

THE SYMPLECTIC GEOMETRY OF POLYGONS IN HYPERBOLIC 3-SPACE*

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Abstract. We study the symplectic geometry of the moduli spaces $M_r = M_r(\mathbb{H}^3)$ of closed n -gons with fixed side-lengths in hyperbolic three-space. We prove that these moduli spaces have almost canonical symplectic structures. They are the symplectic quotients of B^n by the dressing action of $SU(2)$ (here B is the subgroup of the Borel subgroup of $SL_2(\mathbb{C})$ defined below). We show that the hyperbolic Gauss map sets up a real analytic isomorphism between the spaces M_r and the weighted quotients of $(S^2)^n$ by $PSL_2(\mathbb{C})$ studied by Deligne and Mostow. We construct an integrable Hamiltonian system on M_r by bending polygons along nonintersecting diagonals. We describe angle variables and the momentum polyhedron for this system. The results of this paper are the analogues for hyperbolic space of the results of [KM2] for $M_r(\mathbb{S}^3)$, the space of n -gons with fixed side-lengths in \mathbb{E}^3 . We prove $M_r(\mathbb{H}^3)$ and $M_r(\mathbb{E}^3)$ are symplectomorphic.

1. Introduction. An (open) n -gon P in hyperbolic space \mathbb{H}^3 is an ordered $(n+1)$ -tuple (x_1, \dots, x_{n+1}) of points in \mathbb{H}^3 called the vertices. We join the vertex x_i to the vertex x_{i+1} by the unique geodesic segment e_i , called the i -th edge. We let Pol_n denote the space of n -gons in \mathbb{H}^3 . An n -gon is said to be closed if $x_{n+1} = x_1$. We let $CPol_n$ denote the space of closed n -gons. Two n -gons $P = [x_1, \dots, x_{n+1}]$ and $P' = [x'_1, \dots, x'_{n+1}]$ are said to be equivalent if there exists $g \in PSL_2(\mathbb{C})$ such that $gx_i = x'_i$, for all $1 \leq i \leq n+1$. We will either represent an n -gon P by its vertices or its edges, $P = [x_1, \dots, x_{n+1}] = (e_1, \dots, e_n)$.

Let $r = (r_1, \dots, r_n)$ be an n -tuple of positive numbers. This paper is concerned with the symplectic geometry of the space of closed n -gons in \mathbb{H}^3 such that the i -th edge e_i has side-length r_i , $1 \leq i \leq n$, modulo $PSL_2(\mathbb{C})$. We will assume in this paper (with the exception of §3) that r is not on a wall of D_n (see §2), hence M_r is a real-analytic manifold.

The starting point of this paper is (see §4)

THEOREM 1.1. *The moduli spaces M_r are the symplectic quotients obtained from the dressing action of $SU(2)$ on B^n .*

Here $B = AN$ is the subgroup of the Borel subgroup of $SL_2(\mathbb{C})$, $B = \left\{ \begin{pmatrix} \lambda & z \\ 0 & \lambda^{-1} \end{pmatrix} : \lambda \in \mathbb{R}_+, z \in \mathbb{C} \right\}$. B is given the Poisson Lie group structure corresponding to the Manin triple $(\mathfrak{sl}(2, \mathbb{C}), \mathfrak{su}(2), \mathfrak{b})$ with \langle, \rangle on $\mathfrak{sl}(2, \mathbb{C})$ given by the imaginary part of the Killing form.

REMARK 1.2. *As a consequence of Theorem 1.1, the spaces M_r have an almost canonical symplectic structure (the symplectic structure depends on a choice of Iwasawa decomposition of $SL_2(\mathbb{C})$ or a ray in \mathbb{H}^3 , but given two such choices, there exist (infinitely many) $g \in SL_2(\mathbb{C})$ inducing an isomorphism of the two Poisson structures).*

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Our next theorem relates the moduli spaces M_r to the weighted quotients $Q_{sst} = Q_{sst}(r)$ of $(S^2)^n$ constructed by Deligne and Mostow in [DM]. By extending the sides of the n -gon in the positive direction until they meet $S^2 = \partial_\infty \mathbb{H}^3$, we obtain a map, the hyperbolic Gauss map $\gamma : Pol_n \rightarrow (S^2)^n$. We then have (here we assume M_r is smooth)

THEOREM 1.3. *The hyperbolic Gauss map induces a real analytic diffeomorphism $\gamma : M_r \rightarrow Q_{sst}(r)$.*

REMARK 1.4. *In [KM2], the first two authors constructed an analogous analytic isomorphism $\gamma : M_r(\mathbb{E}^3) \rightarrow Q_{sst}(r)$ where $M_r(\mathbb{E}^3)$ is the moduli space of n -gons with the side-lengths $r = (r_1, \dots, r_n)$ in Euclidean space \mathbb{E}^3 . Although they gave a direct proof, this latter result was a consequence of the Kirwan-Kempf-Ness theorem, [K1],[KN], relating Mumford quotients to symplectic quotients. Our new result (Theorem 1.3 above) relates a Mumford quotient to a quotient of a symplectic manifold by a **Poisson** action.*

The key step (surjectivity) in the proof of Theorem 1.3 is of independent interest. We could try to invert $\gamma : M_r \rightarrow Q_{sst}$ as follows. Suppose we are given $\xi = (\xi_1, \dots, \xi_n) \in Q_{sst}$. We wish to construct $P \in M_r$ with $\gamma(P) = \xi$. Choose $x \in \mathbb{H}^3$. Put the first vertex $x_1 = x$. Let σ_1 be the geodesic ray from x_1 to ξ_1 . Let x_2 be the point on σ_1 with $d(x_1, x_2) = r_1$. Let σ_2 be the ray from x_2 to ξ_2 . Cut off σ_2 at x_3 so that $d(x_2, x_3) = r_2$. We continue in this way until we get $P = [x_1, \dots, x_{n+1}]$. However it may not be the case that P closes up (i.e. $x_{n+1} = x_1$).

THEOREM 1.5. *Suppose ξ is a stable configuration (see §3.1) on $(S^2)^n$. Then there is a unique choice of initial point $x = x(r, \xi)$ such that P closes up.*

REMARK 1.6. *Let $\tilde{\nu}$ be the atomic “measure” on S^2 which assigns mass r_i to the point ξ_i , $1 \leq i \leq n$, keeping track of the order of the ξ_i ’s. Then the rule that assigns $x = x(\tilde{\nu}) = x(\xi, r)$ above is $PSL_2(\mathbb{C})$ -equivariant and is a multiplicative analogue of the conformal center of mass, $C(\nu)$, of Douady and Earle [DE], see also [MZ, §4]. Here ν is the measure $\nu = \sum_{i=1}^n r_i \delta(\xi - \xi_i)$.*

REMARK 1.7. *We may use Theorem 1.3 to construct a length-shrinking flow on $CPol_n$. Namely, let $0 \leq t \leq 1$. Replace the weights $r = (r_1, \dots, r_n)$ by $tr = (tr_1, \dots, tr_n)$. We have*

$$M_r \xrightarrow{\gamma_r} Q_{sst}(r) \cong Q_{sst}(tr) \xleftarrow{\gamma_{tr}} M_{tr}.$$

The composition $\gamma_{tr}^{-1} \circ \gamma_r$ is the length-shrinking flow. Note that $Q_{sst}(r)$ and $Q_{sst}(tr)$ are canonically isomorphic as complex analytic spaces. We obtain a curve $x(t) = x(tr, \xi)$. We have

THEOREM 1.8. $\lim_{t \rightarrow 0} x(tr, \xi) = C(\nu)$, *the conformal center of mass of Douady and Earle.*

REMARK 1.9. *We see that $C(\nu)$ is “semi-classical,” it depends only on the limit as the curvature goes to zero (or the speed of light goes to infinity), see §3.3.*

Our final theorems are connected with the study of certain integrable systems on M_r obtained by “bending an n -gon along nonintersecting diagonals” Precisely, we

proceed as follows. We define the diagonal d_{ij} of P to be the geodesic segment joining x_i to x_j . Here we assume $i < j$. We let ℓ_{ij} be the length of d_{ij} . Then ℓ_{ij} is a continuous function on M_r but is not smooth at the points where $\ell_{ij} = 0$. We have the following description of the Hamiltonian flow of ℓ_{ij} (it is defined provided $\ell_{ij} \neq 0$).

THEOREM 1.10. *The Hamiltonian flow Ψ_{ij}^t of ℓ_{ij} applied to an n -gon $P \in M_r$ is obtained as follows. The diagonal d_{ij} separates P into two halves. Leave one half fixed and rotate the other half at constant speed 1 around d_{ij} .*

For obvious reasons we call Ψ_{ij}^t "bending along d_{ij} ."

DEFINITION 1.11. *We say two diagonals d_{ij} and d_{ab} of P do not intersect if the interiors of d_{ij}^* and d_{ab}^* do not intersect, where d_{ij}^* (resp. d_{ab}^*) is the diagonal of a convex planar n -gon P^* corresponding to d_{ij} (resp. d_{ab}). We then have*

THEOREM 1.12. *Suppose d_{ij} and d_{ab} do not intersect, then*

$$\{\ell_{ij}, \ell_{ab}\} = 0.$$

REMARK 1.13. *We give two proofs of this theorem. The first is a direct computation of the Poisson bracket due to Hermann Flaschka. The second is an elementary geometric one depending on the description of the flows in Theorem 1.10. It corresponds to the geometric intuition that we may wiggle flaps of a folded piece of paper independently if the fold lines do not intersect.*

We obtain a maximal collection of commuting flows if we draw a maximal collection of nonintersecting diagonals $\{d_{ij}, (i, j) \in I\}$. Later we will take the collection of all diagonals starting at the first vertex, $I = \{(1, 3), (1, 4), \dots, (1, n - 1)\}$. Each such collection corresponds to a triangulation of a fixed convex planar n -gon P^* . There are $n - 3$ diagonals in such a maximal collection. Since $\dim M_r = 2n - 6$, we obtain

THEOREM 1.14. *For each triangulation of a convex planar n -gon P^* we obtain an integrable system on M_r . Precisely, we obtain a Hamiltonian action of an $(n-3)$ -torus on M_r which is defined on the Zariski open subset M_r' defined by the nonvanishing of the lengths of the diagonals in the triangulation.*

We have a simple description of the angle variables and the momentum polyhedron attached to the above integrable system. Let $M_r^o \subset M_r'$ be the subset such that none of the $n - 2$ triangles in the triangulation are degenerate. Let $\widehat{\theta}_{ij}$ be the dihedral angle at d_{ij} . Put $\theta_{ij} = \pi - \widehat{\theta}_{ij}$. Then the θ_{ij} are angle variables.

To obtain the momentum polyhedron we follow [HK] and note that there are three triangle inequalities associated to each of the $n - 2$ triangles in the triangulation. These are linear inequalities in the ℓ_{ij} 's and the r_{ij} 's. If they are satisfied, we can build the $n - 2$ triangles then glue them together and get an n -gon P with the required side-lengths r_i and diagonal lengths ℓ_{ij} . We obtain

THEOREM 1.15. *The momentum polyhedron of the above torus action (the image of M_r under the ℓ_{ij} 's) is the subset of $(\mathbb{R}_{\geq 0})^{n-3}$ defined by the $3(n - 2)$ triangle inequalities above.*

As a consequence we obtain

COROLLARY 1.16. *The functions ℓ_{ij} , $(i, j) \in I$, are functionally independent.*

Our results on n -gon linkages in \mathbb{H}^3 are the analogues of those of [KM2] for n -gon linkages in \mathbb{E}^3 . We conclude the paper by comparing the symplectic manifolds $M_r(\mathbb{H}^3)$ and $M_r(\mathbb{E}^3)$. Assume henceforth that r is not on a wall of D_n .

Since the Euclidean Gauss map $\gamma_e : M_r(\mathbb{H}^3) \rightarrow Q_{sst}(r)$ is a canonical diffeomorphism as is the hyperbolic Gauss map $\gamma_h : M_r(\mathbb{E}^3) \rightarrow Q_{sst}(r)$ we obtain

THEOREM 1.17. *The hyperbolic and Euclidean Gauss maps induce a canonical diffeomorphism*

$$M_r(\mathbb{E}^3) \simeq M_r(\mathbb{H}^3).$$

The last part of the paper is devoted to proving

THEOREM 1.18. *$M_r(\mathbb{E}^3)$ and $M_r(\mathbb{H}^3)$ are (noncanonically) symplectomorphic.*

This theorem is proved as follows. Let X_κ be the complete simply-connected Riemannian manifold of constant curvature κ . In [Sa], Sargent proved that there exists $\alpha > 0$ and an analytically trivial fiber bundle $\pi : \mathcal{E} \rightarrow (-\infty, \alpha)$ such that $\pi^{-1}(\kappa) = M_r(X_\kappa)$. We construct a closed relative 2-form ω_κ on $\mathcal{E}|_{(-\infty, 0]}$ such that ω_κ induces a symplectic form on each fiber of π and such that the family of cohomology classes $[\omega_\kappa]$ on $\mathcal{E}|_{(-\infty, 0]}$ is parallel for the Gauss-Manin connection. Theorem 1.18 then follows from the Moser technique [Mo].

The results are closely related to but different from those of [GW] and [A].

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2. Criteria for the moduli spaces to be smooth and nonempty. In this chapter we will give necessary and sufficient conditions for the moduli space M_r to be nonempty and sufficient conditions for M_r to be a smooth manifold.

First we need some more notation. Let $*$ be the point in \mathbb{H}^3 which is fixed by $PSU(2)$. We let $Pol_n(*)$ denote the space of n -gons $[x_1, \dots, x_{n+1}]$ with $x_1 = *$ and $CPol_n(*) = CPol_n \cap Pol_n(*)$. We let $\tilde{N}_r \subset Pol_n(*)$ be the subspace of those n -gons $P = [x_1, \dots, x_{n+1}]$ such that $d(x_i, x_{i+1}) = r_i$, $1 \leq i \leq n$. We put $N_r = \tilde{N}_r/PSU(2)$ and $\tilde{M}_r = \tilde{N}_r \cap CPol_n(*)$. Hence, $M_r = \tilde{M}_r/PSU(2)$.

Let $\pi : CPol_n \rightarrow (\mathbb{R}_{\geq 0})^n$ be the map that assigns to an n -gon e its set of side-lengths. $\pi(e) = (r_1, \dots, r_n)$ with $r_i = d(x_i, x_{i+1})$, $1 \leq i \leq n$.

LEMMA 2.1. *The image of π is the closed polyhedral cone D_n defined by the inequalities*

$$r_1 \geq 0, \dots, r_n \geq 0$$

and the triangle inequalities

$$r_i \leq r_1 + \dots + \hat{r}_i + \dots + r_n, \quad 1 \leq i \leq n$$

(here the $\hat{}$ means that r_i is omitted).

Proof. The proof is identical to the proof of the corresponding statement for Euclidean space, [KM1, Lemma 1]. \square

We next give sufficient conditions for M_r to be a smooth manifold. We will use two results and the notation from §4.3 (the reader will check that no circular reasoning is involved here). By Theorem 4.27 we find that M_r is a symplectic quotient.

$$M_r \cong (\varphi|_{\tilde{N}_r})^{-1}(1)/SU(2)$$

By Lemma 4.23, 1 is a regular value of φ unless there exists $P \in \tilde{M}_r$ such that the infinitesimal isotropy $(su_2)|_P = \{x \in SU(2) : \hat{X}(P) = 0\}$ is nonzero.

DEFINITION 2.2. An n -gon P is degenerate if it is contained in a geodesic.

We now have

LEMMA 2.3. M_r is singular only if there exists a partition $\{1, \dots, n\} = I \amalg J$ with $\#(I) > 1, \#(J) > 1$ such that

$$\sum_{i \in I} r_i = \sum_{j \in J} r_j.$$

Proof. Clearly $(su_2)|_P = 0$ unless P is degenerate. But if P is degenerate there exists a partition $\{1, \dots, n\} = I \amalg J$ as above (I corresponds to the back-tracks and J to the forward-tracks of P). \square

REMARK 2.4. In the terminology of [KM1], [KM2], M_r is smooth unless r is on a wall of D_n . Note that if $|I| = 1$ or $|J| = 1$ then $r \in \partial D_n$ and M_r is reduced to a single point.

There is a technical point concerning smoothness. We could also define M_r as the fiber of $\tilde{\pi} : CPol_n/PSL_2(\mathbb{C}) \rightarrow D_n$ over r . It is not quite immediate that smoothness of the symplectic quotient coincides with the smoothness of $\tilde{\pi}^{-1}(r)$. Fortunately, this is the case (note r is a regular value of $\tilde{\pi} \Leftrightarrow r$ is a regular value of π).

LEMMA 2.5. r is a regular value of $\tilde{\pi} \Leftrightarrow 1$ is a regular value of $\varphi|_{\tilde{N}_r}$.

Proof. The lemma follows from a consideration of the diagram

$$\begin{array}{ccccc}
 & & \tilde{N}_r & & \\
 & & \downarrow & \searrow \varphi_r & \\
 CPol_n(*) & \longrightarrow & Pol_n(*) & \xrightarrow{\varphi} & B \\
 & \searrow \pi & \downarrow & & \\
 & & (\mathbb{R}_{\geq 0})^n & &
 \end{array}$$

and the observation that $\varphi : Pol_n(*) \rightarrow B$ (see §4.2) and the side-length map $Pol_n(*) \rightarrow (\mathbb{R}_{\geq 0})^n$ are obviously submersions. Here we have abbreviated $\varphi|_{\tilde{N}_r}$ to φ_r . \square

3. The geometric invariant theory of hyperbolic polygons.

3.1. The hyperbolic Gauss map and weighted quotients of the configuration spaces of points on the sphere. The goal of the next two sections is to construct a natural homeomorphism $\gamma : M_r \rightarrow Q_{sst}$ where Q_{sst} is the r -th weighted quotient of $(S^2)^n$ by $PSL_2(\mathbb{C})$ constructed in [DM] in the case that M_r is smooth. Q_{sst} is a complex analytic space. We now review the construction of Q_{sst} .

Let $M \subset (S^2)^n$ be the set of n -tuples of distinct points. Then $Q = M/PSL_2(\mathbb{C})$ is a (noncompact) Hausdorff manifold.

DEFINITION 3.1. *A point $\vec{u} \in (S^2)^n$ is called r -stable (resp. semi-stable) if*

$$\sum_{u_j=v} r_j < \frac{|r|}{2} \quad (\text{resp. } \leq \frac{|r|}{2})$$

for all $v \in S^2$. Here $|r| = \sum_{j=1}^n r_j$. The set of stable and semi-stable points will be denoted by M_{st} and M_{sst} respectively. A semi-stable point $\vec{u} \in (S^2)^n$ is said to be a nice semi-stable point if it is either stable or the orbit $PSL_2(\mathbb{C})\vec{u}$ is closed in M_{sst} .

