycle $[f^*T^*Y]$ by the section $\tilde{f}^*(\alpha_Y)$. The intersection is considered to be a cycle in f^*T^*Y which is identified with the image of the section.

As for the direct image of the cosegre class by the map f we have the following theorem if we assume for simplicity's sake that $\dim X \leq \dim Y$.

Theorem 2.. *Let $\bar{q}_Y : P(f^*T^*Y \oplus 1) \to X$ be the projective completion of the vector bundle $f^*T^*Y \to X$, $i : f^*T^*Y \to P(f^*T^*Y \oplus 1)$ the canonical injection and $\overline{p_X^*(T_Z^*X) \cdot \tilde{f}^*(\alpha_Y)([f^*T^*Y])}$ the cycle which is obtained as the closure of $p_X^*(T_Z^*X) \cdot \tilde{f}^*(\alpha_Y)([f^*T^*Y])$ in $P(f^*T^*Y \oplus 1)$. Then we have*

$$c(T^*X) \cap s^*(Z, X) = c(f^*T^*Y) \cap \bar{q}_{Y*}(s(L_Y) \cap \overline{[p_X^*(T_Z^*X) \cdot \tilde{f}^*(\alpha_Y)([f^*T^*Y])]}),$$

where L_Y is the tautological line bundle on $P(f^*T^*Y \oplus 1)$.

We have the following corollary which is equivalent to Corollary 5 of Brylinski,Dubson and Kashiwara[1] and the theorem of MacPherson via the Riemann-Hilbert corresopondence.

Corollary. *We have*

$$\check{c}_M(\int_f \mathcal{M}^\cdot) = f_*(\check{c}_M(\mathcal{M}^\cdot)).$$

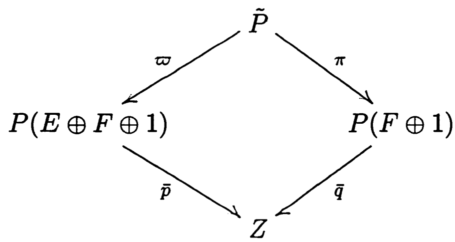
Proof of Theorem 2 and Corollary

For the proof of the theorem we prepare a lemma.

Lemma *Let E, F be vector bundles of rank n, m respectively on an algebraic variety Z. Let $\bar{p} : P(E \oplus F \oplus 1) \to Z(\bar{q} : P(F \oplus 1) \to Z$, resp.) be the projective completion of the vector bundle $E \oplus F \oplus 1 (F \ominus 1$, resp.). We consider $P(F \oplus 1)$ and the projective bundle $P(E)$ to be the subvarieties of $P(E \oplus F \oplus 1)$. Let $\varpi : \tilde{P} \to P(E \oplus F \oplus 1)$ be the blow-up of $P(E \oplus F \oplus 1)$ along $P(E)$. Then the projection of $P(E \oplus F \oplus 1)$ to $P(F \oplus 1)$ with center $P(E)$ induces the morphism $\pi : \tilde{P} \to P(F \oplus 1)$. Let L be the tautological line bundle on the projective bundle $P(F \oplus 1)$. Then $\pi : \tilde{P} \to P(F \oplus 1)$ is equivalent to the projective bundle $P(\bar{q}^* E \oplus L) \to P(F \oplus 1)$. Identifying them, we have*

$$\varpi^* \mathcal{O}_{P(E \oplus F \oplus 1)}(1) = \mathcal{O}_{P(\bar{q}^* E \oplus L)}(1).$$

We identify $P(\bar{q}^* E \oplus L) = P((L^{-1} \otimes \bar{q}^* E) \oplus 1)$. Then the exceptional divisor Θ with respect to the blowing up is equal to the subvariety $P(\bar{q}^* E) = P(L^{-1} \otimes \bar{q}^* E)$ and the line bundle $\mathcal{O}_{\tilde{P}}(\Theta)$ associated to the divisor Θ is $\pi^*(L^{-1}) \otimes \mathcal{O}_{P(\bar{q}^* E \oplus L)}(1)$.



