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LINES ON NON-DEGENERATE SURFACES*

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Abstract. On an affine variety X defined by homogeneous polynomials, every line in the tangent cone of X is a subvariety of X. However there are many other germs of analytic varieties which are not of cone type but contain "lines" passing through the origin. In this paper, we give a method to determine the existence and the "number" of such lines on non-degenerate surface singualrities.

1. Introduction. Let (X, O) be a germ of analytic varieties embedded in $(\mathbb{C}^n, O)$ with a singularity at O. By abuse of language, we say that L is *a line* in (X, O) if (L, O) is a smooth curve germ in (X, O) and $L \setminus \{0\}$ is contained in the regular part of X.

In [3, 5], lines on hypersurfaces with simple singularities are classified by using the classification machinery. All the hypersurfaces of dimension 2 and 3 with simple or simple elliptic singularities passing through x-axis are equivalent to (under the coordinate transformation preserving the x-axis) some surfaces defined by explicit equations. It turns out that the A, D, E singularities split in this classification. This says that different smooth curves on the same surface might have different properties.

Let $\pi : \tilde{X} \to (X, O)$ be a resolution of a surface (X, O) with an isolated singularity at the origin O and let $\{E_1, \dots, E_r\}$ be the exceptional divisors of π. For an exceptional divisor E_i, let $\mathcal{L}_{E_i}$ denote the set of lines on $(X, 0)$ whose strict transform intersect E_i transversally. It is known that $\mathcal{L}_{E_i}$ is non-empty if and only if there exist a function germ h in the maximal ideal $\mathfrak{m}$ such that the multiplicity of $\pi^* h$ along E_i is one and conversely any line in X is contained in some $\mathcal{L}_{E_i}$ ([1, 2]). We call E_i a *normally smooth divisor* if $\mathcal{L}_{E_i} \neq \emptyset$. Geometrically this implies that $d\pi(v) \neq 0$ for any tangent vector $v \in T_P\tilde{X}$ as long as $P \in E_i \setminus \bigcup_{j \neq i} E_j$ and v is not tangent to E_i. If E_i is normally smooth, any germ of a curve intersecting $E_i \setminus \bigcup_{j \neq i} E_j$ transversely defines a line in X. Any two lines in the same $\mathcal{L}_{E_i}$ can be connected by an analytic family of lines in (X, O).

For a given resolution $\pi : \tilde{X} \to X$, we consider the integer $\rho(\pi) := \sharp\{E_i; \mathcal{L}_{E_i} \neq \emptyset\}$. This number depends on the resolution. Put $\rho(X, O)$ to be the minimal value of $\rho(\pi)$. Obviously $\rho(\pi) = \rho(X, O)$ if $\pi : \tilde{X} \to X$ is a minimal resolution. We call $\rho(\pi)$ *the line index of the resolution* $\pi : \tilde{X} \to X$ and we call $\rho(X, O)$ *the line index* of (X, O).

In this paper, we study $\rho(\pi)$ where π is a toric resolution of a non-degenerate surface singularity. Let $(X, 0) \subset (\mathbb{C}^3, 0)$ be a surface defined by $f(z_1, z_2, z_3) = 0$ with isolated singularity at the origin. We assume that f is non-degenerate in the sense of the Newton boundary ([7]). Let Σ^* be a regular simplicial cone subdivision of the dual Newton diagram $\Gamma^*(f)$ and let $\pi : X_{\Sigma^*} \to (X, 0)$ be the associated toric resolution. We denote $\rho(\pi)$ by $\rho(\Sigma^*)$ for simplicity. To each vertex $P = {}^t(p_1, p_2, p_3)$ of Σ^*, there corresponds an exceptional divisor $E(P)$ of π, which may have several components. The multiplicity of $\pi^* z_i$ along $E(P)$ is equal to p_i ([9]). Thus by the result of Gonzalez-Sprinberg and Lejeune-Jalabert ([1]), $E(P)$ is normally smooth if

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and only if $\min(p_1, p_2, p_3) = 1$. We observe that $\rho(\Sigma^*)$ is independent of the choice of Σ^* under certain conditions (see Proposition 6). This allows us to use the canonical toric resolution to determine $\rho(\Sigma^*)$. Note that a toric resolution is not necessarily minimal. So, in general, $\rho(\Sigma^*)$ may be bigger than $\rho(X, O)$ (see Example 28). However to have the equality $\rho(\Sigma^*) = \rho(X, O)$, it is enough that $\pi : X_{\Sigma^*} \to X$ is line-equivalent to the minimal resolution (see § 2 for the definition). The purpose of this paper is to give a method to compute $\rho(\Sigma^*)$.