We denote the space of nice semi-stable points by M_{nsst} . We have the inclusions

$$M_{st} \subset M_{nsst} \subset M_{sst}.$$

Let $M_{cusp} = M_{sst} - M_{st}$. We obtain the points M_{cusp} in the following way. Partition $S = \{1, \dots, n\}$ into disjoint sets $S = S_1 \cup S_2$ with $S_1 = \{i_1, \dots, i_k\}$, $S_2 = \{j_1, \dots, j_{n-k}\}$ in such a way that $r_{i_1} + \dots + r_{i_k} = \frac{|r|}{2}$ (whence $r_{j_1} + \dots + r_{j_{n-k}} = \frac{|r|}{2}$). Then \vec{u} is in M_{cusp} if either $u_{i_1} = \dots = u_{i_k}$ or $u_{j_1} = \dots = u_{j_{n-k}}$. The reader will verify that $\vec{u} \in M_{cusp}$ is a nice semi-stable point if and only if both sets of the equations above hold.

relation \mathcal{R} via:

$\vec{u} \equiv \vec{w} \pmod{\mathcal{R}}$ if either

(a) $\vec{u}, \vec{w} \in M_{st}$ and $\vec{w} \in PSL_2(\mathbb{C})\vec{u}$,

or

(b) $\vec{u}, \vec{w} \in M_{cusp}$ and the partitions of S corresponding to \vec{u}, \vec{w} coincide.

The reader will verify that $\vec{u}, \vec{w} \in M_{nsst} - M_{st}$ then $\vec{u} \equiv \vec{w} \pmod{\mathcal{R}}$ if and only if $\vec{w} \in PSL_2(\mathbb{C})\vec{u}$.

It is clear that \mathcal{R} is an equivalence relation. Set

$$Q_{sst} = M_{sst}/\mathcal{R}, \quad Q_{nsst} = M_{nsst}/\mathcal{R}, \quad Q_{st} = M_{st}/\mathcal{R}, \quad Q_{cusp} = M_{cusp}/\mathcal{R}$$

each with the quotient topology. The elements of Q_{cusp} are uniquely determined by their partitions. Thus Q_{cusp} is a finite set. It is clear that each equivalence class in Q_{cusp} contains a unique $PSL_2(\mathbb{C})$ -orbit of nice semi-stable points whence the inclusion

$$M_{nsst} \subset M_{sst}$$

induces an isomorphism

$$Q_{nsst} = M_{nsst}/PSL_2(\mathbb{C}) \rightarrow Q_{sst}.$$

In case r_1, \dots, r_n are rational then the quotient space Q_{sst} can be given a structure of a complex projective variety by the techniques of geometric invariant theory applied to certain equivariant projective embeddings of $(S^2)^n$, see [DM, §4.6]. This concludes our review of [DM, §4]. We now establish the connection with the moduli space M_r .

For the rest of this section (with one exception) we will use the ball model of hyperbolic space (so $*$ = $(0, 0, 0)$). We will compactify \mathbb{H}^3 by enlarging the open three ball to the closed three ball, thus we add $S^2 = \partial_\infty \mathbb{H}^3$. Each point of S^2 corresponds to an equivalence class of geodesic rays in \mathbb{H}^3 . Two rays α and β are equivalent if they are asymptotic, i.e. $\lim_{t \rightarrow \infty} \alpha(t) = \lim_{t \rightarrow \infty} \beta(t)$ in the closed three ball. Intrinsically the equivalent rays are characterized by the property that they are within finite Hausdorff distance from each other.

In what follows all geodesic segments, geodesics and geodesic rays will be parameterized by arc-length. We now define the hyperbolic Gauss map γ (in various incarnations). Let $\sigma = [x, y]$, $x, y \in \mathbb{H}^3$, be the oriented geodesic segment from x to y . Let $\tilde{\sigma}(0)$ be the ray, $\tilde{\sigma} : [0, \infty) \rightarrow \mathbb{H}^3$ with $\tilde{\sigma}(0) = x$ and $\tilde{\sigma}(\ell) = y$ (here $\ell = \ell(\sigma)$ is the length of the geodesic segment σ). We define the (forward) Gauss map γ on oriented segments by

$$\gamma(\sigma) = \lim_{t \rightarrow \infty} \tilde{\sigma}(t).$$

We may now define $\gamma : \tilde{N}_r \rightarrow (S^2)^n$ by

$$\gamma(e) = (\gamma(e_1), \dots, \gamma(e_n)).$$

One of the main results of this paper is the following theorem – an analogue for Poisson actions of the theorem of Kirwan, Kempf, and Ness, [Ki], [KN].

THEOREM 3.2.

- (i) $\gamma(\tilde{M}_r) \subseteq M_{nssst}$.
- (ii) If P is nondegenerate, then $\gamma(P) \in M_{sst}$.
- (iii) γ induces a real analytic homeomorphism $\gamma : M_r \rightarrow Q_{sst}$.
- (iv) M_r is smooth if and only if $M_{st} = M_{sst}$. In this case Q_{sst} is also smooth and $\gamma : M_r \rightarrow Q_{sst}$ is an analytic diffeomorphism.

Let $\eta \in S^2$. We recall the definition of the geodesic flow ϕ_η^t associated to η . (Strictly speaking, this flow is rather the projection to \mathbb{H}^3 of the restriction of the geodesic flow on $UT(\mathbb{H}^3)$ to the stable submanifold corresponding to η .) Given $z \in \mathbb{H}^3$ there is a unique arc-length parameterized ray σ with $\sigma(0) = z$, $\lim_{t \rightarrow \infty} \sigma(t) = \eta$. By definition,

$$\phi_\eta^t(z) = \sigma(t).$$

We will also need the definition of the Busemann function $b(x, \xi)$, $x \in \mathbb{H}^3, \xi \in \partial_\infty \mathbb{H}^3$. Let σ be an arc-length parameterized geodesic ray from $*$ to ξ . Then

$$b(x, \xi) = \lim_{t \rightarrow \infty} (d(x, \sigma(t)) - t).$$

Note that for $k \in PSU(2) = Stab(*)$ we have

$$b(kx, k\xi) = b(x, \xi).$$

Also, in the upper half space for \mathbb{H}^3 , we have

$$b((x, y, z), \infty) = -\log z.$$

We now prove

LEMMA 3.3. *For fixed ξ , $-\nabla b(x, \xi)$ is the infinitesimal generator of the geodesic flow ϕ_ξ^t .*

Proof. From the first formula above, it suffices to check this statement in the upper half space model (so $*$ = $(0, 0, 1)$) for $\xi = \infty$. By the second formula

$$-\nabla b(x, y, z) = z \frac{\partial}{\partial z}.$$

□

We will now prove (i) and (ii) in the statement of Theorem 3.2 above.

LEMMA 3.4.

- (i) $\gamma(\widetilde{M}_r) \subseteq M_{nsst}$.
- (ii) P is nondegenerate $\Leftrightarrow \gamma(P) \in M_{st}$.

Proof. Let $P \in \widetilde{M}_r$ be a polygon with the vertices $x_1, \dots, x_{n+1} = x_n$, we will use the notation $x_i(t), 0 \leq t \leq r_i$, for the parameterized edge e_i (so that $x_i(0) = x_i$). We test stability of $\gamma(P)$ with respect to a point $\eta \in S^2$. Let $b(x) := b(x, \eta)$ be the corresponding Busemann function. Then for any unit vector $v \in T_x(\mathbb{H}^3)$

$$(3.1) \quad -\nabla b(x) \cdot v \leq 1$$

with the equality if and only if the geodesic ray $\exp(\mathbb{R}_+ v)$ is asymptotic to η . Similarly,

$$(3.2) \quad -\nabla b(x) \cdot v \geq -1$$

with the equality if and only if the geodesic ray $\exp(\mathbb{R}_- v)$ is asymptotic to η . Let $I \subset \{1, \dots, n\}$ be the subset of indices such that $\gamma(e_i) = \eta$. Let J be the complement of I in $\{1, \dots, n\}$. Put $r_I = \sum_{i \in I} r_i$, $r_J = \sum_{j \in J} r_j$. Since the polygon P is closed, using (3.1) and (3.2) we get:

$$\begin{aligned} 0 &= -b \Big|_{x_1}^{x_{n+1}} = \sum_{i=1}^{n-1} \int_0^{r_i} -x'_i(t) \cdot \nabla b(x_i(t)) dt \\ &\geq \sum_{i \in I} r_i - \sum_{j \in J} r_j = r_I - r_J \end{aligned}$$

with the equality if and only if every edge $e_j, j \in J$, is contained in the geodesic through η and x_j . Thus $r_I \leq r_J$, i.e. $\gamma(P)$ is semi-stable. If $\gamma(P)$ is not stable then each edge $e_i, 1 \leq i \leq n$, of P is contained in the geodesic through η and x_i , which implies that this geodesic is the same for all i . Hence P is degenerate in this case. □

In order to prove that $\gamma : M_r \rightarrow Q_{nsst}$ is injective and surjective, we will first need to study a certain dynamical system $f_{r, \xi} \in \text{Diff}(\mathbb{H}^3)$ attached to the configuration of n points $\xi = (\xi_1, \dots, \xi_n)$ on S^2 weighted by $r = (r_1, \dots, r_n)$. The weights r will usually be fixed and we will drop r in $f_{r, \xi}$.

3.2. A dynamical system on \mathbb{H}^3 and the proof that the Gauss map is an isomorphism. Let $\xi = (\xi_1, \dots, \xi_n) \in (S^2)^n$. We define a diffeomorphism $f_\xi : \mathbb{H}^3 \rightarrow \mathbb{H}^3$ as follows. Assume that $r = (r_1, \dots, r_n) \in D_n$ is given. Let $z \in \mathbb{H}^3$ be given. Let σ_1 be the ray emanating from z with $\lim_{t \rightarrow \infty} \sigma_1(t) = \xi_1$. Put $x_1 = z$ and $x_2 = \sigma_1(r_1)$. Now let σ_2 be the ray emanating from x_2 with $\lim_{t \rightarrow \infty} \sigma_2(t) = \xi_2$. Put $x_3 = \sigma_2(r_2)$. We continue in this way until we obtain $x_{n+1} = \sigma_n(r_n)$ where σ_n is the geodesic ray emanating from x_n with $\lim_{t \rightarrow \infty} \sigma_n(t) = \xi_n$. We define f_ξ by $f_\xi(z) = x_{n+1}$. Note that the polygon $P = (x_1, \dots, x_{n+1})$ belongs to \tilde{N}_r .

We now give another description of f_ξ :

$$f_\xi = \phi_{\xi_n}^{r_n} \circ \dots \circ \phi_{\xi_1}^{r_1}$$

where ϕ_ξ^t is the time t geodesic flow towards ξ . We may interpret the previous formula for f_ξ as a product (or multiplicative) integral [DF]. Partition the interval $[0, 1]$ into n equal subintervals, $0 = t_0 < t_1 < \dots < t_n = 1$. Let ν be the atomic measure on $[0, 1]$ given by $\nu(t) = \sum_{i=0}^{n-1} r_{i+1} \delta(t - t_i)$. Let $\lambda : [0, 1] \rightarrow S^2$ be the map given by $\lambda[t_i, t_{i+1}) = v_i$, $0 \leq i \leq n - 1$. Define $A : [0, 1] \rightarrow C^\infty(\mathbb{H}^3, T(\mathbb{H}^3))$ by $A(t)(z) = \nabla b(z, \lambda(t))$. Then in the notation of [DF],

$$f_\nu = \prod_0^1 e^{A(t)d\nu(t)}.$$

in $\text{Diff}(\mathbb{H}^3)$.

REMARK 3.5. *In fact, in [DF] the only integrals considered take values in $GL_n(\mathbb{C})$. We have included the above formula to stress the analogy with the conformal center of mass. The above integral is the multiplicative analogue of the gradient of the averaged Busemann function*

$$\nabla b_\nu(z) = \int_{S^2} \nabla b(z, \eta) (\lambda_* d\nu)(\eta)$$

used to define the conformal center of mass (see §3.3).

We will first prove

PROPOSITION 3.6. *Suppose ξ consists of three or more distinct points. Then f_ξ is a strict contraction. Hence, if ξ is stable, f_ξ is a strict contraction.*

We will need the following lemma

LEMMA 3.7. *Let $\xi \in S^2$ and ϕ_ξ^t be the geodesic flow towards ξ . Then, for each $t > 0$,*

- (i) $d(\phi_\xi^t(z_1), \phi_\xi^t(z_2)) \leq d(z_1, z_2)$ with equality if and only if z_1 and z_2 belong to the same geodesic η with end-point ξ .
- (ii) If $Z \in T_z(\mathbb{H}^3)$ is a tangent vector, then

$$\|D\phi_\xi^t(Z)\| \leq \|Z\|$$

with equality if and only if Z is tangent to the geodesic η through z which is asymptotic to ξ .

Proof. We prove (ii) noting that (ii) implies (i). Use the upper half space model for \mathbb{H}^3 and send ξ to ∞ . Then, if $Z = (a, b, c)$ is tangent to \mathbb{H}^3 at (x, y, z) , we have

$$\phi_\infty^t(x, y, z) = (x, y, e^t z)$$

and

$$\|D\phi_\infty^t(a, b, c)\| \Big|_{(x, y, e^t z)} = \sqrt{e^{-2t} a^2 + e^{-2t} b^2 + c^2}.$$

□

REMARK 3.8. 1. In case of equality in (i), the points $z_1, z_2, \phi_\xi^t(z_1), \phi_\xi^t(z_2)$ all belong to η .

2. The above lemma also follows from the fact that $D\phi_\xi^t(Z)$ is a stable Jacobi field along η .

We can now prove the proposition. By the previous lemma, $f_\xi = \phi_{\xi_n}^{r_n} \circ \dots \circ \phi_{\xi_1}^{r_1}$ does not increase distance. Suppose then that $d(f_\xi(z_1), f_\xi(z_2)) = d(z_1, z_2)$. Then,

$$d(\phi_{\xi_1}^{r_1}(z_1), \phi_{\xi_1}^{r_1}(z_2)) = d(z_1, z_2).$$

Hence, $z_1, z_2, \phi_{\xi_1}^{r_1}(z_1), \phi_{\xi_1}^{r_1}(z_2)$ are all on the geodesic η_1 joining z_1 to ξ_1 .

Next,

$$d(\phi_{\xi_2}^{r_2}(\phi_{\xi_1}^{r_1}(z_1)), \phi_{\xi_2}^{r_2}(\phi_{\xi_1}^{r_1}(z_2))) = d(\phi_{\xi_1}^{r_1}(z_1), \phi_{\xi_1}^{r_1}(z_2)).$$

Hence, $\phi_{\xi_1}^{r_1}(z_1), \phi_{\xi_1}^{r_1}(z_2), \phi_{\xi_2}^{r_2}(\phi_{\xi_1}^{r_1}(z_1)), \phi_{\xi_2}^{r_2}(\phi_{\xi_1}^{r_1}(z_2))$ are all on the same geodesic. This geodesic is necessarily η_2 , the geodesic joining $\phi_{\xi_1}^{r_1}(z_1)$ to ξ_2 , since it contains $\phi_{\xi_1}^{r_1}(z_1)$ and $\phi_{\xi_2}^{r_2}(\phi_{\xi_1}^{r_1}(z_1))$. But since η_2 contains $\phi_{\xi_1}^{r_1}(z_1)$ and $\phi_{\xi_1}^{r_1}(z_2)$ it also coincides with η_1 . Hence, either $\xi_2 = \xi_1$ or ξ_1 is the opposite end ξ_1 of the geodesic η_1 . We continue in this way and find that either $\xi_i = \xi_1$ or $\xi_i = \xi_1$, for all $1 \leq i \leq n$. □

Our next goal is to prove that f_ξ has a fixed-point in \mathbb{H}^3 . Let H be the convex hull of $\{\xi_1, \dots, \xi_n\}$. Let β_i be the negative of the Busemann function associated to ξ_i so β_i increases along geodesic rays directed toward ξ_i . Fix a vector $r = (r_1, \dots, r_n)$ and r -stable configuration $\xi = (\xi_1, \dots, \xi_n) \in S^2$. For $1 \geq h > 0$, we shall consider $f_{hr, \xi} : \mathbb{H}^3 \rightarrow \mathbb{H}^3$ where $hr = (hr_1, \dots, hr_n)$. Note that $f_{hr, \xi}(H) \subset H$.

LEMMA 3.9. There exist open horoballs, \mathcal{O}_i , $1 \leq i \leq n$, centered at ξ_i , which depend only on r and ξ , such that for each $1 \leq i \leq n$, if $x \in \mathcal{O}_i \cap H$, then

$$\beta_i(f_{hr, \xi}(x)) < \beta_i(x)$$

(so $f_{hr, \xi}(x)$ is “further away from” ξ_i than x).

Proof. The angle between any two geodesics asymptotic to ξ_i is zero, thus by continuity, for each $\epsilon > 0$, there exists a horoball $\mathcal{O}_i(\epsilon)$ centered at ξ_i so that for each $x \in \mathcal{O}_i(\epsilon) \cap H$ and for each point ξ_j which is different from ξ_i , the angle between the geodesic ray from x to ξ_j and $\nabla b(x, \xi_i)$ is $\leq \epsilon$. Let $I = \{\ell \in \{1, \dots, n\} : \xi_\ell = \xi_i\}$, $J := \{1, \dots, n\} - I$. Recall the stability condition means:

$$r_I := \sum_{\ell \in I} r_\ell < r_J := \sum_{j \in J} r_j$$

thus we can choose $\pi/2 > \epsilon > 0$ so that

$$r_I - \cos(\epsilon)r_J < 0.$$

We define L by $\mathcal{O}_i(\epsilon) = \{\beta_i > L\}$ for this choice of ϵ , then we define \mathcal{O}_i by $\mathcal{O}_i := \{\beta_i > L + |r|\}$. Pick $x \in H \cap \mathcal{O}_i$, this point is the initial vertex of the linkage P with vertices

$$x_1 = x, x_2 = \phi_{\xi_1}^{hr_1}(x_1), \dots, x_{n+1} = \phi_{\xi_n}^{hr_n}(x_n) = f_{hr,\xi}(x).$$

Note that since the length of P equals $h|r|$, (and $h \leq 1$), the whole polygon P is contained in $H \cap \mathcal{O}_i(\epsilon)$. We let $x_j(t), t \in [0, hr_j]$ be the geodesic segment connecting x_j to x_{j+1} (parameterized by the arc-length). Then,

$$\begin{aligned} \beta_i(x_{n+1}) - \beta_i(x_1) &= \sum_{k=1}^n \beta_i(x) \Big|_{x_k}^{x_{k+1}} \\ &= \sum_{k=1}^n \int_0^{hr_k} \nabla \beta_i(x_k(t)) \cdot x'_k(t) dt \\ &= \sum_{\ell \in I} \int_0^{hr_\ell} \nabla \beta_i(x_\ell(t)) \cdot x'_\ell(t) dt + \sum_{j \in J} \int_0^{hr_j} \nabla \beta_i(x_j(t)) \cdot x'_j(t) dt. \end{aligned}$$

Recall that $\|\nabla \beta_i(x_k(t))\| = 1, \|x'_k(t)\| = 1$, if $\ell \in I$ then

$$\nabla \beta_i(x_\ell(t)) \cdot x'_\ell(t) = 1,$$

if $j \in J$ then

$$\nabla \beta_i(x_j(t)) \cdot x'_j(t) \leq -\cos(\epsilon)$$

since $x_j(t) \in \mathcal{O}_i(\epsilon)$ for each $0 \leq t \leq hr_j$. Thus,

$$\beta_i(x_{n+1}) - \beta_i(x_1) \leq h \sum_{\ell \in I} r_\ell - h \cos(\epsilon) \sum_{j \in J} r_j = h(r_I - \cos(\epsilon)r_J) < 0.$$

□

We let $\mathcal{O}'_i := \{\beta_i > L + 2|r|\}$, then

PROPOSITION 3.10. $f_{hr,\xi}$ has a fixed point in $K := H - \bigcup_{i=1}^n \mathcal{O}'_i$.