Proof. Let $\mathcal{O}_{P(E)}(1)$ be the canonical line bundle of $P(E)$. The normal bundle to $P(E)$ in $P(E \oplus F \oplus 1)$ is isomorphic to $\mathcal{O}_{P(E)}(1) \otimes (F \oplus 1)$. Hence we may consider $\tilde{P}$ to be the subvariety of $P(E \oplus F \oplus 1) \times_Z P(\mathcal{O}_{P(E)}(1) \otimes (F \oplus 1)) = P(E \oplus F \oplus 1) \times_Z P(F \oplus 1)$ and $\pi : \tilde{P} \to P(F \oplus 1)$ to be the projection $P(E \oplus F \oplus 1) \times_Z P(F \oplus 1) \to P(F \oplus 1)$. Let U_α be an open subset of Z and $E|U_\alpha = U_\alpha \times \mathbb{C}^n$ with fiber coordinates $(x_{\alpha,0}, \dots, x_{\alpha,n-1})$ ($F|U_\alpha = U_\alpha \times \mathbb{C}^m$ with fiber coordinates $(x_{\alpha,n}, \dots, x_{\alpha,n+m-1})$, $1|U_\alpha = U_\alpha \times \mathbb{C}$ with fiber coordinates $x_{\alpha,n+m}$, respectively) a local trivialization of the vector bundle E (F, 1, respectively). We consider a copy of vector bundle $F \oplus 1$ and a local trivialization $U_\alpha \times \mathbb{C}^{m+1}$ of the copy with fiber coordinates $(y_{\alpha,0}, \dots, y_{\alpha,m})$. Then $\tilde{P}$ is considered to be the subvariety of $U_\alpha \times P(\mathbb{C}^n \times \mathbb{C}^m \times \mathbb{C}) \times P(\mathbb{C}^{m+1})$ defined by the equations

$$x_{\alpha,n+i}\, y_{\alpha,j} = x_{\alpha,n+j}\, y_{\alpha,i}, \quad 0 \leq i, j \leq m.$$

Let $V_{\alpha,j}$ be the subset of $P(E \oplus F \oplus 1)|U_\alpha$ such that $x_{\alpha,j} \neq 0$. A system of coordinate transformations $\{g_{(\beta,j),(\alpha,i)}\}$ of the canonical line bundle $\mathcal{O}_{P(E \oplus F \oplus 1)}(1)$ with respect to the covering $\{V_{\alpha,i}\}$ is given by $g_{(\beta,j),(\alpha,i)} = x_{\alpha,i}/x_{\beta,j}$ for $V_{\alpha,i} \cap V_{\beta,j} \neq \emptyset$. Let $W_{\alpha,i}$ be the open subset of $P(F \oplus 1)|U_\alpha$ such that $y_{\alpha,i} \neq 0$. Then the map $\psi_{\alpha,i} : \pi^{-1}(W_{\alpha,i}) \to W_{\alpha,i} \times \mathbb{P}^n$, where $\mathbb{P}^n$ is a projective space with homogeneous coordinates $(\lambda_{\alpha,0}, \dots, \lambda_{\alpha,n})$, defined to be

$$(\lambda_{\alpha,0}, \dots, \lambda_{\alpha,n}) = (x_{\alpha,0}, \dots, x_{\alpha,n-1}, x_{\alpha,n+i})$$

is a well-defined morphism. In fact, if $x_{\alpha,n+i} = 0$, by the above defining equations of $\tilde{P}|U_\alpha$ we have $x_{\alpha,n+j} = x_{\alpha,n+i} \dfrac{y_{\alpha,j}}{y_{\alpha,i}}$ and hence $x_{\alpha,n+j} = 0$, for $j = 0, \dots, m$. Therefore we have $(x_{\alpha,0}, \dots, x_{\alpha,n-1}) \neq 0$. We infer readily that the morphism $\psi_{\alpha,i}$

is an isomorphism and $\pi : \tilde{P} \to P(F \oplus 1)$ is equivalent to the projective bundle $P(\bar{q}^*E \oplus L) \to P(F \oplus 1)$. We identify them. If $x_{\alpha,n+i} \neq 0$ and $y_{\alpha,i} = 0$, then $y_{\alpha,j} = 0$ for $j = 0, \ldots, m$. Hence we have $\pi(V_{\alpha,n+i}) \subset W_{\alpha,i}$, from which we infer readily that

$$\pi^{-1}(W_{\alpha,i}) = V_{\alpha,n+i} \cup \bigcup_{j=0}^{n-1} V_{\alpha,j} \cap \pi^{-1}(W_{\alpha,i}).$$