2. Line-admissible blowing-ups. Let (X, O) be a germ of a surface with an isolated singularity at O. Suppose that we have a good resolution $\pi_1 : X_1 \to X$ and let $E_1, \dots, E_r$ be the exceptional divisors of π_1. Take a divisor E_{i_0} and a point Q on E_{i_0} and let $\pi_Q : \tilde{X}_1 \to X_1$ be the blowing-up at Q and let E_Q be the exceptional divisor of π_Q. The following statements are obvious.

PROPOSITION 1. *Take a function $h \in \mathfrak{m}$ and let m_i be the multiplicity of $\pi_1^* h$ along E_i. Then the multiplicity m_Q of the pull-back $\pi_Q^*(\pi_1^* h)$ along E_Q is the sum of m_i for all i such that $Q \in E_i$. In particular, $m_Q \geq 1$, and $m_Q = 1$ if and only if $m_{i_0} = 1$ and $Q \in E_{i_0} \setminus \bigcup_{i \neq i_0} E_i$.*

COROLLARY 2. *Under the situation of Proposition 1, E_Q is a normally smooth divisor of the composition $\pi_1 \circ \pi_Q : \tilde{X}_1 \to X$ if and only if E_{i_0} is a normally smooth divisor of $\pi_1 : X_1 \to X$ and Q is contained in $E_{i_0} \setminus \cup_{j \neq i_0} E_j$.*

We call $\pi_Q : \tilde{X}_1 \to X_1$ a *line-admissible* blowing-up if either the center Q is at the intersection of two exceptional divisor or the supporting divisor is not normally smooth. Suppose that we have another good resolution $\pi_2 : X_2 \to X$. We say that $\pi_2 : X_2 \to X$ is *line-equivalent* to $\pi_1 : X_1 \to X$ if there exist a finite chain of resolutions $\pi_i' : Y_i \to X, i = 1, \dots, s$ such that (1) $Y_1 = X_1$ and $\pi_1' = \pi_1$ and $Y_s = X_2$ and $\pi_s' = \pi_2$ and (2) any consecutive resolutions factor by either $\sigma_i : Y_i \to Y_{i+1}$ or $\sigma_i' : Y_{i+1} \to Y_i$, where σ_i and σ_i' are line-admissible blowing-ups.

An immediate consequence of the definition and Corollary 2 is:

COROLLARY 3. *Assume that $\pi_i : X_1 \to X, i = 1, 2$ are line-equivalent. Then $\rho(\pi_1) = \rho(\pi_2)$.*

3. Toric resolution and the computation of $\rho(\Sigma^*)$.

3.1. Non-degenerate surfaces. We begin with recalling the toric resolutions of surface singularities since this also helps us to fix some notations. We use the notations of [9]. Let (X, O) be the germ of a surface in $(\mathbb{C}^3, O)$ defined by a function $f : (\mathbb{C}^3, O) \to (\mathbb{C}, O)$. Hereafter we always assume that X has an isolated singularity at O. Let $\sum_\nu a_\nu z^\nu$ be the Taylor expansion of f. The *Newton polyhedron* $\Gamma_+(f)$ is by definition the convex hull of $\bigcup_{\{\nu; a_\nu \neq 0\}} \{\nu + \mathbb{R}^3\}$. The *Newton boundary* $\Gamma(f)$ is by definition the union of the compact faces of $\Gamma_+(f)$.

Let $N := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^3, \mathbb{Z})$ be the set of covectors. We identify N with $\mathbb{Z}^3$ and we denote the elements of N by column vectors. Let N_+ be the set of covectors $P = {}^t(p_1, p_2, p_3) \in N$ with $p_i \geq 0, i = 1, 2, 3$. Put $E_1 := {}^t(1, 0, 0), E_2 := {}^t(0, 1, 0)$ and $E_3 := {}^t(0, 0, 1)$. P is called *strictly positive covector* if $p_j > 0$ for all j. We denote the minimal value of the linear function P on $\Gamma_+(f)$ by $d(P; f)$. Put $\Delta(P; f) = \{z \in \Gamma_+(f) \mid P(z) = d(P; f)\}$. The *face function* of f with respect to P is by definition $f_P(z) = f_{\Delta(P;f)} := \sum_{\nu \in \Delta(P;f)} a_\nu z^\nu$. Two covectors $P, P' \in N_+$ are equivalent if and only if $\Delta(P; f) = \Delta(P'; f)$. The *dual Newton diagram* $\Gamma^*(f)$ of X is a conical polyhedral subdivision of N_+ given by the above equivalent classes.