Proof. We claim that if $x \in H - \bigcup_{i=1}^n \mathcal{O}_i$ then for all $m \geq 0, f_\xi^{(m)}(x) \notin \bigcup_{i=1}^n \mathcal{O}'_i$. We first treat the case $m = 1$. Since $d(x, f_\xi(x)) \leq |r|$ we see that $x \in H - \bigcup_{i=1}^n \mathcal{O}_i$ implies $f_\xi(x) \notin \bigcup_{i=1}^n \mathcal{O}'_i$. But if there exist an $m - 1$ such that $y = f_\xi^{(m-1)}(x) \in \bigcup_{i=1}^n (\mathcal{O}_i - \mathcal{O}'_i)$, then $f_\xi^{(m)}(x) = f_\xi(y) \notin \bigcup_{i=1}^n \mathcal{O}'_i$ by Lemma 3.9 and the claim is proved.

We find that the sequence $\{f_\xi^{(m)}(x)\}$ is relatively compact and contained in K . Let $A \subset K$ be the accumulation set for this sequence. This is a compact subset such that $f_\xi(A) \subset A$. If f_ξ does not have a fixed point in A then the continuous function $\theta(x) := d(x, f_\xi(x)), x \in A$ is bounded away from zero. Let $x_0 \in A$ be a point where θ attains its minimum. However (since f_ξ is a strict contraction)

$$\theta(f_\xi(x_0)) = d(f_\xi(x_0), f_\xi^2(x_0)) < d(x_0, f(x_0)) = \theta(x_0),$$

contradiction. \square

We can now prove Theorem 3.2. We first prove that $\gamma : M_r \rightarrow Q_{sst}$ is injective. This easily reduces to proving that if $P, Q \in M_r$ with $\gamma(P) = \gamma(Q)$, then $P = Q$. Let x_1 be the first vertex of P , x'_1 be the first vertex of Q , and $\xi = \gamma(P) = \gamma(Q)$. Since P closes up, we have $f_\xi(x_1) = x_1$. Since Q closes up, we have $f_\xi(x'_1) = x'_1$. But, f_ξ is a strict contraction, hence $x_1 = x'_1$. It follows immediately that $P = Q$.

We now prove that γ is surjective. Let $\xi \in M_{st}$. There exists $x \in \mathbb{H}^3$ with $f_\xi(x) = x$. Let P be the n -gon with $\gamma(P) = \xi$ and first vertex x . Then P closes up and we have proved that γ is onto the stable points. If ξ is nice semi-stable but not stable, then $\xi = \gamma(P)$ for a suitable degenerate n -gon. Hence, γ is surjective and Theorem 3.2 is proved.

REMARK 3.11. *We have left the proof that the inverse map to $\gamma : M_r \rightarrow Q_{st}$ is smooth (resp. analytic) in the case M_r is smooth to the reader. This amounts to checking that the fixed-point of f_ξ depends smoothly (resp. analytically) on ξ .*

3.3. Connection with the conformal center of mass of Douady and Earle. In this section, we prove Theorem 1.8 of the Introduction. We begin by reviewing the definition of the conformal center of mass $C(\nu) \in \mathbb{H}^3$, where ν is a stable measure on $S^2 = \partial_\infty \mathbb{H}^3$. Here we are using

DEFINITION 3.12. *A measure ν on S^2 is stable if*

$$\nu(\{x\}) < \frac{|\nu|}{2}, x \in S^2.$$

Here, $|\nu|$ is the total mass of ν .

We define the averaged Busemann function, $b_\nu : \mathbb{H}^3 \rightarrow \mathbb{R}$, by

$$b_\nu(x) = \int_{S^2} b(x, \xi) d\nu(\xi).$$

We recall the following proposition ([DE], [MZ, Lemma 4.11]):

PROPOSITION 3.13. *Suppose ν is stable. The b_ν is strictly convex and has a unique critical point (necessarily a minimum).*

DEFINITION 3.14. *The conformal center of mass $C(\nu)$ is defined to be the above critical point. Thus,*

$$\nabla b_\nu|_{C(\nu)} = 0.$$

The main point is the following,

LEMMA 3.15. *The assignment $\nu \rightarrow C(\nu)$ is $PSL_2(\mathbb{C})$ -equivariant,*

$$C(g_*\nu) = gC(\nu).$$

Here $g_*\nu$ is the push-forward of ν by $g \in PSL(2, \mathbb{C})$.

We now return to the set-up of the previous sections. We are given $r = (r_1, \dots, r_n)$ and a stable configuration $\xi = (\xi_1, \dots, \xi_n) \in (S^2)^n$. We have the dynamical system $f_{tr, \xi}$

of the previous chapter, with fixed-point $x = x(tr, \xi)$. We put $\nu = \sum_{i=1}^n r_i \delta(\xi - \xi_i)$, where δ is the Dirac probability measure supported on the origin in \mathbb{R}^3 . We now have,

LEMMA 3.16.

$$\frac{d}{dt} f_{tr, \xi} \Big|_{t=0} = -\nabla b_\nu .$$

Proof. We abbreviate $-\nabla b(x, \xi_i)$, the infinitesimal generator of the geodesic flow associated to ξ_i , to X_i . Thus we want to prove

$$\frac{d}{dt} f_{tr, \xi} \Big|_{t=0} = \sum_{i=1}^n r_i X_i .$$

But if φ_t and ψ_t are flows with infinitesimal generators X and Y respectively, then

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} \varphi \circ \psi(x) &= \frac{\partial^2}{\partial t_1 \partial t_2} \Big|_{t_1=t_2=0} \varphi_{t_1} \circ \psi_{t_2}(x) \\ &= \frac{\partial}{\partial t_2} \Big|_{t_2=0} \varphi \circ \psi_{t_2}(x) + \frac{\partial}{\partial t_1} \Big|_{t_1=0} \varphi_{t_1} \circ \psi_0(x) \\ &= \frac{\partial}{\partial t_2} \Big|_{t_2=0} \psi_{t_2}(x) + \frac{\partial}{\partial t_1} \Big|_{t_1=0} \varphi_{t_1}(x) \\ &= Y(x) + X(x) . \end{aligned}$$

Recall that

$$f_{tr, \xi}(x) = \phi_{\xi_n}^{tr_n} \circ \phi_{\xi_{n-1}}^{tr_{n-1}} \circ \dots \circ \phi_{\xi_1}^{tr_1}(x) .$$

Hence,

$$\frac{d}{dt} \Big|_{t=0} f_{tr, \xi}(x) = \sum_{i=1}^n X_i(x) .$$

□

We are ready to prove Theorem 1.8 of the Introduction. We abbreviate $-\nabla b_\nu$ by X .

THEOREM 3.17. *Let $x(tr, \xi)$ be the unique fixed point of $f_{tr, \xi}$, $0 < t \leq 1$. Then*

$$\lim_{t \rightarrow 0} x(tr, \xi) = C(\nu) .$$

Proof. We note that $f_{0r, \xi} = \text{id}$. Hence, applying Lemma 3.16, the Taylor approximation of $f_{tr, \xi}(x)$ around $t = 0$ is

$$f_{tr, \xi}(x) = x + tX(x) + t^2R(x, t)$$

(where $R(x, t)$ is smooth). Let $\varphi(x, t) = X(x) + tR(x, t)$. By definition, the conformal center of mass $C(\nu)$ is the unique solution of

$$\varphi(x, 0) = X(x) = 0.$$

Since b_ν is strictly convex, if ν is stable, [MZ, Corollary 4.6], $C(\nu)$ is a nondegenerate zero of X and we may apply the implicit function theorem to solve

$$\varphi(x, t) = 0$$

for x as a function of t near $(C(\nu), 0)$. Thus, there exists $\delta > 0$ and smooth curve, $\hat{x}(t)$, defined for $|t| < \delta$, satisfying

- (i) $\varphi(\hat{x}(t), t) = 0$
- (ii) $\hat{x}(0) = C(\nu)$.

But clearly, (i) implies $f_{tr, \xi}(\hat{x}(t)) = \hat{x}(t)$, $0 < t < \delta$. Since the fixed-point of $f_{tr, \xi}$, $0 < t \leq 1$, is unique, we conclude $\hat{x}(t) = x(tr, \xi)$, $0 < t < \delta$. Hence,

$$\lim_{t \rightarrow 0} x(tr, \xi) = \lim_{t \rightarrow 0} \hat{x}(t) = C(\nu).$$

□

4. The Symplectic geometry of $M_r(\mathbb{H}^3)$.

4.1. The Poisson-Lie group structure on B^n . In this section we let G be any (linear) complex simple group, $B = AN$ be the subgroup of the Borel subgroup such that N is its unipotent radical and A is the connected component of the identity in a maximal split torus over \mathbb{R} , and K be a maximal compact subgroup. We will construct a Poisson Lie group structure on G which will restrict to a Poisson Lie group structure on B . For the basic notions of Poisson Lie group, Poisson action, etc. we refer the reader to [Lu1], [GW], [LW], and [CP].

Let R_g and L_g be the action of g on G by the right and left multiplication respectively. Let \mathfrak{g} denote the Lie algebra of G , \mathfrak{k} be the Lie algebra of K and \mathfrak{b} be the Lie algebra of B . Then $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{k}$ and $G = BK$. Let $\rho_{\mathfrak{k}}$ (resp. $\rho_{\mathfrak{b}}$) be the projection on \mathfrak{k} (resp. on \mathfrak{b}). We define $\mathcal{R} := \rho_{\mathfrak{k}} - \rho_{\mathfrak{b}}$ and let $\frac{1}{2}\langle, \rangle$ be the imaginary part of the Killing form on \mathfrak{g} . In the direct sum splitting $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{b}$ we see that \mathfrak{k} and \mathfrak{b} are totally-isotropic subspaces dually paired by \langle, \rangle .

Let $\varphi \in C^\infty(G)$. Define $D\varphi : G \rightarrow \mathfrak{g}$ and $D'\varphi : G \rightarrow \mathfrak{g}$ by

$$\langle D'\varphi(g), \nu \rangle = \frac{d}{dt} \Big|_{t=0} \varphi(ge^{t\nu})$$

$$\langle D\varphi(g), \nu \rangle = \frac{d}{dt} \Big|_{t=0} \varphi(e^{t\nu}g)$$

for $\nu \in \mathfrak{g}$.

We extend \langle, \rangle to a biinvariant element of $C^\infty(G, S^2T^*(G))$ again denoted \langle, \rangle . Now define $\nabla\varphi \in C^\infty(G, T^*(G))$ by

$$\langle \nabla\varphi(g), x \rangle = d\varphi_g(x), x \in T_g(G)$$

We have

$$D\varphi(g) = dR_g^{-1} \nabla\varphi(g)$$

$$D'\varphi(g) = dL_g^{-1}\nabla\varphi(g) = Ad_{g^{-1}}D\varphi(g)$$

The Sklyanin bracket $\{\varphi, \psi\}$ is defined for $\varphi, \psi \in C^\infty(G)$ by

$$\{\varphi, \psi\}(g) = \frac{1}{2}(\langle \mathcal{R}D'\varphi(g), D'\psi(g) \rangle - \langle \mathcal{R}D\varphi(g), D\psi(g) \rangle)$$

where $\mathcal{R} = \rho_{\mathfrak{k}} - \rho_{\mathfrak{b}}$.

LEMMA 4.1. *The bracket $\{\varphi, \psi\}$ is a Poisson bracket on $C^\infty(G)$.*

Proof. See Theorem 1 of [STS].

Let $w \in C^\infty(G, \Lambda^2 T(G))$ be the bivector field corresponding to $\{\cdot, \cdot\}$. We now show that $\{\cdot, \cdot\}$ induces a Poisson bracket on $C^\infty(B)$.

LEMMA 4.2. *$w(b)$ is tangent to B , i.e. $w(b) \in \Lambda^2 T_b(B) \subset \Lambda^2 T(G)$ for all $b \in B$.*

Proof. It suffices to prove that if φ vanishes identically on B then $\{\varphi, \psi\}$ vanishes identically on B for each ψ . However if φ vanishes identically on B then $\nabla\varphi(b) \in T_b B$ for all $b \in B$. Hence $D\varphi(b) \in \mathfrak{b}$, $D'\varphi \in \mathfrak{b}$ for each $b \in B$. This implies that $\mathcal{R}D\varphi = -D\varphi$, $\mathcal{R}D'\varphi = -D'\varphi$ and

$$2\{\varphi, \psi\}(b) = -\langle D'\varphi(b), D'\psi(b) \rangle + \langle D\varphi(b), D\psi(b) \rangle$$

But $D'\varphi(b) = Ad_{b^{-1}}D\varphi(b)$, $D'\psi(b) = Ad_{b^{-1}}D\psi(b)$ and $\langle \cdot, \cdot \rangle$ is Ad -invariant. \square

For the next corollary note that $T_b^*(B)$ is a quotient of $T_b^*(G)$.

COROLLARY 4.3. *Let π be the skew-symmetric 2-tensor on $T^*(G)$ corresponding to w . Pick $b \in B$ and $\alpha, \beta \in T_b^*(G)$. Then $\pi|_b$ depends only on the images of α and β in $T_b^*(B)$.*

We will continue to use π for the skew-symmetric 2-tensor on $T^*(B)$ induced by π above.

REMARK 4.4. *An argument identical to that above proves that $w(k)$ is tangent to K . Hence $\{\cdot, \cdot\}$ induces a Poisson structure on K . With the above structures K and B are sub Poisson Lie subgroups of the Poisson Lie group G .*

We will need a formula for the Poisson tensor π on B . We will use $\langle \cdot, \cdot \rangle$ to identify $T^*(G)$ and $T(G)$. Under this identification $T_b^*(B)$ is identified to $T_b(G)/T_b(B)$. We will identify this quotient with $dR_b\mathfrak{k}$. We let $\tilde{\pi}|_b$ denote the resulting skew-symmetric 2-tensor on $dR_b\mathfrak{k}$. Finally we define $\tilde{\pi}^r \in C^\infty(B, (\Lambda^2 \mathfrak{k})^*)$ by

$$\tilde{\pi}|_b^r(x, y) = \tilde{\pi}|_b(dR_b x, dR_b y), \quad x, y \in \mathfrak{k}$$

We now recover formulae (2.25) of [FR] (or [LR, Definition 4.2]) for $\tilde{\pi}^r$.

LEMMA 4.5. $\tilde{\pi}|_b^r(x, y) = \langle \rho_{\mathfrak{k}}(Ad_{b^{-1}}x), \rho_{\mathfrak{b}}(Ad_{b^{-1}}y) \rangle$.

Proof. Choose $\varphi, \psi \in C^\infty(G)$ with $\nabla\varphi(b) = dR_b x$ and $\nabla\psi(b) = dR_b y$. Then

$$\begin{aligned} \tilde{\pi}|_b^r(x, y) &= \tilde{\pi}|_b(dR_b x, dR_b y) = \tilde{\pi}|_b(\nabla\varphi(b), \nabla\psi(b)) \\ &= \pi|_b(d\varphi(b), d\psi(b)) = \{\varphi, \psi\}(b) \\ &= \frac{1}{2}\langle \mathcal{R}dL_b^{-1}\nabla\varphi(b), dL_b^{-1}\nabla\psi(b) \rangle - \frac{1}{2}\langle \mathcal{R}dR_b^{-1}\nabla\varphi(b), dR_b^{-1}\nabla\psi(b) \rangle \\ &= \frac{1}{2}\langle \mathcal{R}Ad_{b^{-1}}x, Ad_{b^{-1}}y \rangle - \frac{1}{2}\langle \mathcal{R}x, y \rangle. \end{aligned}$$

But $x, y \in \mathfrak{k}$ implies $\langle \mathcal{R}x, y \rangle = 0$. Hence,

$$\begin{aligned} \tilde{\pi}|_b^r(x, y) &= \frac{1}{2} \langle \rho_{\mathfrak{k}}(Ad_{b^{-1}}x), Ad_{b^{-1}}y \rangle - \frac{1}{2} \langle \rho_{\mathfrak{b}}(Ad_{b^{-1}}x), Ad_{b^{-1}}y \rangle \\ &= \frac{1}{2} \langle \rho_{\mathfrak{k}}(Ad_{b^{-1}}x), \rho_{\mathfrak{b}}Ad_{b^{-1}}y \rangle - \frac{1}{2} \langle \rho_{\mathfrak{b}}(Ad_{b^{-1}}x), \rho_{\mathfrak{k}}Ad_{b^{-1}}y \rangle \\ &= \langle \rho_{\mathfrak{k}}(Ad_{b^{-1}}x), \rho_{\mathfrak{b}}(Ad_{b^{-1}}y) \rangle. \end{aligned}$$

The last equality holds by skew-symmetry, see [LR, Lemma 4.3]. \square

We will abuse notation and drop the $\tilde{}$ and r in the notation for $\tilde{\pi}^r$ henceforth.

REMARK 4.6. *The Poisson tensor on K , π_K , induced from the Skymanin bracket on G is the negative of the usual Poisson tensor on K (see [FR], [Lu1]). Throughout this paper we let $\pi_K(k) = dL_k X \wedge Y - dR_k X \wedge Y$, where $X = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $Y = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$.*

We now give G^n the product Poisson structure, hence B^n inherits the product structure. We introduce more notation to deal with the product. We let $\mathfrak{g}_i \subset \mathfrak{g}^n = \mathfrak{g} \oplus \dots \oplus \mathfrak{g}$ be the image of \mathfrak{g} under the embedding into i -th summand. For $\varphi \in C^\infty(G^n)$ we define

$$D_i\varphi : G^n \rightarrow \mathfrak{g}_i, \quad D'_i\varphi : G^n \rightarrow \mathfrak{g}_i$$

as follows. Let $g = (g_1, \dots, g_n) \in G^n$ and $\nu \in \mathfrak{g}_i$, then

$$\langle D_i\varphi, \nu \rangle = \frac{d}{dt} \Big|_{t=0} \varphi(g_1, \dots, e^{t\nu} g_i, \dots, g_n)$$

$$\langle D'_i\varphi, \nu \rangle = \frac{d}{dt} \Big|_{t=0} \varphi(g_1, \dots, g_i e^{t\nu}, \dots, g_n)$$

Here we extend \langle, \rangle to \mathfrak{g}^n by

$$\langle \delta, \gamma \rangle = \sum_{i=1}^n \langle \delta_i, \gamma_i \rangle$$

for $\delta = (\delta_1, \dots, \delta_n), \gamma = (\gamma_1, \dots, \gamma_n)$. We define d_i, ∇_i in an analogous fashion. Finally define the Poisson bracket on $C^\infty(G^n)$ by

$$\{\varphi, \psi\}(g) = \frac{1}{2} \sum_{i=1}^n [\langle \mathcal{R}D'_i\varphi(g), D'_i\psi(g) \rangle - \langle \mathcal{R}D_i\varphi(g), D_i\psi(g) \rangle].$$

As expected we obtain an induced Poisson bracket on $C^\infty(B^n)$ using the above formula with $g \in G$ replaced by $b \in B$.