If we check the system of coordinate transformations of the line bundles $\mathcal{O}_{P(E\oplus F\oplus 1)}(1)$ and $\mathcal{O}_{P(\bar{q}^*E\oplus L)}(1)$ with respect to the refined open covering $\{V_{\alpha,n+i}, V_{\alpha,j}\cap\pi^{-1}(W_{\alpha,i})\}$, then we have $\varpi^*\mathcal{O}_{P(E\oplus F\oplus 1)}(1) = \mathcal{O}_{P(\bar{q}^*E\oplus L)}(1)$.

Now we prove the theorem. Let $\varpi_X : \tilde{P}_X \to P(T^*X \oplus f^*T^*Y \oplus 1)(\varpi_Y : \tilde{P}_Y \to P(T^*X \oplus f^*T^*Y \oplus 1)$, resp.) be the blow-up of $P(T^*X \oplus f^*T^*Y \oplus 1)$ along $P(f^*T^*Y)(P(T^*X)$, resp.) and $\pi_X : \tilde{P}_X \to P(T^*X \oplus 1)(\pi_Y : \tilde{P}_Y \to P(f^*T^*Y \oplus 1)$, resp.) the morphism induced by the projection.

$$\begin{array}{ccccc}
\tilde{P}_X & \xrightarrow{\varpi_X} & P(T^*X \oplus f^*T^*Y \oplus 1) & \xleftarrow{\varpi_Y} & \tilde{P}_Y \\
\downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \bar{p}} & & \downarrow{\scriptstyle \pi_Y} \\
P(T^*X \oplus 1) & \xrightarrow[\bar{q}_X]{} & X & \xleftarrow[\bar{q}_Y]{} & P(f^*T^*Y \oplus 1)
\end{array}$$

Then by Lemma $\pi_X : \tilde{P}_X \to P(T^*X \oplus 1)(\pi_Y : \tilde{P}_Y \to P(f^*T^*Y \oplus 1)$, resp.) is considered to be the projective bundle $P(\bar{q}_X{}^*f^*T^*Y \oplus L_X) \to P(T^*X \oplus 1)(P(\bar{q}_Y{}^*T^*X \oplus L_Y) \to P(f^*T^*Y \oplus 1)$, resp.), where $L_X(L_Y$, resp.) is the tautological line bundle on $P(T^*X \oplus 1)(P(f^*T^*Y \oplus 1)$, resp.). The vector bundle $T^*X \oplus f^*T^*Y$ is an open dense subset of $P(T^*X \oplus f^*T^*Y \oplus 1)$ and the subvarieties $P(f^*T^*Y), P(T^*X)$ are contained in the complement of $T^*X \oplus f^*T^*Y$. Hence we may consider $T^*X \oplus f^*T^*Y = T^*X \times_X f^*T^*Y$ to be an open dense subset of $\tilde{P}_X$ and $\tilde{P}_Y$, respectively. It is clear that $\varpi_{X*}\pi_X^*([\overline{T_Z^*X}]) = [\overline{T_Z^*X \times_X f^*T^*Y}]$ on $P(T^*X \oplus f^*T^*Y \oplus 1)$. Hence by Fulton[2, Example 3.2.1.] we have

$$\bar{q}_{X*}\pi_{X*}(s(\tilde{L}_X) \cap \pi_X{}^*[\overline{T_Z^*X}]) = \bar{q}_{X*}(s(\bar{q}_X{}^*f^*T^*Y \oplus L_X)) \cap [\overline{T_Z^*X}]).$$

Hence we have

$$s(f^*T^*Y) \cap s^*(Z, X) = \bar{q}_{X*}\pi_{X*}(s(\tilde{L}_X) \cap \pi_X{}^*[\overline{T_Z^*X}])$$

$$= \bar{p}_*\varpi_{X*}(s(\tilde{L}_X) \cap \pi_X^*[\overline{T_Z^*X}]) = \bar{p}_*(s(L) \cap \varpi_{X*}\pi_X{}^*[\overline{T_Z^*X}])$$

$$= \bar{p}_*(s(L) \cap [\overline{T_Z^*X \times_X f^*T^*Y}])$$

in the Borel-Moore homology group $H.(Z, \mathbb{Z})$.