Now let π be the Poisson tensor on G^n corresponding to the above Poisson bracket. Let $\pi^\# \in \text{Hom}(T^*(G^n), T(G^n))$ be defined by $\beta(\pi^\#(\alpha)) = \pi(\alpha, \beta)$. Let $\varphi \in C^\infty(G^n)$. We have

DEFINITION 4.7. *The Hamiltonian vector field associated to φ is the vector field $X_\varphi \in C^\infty(G^n, T(G^n))$ given by*

$$X_\varphi = \pi^\# d\varphi$$

We will need a formula for X_φ .

LEMMA 4.8. *Let $g = (g_1, \dots, g_n)$. Then $X_\varphi(g) = (X_1(g), \dots, X_n(g))$ where*

$$X_i(g) = \frac{1}{2}[dR_{g_i} \mathcal{R}D_i \varphi(g) - dL_{g_i} \mathcal{R}D'_i \varphi(g)].$$

Proof. We will use the formula

$$\{\varphi, \psi\} = -d\psi(X_\varphi) = -\langle X_\varphi, \nabla\psi \rangle$$

Here ∇ is the gradient with respect to \langle, \rangle on g^n , see above, hence $\nabla\psi = (\nabla_1\psi, \dots, \nabla_n\psi)$. We have

$$\begin{aligned} \{\varphi, \psi\}(g) &= \frac{1}{2} \sum_{i=1}^n [\langle \mathcal{R}D'_i \varphi(g), D'_i \psi(g) \rangle - \langle \mathcal{R}D_i \varphi(g), D_i \psi(g) \rangle] \\ &= \frac{1}{2} \sum_{i=1}^n [\langle \mathcal{R}D'_i \varphi(g), dL_{g_i}^{-1} \nabla_i \psi(g) \rangle - \langle \mathcal{R}D_i \varphi(g), dR_{g_i}^{-1} \nabla_i \psi(g) \rangle] \\ &= \frac{1}{2} \sum_{i=1}^n [\langle dL_{g_i} \mathcal{R}D'_i \varphi(g), \nabla_i \psi(g) \rangle - \langle dR_{g_i} \mathcal{R}D_i \varphi(g), \nabla_i \psi(g) \rangle] \\ &= \frac{1}{2} \langle (X_1(g), \dots, X_n(g)), (\nabla_1\psi(g), \dots, \nabla_n\psi(g)) \rangle \\ &= \frac{1}{2} \langle X_\varphi(g), \nabla\psi(g) \rangle \end{aligned}$$

□

REMARK 4.9. *Since $w(b)$ is tangent to B^n the field $X_\varphi(b)$ will also be tangent to B^n .*

4.2. The dressing action of K on B^n and the action on n -gons in G/K .

The basic reference for this section is [FR]. In that paper the authors take $n = 2$ and write $G = KB$. We will leave to the reader the task of comparing our formulae with theirs.

In what follows we let $G = SL_2(\mathbb{C})$, $K = SU(2)$, and B be the subgroup of G consisting of upper-triangular matrices with positive diagonal entries. We let ρ_B, ρ_K be the projections relative to the decomposition $G = BK$. For the next theorem (in the case $n = 2$) see [FR, Formula 2.15].

THEOREM 4.10. *There is a Poisson action of K on the Poisson manifold B^n given by*

$$k \cdot (b_1, \dots, b_n) = (b'_1, \dots, b'_n)$$

with $b'_i = \rho_B(\rho_K(kb_1 \cdots b_{i-1})b_i)$, $1 \leq i \leq n$.

DEFINITION 4.11. *The above action is called the dressing action of K on B^n .*

DEFINITION 4.12. *For $n = 1$, we denote by $B_r = Kb$ the dressing orbit of b , where $b \in B$ and $d(b*, *) = r$. We will also need the formula for the infinitesimal dressing action of \mathfrak{k} on B^n . This action is given for $x \in \mathfrak{k}$ by*

$$x \cdot (b_1, \dots, b_n) = (\xi_1, \dots, \xi_n) \in T_b(B^n)$$

with $\xi_i = dL_{b_i} \rho_b Ad_{b_i^{-1}} \rho_{\mathfrak{k}} Ad_{(b_1 \cdots b_{i-1})^{-1}} x$. Note that $\xi_i \in T_{b_i}(B)$.

REMARK 4.13. *In order to pass from the K -action to the \mathfrak{k} -action observe that $\rho_K(bg) = \rho_K(g)$ and $\rho_B(bg) = b\rho_B(g)$. Accordingly we may rewrite the K -action as $k \cdot (b_1, \dots, b_n) = (b'_1, \dots, b'_n)$ with*

$$b'_i = b_i \rho_B(b_i^{-1} \rho_K((b_1 \cdots b_{i-1})^{-1} k b_1 \cdots b_{i-1}) b_i), \quad 1 \leq i \leq n.$$

Recall, $* \in \mathbb{H}^3$ is the element fixed by the action of K , $K \cdot * = *$. Since B acts simply-transitively on G/K we have

LEMMA 4.14. (i) *The map $\Phi : B^n \rightarrow Pol_n(*)$ given by*

$$\Phi(b_1, \dots, b_n) = (*, b_1*, \dots, b_1 \cdots b_n*)$$

is a diffeomorphism.

(ii) *The map Φ induces a diffeomorphism from $\{b \in B^n : b_1 \cdots b_n = 1\}$ onto $CPol_n(*)$.*

We now have

LEMMA 4.15. *Φ is a K -equivariant diffeomorphism where K acts on B^n by the dressing action and on $Pol_n(*)$ by the natural (diagonal) action.*

Proof. Let $k \cdot (b_1, \dots, b_n) = (b'_1, \dots, b''_n)$ be the pull-back to B^n of the action of K on $Pol_n(*)$. Then

$$b'_1* = kb_1*$$

$$b''_1 b''_2* = kb_1 b_2*$$

$$\vdots$$

$$b''_1 \cdots b''_n* = kb_1 \cdots b_n*$$

We obtain

$$b''_1 \cdots b''_i = \rho_B(kb_1 \cdots b_i)$$

$$b''_i = \rho_B((b''_1 \cdots b''_{i-1})^{-1} k(b_1 \cdots b_i)) = \rho_B((\rho_B(b''_1 \cdots b''_{i-1}))^{-1} kb_1 \cdots b_{i-1} b_i)$$

$$= \rho_B((\rho_B(kb_1 \cdots b_{i-1}))^{-1} kb_1 \cdots b_{i-1} b_i) = \rho_B(\rho_K(kb_1 \cdots b_{i-1}) b_i)$$

□

There is another formula for the dressing action of K on B^n that will be useful.

LEMMA 4.16. *With the above notation*

$$b'_i = \rho_B(\underbrace{\rho_K(\cdots(\rho_K(kb_1)b_2)\cdots b_{i-1})}_{i-1 \text{ times}} b_i)$$

Proof. Induction on i . □

We obtain a corresponding formula for the infinitesimal dressing action of \mathfrak{k} on B^n .

LEMMA 4.17. $x \cdot (b_1, \dots, b_n) = (\xi_1, \dots, \xi_n)$ where

$$\xi_i = dL_{b_i} \rho_{\mathfrak{b}} Ad_{b_i^{-1}} \rho_{\mathfrak{k}} Ad_{b_{i-1}^{-1}} \cdots \rho_{\mathfrak{k}} Ad_{b_1^{-1}} x.$$

We now draw an important consequence.

LEMMA 4.18. *The map Φ induces a diffeomorphism between $B_r^n = B_{r_1} \times \cdots \times B_{r_n}$ and the configuration space of open based n -gon linkages \tilde{N}_r , where if $b \in B_r^n$, then $r = (r_1, \dots, r_n)$ and $d(b_1 \cdots b_i^*, b_1 \cdots b_{i-1}^*) = r_i$, for all $1 \leq i \leq n$.*

Proof. Let $b \in B_r^n$ be given. Then $\Phi(b) = (*, b_1^*, \dots, b_1 \cdots b_n^*)$. The K -orbit of $\Phi(b)$ is

$$[* , kb_1^* , \dots , k(b_1 \cdots b_n)^*]$$

The i -th edge e_i of $\Phi(b)$ is the geodesic segment joining $kb_1 \cdots b_{i-1}^*$ to $kb_1 \cdots b_i^*$. Clearly this is congruent (by $kb_1 \cdots b_{i-1}$) to the segment connecting $*$ to b_i^* . □

COROLLARY 4.19. *The symplectic leaves of B^n map to the configuration spaces \tilde{N}_r under Φ .*

Proof. The B_r^n are the symplectic leaves of B^n . □

4.3. The moduli space M_r as a symplectic quotient. We have seen that Φ induces a diffeomorphism from $\{(b_1, \dots, b_n) \in B_r^n : b_1 \cdots b_n = 1\}/K$ to the moduli space M_r of closed n -gon linkages in \mathbb{H}^3 modulo isometry. In this section we will prove that the map $\varphi : B^n \rightarrow B$ given by $\varphi(b_1, \dots, b_n) = b_1 \cdots b_n$ is a momentum map for the (dressing) K action on B^n . Hence, M_r is a symplectic quotient, in particular it is a symplectic manifold if 1 is a regular value by Lemma 4.24 below.

The definition of a momentum map for a Poisson action of a Poisson Lie group was given in [Lu1].

DEFINITION 4.20. *Suppose that K is a Poisson Lie group, (M, π) is a Poisson manifold, and $K \times M \rightarrow M$ is a Poisson action. Let $x \in \mathfrak{k}$, α_x be the extension of*

$x \in \mathfrak{k} = (\mathfrak{k}^*)^*$ to a right-invariant 1-form on K^* , and \hat{x} be the induced vector field on M . Then a map $\varphi : M \rightarrow K^*$ is a **momentum map** if it satisfies the equation

$$-\pi^\# \varphi^* \alpha_x = \hat{x}$$

REMARK 4.21. For the definition of K^* , the **dual Poisson Lie group**, see [Lu1]. In our case $K^* = B$.

The next lemma is proved in [Lu1] and [Lu3]. We include a proof here for completeness.

LEMMA 4.22. *Suppose that M is a symplectic manifold, K is a Poisson Lie group and $K \times M \rightarrow M$ is a Poisson action with an equivariant momentum map $\varphi : M \rightarrow K^*$. Assume 1 is a regular value of φ . Then $\varphi^{-1}(1)/K$ is a symplectic orbifold with the symplectic structure given by taking restriction and quotient of the symplectic structure on M . If we assume further that the isotropy subgroups of all $x \in \varphi^{-1}(1)$ are trivial then $\varphi^{-1}(1)/K$ is a manifold.*

Proof. Let ω be the symplectic form on M and $m \in \varphi^{-1}(1) \subset M$. Let V_m be a complement to $T_m \varphi^{-1}(1)$ in $T_m M$. Let $\mathfrak{k} \cdot m \subset T_m M$ be the tangent space to the orbit $K \cdot m$. Hence $\mathfrak{k}_m = \{\hat{x}(m) : x \in \mathfrak{k}\}$. We first prove the identity

$$\omega_m \hat{x}(m) = -\varphi^* \alpha_x|_m$$

Here we use ω_m to denote the induced map $T_m M \rightarrow T_m^* M$ as well as the symplectic form evaluated at m . Indeed we have the identity

$$-\pi^\# \varphi^* \alpha_x = \hat{x}$$

Applying ω we get

$$\omega \hat{x} = -\varphi^* \alpha_x$$

We claim that if $m \in \varphi^{-1}(1)$ the $\mathfrak{k} \cdot m$ is orthogonal (under ω_m) to $T_m(\varphi^{-1}(1))$, in particular it is totally-isotropic. Let $\hat{x}(m) \in \mathfrak{k} \cdot m$ and $u \in T_m(\varphi^{-1}(1))$. Then

$$\omega_m(\hat{x}(m), u) = -\varphi^* \alpha_x|_m(u) = -\alpha_x|_m(d\varphi_m u)$$

But $d\varphi_m u = 0$ and the claim is proved. Hence, the restriction of ω_m to $T_m(\varphi^{-1}(1))$ descends to $T_m(\varphi^{-1}(1))/\mathfrak{k} \cdot m = T_{K \cdot m}(\varphi^{-1}(1)/K)$. We now prove that the induced form is nondegenerate.

To this end we claim that $\mathfrak{k} \cdot m$ and V_m are dually paired by ω_m . We draw two conclusions from the hypothesis that 1 is a regular value for φ . First by [FR, Lemma 4.2] the map $\mathfrak{k} \rightarrow \mathfrak{k} \cdot m$ given by $x \mapsto \hat{x}(m)$ is an isomorphism. Second, $d\varphi_m : V_m \rightarrow \mathfrak{k}^*$ is an isomorphism. Let $\{x_1, \dots, x_N\}$ be a basis for \mathfrak{k} , whence $\{\hat{x}_1(m), \dots, \hat{x}_N(m)\}$ is a basis for $\mathfrak{k} \cdot m$. We want to find a basis $\{v_1, \dots, v_N\}$ for V_m so that $\omega_m(\hat{x}_i(m), v_j) = \delta_{ij}$. Choose a basis $\{v_1, \dots, v_N\}$ for V_m such that $\{d\varphi_m v_1, \dots, d\varphi_m v_N\} \subset \mathfrak{k}^*$ is dual to $\{x_1, \dots, x_N\}$. Then

$$\omega_m(\hat{x}_i(m), v_j) = -\varphi^* \alpha_{x_i}|_m(v_j) = -\alpha_{x_i}|_m(d\varphi_m v_j) = -d\varphi_m v_j(x_i) = -\delta_{ij}$$

As a consequence of the previous claim, the restriction of ω_m to $\mathfrak{k} \cdot m \oplus V_m$ is nondegenerate. We then have the orthogonal complement $(\mathfrak{k} \cdot m \oplus V_m)^\perp$ is a complement to $\mathfrak{k} \cdot m \oplus V_m$ and $\omega_m|_{(\mathfrak{k} \cdot m \oplus V_m)^\perp}$ is nondegenerate. But then $(\mathfrak{k} \cdot m \oplus V_m)^\perp$ maps isomorphically to $T_{K \cdot m}(\varphi^{-1}(1)/K)$. \square

We will also need

LEMMA 4.23. $\varphi(m)$ is a regular value for φ if and only if $\mathfrak{k}_m = \{x \in \mathfrak{k} : \hat{x}(m) = 0\} = 0$.

Proof. Let $x \in \mathfrak{k}$. Then $x \in (Im d\varphi|_m)^\perp \Leftrightarrow \varphi^* \alpha_x = 0 \Leftrightarrow 0 = -\pi \varphi^* \alpha_x = \hat{x}(m)$. \square

We now begin the proof that φ is a momentum map for the dressing action of K on B^n . We will need some notation. Let $x \in \mathfrak{k} = \mathfrak{b}^*$. Recall that α_x is the extension of x to a right-invariant 1-form on B . Thus if $\zeta \in T_b(B)$ we have

$$\alpha_x|_b(\zeta) = \langle dR_b x, \zeta \rangle.$$

LEMMA 4.24. $-\pi^\# \alpha_x|_b = dL_b \rho_b Ad_{b^{-1}} x$.

Proof. Let $y \in \mathfrak{k}$. It suffices to prove that

$$\pi|_b(\alpha_x(b), \alpha_y(b)) = -\alpha_y|_b(dL_b \rho_b Ad_{b^{-1}} x)$$

Now

$$\pi|_b(\alpha_x(b), \alpha_y(b)) = \pi|_b(dR_b x, dR_b y) = \pi^r|_b(x, y) = \langle \rho_\mathfrak{k}(Ad_{b^{-1}} x), \rho_b(Ad_{b^{-1}} y) \rangle$$

according to Lemma 4.5. Also

$$\begin{aligned} -\alpha_y|_b(dL_b \rho_b Ad_{b^{-1}} x) &= -\langle dR_b y, dL_b \rho_b Ad_{b^{-1}} x \rangle \\ &= -\langle Ad_{b^{-1}} y, \rho_b Ad_{b^{-1}} x \rangle = -\langle \rho_\mathfrak{k} Ad_{b^{-1}} y, \rho_b Ad_{b^{-1}} x \rangle = \langle \rho_\mathfrak{k} Ad_{b^{-1}} x, \rho_b Ad_{b^{-1}} y \rangle \end{aligned}$$

\square

LEMMA 4.25. $\varphi^* \alpha_x|_b = (\alpha_{x_1}|_{b_1}, \dots, \alpha_{x_n}|_{b_n})$, where $x_1 = x$ and $x_i = \rho_\mathfrak{k}(Ad_{(b_1 \dots b_{i-1})^{-1}} x)$, $2 \leq i \leq n$.

Proof. We will use the following formula (the product rule). Let $b = (b_1, \dots, b_n)$ and $\zeta = (\zeta_1, \dots, \zeta_n) \in T_b(B^n)$. Then

$$d\varphi_b(\zeta) = (dR_{b_2 \dots b_n} \zeta_1 + dL_{b_1} dR_{b_3 \dots b_n} \zeta_2 + \dots + dL_{b_1 \dots b_{n-1}} \zeta_n)$$

Hence

$$\begin{aligned} (\varphi^* \alpha_x)|_b(\zeta) &= \alpha_x|_{b_1 \dots b_n}(dR_{b_2 \dots b_n} \zeta_1 + dL_{b_1} dR_{b_3 \dots b_n} \zeta_2 + \dots + dL_{b_1 \dots b_{n-1}} \zeta_n) \\ &= \langle dR_{b_1 \dots b_n} x, dR_{b_2 \dots b_n} \zeta_1 \rangle + \langle dR_{b_1 \dots b_n} x, dL_{b_1} dR_{b_3 \dots b_n} \zeta_2 \rangle + \dots \\ &\quad + \langle dR_{b_1 \dots b_n} x, dL_{b_1 \dots b_{n-1}} \zeta_n \rangle \\ &= \langle dR_{b_1} x, \zeta_1 \rangle + \langle dR_{b_2} Ad_{b_1^{-1}} x, \zeta_2 \rangle + \dots + \langle dR_{b_n} Ad_{(b_1 \dots b_{n-1})^{-1}} x, \zeta_n \rangle \\ &= \sum_{i=1}^n \alpha_{x_i}(\zeta_i) \end{aligned}$$

PROPOSITION 4.26. φ is an equivariant momentum map for the dressing action of K on B^n .

Proof. To show that φ is a momentum map we have to check that

$$-\pi^\# \varphi^* \alpha_x|_b = x \cdot (b_1, \dots, b_n)$$

But $-\pi^\# \varphi^* \alpha_x = (-\pi^\# \alpha_{x_1}, \dots, -\pi^\# \alpha_{x_n})$ and the result follows from the previous two lemmas. To show that φ is equivariant we have to check that $\varphi(k \cdot (b_1, \dots, b_n)) = k \cdot b_1 \dots b_n$. This is obvious from the point of view of polygons. \square

As a consequence of the above proposition we obtain

THEOREM 4.27. The map Φ carries the symplectic quotient $(\varphi|_{B^n})^{-1}(1)/K$ diffeomorphically to the moduli space of n -gon linkages M_r .

REMARK 4.28. We obtain a symplectic structure on M_r by transport of structure.

4.4. The bending Hamiltonians. In this section we will compute the Hamiltonian vector fields X_{f_j} of the functions

$$f_j(b) = \operatorname{tr}((b_1 \cdots b_j)(b_1 \cdots b_j)^*), \quad 1 \leq j \leq n.$$

Throughout the rest of the paper, we will assume $G = SL_2(\mathbb{C})$. Then $G = BK$, where $B = \{ \begin{pmatrix} a & z \\ 0 & a^{-1} \end{pmatrix} \in SL_2(\mathbb{C}) \mid a \in \mathbb{R}_+, z \in \mathbb{C} \}$ and $K = SU(2)$.

We will use the following notation. If $A \in M_m(\mathbb{C})$ then $A^0 = A - \frac{1}{m} \operatorname{tr}(A)I$ will be its projection to the traceless matrices.

THEOREM 4.29. Define $F_j : B^n \rightarrow \mathfrak{k}$ for $b = (b_1, b_2, \dots, b_n)$ by

$$F_j(b) = \sqrt{-1} [(b_1 \cdots b_j)(b_1 \cdots b_j)^*]^0$$

Then $X_{f_j}(b) = (F_j(b) \cdot (b_1, \dots, b_j), 0, \dots, 0)$ where \cdot is the infinitesimal dressing action of \mathfrak{k} on B^j , see §4.2.

Proof. It will be convenient to work on G^n and then restrict to B^n . By the formula for X_φ of Lemma 4.8 it suffices to compute $D_i f_j$ and $D'_i f_j$. We recall that

$$D'_i \varphi(g) = \operatorname{Ad}_{g_i^{-1}} D_i \varphi(g)$$

hence it suffices to compute $D_i f_j(g)$. We first reduce to computing $D_1 f_j$ by

$$\text{LEMMA 4.30. } D_i f_j(g) = \operatorname{Ad}_{(g_1 \cdots g_{i-1})^{-1}} D_1 f_j(g).$$

Proof. By definition

$$\langle D_i f_j(g), \nu \rangle = \frac{d}{dt} \Big|_{t=0} f_j(g_1, \dots, e^{t\nu} g_i, \dots, g_n)$$

But it is elementary that

$$f_j(g_1, \dots, e^{t\nu} g_i, \dots, g_n) = f_j((\operatorname{Ad}_{g_1 \cdots g_{i-1}} e^{t\nu}) g_1, \dots, g_n)$$

Differentiating at $t = 0$ we obtain

$$\langle D_1 f_j(g), \nu \rangle = \langle D_1 f_j(g), Ad_{g_1 \cdots g_{i-1}} \nu \rangle = \langle Ad_{(g_1 \cdots g_{i-1})^{-1}} D_1 f_j(g), \nu \rangle$$

□

We next have

LEMMA 4.31. $D_1 f_j(g) = F_j(g)$.

Proof. By definition

$$\begin{aligned} \langle D_1 f_j(g), \nu \rangle &= \frac{d}{dt} \Big|_{t=0} tr[(e^{t\nu} g_1 \cdots g_j)(e^{t\nu} g_1 \cdots g_j)^*] \\ &= tr[(\nu g_1 \cdots g_j)(g_1 \cdots g_j)^* + (g_1 \cdots g_j)(\nu g_1 \cdots g_j)^*] \\ &= tr[(\nu g_1 \cdots g_j)(g_1 \cdots g_j)^*] + tr[(\nu g_1 \cdots g_j)(g_1 \cdots g_j)^*]^* \\ &= tr[(\nu g_1 \cdots g_j)(g_1 \cdots g_j)^*] + \overline{tr[(\nu g_1 \cdots g_j)(g_1 \cdots g_j)^*]} \\ &= 2Retr[(\nu g_1 \cdots g_j)(g_1 \cdots g_j)^*] \\ &= 2Im\sqrt{-1}tr[\nu(g_1 \cdots g_j)(g_1 \cdots g_j)^*] \end{aligned}$$

Since $\nu \in sl_2(\mathbb{C})$ we may replace

$$(g_1 \cdots g_j)(g_1 \cdots g_j)^*$$

by its traceless projection

$$[(g_1 \cdots g_j)(g_1 \cdots g_j)^*]^0.$$

Since tr is complex bilinear we obtain

$$\begin{aligned} \langle D_1 f_j(g), \nu \rangle &= 2Imtr(\nu\sqrt{-1}[(g_1 \cdots g_j)(g_1 \cdots g_j)^*]^0) \\ &= \langle \sqrt{-1}[(g_1 \cdots g_j)(g_1 \cdots g_j)^*]^0, \nu \rangle. \end{aligned}$$

□

Now we restrict to B^n and substitute into our formula for $X_{f_j}(b)$ in Lemma 4.8. We obtain

LEMMA 4.32. $(X_{f_j})_i = (D_1 f_j(b) \cdot (b_1, \dots, b_n))_i =$ the i -th component of the infinitesimal dressing action of $D_1 f_j(b) \in \mathfrak{k}$.

Proof. By Lemma 4.8 we have

$$\begin{aligned} (X_{f_j})_i &= \frac{1}{2}[dR_{b_i} \mathcal{R} D_i f_j(b) - dL_{b_i} \mathcal{R} D'_i f_j(b)] \\ &= \frac{1}{2}[dR_{b_i} \mathcal{R} Ad_{(b_1 \cdots b_{i-1})^{-1}} D_1 f_j(b) - dL_{b_i} \mathcal{R} Ad_{b_i^{-1}} Ad_{(b_1 \cdots b_{i-1})^{-1}} D_1 f_j(b)] \end{aligned}$$

We write

$$Ad_{(b_1 \cdots b_{i-1})^{-1}} D_1 f_j(b) = X_1 + \eta_1$$

with $X_1 \in \mathfrak{k}$ and $\eta \in \mathfrak{b}$. Hence

$$X_1 = \rho_{\mathfrak{k}}(Ad_{(b_1 \cdots b_{i-1})^{-1}} D_1 f_j(b))$$

$$\eta_1 = \rho_{\mathfrak{b}}(Ad_{(b_1 \cdots b_{i-1})^{-1}} D_1 f_j(b))$$

Then

$$\mathcal{R}Ad_{(b_1 \cdots b_{i-1})^{-1}} D_1 f_j(b) = X_1 - \eta_1$$

We write

$$Ad_{b_i^{-1}} X_1 = X_2 + \eta_2, X_2 \in \mathfrak{k}, \eta_2 \in \mathfrak{b}$$

Then $X_2 = \rho_{\mathfrak{k}}(Ad_{b_i^{-1}} X_1)$, $\eta_2 = \rho_{\mathfrak{b}}(Ad_{b_i^{-1}} X_1)$ and

$$Ad_{b_i^{-1}}(X_1 + \eta_1) = X_2 + \eta_2 + Ad_{b_i^{-1}} \eta_1$$

Hence

$$\begin{aligned} \mathcal{R}Ad_{b_i^{-1}} Ad_{(b_1 \cdots b_{i-1})^{-1}} D_1 f_j(b) &= X_2 - \eta_2 - Ad_{b_i^{-1}} \eta_1 \\ &= X_2 + \eta_2 - 2\eta_2 - Ad_{b_i^{-1}} \eta_1 \\ &= Ad_{b_i^{-1}} X_1 - 2\eta_2 - Ad_{b_i^{-1}} \eta_1 \end{aligned}$$

Hence

$$(X_{f_j})_i = \frac{1}{2} [dR_{b_i} X_1 - dR_{b_i} \eta_1 - dL_{b_i} Ad_{b_i^{-1}} X_1 + 2dL_{b_i} \eta_2 + dL_{b_i} Ad_{b_i^{-1}} \eta_1]$$

But $dL_{b_i} Ad_{b_i^{-1}} = dR_{b_i}$ and we obtain

$$(X_{f_j})_i = dL_{b_i} \eta_2$$

Since $\eta_2 = \rho_{\mathfrak{b}}(Ad_{b_i^{-1}} \rho_{\mathfrak{k}}(Ad_{(b_1 \cdots b_{i-1})^{-1}} D_1 f_j(b)))$ the lemma follows. \square

With this Theorem 4.29 is proved. \square

4.5. Commuting Hamiltonians. In this section we will show the functions

$$f_j(b) = \text{tr}((b_1 \cdots b_j)(b_1 \cdots b_j)^*), \quad 1 \leq j \leq n$$

Poisson commute. The proof is due to Hermann Flaschka.

PROPOSITION 4.33. $\{f_j, f_k\} = 0$ for all j, k .

Proof. Again we will work on G^n and then restrict to B^n . Without loss of generality we let $j \leq k$.

Recall from Lemma 4.30

$$D_i f_j(g) = Ad_{(g_1 \cdots g_{i-1})^{-1}} D_1 f_j(g).$$

It is easily seen that

$$D_i f_j(g) = D'_{i-1} f_j(g), \quad \text{for } 1 \leq i \leq j.$$

We now have,

$$\begin{aligned}
 \{f_j, f_k\}(g) &= \frac{1}{2} \sum_{i=1}^n [\langle \mathcal{R}D'_i f_j(g), D'_i f_k(g) \rangle - \langle \mathcal{R}D_i f_j(g), D_i f_k(g) \rangle] \\
 &= \frac{1}{2} \sum_{i=1}^j [\langle \mathcal{R}D'_i f_j(g), D'_i f_k(g) \rangle - \langle \mathcal{R}D_i f_j(g), D_i f_k(g) \rangle] \\
 &= \frac{1}{2} [\langle \mathcal{R}D'_j f_j(g), D'_j f_k(g) \rangle - \langle \mathcal{R}D_1 f_j(g), D_1 f_k(g) \rangle] \\
 &= \frac{1}{2} [\langle \mathcal{R}Ad_{(g_1 \cdots g_j)^{-1}} D_1 f_j(g), Ad_{(g_1 \cdots g_j)^{-1}} D_1 f_k(g) \rangle - \langle \mathcal{R}D_1 f_j(g), D_1 f_k(g) \rangle] \\
 &= \frac{1}{2} \langle \mathcal{R}Ad_{(g_1 \cdots g_j)^{-1}} D_1 f_j(g), Ad_{(g_1 \cdots g_j)^{-1}} D_1 f_k(g) \rangle
 \end{aligned}$$

since $D_1 f_i(g) \in \mathfrak{k}$ for all i . The proposition follows if we can show

$$\langle \mathcal{R}Ad_{(g_1 \cdots g_j)^{-1}} D_1 f_j(g), Ad_{(g_1 \cdots g_j)^{-1}} D_1 f_k(g) \rangle = 0.$$

It follows from the proof of Theorem 4.29 that

$$\begin{aligned}
 Ad_{(g_1 \cdots g_j)^{-1}} D_1 f_j(g) &= \sqrt{-1} Ad_{(g_1 \cdots g_j)^{-1}} [(g_1 \cdots g_j)(g_1 \cdots g_j)^*]^0 \\
 &= \sqrt{-1} [(g_1 \cdots g_j)^*(g_1 \cdots g_j)]^0 \in \mathfrak{k}.
 \end{aligned}$$

Hence,

$$\begin{aligned}
 \{f_j, f_k\}(g) &= \langle \mathcal{R}Ad_{(g_1 \cdots g_j)^{-1}} D_1 f_j(g), Ad_{(g_1 \cdots g_j)^{-1}} D_1 f_k(g) \rangle \\
 &= \langle Ad_{(g_1 \cdots g_j)^{-1}} D_1 f_j(g), Ad_{(g_1 \cdots g_j)^{-1}} D_1 f_k(g) \rangle \\
 &= \langle D_1 f_j(g), D_1 f_k(g) \rangle \\
 &= 0
 \end{aligned}$$

since $\langle \cdot, \cdot \rangle$ is Ad -invariant. This proves the proposition on G^n . The result then holds when we restrict to B^n . \square

4.6. The Hamiltonian flow. In this section we compute the Hamiltonian flow, φ_k^t , associated to f_k .

Recall from Theorem 4.29 the Hamiltonian field for f_k is given by $X_{f_j}(b) = (F_j(b) \cdot (b_1, \dots, b_j), 0, \dots, 0)$ where \cdot is the infinitesimal dressing action of K on B^n . We now need to solve the system of ordinary differential equations

$$(*) \begin{cases} \frac{db_i}{dt} = (F_j(b) \cdot (b_1, \dots, b_j))_i, & 1 \leq i \leq j \\ \frac{db_i}{dt} = 0, & j+1 \leq i \leq n \end{cases}$$

LEMMA 4.34. $D_1 f_j(b) = F_j(b)$ is invariant along solution curves of $(*)$.

Proof. It suffices to show $\varphi_j(b) = b_1 \cdots b_j$ is constant along solution curves. Let $b(t) = (b_1(t), \dots, b_n(t))$ be a solution of X_{f_j} . Then

$$\begin{aligned}
\frac{d}{dt}\varphi_j(b(t)) &= \frac{db_1}{dt}(t)b_2(t)\cdots b_j(t) + b_1(t)\frac{db_2}{dt}(t)\cdots b_j(t) + \cdots + b_1(t)b_2(t)\cdots \frac{db_j}{dt}(t) \\
&= \frac{1}{2}[(\mathcal{R}D_1 f_j(b(t)))b_1(t) - b_1(t)\mathcal{R}D'_1 f_j(b(t))]b_2(t)\cdots b_j(t) \\
&\quad + b_1(t)\frac{1}{2}[(\mathcal{R}D_2 f_j(b(t)))b_2(t) - b_2(t)\mathcal{R}D'_2 f_j(b(t))]b_3(t)\cdots b_j(t) + \cdots + \\
&\quad + b_1(t)b_2(t)\cdots b_{j-1}(t)\frac{1}{2}[(\mathcal{R}D_j f_j(b(t)))b_j(t) - b_j(t)\mathcal{R}D'_j f_j(b(t))] \\
&= \frac{1}{2}[\mathcal{R}(D_1 f_j(b(t)))b_1(t)\cdots b_j(t) - b_1(t)\cdots b_j(t)\mathcal{R}(D'_j f_j(b(t)))] \\
&= \frac{1}{2}[(D_1 f_j(b(t)))b_1(t)\cdots b_j(t) - b_1(t)\cdots b_j(t)(D'_j f_j(b(t)))] \\
&= \frac{1}{2}[(D_1 f_j(b(t)))b_1(t)\cdots b_j(t) - b_1(t)\cdots b_j(t)(Ad_{(b_1\cdots b_j)^{-1}}D_1 f_j(b(t)))] \\
&= \frac{1}{2}[(D_1 f_j(b(t)))b_1(t)\cdots b_j(t) - (D_1 f_j(b(t)))b_1(t)\cdots b_j(t)] = 0
\end{aligned}$$

Thus $\varphi(b)$ is constant along solution curves of X_{f_j} , proving the lemma. \square

REMARK 4.35. *It also follows from the previous proof that $f_j(b)$ is constant along solution curves of (*)*.

Let $b = \begin{pmatrix} a & z \\ 0 & a^{-1} \end{pmatrix}$ with $a \in \mathbb{R}_+$ and $z \in \mathbb{C}$, then it follows from a simple calculation that

$$\det(F_1(b)) = \frac{1}{4}(a^4 + a^{-4} + |z|^4 - 2 + 2a^2|z|^2 + 2a^{-2}|z|^2).$$

Since $a > 0$ we see that $a^4 + a^{-4} \geq 2$ with equality if $a = 1$. Therefore, $\det(F_1(b)) \geq 0$ with equality iff $b = 1$. From the above argument it follows that $\det(F_j(b)) \geq 0$ with equality iff $b_1 \cdots b_j = 1$.

It is also an easy calculation to show

$$\det(F_j(b)) = \frac{1}{4}f_j(b)^2 - 1, \quad \forall b \in B^n$$

LEMMA 4.36. *The curve $\exp(tF_j(b))$ is periodic with period $2\pi/\sqrt{\frac{1}{4}f_j(b)^2 - 1}$*

Proof. To simplify notation, let $X = F_j(b) \in \mathfrak{k}$. Then

$$X^{-1} = -\frac{1}{\det(X)}X$$

giving us

$$X^2 = -(\det(X))X^{-1}X = -\det(X)I$$

So,

$$\begin{aligned} \exp tX &= \sum_{n=0}^{\infty} \frac{t^n X^n}{n!} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n (t \det(X))^n}{(2n)!} I + \sum_{n=1}^{\infty} \frac{(-1)^n (t \det(X))^n}{(2n+1)!} \frac{X}{\sqrt{\det(X)}} \\ &= \cos\left(t\sqrt{\det(X)}\right) I + \frac{\sin\left(t\sqrt{\det(X)}\right)}{\sqrt{\det(X)}} X \\ &= \cos\left(t\sqrt{\frac{1}{4}f_j(b)^2 - 1}\right) I + \frac{\sin\left(t\sqrt{\frac{1}{4}f_j(b)^2 - 1}\right)}{\sqrt{\frac{1}{4}f_j(b)^2 - 1}} F_j(b) \end{aligned}$$

Therefore the curve is periodic with period $2\pi/\sqrt{\frac{1}{4}f_j(b)^2 - 1}$. \square

We can now find a solution to the system (*)

PROPOSITION 4.37. *Suppose $P \in M_\tau$ has vertices given by b_1, \dots, b_n . Then $P(t) = \varphi_k^t(P)$ has vertices given by $b_1(t), \dots, b_n(t)$ where*

$$b_i(t) = (\exp(tF_j(b)) \cdot (b_1, \dots, b_j))_i, \quad 1 \leq i \leq k$$

$$b_i(t) = b_i, \quad k+1 \leq i \leq n.$$

Here \cdot is the dressing action of K on B^j .

Proof. This follows from $F_j(b)$ being constant on solution curves of (*). We can see immediately that the b_i 's are solutions curves of our system of ordinary differential equations. \square

COROLLARY 4.38. *The flow $\varphi_k^t(P)$ is periodic with period $2\pi/\sqrt{\frac{1}{4}f_j(b)^2 - 1}$.*

REMARK 4.39. *If the k -th diagonal is degenerate ($b_1 \cdots b_k = 1$) then P is a fixed point of φ_k^t . In this case the flow has infinite period.*

Let $\ell_k(b) = 2 \cosh^{-1}(\frac{1}{2}f_k(b))$, then

$$d\ell_k = \frac{1}{\sqrt{\frac{1}{4}f_k(b)^2 - 1}} df_k$$

and consequently

$$X_{\ell_k} = X_{f_k} / \sqrt{\frac{1}{4}f_k^2 - 1}$$

where X_{ℓ_k} is the Hamiltonian vector field associated to ℓ_k . Since f_j is a constant of motion, X_{ℓ_k} is constant along solutions of (*) as well. Let Ψ_k^t be the flow of X_{ℓ_k} . We have the following

PROPOSITION 4.40. *Suppose $P \in M'_r$ has vertices b_1, \dots, b_n . Then $P(t) = \Psi_k^t(P)$ has vertices $b_1(t), \dots, b_n(t)$ given by*

$$b_i(t) = \exp\left((tF_j(b))/\sqrt{\frac{1}{4}f_k(b)^2 - 1}\right) \cdot (b_1, \dots, b_j)_i, \quad 1 \leq i \leq k$$

$$b_i(t) = b_i, \quad k+1 \leq i \leq n$$

where \cdot is the dressing action of K on B^n .

Thus Ψ_k^t is periodic with period 2π and rotates a part of P around the k -th diagonal with constant angular velocity 1 and leaves the other part fixed.