Meanwhile we show that the section $\tilde{f}^*\alpha_Y : f^*T^*Y \to T^*X \oplus f^*T^*Y = q_Y{}^*T^*X$, where $q_Y : f^*T^*Y \to X$ is the projection, is extensible to the section σ to the vector bundle $L_Y^{-1} \otimes \bar{q}_Y{}^*T^*X$ over $P(f^*T^*Y \oplus 1)$. It is sufficient to see this locally. Hence we assume that $X(Y$, resp.) is an open subset of $\mathbb{C}^n$ ($\mathbb{C}^m$, resp.) with coordinates $(x_i)((y_j)$, resp.) and the morphism f is given by $(y_j) = f((x_i)) = (y_j(x_i))$. Let

$(\xi_i)((\eta_j)$, resp.) the associated fiber coordinates of $T^*X(T^*Y$, resp.). Then $P(f^*T^*Y \oplus 1) = X \times \mathbb{P}(\mathbb{C}^m \times \mathbb{C})$, where $\mathbb{P}(\mathbb{C}^m \times \mathbb{C})$ is a projective space with the homogeneous coordinates $(\eta_0, \ldots, \eta_{m-1}, \eta_m)$. Let V_j be the open subset of $\mathbb{P}(\mathbb{C}^m \times \mathbb{C})$ such that $\eta_j \neq 0$. We consider $\pi_Y{}^{-1}(X \times V_j)$ to be $X \times V_j \times \mathbb{P}(\mathbb{C}^n \times \mathbb{C})$, where $\mathbb{P}(\mathbb{C}^n \times \mathbb{C})$ is a projective space with homogeneous coordinate $((\xi_i), \zeta_j)$. Then $T^*X \oplus f^*T^*Y$ is the open subset of $\pi_Y{}^{-1}(X \times V_j)$ such that $\zeta_j \neq 0$. The 1-form $\tilde{f}^*\alpha_Y$ is expressed as

$$\tilde{f}^*\alpha_Y = \sum \eta_j \frac{\partial y_j}{\partial x_i} dx_i.$$

Hence the section $\tilde{f}^*\alpha_Y$ is defined by the equations

$$\xi_i = \sum_{j=0}^{m-1} \frac{\partial y_j}{\partial x_i}\eta_j, \quad i = 0, \ldots, n-1.$$

We define $\sigma|V_j : X \times V_j \to X \times V_j \times \mathbb{P}(\mathbb{C}^n \times \mathbb{C})$ to be

$$(\sigma|V_j)((x_i), (\eta_0, \ldots, \eta_m)) = ((x_i), (\eta_0, \ldots, \eta_m), ((\sum_{\nu=0}^{\nu=m-1} \frac{\partial y_\nu}{\partial x_i}\eta_\nu), \eta_j)).$$

These $\{\sigma|V_j\}$ give the desired extension.

By the above defining equations of the section we see easily that the image of the section σ is the cone over the image of the restriction of σ to $P(f^*T^*Y)$.

$$\begin{array}{ccc} \pi_Y^{-1}(P(f^*T^*Y)) & \longrightarrow & (L_Y^{-1} \otimes \bar{q}_Y{}^*T^*X) \oplus 1 \\ \downarrow & & \downarrow \pi_Y \\ P(f^*T^*Y) & \longrightarrow & P(f^*T^*Y \oplus 1) \end{array}$$

Hereafter we often use the smame letter for the fundamental cycle of a subvariety. It is clear that the closure $\overline{p_X^*T_Z^*X}$ of $p_X^*T_Z^*X$ in $\tilde{P}_Y$ is a cone over a subvariety of $\pi_Y^{-1}(P(f^*T^*Y))$, where p_X is the projecton of $q_X^*(T^*X)$ to T^*X, for the defining equations of the subvariety $p_X^*T_Z^*X$ in $\tilde{P}_Y$ do not contain (η_i) of the above coordinate systems. Therefore the intersection of $\overline{p_X^*T_Z^*X}$ and the image of the section σ is the cone over the intersection of the corresponding subvarieties in $\pi_Y^{-1}(P(f^*T^*Y))$. Hence we have

$$\overline{p_X^*(T_Z^*X) \cdot \tilde{f}^*(\alpha_Y)([f^*T^*Y])} = \overline{p_X^*T_Z^*X} \cdot \sigma(P(f^*T^*Y \oplus 1)).$$

By Fulton[2, Proposition 3.3.] we have

$$\overline{p_X^*T_Z^*X} \cdot \sigma(P(f^*T^*Y \oplus 1)) = \pi_*(c_n(\zeta) \cap \overline{p_X^*T_Z^*X}),$$

where ζ is the universal rank n quotient bundle of $(L_Y^{-1} \otimes \bar{q}_Y{}^*T^*X) \oplus 1$. Since $\varpi_{Y*}(\Theta \cdot C) = 0$ for any cycle C on $\tilde{P}_Y$, we have

$$\varpi_{Y*}(c_n(\zeta) \cap \overline{p_X^*T_Z^*X}) = \varpi_{Y*}(c_n((L_Y^{-1} \otimes \bar{q}_Y{}^*T^*X) \oplus 1) \cap \overline{p_X^*T_Z^*X}),$$

for, letting $\tilde{L}_Y$ be the tautological line bundle on the projective bundle $P((L_Y^{-1} \otimes \bar{q}_Y{}^*T^*X) \oplus 1)$, we have the exact sequence

$$0 \to \tilde{L}_Y \to \pi^*((L_Y^{-1} \otimes \bar{q}_Y{}^*(T^*X) \oplus 1) \to \zeta \to 0,$$

and by Lemma we have $c_1(\tilde{L}_Y^{-1}) = \Theta$. From $\tilde{L}_Y^{-1} = \mathcal{O}_{P(\bar{q}_Y{}^*T^*X \oplus L_Y)}(1) \otimes L_Y$ we have

$$\varpi_{Y*}(c_1(L_Y^{-1})^i \cap C) = \varpi_{Y*}(c_1(\mathcal{O}_{P(\bar{q}_Y{}^*T^*X \oplus L_Y)}(1))^i \cap C)$$

for a positive intager i and a cycle C on $\tilde{P}_Y$ and hence we have

$$\varpi_{Y*}(c_n((L_Y^{-1} \otimes \bar{q}_Y{}^*T^*X) \oplus 1) \cap \overline{p_X^* T_Z^* X})$$

$$= \varpi_{Y*}(\sum_{i=0}^{n} c_1(\mathcal{O}_{P(\bar{q}_Y{}^*T^*X \oplus L_Y)}(1))^i c_{n-i}(\bar{q}^*T^*X) \cap \overline{p_X^* T_Z^* X})$$

$$= \sum_{i=0}^{n} c_1(\mathcal{O}_{P(T^*X \oplus f^*T^*Y \oplus 1)}(1))^i \bar{p}^* c_{n-i}(T^*X) \cap \overline{T_Z^*X \times_X f^*T^*Y}.$$

Therefore we have

$$\bar{q}_{Y*}(s(L_Y) \cap \overline{[p_X^*(T_Z^*X) \cdot \tilde{f}^*(\alpha_Y)([f^*T^*Y])]})$$

$$= \bar{p}^*(\sum_{k \geq 0, 0 \leq i \leq n} c_1(\mathcal{O}_{P(\bar{q}_Y{}^*T^*X \oplus L_Y)}(1))^{k+i} \bar{p}^* c_{n-i}(T^*X) \cap \overline{T_Z^*X \times_X f^*T^*Y}).$$

By the assumption that $n \leq m$, noting that

$$\pi_{X*}(c_1(\mathcal{O}_{P(f^*T^*Y \oplus 1)}(1))^i \cap \pi_X^*(C))) = 0$$

for $i \leq m$ and a cycle on $P(T^*X \oplus 1)$, we have

$$\pi_{X*}(\sum_i \sum_{k \geq 0} c_1(\mathcal{O}_{P(\bar{q}_Y{}^*T^*X \oplus L_Y)}(1))^{k+i} \pi_X^*(c_{n-i}(\bar{q}_X^* T^*X) \cap T_Z^*X)$$

$$= \pi_{X*}(\sum_i \sum_{k \geq 0} c_1(\mathcal{O}_{P(\bar{q}_Y{}^*T^*X \oplus L_Y)}(1))^k \pi_X^*(c_{n-i}(\bar{q}_X^* T^*X) \cap T_Z^*X)$$

$$= \pi_{X*}(s(\tilde{L}_X) \cap \pi_X^*(\bar{q}_X^* c(T^*X) \cap T_Z^*X)) = s(f^*T^*Y \oplus L_X)\bar{q}_X^* c(T^*X) \cap T_Z^*X.$$