4.7. Angle variables, the momentum polyhedron and a new proof of involutivity. We continue to assume that our n -gons are triangulated by the diagonals $\{d_{1i}, 3 \leq i \leq n-1\}$. We assume $P \in M_r^\circ$ so none of the $n-2$ triangles, $\Delta_1, \Delta_2, \dots, \Delta_{n-2}$, created by the above diagonals are degenerate. We construct a polyhedral surface S bounded by P by filling in the triangles $\Delta_1, \Delta_2, \dots, \Delta_{n-2}$. Hence, Δ_1 has edges e_1, e_2 , and d_{13} , Δ_2 has edges d_{13}, e_3 , and d_{14} , ..., and Δ_{n-2} has edges $d_{1,n-1}, e_{n-1}$, and e_n .

We define $\hat{\theta}_i$ to be the oriented dihedral angle measured from Δ_i to Δ_{i+1} , $1 \leq i \leq n-3$. We define the i -th angle variable θ_i by

$$\theta_i = \pi - \hat{\theta}_i, \quad 1 \leq i \leq n-3.$$

THEOREM 4.41. $\{\theta_1, \dots, \theta_{n-3}\}$ are angle variables, that is we have

- (i) $\{\ell_i, \theta_j\} = \delta_{ij}$
- (ii) $\{\theta_i, \theta_j\} = 0$.

Proof. The proof is identical to that of [KM2, §4]. \square

We next describe the momentum polyhedron B_r for the action of the above $(n-3)$ -torus by bendings. Hence,

$$B_r = \{\ell(M_r) \subset (\mathbb{R}_{\geq 0})^{n-3} : \ell = (\ell_1, \dots, \ell_{n-3})\}.$$

Let $(\ell_1, \dots, \ell_{n-3}) \in (\mathbb{R}_{\geq 0})^{n-3}$ be given. We first consider the problem of constructing the triangles, $\Delta_1, \Delta_2, \dots, \Delta_{n-2}$ above. We note that there are three triangle inequalities $E_i(\ell, r)$, $1 \leq i \leq n-2$, among the r_i 's and ℓ_j 's that give necessary and sufficient conditions for the existence of Δ_i . Once we have obtained the triangles $\Delta_1, \Delta_2, \dots, \Delta_{n-2}$, we can glue them along the diagonals d_{1i} , $3 \leq i \leq n-1$, and obtain a polyhedron surface S and a n -gon P . We obtain

THEOREM 4.42. *The momentum polyhedron $B_r \subset (\mathbb{R}_{\geq 0})^{n-3}$ is defined by the $3(n-2)$ triangle inequalities*

$$\begin{aligned} |r_1 - r_2| &\leq \ell_1 \leq r_1 + r_2 \\ |\ell_1 - r_3| &\leq \ell_2 \leq \ell_1 + r_3 \\ &\vdots \\ |\ell_{n-4} - r_{n-2}| &\leq \ell_{n-3} \leq \ell_{n-4} + r_{n-2} \\ |r_{n-1} - r_n| &\leq \ell_{n-3} \leq r_{n-1} + r_n \end{aligned}$$

Here $r = (r_1, \dots, r_n)$ is fixed, the ℓ_i 's, $1 \leq i \leq n - 3$, are the variables.

As a consequence we have

THEOREM 4.43. *The functions $\ell_1, \ell_2, \dots, \ell_{n-3}$ on M_r are functionally independent.*

The theorem follows from Corollary 4.45. We will apply the next lemma with $M = \ell^{-1}(B_r^\circ)$, the inverse image of the interior of the momentum polyhedron under $\ell = (\ell_1, \ell_2, \dots, \ell_{n-3})$. Then $M \simeq B_r^\circ \times (S^1)^{n-3}$.

LEMMA 4.44. *Suppose $M = M_r^\circ$ is a connected real-analytic manifold and $F = (f_1, \dots, f_k) : M^n \rightarrow \mathbb{R}^k$, $n \geq k$, is a real-analytic map such that $F(M)$ contains a k -ball. Then the 1-forms df_1, \dots, df_k are linearly independent over $C^\infty(M)$.*

Proof. Since the 1-forms df_1, df_2, \dots, df_k are real-analytic, the set of points $x \in M$ such that $df_1|_x, \dots, df_k|_x$ are not independent over \mathbb{R} is an analytic subset W of M . Let $M^\circ = M - W$. Hence either M° is empty or it is open and dense. But by Sard's Theorem, $F(W)$ has measure zero. Since $F(M)$ does not have measure zero, $M \neq W$ and M° is nonempty, hence open and dense. Therefore, if there exists $\varphi_1, \dots, \varphi_k \in C^\infty(M)$ such that $\sum_{i=1}^k \varphi_i df_i = 0$ then $\varphi_i|_{M^\circ} \equiv 0$, $1 \leq i \leq k$, and by density $\varphi_i \equiv 0$, $1 \leq i \leq k$. \square

COROLLARY 4.45. *The restrictions of $d\ell_1, d\ell_2, \dots, d\ell_{n-3}$ to $M \subset M_r$ are independent over $C^\infty(M)$.*

REMARK 4.46. *Since ℓ is onto, if there exists $\Phi \in C^\infty(B_r)$ such that $\Phi(\ell_1(x), \dots, \ell_k(x)) \equiv 0$, then $\Phi \equiv 0$.*

We conclude this chapter by giving a second proof that the bending flows on disjoint diagonals commute. Since M_r° is dense in M_r , it suffices to prove

LEMMA 4.47. $\Psi_i^s(\Psi_j^t(P)) = \Psi_j^t(\Psi_i^s(P))$, for $P \in M_r^\circ$.

Proof. We assume $i > j$. We observe that the diagonals d_{1i} and d_{1j} divide the surface S into three polyhedral "flaps", I , II , III (the boundary of I contains e_1 , the boundary of II contains e_i , and the boundary of III contains e_j). Let R_i^s and R_j^t be the one parameter groups of rotations around d_{1i} and d_{1j} , respectively. We first record what $\Psi_i^s \circ \Psi_j^t$ does to the flaps.

$$\begin{aligned} \Psi_i^s \circ \Psi_j^t(I) &= R_i^s R_j^t(I) \\ \Psi_i^s \circ \Psi_j^t(II) &= R_i^s(II) \\ \Psi_i^s \circ \Psi_j^t(III) &= III \end{aligned}$$

Now we compute what $\Psi_j^t \circ \Psi_i^s$ does to the flaps. The point is, after the bending on d_{1i} , the diagonal d_{1j} moves $R_i^s d_{1j}$. Hence, the next bending rotates I around $R_i^s d_{1j}$. Hence, the next bending curve is $R_i^s \circ R_j^t \circ R_i^{-s}$. We obtain

$$\begin{aligned} \Psi_j^t \circ \Psi_i^s(I) &= (R_i^s R_j^t R_i^{-s}) R_i^s(I) \\ \Psi_j^t \circ \Psi_i^s(II) &= R_i^s(II) \\ \Psi_j^t \circ \Psi_i^s(III) &= III. \end{aligned}$$

\square

5. Symplectomorphism of $M_r(\mathbb{E}^3)$ and $M_r(\mathbb{H}^3)$. Recall r is not on a wall of D_n . Then by Theorem 3.2 of this paper, the hyperbolic Gauss map $\gamma = \gamma_h : M_r(\mathbb{H}^3) \rightarrow Q_{sst}(r)$ is a diffeomorphism. Moreover by Theorem 2.3 of [KM2], the Euclidean Gauss map $\gamma_e : M_r(\mathbb{E}^3) \rightarrow Q_{sst}(r)$ is also a diffeomorphism. We obtain

THEOREM 5.1. *Suppose r is not on a wall of D_r , then the composition $\gamma_h^{-1} \circ \gamma_e : M_r(\mathbb{E}^3) \rightarrow M_r(\mathbb{H}^3)$ is a diffeomorphism.*

REMARK 5.2. *The result that $M_r(\mathbb{E}^3)$ and $M_r(\mathbb{H}^3)$ are (noncanonically) diffeomorphic was obtained by [Sa].*

It does not appear to be true that $\gamma_h^{-1} \circ \gamma_e$ is a symplectomorphism.

5.1. A Formula of Lu. In the next several sections we will prove that $M_r(\mathbb{H}^3)$ is symplectomorphic to $M_r(\mathbb{E}^3)$.

We first define a family of nondegenerate Poisson structures π_ϵ , $\epsilon \in [0, 1]$, on the 2-sphere, $S^2 \simeq K/T$. Letting ω_ϵ be the corresponding family of symplectic forms we show the cohomology classes $[\omega_\epsilon]$ of ω_ϵ in $H^2(S^2)$ are constant.

Fix $\lambda \in \mathbb{R}_+$ and $\Lambda = X \wedge Y \in \wedge^2 \mathfrak{k}$, where $X = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $Y = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. The following family of Poisson structures π_ϵ on $K/T \simeq S^2$ for $\epsilon \in (0, 1]$ are due to J.-H. Lu [Lu2].

$$\pi_\epsilon = \epsilon[\pi_\infty - \tau(\epsilon)\pi_0]$$

where $\pi_\infty = p_*\pi_K = p_*(dL_k\Lambda - dR_k\Lambda)$, $\tau(\epsilon) = \frac{1}{1-e^{4\epsilon\lambda}}$, and $\pi_0 = 2dL_k\Lambda$. Here $p : K \rightarrow K/T$ is the projection map. Then

$$\pi_\epsilon(k) = \epsilon(dL_k\Lambda - dR_k\Lambda) - \frac{2\epsilon}{1-e^{4\epsilon\lambda}} dL_k\Lambda.$$

LEMMA 5.3.

$$\lim_{\epsilon \rightarrow 0} \pi_\epsilon = \frac{1}{4\lambda} \pi_0$$

Proof. The proof of the lemma is a simple application of L'Hôpital's rule.

LEMMA 5.4. π_ϵ is nondegenerate for $\epsilon \in [0, 1]$.

We will prove Lemma 5.4 in Proposition 5.23, where we show $(K/T, \pi_\epsilon)$ is symplectomorphic to a symplectic leaf of the Poisson Lie group $(B_\epsilon, \hat{\pi}_{B_\epsilon})$

We leave it to the reader to verify the Poisson structures on S^2 can be written

$$\pi_\infty = \frac{1}{2}(1 + \alpha^2 + \beta^2) \frac{\partial}{\partial \alpha} \wedge \frac{\partial}{\partial \beta}.$$

and

$$\pi_0 = \frac{1}{2}(1 + \alpha^2 + \beta^2)^2 \frac{\partial}{\partial \alpha} \wedge \frac{\partial}{\partial \beta}$$

where (α, β) are coordinates obtained by stereographic projection with respect to the north pole (see [LW]). π_∞ is the Bruhat-Poisson structure on K/T . We now

let ω_ϵ be the symplectic form obtained by inverting π_ϵ (this is possible since π_ϵ is nondegenerate).

$$\omega_\epsilon = \frac{-d\alpha \wedge d\beta}{\epsilon(\frac{1}{2}(1 + \alpha^2 + \beta^2) - \frac{1}{2}\tau(\epsilon)(1 + \alpha^2 + \beta^2)^2)}, \epsilon \in (0, 1]$$

Let ω_0 be the limiting symplectic structure

$$\omega_0 = -8\lambda \frac{d\alpha \wedge d\beta}{(1 + \alpha^2 + \beta^2)^2}$$

LEMMA 5.5.

$$\int_{\mathbb{R}^2} \omega_0 = -8\pi\lambda$$

Proof.

$$\begin{aligned} \int_{\mathbb{R}^2} \omega_0 &= \int_{\mathbb{R}^2} -8\lambda \frac{d\alpha \wedge d\beta}{(1 + \alpha^2 + \beta^2)^2} \\ &= -8\lambda \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=\infty} \frac{r \, dr \wedge d\theta}{(1 + r^2)^2} \\ &= -16\pi\lambda \int_{u=1}^{u=\infty} \frac{(1/2)du}{u^2} \\ &= -8\pi\lambda \end{aligned}$$

□

LEMMA 5.6.

$$\int_{\mathbb{R}^2} \omega_\epsilon = -8\pi\lambda, \epsilon \in (0, 1]$$

Proof. Note that $\tau(\epsilon) < 0$. Then

$$\begin{aligned} \int_{\mathbb{R}^2} \omega_\epsilon &= -\frac{2}{\epsilon} \int_{\mathbb{R}^2} \frac{d\alpha \wedge d\beta}{(1 + \alpha^2 + \beta^2) - \tau(\epsilon)(1 + \alpha^2 + \beta^2)^2} \\ &= -\frac{2}{\epsilon} \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=\infty} \frac{r \, dr \wedge d\theta}{(1 + r^2) - \tau(\epsilon)(1 + r^2)^2} \\ &= -\frac{4\pi}{\epsilon} \int_{u=1}^{u=\infty} \frac{(1/2)du}{u - \tau(\epsilon)u^2} \\ &= -\frac{2\pi}{\epsilon} \log \left| \frac{\tau(\epsilon) - 1}{\tau(\epsilon)} \right| \\ &= -\frac{2\pi}{\epsilon} \log(e^{4\epsilon\lambda}) \\ &= -8\pi\lambda \end{aligned}$$

We have proved the following

LEMMA 5.7. *The cohomology classes $[\omega_\epsilon]$ of ω_ϵ in $H^2(S^2)$ are constant.*

REMARK 5.8. *The previous lemma is a special case of Lemma 5.1 of [GW].*

5.2. Symplectomorphism of $(\Sigma_\lambda(\epsilon), \pi_{B_\epsilon})$ and $(K/T, \pi_{\lambda, \epsilon})$. In this section we obtain the Poisson structure π_ϵ from a deformed Manin triple $(\mathfrak{g}_\epsilon, \mathfrak{k}, \mathfrak{b}_\epsilon)$.

For $\epsilon > 0$, we define the isomorphism $f_\epsilon : \mathfrak{g}_\epsilon \rightarrow \mathfrak{g}$ by $f_\epsilon = \rho_\mathfrak{k} + \epsilon\rho_\mathfrak{b}$, so that $f_\epsilon(X + \xi) = X + \epsilon\xi$ for $X \in \mathfrak{k}$ and $\xi \in \mathfrak{b}$. We will define a Lie bracket on \mathfrak{g}_ϵ by the pullback of the Lie bracket on \mathfrak{g} , $[u, v]_\epsilon = f_{1/\epsilon}[f_\epsilon u, f_\epsilon v]$. We also define $\langle, \rangle_\epsilon$ as the pullback of \langle, \rangle . Here $[,]$ and \langle, \rangle are the usual structures on \mathfrak{g} . We define $\mathcal{B}_\epsilon : \mathfrak{g} \rightarrow \mathfrak{g}^*$ as the map induced by $\langle, \rangle_\epsilon$. To simplify notation, the subscripts will be dropped when $\epsilon = 1$.

The following lemma gives us a formula for the Lie bracket on \mathfrak{g}_ϵ .

LEMMA 5.9. $[X + \alpha, Y + \beta]_\epsilon = [X, Y] + \epsilon\rho_\mathfrak{k}[X, \beta] + \epsilon\rho_\mathfrak{k}[\alpha, Y] + \rho_\mathfrak{b}[X, \beta] + \rho_\mathfrak{b}[\alpha, Y] + \epsilon[\alpha, \beta]$, where $X, Y \in \mathfrak{k}$ and $\alpha, \beta \in \mathfrak{b}$.

Proof.

$$\begin{aligned} [X + \alpha, Y + \beta]_\epsilon &= f_{1/\epsilon}[f_\epsilon(X + \alpha), f_\epsilon(Y + \beta)] \\ &= f_{1/\epsilon}[X + \epsilon\alpha, Y + \epsilon\beta] \\ &= f_{1/\epsilon}\{[X, Y] + \epsilon[X, \beta] + \epsilon[\alpha, Y] + \epsilon^2[\alpha, \beta]\} \\ &= f_{1/\epsilon}\{[X, Y] + \epsilon\rho_\mathfrak{k}[X, \beta] + \epsilon\rho_\mathfrak{b}[X, \beta] + \epsilon\rho_\mathfrak{k}[\alpha, Y] + \epsilon\rho_\mathfrak{b}[\alpha, Y] + \epsilon^2[\alpha, \beta]\} \\ &= [X, Y] + \epsilon\rho_\mathfrak{k}[X, \beta] + \epsilon\rho_\mathfrak{k}[\alpha, Y] + \rho_\mathfrak{b}[X, \beta] + \rho_\mathfrak{b}[\alpha, Y] + \epsilon[\alpha, \beta] \end{aligned}$$

□

We leave it to the reader to check

LEMMA 5.10. $\langle, \rangle_\epsilon = \epsilon\langle, \rangle$

Let G_ϵ be the simply-connected Lie group with Lie algebra \mathfrak{g}_ϵ . Let $F_\epsilon : G_\epsilon \rightarrow G$ be the isomorphism induced by f_ϵ . We have a commutative diagram of isomorphisms.

$$\begin{array}{ccc} \mathfrak{g}_\epsilon & \xrightarrow{f_\epsilon} & \mathfrak{g} \\ \exp^\epsilon \downarrow & & \downarrow \exp \\ G_\epsilon & \xrightarrow{F_\epsilon} & G \end{array}$$

Let $x \in \mathfrak{g}_\epsilon$. We use the identity map to identify \mathfrak{g} and \mathfrak{g}_ϵ as vector spaces. In what follows, we will make frequent use of

LEMMA 5.11. $\widehat{Ad}(\exp^\epsilon x) = Ad(\exp \epsilon x)$ as elements in $GL(\mathfrak{b})$ for all $x \in \mathfrak{b}_\epsilon = \mathfrak{b}$. Here \widehat{Ad} denotes the adjoint action of G_ϵ on \mathfrak{g}_ϵ .

Proof. By [Wa, pg. 114],

$$\begin{aligned} \widehat{Ad}(\exp^\epsilon x) &= e^{\widehat{ad}x} \\ &= e^{\epsilon adx} \\ &= e^{ad(\epsilon x)} \\ &= Ad(\exp \epsilon x) \end{aligned}$$

□

Given our deformed Manin triple on G_ϵ , $(\mathfrak{g}_\epsilon, \mathfrak{k}, \mathfrak{b}_\epsilon)$, we will construct a Poisson structure $\widehat{\pi}_{B_\epsilon}$ on B_ϵ , the simply-connected Lie group with Lie algebra, \mathfrak{b}_ϵ . We will denote all quantities associated to the deformed Manin triple with a hat $\widehat{}$.

We define the Poisson Lie structure on B_ϵ by the Lu-Weinstein Poisson tensor [LW]

$$\widehat{\pi}_{B_\epsilon}(b)(\widehat{dR}_{b^{-1}}^* \alpha_X, \widehat{dR}_{b^{-1}}^* \alpha_Y) = \langle \rho_{\mathfrak{k}}(\widehat{Ad}_{b^{-1}} B_\epsilon^{-1}(\alpha_X)), \rho_{\mathfrak{b}}(\widehat{Ad}_{b^{-1}} B_\epsilon^{-1}(\alpha_Y)) \rangle_\epsilon$$

where $\alpha_X, \alpha_Y \in \mathfrak{b}_\epsilon^*$, $\alpha_X = \langle X, \cdot \rangle_1$ and $\alpha_Y = \langle Y, \cdot \rangle_1$.

REMARK 5.12. *Since $\lim_{\epsilon \rightarrow 0} \langle \cdot, \cdot \rangle_\epsilon = 0$, it appears as if the limiting Poisson structure $\lim_{\epsilon \rightarrow 0} \widehat{\pi}_{B_\epsilon}$ will vanish. However, we will see in Proposition 5.14 that the limiting Poisson structure is associated to the Manin triple $(\mathfrak{g}_0, \mathfrak{k}, \mathfrak{b}_0)$ and $\langle \cdot, \cdot \rangle_0 = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \langle \cdot, \cdot \rangle_\epsilon$.*

We denote by π_{B_ϵ} the Poisson structure on B_ϵ using the scaled bilinear form $\frac{1}{\epsilon} \langle \cdot, \cdot \rangle_\epsilon = \langle \cdot, \cdot \rangle$. Then

$$\pi_{B_\epsilon}(b)(\widehat{dR}_{b^{-1}}^* \alpha_X, \widehat{dR}_{b^{-1}}^* \alpha_Y) = \langle \rho_{\mathfrak{k}}(\widehat{Ad}_{b^{-1}} B_1^{-1}(\alpha_X)), \rho_{\mathfrak{b}}(\widehat{Ad}_{b^{-1}} B_1^{-1}(\alpha_Y)) \rangle$$

where $\alpha_X, \alpha_Y \in \mathfrak{b}_\epsilon^*$. For the following we will let $X_\epsilon = B_\epsilon^{-1}(\alpha_X) \in \mathfrak{k}$, again dropping the subscript when $\epsilon = 1$, so that $X_\epsilon = \frac{1}{\epsilon} X$.

LEMMA 5.13. $\widehat{\pi}_{B_\epsilon} = \frac{1}{\epsilon} \pi_{B_\epsilon}$

Proof.

$$\begin{aligned} \widehat{\pi}_{B_\epsilon}(b)(\widehat{dR}_{b^{-1}}^* \alpha_X, \widehat{dR}_{b^{-1}}^* \alpha_Y) &= \langle \rho_{\mathfrak{k}}(\widehat{Ad}_{b^{-1}} X_\epsilon), \rho_{\mathfrak{b}}(\widehat{Ad}_{b^{-1}} Y_\epsilon) \rangle_\epsilon \\ &= \langle \rho_{\mathfrak{k}}(\widehat{Ad}_{b^{-1}} \frac{1}{\epsilon} X), \rho_{\mathfrak{b}}(\widehat{Ad}_{b^{-1}} \frac{1}{\epsilon} Y) \rangle_\epsilon \\ &= \frac{1}{\epsilon} \langle \rho_{\mathfrak{k}}(\widehat{Ad}_{b^{-1}} X), \rho_{\mathfrak{b}}(\widehat{Ad}_{b^{-1}} Y) \rangle \end{aligned}$$

□

PROPOSITION 5.14. $\lim_{\epsilon \rightarrow 0} \widehat{\pi}_{B_\epsilon}(b)(\widehat{dR}_{b^{-1}}^* \alpha_X, \widehat{dR}_{b^{-1}}^* \alpha_Y) = -\langle \log b, [X, Y] \rangle$.

Proof.

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \widehat{\pi}_{B_\epsilon}(b)(\widehat{dR}_{b^{-1}}^* \alpha_X, \widehat{dR}_{b^{-1}}^* \alpha_Y) &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \langle \rho_{\mathfrak{k}}(\widehat{Ad}_{b^{-1}} X), \rho_{\mathfrak{b}}(\widehat{Ad}_{b^{-1}} Y) \rangle \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} 2Im \operatorname{tr}(\rho_{\mathfrak{k}}(Ad_{e^{-\epsilon \log b}} X) \rho_{\mathfrak{b}}(Ad_{e^{-\epsilon \log b}} Y)) \\ &= \lim_{\epsilon \rightarrow 0} 2Im \operatorname{tr}(\rho_{\mathfrak{k}}(Ad_{e^{-\epsilon \log b}} X) \rho_{\mathfrak{b}}(-\log b Y + Y \log b)) \\ &= -2Im \operatorname{tr}(X \rho_{\mathfrak{b}}[\log b, Y]) \\ &= -2Im \operatorname{tr}(X[\log b, Y]) \\ &= -2Im \operatorname{tr}(\log b[X, Y]) \\ &= -\langle \log b, [X, Y] \rangle \end{aligned}$$

□

REMARK 5.15. *Before stating the next corollary, note that the limit Lie algebra \mathfrak{b}_0 is abelian whence the limit Lie group B_0 is abelian. Hence, $\exp_0 : \mathfrak{b}_0 = T_0(B_0) \rightarrow B_0$ is*

the canonical identification of the vector space B_0 with its tangent space at the origin. Hence, \exp_0 is an isomorphism of Lie groups and \exp_0^* carries invariant 1-forms on B_0 to invariant 1-forms on \mathfrak{b}_0 .

COROLLARY 5.16. $\lim_{\epsilon \rightarrow 0} \widehat{\pi}_{B_\epsilon}$ is the negative of the Lie Poisson structure on $\mathfrak{k}^* \simeq \mathfrak{b}_0$ transferred to B_0 using the exponential map on the vector space B_0 .

Proof. The proof is left to the reader.

REMARK 5.17. $(K, \epsilon\pi_K)$ is the dual Poisson Lie group of $(B_\epsilon, \widehat{\pi}_{B_\epsilon})$.

We will denote the dressing action of K on B_ϵ by $\widehat{D}_\epsilon^\ell$ and the infinitesimal dressing action of \mathfrak{k} on B_ϵ by $\widehat{d}_\epsilon^\ell$. By definition $\widehat{d}_\epsilon^\ell(b)(X) = \widehat{\pi}_{B_\epsilon}(\cdot, \alpha_X)$. We then have the following.

LEMMA 5.18. $\widehat{d}_\epsilon^\ell(b)(X) = \frac{1}{\epsilon} d^\ell(b)(X)$

Proof. Follows immediately from Lemma 5.13 and the definition of dressing action. \square

REMARK 5.19. $\lim_{\epsilon \rightarrow 0} \widehat{d}_\epsilon^\ell(b)(X) = ad^*(X)(\log b)$

For the remainder of the section, fix $\lambda \in \mathbb{R}_+$ and $a = \exp^\epsilon \lambda H \in B_\epsilon$, where $H = \text{diag}(1, -1) \in \mathfrak{a}_\epsilon$. Let $\varphi_\epsilon : K \rightarrow \Sigma_\lambda^\epsilon \subset B_\epsilon$ be the map defined by $\varphi_\epsilon(k) = \widehat{D}_\epsilon^\ell(k)(a) = \rho_{B_\epsilon}(k * a)$, where Σ_λ^ϵ is the symplectic leaf through the point $a \in B_\epsilon$. The map φ_ϵ induces a diffeomorphism from K/T onto Σ_λ^ϵ which we will also denote by φ_ϵ . Recall the family of Poisson tensors on K/T given in §5.1

$$\pi_{\lambda, \epsilon} = \epsilon(\pi_\infty - \tau(\epsilon\lambda)\pi_0).$$

LEMMA 5.20. The map $\varphi_\epsilon : K/T \rightarrow B_\epsilon$ is K -equivariant, where $(K, \epsilon\pi_K)$ acts on $(K/T, \pi_{\lambda, \epsilon})$ by left multiplication and B_ϵ by the dressing action.

Proof. $\varphi_\epsilon(g \cdot k) = \widehat{D}_\epsilon^\ell(gk)(a) = \widehat{D}_\epsilon^\ell(g)(\widehat{D}_\epsilon^\ell(k)(a)) = g \cdot \varphi_\epsilon(k)$. \square

REMARK 5.21. The action of $(K, \epsilon\pi_K)$ on $(K/T, \pi_{\lambda, \epsilon})$ by left multiplication is a Poisson action.

Since K/T is a symplectic manifold, there is a momentum map for the action of K on K/T , see [Lu1, Theorem 3.16]. We will see as a consequence of Proposition 5.23

LEMMA 5.22. The momentum map for the action of $(K, \epsilon\pi_K)$ on $(K/T, \pi_{\lambda, \epsilon})$ is φ_ϵ .

PROPOSITION 5.23. The map φ_ϵ induces a symplectomorphism from $(K/T, \pi_{\lambda, \epsilon})$ to $(\Sigma_\lambda^\epsilon, \widehat{\pi}_{B_\epsilon})$.

Proof. Since the K -actions on K/T and Σ_λ^ϵ are Poisson and the map $\varphi_\epsilon : K/T \rightarrow \Sigma_\lambda^\epsilon$ is a K -equivariant diffeomorphism, if $(d\varphi_\epsilon)_e(\pi_{\lambda, \epsilon}(e)) = \widehat{\pi}_{B_\epsilon}(a)$ then it follows that $(d\varphi_\epsilon)_k(\pi_{\lambda, \epsilon}(k)) = \widehat{\pi}_{B_\epsilon}(\varphi(k))$ for all $k \in K/T$.

We will need the following lemmas to prove the proposition. We let $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{b}$ and $\Lambda = E \wedge iE \in \mathfrak{b} \wedge \mathfrak{b}$. If we set $\widehat{\pi}_\Lambda(b) = \frac{1}{\epsilon}(\widehat{dL}_b \Lambda - \widehat{dR}_b \Lambda)$, we then have the following.

LEMMA 5.24. $\widehat{\pi}_{B_\epsilon}|_a = \frac{1}{2}\widehat{\pi}_\Lambda|_a$ for $a = \exp^\epsilon \lambda H$.

Proof. Let $X = \begin{pmatrix} si & u \\ -\bar{u} & -si \end{pmatrix}$, $Y = \begin{pmatrix} ti & v \\ -\bar{v} & -ti \end{pmatrix} \in \mathfrak{k}$

$$\begin{aligned} \widehat{\pi}_{B_\epsilon}(a)(\widehat{dR}_{a^{-1}}^* \alpha_X, \widehat{dR}_{a^{-1}}^* \alpha_Y) &= \frac{1}{\epsilon} \langle \rho_{\mathfrak{k}}(\widehat{Ad}_{a^{-1}} X), \rho_{\mathfrak{b}}(\widehat{Ad}_{a^{-1}} Y) \rangle \\ &= \frac{1}{\epsilon} \langle X, \widehat{Ad}_a \rho_{\mathfrak{b}}(\widehat{Ad}_{a^{-1}} Y) \rangle \end{aligned}$$

We can see,

$$\begin{aligned} \widehat{Ad}_a \rho_{\mathfrak{b}}(\widehat{Ad}_{a^{-1}} Y) &= \widehat{Ad}_a \rho_{\mathfrak{b}} \left(\widehat{Ad}_{a^{-1}} \begin{pmatrix} ti & v \\ -\bar{v} & -ti \end{pmatrix} \right) \\ &= \widehat{Ad}_a \rho_{\mathfrak{b}} \begin{pmatrix} ti & e^{-2\epsilon\lambda} v \\ -e^{2\epsilon\lambda} \bar{v} & -ti \end{pmatrix} \\ &= \widehat{Ad}_a \begin{pmatrix} 0 & (e^{-2\epsilon\lambda} - e^{2\epsilon\lambda})v \\ 0 & 0 \end{pmatrix} \\ &= (1 - e^{4\epsilon\lambda}) \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} \end{aligned}$$

so that

$$\begin{aligned} \frac{1}{\epsilon} \langle X, \widehat{Ad}_a \rho_{\mathfrak{b}}(\widehat{Ad}_{a^{-1}} Y) \rangle &= \frac{2}{\epsilon} \text{Imtr} \left[\begin{pmatrix} si & u \\ -\bar{u} & -si \end{pmatrix} \begin{pmatrix} 0 & (1 - e^{4\epsilon\lambda})v \\ 0 & 0 \end{pmatrix} \right] \\ &= -\frac{2}{\epsilon} (1 - e^{4\epsilon\lambda}) \text{Im}(\bar{u}v) \\ &= \frac{2}{\epsilon} (e^{4\epsilon\lambda} - 1) \text{Im}(\bar{u}v). \end{aligned}$$

If we evaluate the right-hand side of the above formula we see

$$\begin{aligned} \frac{1}{2} \widehat{\pi}_\Lambda(a)(\widehat{dR}_{a^{-1}}^* \alpha_X, \widehat{dR}_{a^{-1}}^* \alpha_Y) &= \frac{1}{2\epsilon} [\alpha_X \wedge \alpha_Y(\widehat{Ad}_a E, \widehat{Ad}_a iE) - \alpha_X \wedge \alpha_Y(E, iE)] \\ &= \frac{2}{\epsilon} [e^{4\epsilon\lambda} \text{Im}(\bar{u}v) - \text{Im}(\bar{u}v)] \\ &= \frac{2}{\epsilon} (e^{4\epsilon\lambda} - 1) \text{Im}(\bar{u}v) \\ &= \widehat{\pi}_{B_\epsilon}(a)(\widehat{dR}_{a^{-1}}^* \alpha_X, \widehat{dR}_{a^{-1}}^* \alpha_Y) \end{aligned}$$

□

We then have the following.

COROLLARY 5.25. $\widehat{\pi}_{B_\epsilon}(a) = \frac{1}{2\epsilon} (1 - e^{-4\epsilon\lambda}) \widehat{dL}_a(E \wedge iE)$

Proof.

$$\begin{aligned} \widehat{\pi}_{B_\epsilon}(a) &= \frac{1}{2\epsilon} [\widehat{dL}_a(E \wedge iE) - \widehat{dR}_a(E \wedge iE)] \\ &= \frac{1}{2\epsilon} \widehat{dL}_a[E \wedge iE - \widehat{Ad}_{a^{-1}}(E \wedge iE)] \\ &= \frac{1}{2\epsilon} \widehat{dL}_a[E \wedge iE - e^{-4\epsilon\lambda}(E \wedge iE)] \\ &= \frac{1}{2\epsilon} (1 - e^{-4\epsilon\lambda}) \widehat{dL}_a(E \wedge iE). \end{aligned}$$

□

The diffeomorphism $\varphi_\epsilon : K/T \rightarrow \Sigma_\lambda^\epsilon$ gives us $(d\varphi_\epsilon)_e : \mathfrak{k}/\mathfrak{t} \rightarrow T_a \Sigma_\lambda^\epsilon \subset T_a B_\epsilon$ defined by $(d\varphi_\epsilon)_e(\xi) = \frac{1}{\epsilon} \widehat{dL}_a \rho_b(\widehat{Ad}_{a^{-1}} \xi)$. Now let $X = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $Y = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ as in §5.1, then

$$(d\varphi_\epsilon)_e(X) = \frac{1}{2\epsilon} (e^{-2\epsilon\lambda} - e^{2\epsilon\lambda}) \widehat{dL}_a E \quad \text{and} \quad (d\varphi_\epsilon)_e(Y) = \frac{1}{2\epsilon} (e^{-2\epsilon\lambda} - e^{2\epsilon\lambda}) \widehat{dL}_a iE.$$

It then follows that

$$\text{LEMMA 5.26. } (d\varphi_\epsilon)_e(\pi_{\lambda,\epsilon}(e)) = \widehat{\pi}_{B_\epsilon}(a)$$

Proof.

$$\begin{aligned} (d\varphi_\epsilon)_e(\pi_{\lambda,\epsilon}(e)) &= (d\varphi_\epsilon)_e(\epsilon(\pi_\infty(e) - \tau(\epsilon\lambda)\pi_0(e))) \\ &= \epsilon(d\varphi_\epsilon)_e(\pi_\infty(e)) - \epsilon\tau(\epsilon\lambda)(d\varphi_\epsilon)_e(\pi_0(e)) \\ &= 0 - 2\epsilon\tau(\epsilon\lambda)(d\varphi_\epsilon)_e(X \wedge Y) \\ &= -\frac{1}{2\epsilon} \tau(\epsilon\lambda) (e^{-2\epsilon\lambda} - e^{2\epsilon\lambda})^2 \widehat{dL}_a (E \wedge iE) \\ &= -\frac{1}{2\epsilon} (e^{-4\epsilon\lambda} - 1) \widehat{dL}_a (E \wedge iE) \\ &= \widehat{\pi}_{B_\epsilon}(a). \end{aligned}$$

□

This completes the proof of Proposition 5.23. □

We can next look at the product $(K/T)^n$. We give $(K/T)^n$ the product Poisson structure $\pi_{\lambda,\epsilon} = \pi_{\lambda_1,\epsilon} + \cdots + \pi_{\lambda_n,\epsilon}$. Define the map

$$\widetilde{\Phi}_\epsilon : (K/T)^n \rightarrow \Sigma_{\lambda_1}^\epsilon \times \cdots \times \Sigma_{\lambda_n}^\epsilon$$

given by

$$\widetilde{\Phi}_\epsilon(k_1, \dots, k_n) = (\varphi_1^\epsilon(k_1), \dots, \varphi_n^\epsilon(k_n)) = (\widehat{D}_\epsilon^\ell(k_1)(a_{\lambda_1}), \dots, \widehat{D}_\epsilon^\ell(k_n)(a_{\lambda_n}))$$

where $\varphi_i^\epsilon(k_i) = \widehat{D}_\epsilon^\ell(k_i)(a_{\lambda_i})$ and $a_{\lambda_i} = \exp^\epsilon(\lambda_i H) \in B_\epsilon$. We note the map $\widetilde{\Phi}_\epsilon : (K/T)^n \rightarrow \Sigma_{\lambda_1}^\epsilon \times \cdots \times \Sigma_{\lambda_n}^\epsilon$ is a symplectomorphism.

We leave the proof of the following lemma to the reader.

LEMMA 5.27. *The action of K on $(K/T)^n$ given by*

$$k \circ (k_1, \dots, k_n) = (kk_1, \rho_K(k\varphi_1^\epsilon(k_1))k_2, \dots, \rho_K(k\varphi_1^\epsilon(k_1)) \cdots \varphi_{n-1}^\epsilon(k_{n-1}))k_n$$

is the pull back under $\widetilde{\Phi}_\epsilon$ of the ϵ -dressing action on $\Sigma_{\lambda_1}^\epsilon \times \cdots \times \Sigma_{\lambda_n}^\epsilon \subset B_\epsilon^n$.

The momentum map for the action of $(K, \epsilon\pi_K)$ on $((K/T)^n, \pi_{\lambda,\epsilon})$ is

$$\widetilde{\Psi}_\epsilon : (K/T)^n \rightarrow B_\epsilon$$

where

$$\widetilde{\Psi}_\epsilon(k_1, \dots, k_n) = \varphi_1^\epsilon(k_1) * \cdots * \varphi_{n-1}^\epsilon(k_{n-1}).$$

5.3. The ϵ -dressing orbits are small spheres in hyperbolic 3-space. Let b be the Killing form on \mathfrak{g} divided by 8. We have normalized b so that the induced Riemannian metric $(,)$ on G/K has constant curvature -1. We let $b_\epsilon = f_\epsilon^* b$, hence b_ϵ is the Killing form on \mathfrak{g}_ϵ . Then $(,)_\epsilon = F_\epsilon^*(,)$ is the induced Riemannian metric on G_ϵ/K and G_ϵ/K has constant curvature -1 (since F_ϵ is an isometry). We will call $(,)_\epsilon$ the hyperbolic metric on G_ϵ/K .