Thus we have

$$\bar{q}_{Y*}(s(L_Y) \cap \overline{[p_X^*(T_Z^*X) \cdot \tilde{f}^*(\alpha_Y)([f^*T^*Y])]}) = s(f^*T^*Y) c(T^*X) \cap s^*(Z, X),$$

which completes the proof of the theorem.

Remark If, for example, $m = 0$, we have the following formula

$$\check{c}_{M,0}(Z) = \bar{q}_{X*}(s_0(X) \cdot [\bar{T_Z^*X}]),$$

where s_0 is the zero section to the projective completion $P(T^*X \oplus 1)$.

Finally we prove the corollary. In a similar notations to the above we have the following fiber square

$$\begin{array}{ccc} P(f^*T^*Y \oplus 1) & \xrightarrow{\bar{f}} & P(T^*Y \oplus 1) \\ \bar{q}_X \downarrow & & \downarrow \bar{q}_Y \\ X & \xrightarrow[f]{} & Y. \end{array}$$

Let L_Y be the tautological line bundle on $P(T^*Y \oplus 1)$. Then the induced bundle $\bar{f}^* L_Y$ is the tautological line bundle on $P(f^*T^*Y \oplus 1)$. Hence by Theorem 2 we have

$$f_*(\check{c}_M(\mathcal{M}^{\cdot})) = f_*(\sum m_j c(T^*X) \cap s^*(\bar{V_j}, X))$$

$$= f_*(\sum m_j c(f^*T^*Y) \cap \bar{q}_{X*}(s(\bar{f}^* L_Y) \cap \overline{[p_X^*(T^*_{\bar{V}_j} X) \cdot \tilde{f}^*(\alpha_Y)([f^*T^*Y])]})$$

$$= c(T^*Y) \cap \bar{q}_{Y*} \bar{f}_*(\sum m_j s(\bar{f}^* L_Y) \cap \overline{[p_X^*(T^*_{\bar{V}_j} X) \cdot \tilde{f}^*(\alpha_Y)([f^*T^*Y])]}).$$

By Theorem 1 we have

$$\bar{f}_*(\sum m_j \overline{[p_X^*(T^*_{\bar{V}_j} X) \cdot \tilde{f}^*(\alpha_Y)([f^*T^*Y])]}) = \overline{\mathrm{Ch}(\int_f \mathcal{M}^*)}.$$

Hence we have

$$f_*(\check{c}_M)(\mathcal{M}^{\cdot}) = c(T^*Y) \cap \bar{q}_{Y*}(s(L_Y) \cap \overline{[\mathrm{Ch}(\int_f \mathcal{M}^{\cdot})]})$$

$$= c(T^*Y) \cap s^*(\int_f \mathcal{M}^{\cdot}, Y) = \check{c}_M(\int_f \mathcal{M}^{\cdot}).$$

REFERENCES

[1] J. L. Brylinski, A. Dubson, and M. Kashiwara, *Formule de l'indice pour les modules holonomes et obstruction d'Euler locale*, C. R. Acad. Sci., 293 (1981), pp. 573–576.
[2] W. Fulton, *Intersection Theory*, Springer-Verlag, 1984.
[3] V. Ginsburg, *Characteristic cycles and vanishing cycles*, Invent. Math., 84 (1986), pp. 327–402.
[4] M. Kashiwara and P. Schapira, *Sheaves on Manifolds*, Springer-Verlag, 1990.
[5] S. Kawai, *A note on the characteristic cycle of the image of the constant sheaf*, Comm. Math. Univ. Snacti Pauli, 48 (1999), pp. 119–128.
[6] R. MacPherson, *Chern class for singular varieties*, Ann. Math., 100 (1974), pp. 423–432.
[7] C. Sabbah, *Quelques remarques sur la géométrie des espaces conormaux*, in Systèmes différentiels et singularités, Astérisque 130, 1985, pp. 161–192.