The map $\zeta : B_\epsilon \rightarrow G_\epsilon/K$ given by $\zeta(b) = b * K$ is a diffeomorphism that intertwines the ϵ -dressing orbits of K on B_ϵ with the natural K action on G_ϵ/K given by left multiplication (using the multiplication in G_ϵ). We abbreviate the identity coset K in G_ϵ/K to x_0 and use the same letter for the corresponding point in G/K . We have

LEMMA 5.28. *The image of the ϵ -dressing orbit Σ_λ^ξ under ζ is the sphere around x_0 of radius $\epsilon\lambda$.*

Proof. Let d_ϵ be the Riemannian distance function on G_ϵ/K and d the Riemannian distance function on G/K . We have

$$\begin{aligned} d_\epsilon(x_0, \exp_{x_0}^\epsilon \lambda H) &= d(x_0, F_\epsilon \exp_{x_0}^\epsilon \lambda H) \\ &= d(x_0, \exp_{x_0} f_\epsilon(\lambda H)) \\ &= d(x_0, \exp_{x_0} \epsilon \lambda H) \\ &= \epsilon \lambda \end{aligned}$$

□

5.4. The family of symplectic quotients. In this section we will continue to use the notation of §5.2. Let $p : E = (K/T)^n \times I \rightarrow I$ be a projection. Here we define $I = [0, 1]$. We let $T^{vert}(E) \subset T(E)$ be the tangent space to the fibers of p . Hence $\bigwedge^2 T^{vert}(E)$ is a subbundle of $\bigwedge^2 T(E)$. We define a Poisson bivector π on E by $\pi(u, \epsilon) = \pi_{\lambda, \epsilon}|_u$. π is a section of $\bigwedge^2 T^{vert}(E)$.

Let $S \subset B^n \times I$ be defined by $S = \{(b, \epsilon) | b \in \Sigma_\lambda^\xi\}$. We let $(K, \epsilon\pi_K)$ act on $(K/T)^n \times I$ by $k \cdot (u, \epsilon) = (k \circ u, \epsilon)$, where \circ is the action given in Lemma 5.27, and act on S by $k \cdot (b, \epsilon) = (\widehat{D}_\epsilon^k(k)(b), \epsilon)$. We then define the map $\Phi : E \rightarrow S$ by $\Phi(u, \epsilon) = (\widehat{\Phi}_\epsilon(u), \epsilon)$ which is a K -equivariant diffeomorphism. We also define $\Psi : E \rightarrow B$ by $\Psi(u, \epsilon) = \widehat{\Psi}_\epsilon(u)$.

REMARK 5.29. $\Psi|_{p^{-1}(\epsilon)}$ is the momentum map for the Poisson action of $(K, \epsilon\pi_K)$ on $(K/T)^n \times \{\epsilon\}$.

We need some notation. Suppose $n \geq m$, $F : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^m$ is a smooth map, and $0 \in \mathbb{R}^m$ is a regular value of F . Let $M = F^{-1}(0)$. Write $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ with $x \in \mathbb{R}^n$, $t \in \mathbb{R}$. Let $p : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ be the projection onto the t -line. The next lemma is taken from [Sa].

LEMMA 5.30. *Let $(x, t) \in M$. Suppose $\frac{\partial F}{\partial x}|_{(x, t)}$ has maximal rank m . Then $dp|_{(x, t)} : T_{(x, t)}(M) \rightarrow T_t(\mathbb{R})$ is onto.*

Proof. It suffices to construct a tangent vector $v \in T_{(x, t)}(\mathbb{R}^{n+1})$ satisfying

- (i) $v \in \ker dF|_{(x, t)}$
- (ii) $v = \sum_{i=1}^n c_i \frac{\partial}{\partial x_i} + \frac{\partial}{\partial t}$

Put $c = (c_1, \dots, c_n)$ and write the Jacobian matrix $dF|_{(x,t)}$ as (A, b) where A is the m by n matrix given by $A = \frac{\partial F}{\partial x}|_{(x,t)}$ and b is the column vector of length m given by $b = \frac{\partial F}{\partial t}|_{(x,t)}$. We are done if we can solve

$$Ac + b = 0.$$

But since $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is onto we can solve this equation. \square

REMARK 5.31. *We need to generalize to the case in which \mathbb{R}^{n+1} is replaced by the closed half-space $\bar{H} = \{(x, t) : x \in \mathbb{R}^n, t \geq 0\}$ and the t -line by the closed half-line. Given $(x, 0) \in \partial M$ we wish to find $v = \sum_{i=1}^n c_i \frac{\partial}{\partial x_i} + \frac{\partial}{\partial t}$ with $dF|_{(x,0)}(v) = 0$ (so v is in the tangent half-space to M at $(x, 0) \in \partial M$). The argument is analogous to that of the lemma and is left to the reader.*

COROLLARY 5.32. *Suppose M is compact and for all (x, t) and further that $\frac{\partial F}{\partial x}|_{(x,t)}$ has maximal rank for all $(x, t) \in M$. Then $p : M \rightarrow \mathbb{R}$ is a trivial fiber bundle.*

Proof. p is proper since M is compact. This is the Ehresmann fibration theorem [BJ, 8.12]. \square

REMARK 5.33. *We leave to the reader the task of extending the corollary to the case where the t -line is replaced by the closed t half-line.*

We now return to our map $\Psi : (K/T)^n \times I \rightarrow B$. We have

$$\Psi(u, \epsilon) = \varphi_1^\epsilon(u_1) * \dots * \varphi_n^\epsilon(u_1).$$

Let $\epsilon > 0$. We apply Lemma 2.5 to deduce that $1 \in B$ is a regular value for $u \rightarrow \Psi(u, \epsilon)$ (recall we have assumed r is not on a wall of D_n). Now let $\epsilon = 0$. It is immediate (see [KM2]) that $0 \in \mathbb{R}^3$ is a regular value of $u \rightarrow \Psi(u, \epsilon)$ (again because r is not on a wall of D_n). We obtain

LEMMA 5.34. *$p : \Psi^{-1}(1) \rightarrow I$ is a trivial fiber bundle.*

Now let $\mathcal{M} = \Psi^{-1}(1)/K$. We note that p factors through the free action of K on $\Psi^{-1}(1)$ and we obtain a fiber bundle $\bar{p} : \mathcal{M} \rightarrow I$. This gives the required family of symplectic quotients.

PROPOSITION 5.35. *$\bar{p} : \mathcal{M} \rightarrow I$ is a trivial fiber bundle.*

REMARK 5.36. *$\bar{p}^{-1}(0) = M_r(\mathbb{E}^3)$ and $\bar{p}^{-1}(1) = M_r(\mathbb{H}^3)$ and we may identify \mathcal{M} with the product $M_r(\mathbb{E}^3) \times I$.*

We now give a description of the symplectic form along the fibers of \bar{p} . Recall that if $\pi : E \rightarrow B$ is a smooth fiber bundle then the relative forms on E are the elements of the quotient of $A^*(E)$ by the ideal generated by elements of positive degree in $\pi^*A^*(B)$. Note the restriction of a relative form to a fiber of π is well-defined and the relative forms are a differential graded-commutative algebra with product and differential induced by those of $A^*(E)$.

Lu's one parameter family of forms ω_ϵ of §5.1 induces a relative 2-form ω_ϵ on $K/T \times I$ which is relatively closed. By taking sums we obtained a relative 2-form $\bar{\omega}_\epsilon$ on $(K/T)^n \times I$ and by restriction and projection a relative 2-form $\bar{\omega}_\epsilon$ on \mathcal{M} . Clearly $\bar{\omega}_\epsilon$ is relatively closed and induces the symplectic form along the fibers of $\bar{p} : \mathcal{M} \rightarrow I$.

We let $[\bar{\omega}_\epsilon]$ be the class in $H^2(p^{-1}(\epsilon))$ determined by $\bar{\omega}_\epsilon$.

5.5. $[\bar{\omega}_\epsilon]$ is constant and Moser’s Theorem. To complete our proof we need to review the i -th cohomology bundle associated to a smooth fiber bundle $\pi : E \rightarrow B$. The total space \mathcal{H}_E^i of the i -th cohomology bundle is given by $\mathcal{H}_E^i = \{(b, z) : b \in B, z \in H^i(p^{-1}(b))\}$. We note that a trivialization of $E|_U$ induces an isomorphism between $\mathcal{H}_{E|_U}^i$ and $\mathcal{H}_{U \times F}^i$. But $\mathcal{H}_{U \times F}^i = \{(x, z) : x \in U, z \in H^i(F)\}$, whence $\mathcal{H}_{U \times F}^i = U \times H^i(F)$. It is then clear that \mathcal{H}_E^i is a vector bundle over B with typical fiber $H^i(F)$. We next observe that the action of the transition functions of E on $H^i(F)$ induce the transition functions of \mathcal{H}_E^i . Hence if we trivialize E relative to a covering $\mathcal{U} = \{U_i : i \in I\}$ such that all pairwise intersections are contractible then the corresponding transition functions of \mathcal{H}_E^i are constant. Hence \mathcal{H}_E^i admits a flat connection called the Gauss-Manin connection. We observe that a cross-section of \mathcal{H}_E^i is parallel for the Gauss-Manin connection if when expressed locally as an element of $\mathcal{H}_{U \times F}^i$ as above it corresponds to a constant map from U to $H^i(F)$.

REMARK 5.37. *If τ is a relative i -form on E which is relatively closed then it gives rise to a cross-section $[\tau]$ of \mathcal{H}_E^i such that $[\tau](b)$ is the de Rham cohomology class of $\tau(b)|_{\pi^{-1}(b)}$.*

We now consider the relative 2-form $\bar{\omega}_\epsilon$ on \mathcal{M} . The form $\bar{\omega}_\epsilon$ is obtained from the corresponding form $\tilde{\omega}_\epsilon$ on $(K/T)^n \times I$ by first pulling $\tilde{\omega}_\epsilon$ back to $\Psi^{-1}(1)$ then using the invariance of $\tilde{\omega}_\epsilon$ under K to descend $\tilde{\omega}_\epsilon$ to $\bar{\omega}_\epsilon$. We observe that $[\bar{\omega}_\epsilon]$ (reps. $[\tilde{\omega}_\epsilon]$) is a smooth section of $\mathcal{H}_{(K/T)^n \times I}^2$ (resp. $\mathcal{H}_{\mathcal{M}}^2$).

We obtain a diagram of second cohomology bundles with connection

$$\mathcal{H}_{(K/T)^n \times I}^2 \xrightarrow{i^*} \mathcal{H}_{\Psi^{-1}(1)}^2 \xleftarrow{\pi^*} \mathcal{H}_{\mathcal{M}}^2$$

where $i : \Psi^{-1}(1) \rightarrow (K/T)^n \times I$ is the inclusion and $\pi : \Psi^{-1}(1) \rightarrow \mathcal{M}$ is the quotient map. We have

$$\pi^*[\bar{\omega}_\epsilon] = i^*[\tilde{\omega}_\epsilon].$$

PROPOSITION 5.38. *$[\bar{\omega}_\epsilon]$ is parallel for the Gauss-Manin connection on $\mathcal{H}_{\mathcal{M}}^2$.*

Proof. By Lemma 5.7, $[\tilde{\omega}_\epsilon]$ is parallel for the Gauss-Manin connection on $(K/T)^n \times I$. Hence $i^*[\tilde{\omega}_\epsilon]$ is parallel for the Gauss-Manin connection on $\mathcal{H}_{\Psi^{-1}(1)}^2$. But an elementary spectral sequence argument for the bundle $K \rightarrow \Psi^{-1}(1) \rightarrow \mathcal{M}$ shows that $\pi^* : \mathcal{H}_{\mathcal{M}}^2 \rightarrow \mathcal{H}_{\Psi^{-1}(1)}^2$ is a bundle monomorphism. Hence if $\pi^*[\bar{\omega}_\epsilon]$ is parallel, so is $[\bar{\omega}_\epsilon]$. \square

COROLLARY 5.39. *The cohomology class of $\bar{\omega}_\epsilon$ is constant relative to any trivialization of $p : \mathcal{M} \rightarrow I$.*

We now complete the proof of symplectomorphism by applying a version of Moser’s Theorem [Mo] with $M = M_\tau(\mathbb{E}^3)$. For the benefit of the reader we will state and prove the version of Moser’s Theorem we need here.

THEOREM 5.40. *Suppose ω_ϵ is a smooth one-parameter family of symplectic forms on a compact smooth manifold M . Suppose the cohomology class $[\omega_\epsilon]$ of ω_ϵ in $H^2(M)$ is constant. Then there is a smooth curve ϕ_ϵ in $\text{Diff}(M)$ with $\phi_0 = \text{id}_M$ such that*

$$\omega_\epsilon = \phi_\epsilon^* \omega_0.$$

Proof. Choose a smooth one-parameter family of 1-forms τ_ϵ such that

$$\frac{d\omega_\epsilon}{d\epsilon} = -d\tau_\epsilon.$$

(We may choose τ_ϵ smoothly by first choosing a Riemannian metric then taking τ_ϵ to be the coexact primitive of $\frac{d\omega_\epsilon}{d\epsilon}$ - here we use the compactness of M).

Let ζ_ϵ be the one parameter family of vector fields such that

$$i_{\zeta_\epsilon}\omega_\epsilon = \tau_\epsilon.$$

Now we integrate the time dependent vector field ζ_ϵ to a family Ψ_ϵ of diffeomorphisms (again we use that M is compact). We have

$$\begin{aligned} \frac{d}{d\epsilon}\Psi_\epsilon^*\omega_\epsilon &= \Psi_\epsilon^*\mathcal{L}_{\zeta_\epsilon}\omega_\epsilon + \Psi_\epsilon^*\frac{d\omega_\epsilon}{d\epsilon} \\ &= \Psi_\epsilon^*[di_{\zeta_\epsilon}\omega_\epsilon - d\tau_\epsilon] \\ &= \Psi_\epsilon^*[d\tau_\epsilon - d\tau_\epsilon] \\ &= 0. \end{aligned}$$

Hence $\Psi_\epsilon^*\omega_\epsilon$ is constant so $\Psi_\epsilon^*\omega_\epsilon = \omega_0$ and $\omega_\epsilon = (\Psi_\epsilon^{-1})^*\omega_0$. \square

5.6. The geometric meaning of the family \mathcal{M} of symplectic quotients - shrinking the curvature. We recall that X_κ denotes the complete simply-connected Riemannian manifold of constant curvature κ . Let $r = (r_1, r_2, \dots, r_n) \in (\mathbb{R}_+)^n$ with r not on a wall of D_n . Let $M_r(X_\kappa)$ be the moduli space of n -gon linkages with side-lengths r in the space X_κ . The following theorem is the main result of [Sa].

THEOREM 5.41. *There exists $\alpha > 0$ and an analytically trivial fiber bundle $\pi : \mathcal{E} \rightarrow (-\infty, \alpha)$ such that $\pi^{-1}(\kappa) = M_r(X_\kappa)$.*

Let \mathcal{M} be the family of symplectic quotients just constructed (except we will take $(-\infty, 0]$ as base instead of $[1, 0]$). We then have

THEOREM 5.42. *We have an isomorphism of fiber bundles*

$$\mathcal{E}|_{(-\infty, 0]} \simeq \mathcal{M}.$$

We will need

LEMMA 5.43. *Let $\lambda > 0$. Then we have a canonical isomorphism*

$$M_r(X_\kappa) \simeq M_{\lambda r}(X_{\kappa/\lambda}).$$

Proof. Multiply the Riemannian metric on X_κ by λ . Then the Riemannian distance function is multiplied by λ and the sectional curvature is multiplied by $\frac{1}{\lambda}$. \square

REMARK 5.44. *There is a good way to visualize the above isomorphism by using the embedding of X_κ , $\kappa < 0$, in Minkowski space (as the upper sheet of the hyperboloid*

$x^2 + y^2 + z^2 - t^2 = \frac{-1}{\kappa^2}$) or X_κ , $\kappa > 0$ in \mathbb{R}^4 (as the sphere $x^2 + y^2 + z^2 + t^2 = \frac{1}{\kappa^2}$). The dilation map $v \mapsto \lambda v$ of the ambient vector space maps X_κ to $X_{\kappa/\lambda}$ and multiplies the side-lengths by λ .

Now we can prove the theorem. Let $\bar{p} : \mathcal{M} \rightarrow (-\infty, 0]$ be the family constructed in §5.4. By Lemma 5.28 we see that $\bar{p}^{-1}(\epsilon) \simeq M_{\epsilon r}(X_{-1})$. Thus we are shrinking the side-lengths of the n -gons as $\epsilon \rightarrow 0$. But we have just constructed a canonical isomorphism

$$M_{\epsilon r}(X_{-1}) \simeq M_r(X_{-\epsilon}).$$

So we may regard the deformation of §5.4 as keeping the side-lengths fixed and shrinking the curvature to zero.

To give a formal proof we will construct an explicit diffeomorphism

$$\begin{array}{ccc} \mathcal{M} & \longrightarrow & \mathcal{E}|_{(-\infty, 0]} \\ & \searrow & \downarrow \\ & & (-\infty, 0] \end{array}$$

To this end, observe that the map $B \times (-\infty, 0] \rightarrow G/K \times (-\infty, 0]$ given by $(b, \kappa) \mapsto (b * K, \kappa)$ induces a K -equivariant diffeomorphism

$$\begin{array}{ccc} B^n \times (-\infty, 0] & \xrightarrow{F} & (G/K)^n \times (-\infty, 0] \\ & \searrow p & \downarrow \pi \\ & & (-\infty, 0] \end{array}$$

given by $F((b_1, \dots, b_n), \kappa) = ((K, b_1 * K, \dots, b_1 * \dots * b_{n-1} * K), \kappa)$.

We give $\pi^{-1}(\kappa)$ the Riemannian metric $|\kappa|(\cdot, \cdot)_\kappa$. Let $\tilde{\mathcal{E}}' \subset (G/K)^n \times (-\infty, 0]$ be defined by

$$\tilde{\mathcal{E}}' = \{(y_1, \dots, y_n, \kappa) : y_1 = x_0, d_\kappa(y_i, y_{i+1}) = r_i, 1 \leq i \leq n\}$$

Here d_κ is the distance function on $\pi^{-1}(\kappa)$ associated to the Riemannian metric $|\kappa|(\cdot, \cdot)_\kappa$. Let $\Sigma_r \times (-\infty, 0]$ be the dressing orbit through $(e^{r_1 H}, \dots, e^{r_n H})$ for the K -dressing action of K on $p^{-1}(\kappa)$. We let $\tilde{\mathcal{M}} \subset \Sigma_r \times (-\infty, 0]$ be the subset $\tilde{\mathcal{M}} = \{(b_1, \dots, b_n, \kappa) : b_1 * b_2 * \dots * b_n = 1\}$. Then F carries $\tilde{\mathcal{M}}$ diffeomorphically onto $\tilde{\mathcal{E}}'$ and induces the required diffeomorphism $\mathcal{M} \rightarrow \mathcal{E}|_{(-\infty, 0]}$. \square

REMARK 5.45. The relative 2-form $\bar{\omega}_\epsilon$ is a symplectic form along the fibers of \bar{p} . Thus we have made the restriction of the family of $[Sa]$ to $(-\infty, 0]$ into a family of symplectic manifolds. Can ω_ϵ be extended to $(-\infty, \alpha)$ for some $\alpha > 0$?

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