A surface X is called *non-degenerate* (with respect to the local coordinate z) if for any strictly positive covector $P \in N_+$, $X^*(P) := \{z \in \mathbb{C}^{*3} \mid f_P(z) = 0\}$ is a reduced non-singular surface in the complex torus $\mathbb{C}^{*3}$. The notion of non-degeneracy can be extended to complete intersection varieties (cf. [6, 9]).

3.2. Canonical subdivisions. We assume that X is defined by $f(z_1, z_2, z_3) = 0$ and f is non-degenerate. Let $\Gamma^*(f)_2^+$ be the union of the two-dimensional cones $\mathrm{Cone}(P, Q)$ of $\Gamma^*(f)$ such that the interior points are strictly positive. Let Σ^* be a regular simplicial subdivision of the dual Newton diagram $\Gamma^*(f)$ and let $\pi : X_{\Sigma^*} \to X$ be the associated toric modification. Let $\mathcal{V}(\Sigma^*)$ be the set of strictly positive vertices P's of Σ^* such that $\dim \Delta(P; f) \geq 1$. The exceptional divisors correspond bijectively to $\mathcal{V}(\Sigma^*)$ and for each $P \in \mathcal{V}(\Sigma^*)$ we denote the corresponding divisor by $E(P)$. Note that $E(P)$ need not to be irreducible but it is a disjoint union of rational spheres if $\dim \Delta(P; f) = 1$. The number of connected components is given by $r(P) + 1$, where $r(P)$ is the number of integral points on the interior of $\Delta(P; f)$ ([9, III§6]). The structure of this resolution $\pi : X_{\Sigma^*} \to X$ depends only on the restriction of Σ^* to $\Gamma^*(f)_2^+$. This follows from the following observation:

PROPOSITION 4. *Assume that Σ_1^* is a regular subdivision of Σ^* such that $\mathcal{V}(\Sigma_1^*) = \mathcal{V}(\Sigma^*)$. Then the canonical morphism $\psi : X_{\Sigma_1^*} \to X_{\Sigma^*}$, which is induced by the morphism of the ambient toric varieties, is an isomorphism.*

For any two dimensional cone $\sigma = \mathrm{Cone}\,(P, Q) \in \Gamma^*(f)$, there exists a canonical regular subdivision of σ which is described as follows. Denote by $d := \det(P, Q)$ the greatest common divisor of the absolute values of the 2×2 minors of the matrix (P, Q). If $d > 1$, there exists a unique integer $d_1, 1 \leq d_1 < d$ such that $Q_1 := (P + d_1 Q)/d$ is an integral covector. If $d_1 > 1$, repeat the process for $\mathrm{Cone}\,(P, Q_1)$, until a regular subdivision of $\mathrm{Cone}\,(P, Q)$ is obtained. Let $Q_1, \ldots, Q_k$ be the covectors obtained in this way. Let $d/d_1 = [m_1, \ldots, m_\ell]$ be the continuous fraction expansion. Then $\ell = k$ and the self-intersection number of each component of $E(Q_i)$ is $-m_i$ (cf. [9, III]). Note that $\Delta(Q_i; f) = \Delta(P; f) \cap \Delta(Q; f)$. This implies $r(Q_i)$ is independent of $i = 1, \ldots, k$ and we denote this number by $r(P, Q)$. Recall that the continuous fraction is defined inductively by $[m_1] = m_1$ and $[m_1, m_2, \ldots, m_k] = m_1 - 1/[m_2, \ldots, m_k]$.

A regular simplicial cone subdivision of $\Gamma^*(f)$ is called a *canonical regular subdivision* if its restriction to each cone σ in $\Gamma^*(f)_2^+$ is canonical in the above sense, and we denote it by Σ^*_{can}. The associated toric resolution is called the *canonical toric resolution* of X.

Let $Q = {}^t(q_1, q_2, q_3)$ and $P = {}^t(p_1, p_2, p_3)$. Put $Q_0 = Q$ and $Q_{k+1} = P$ and let $Q_j := {}^t(q_{1,j}, q_{2,j}, q_{3,j}), j = 0, \ldots, k+1$. The canonical subdivision enjoys the following property:

LEMMA 5. *Assume that* $\mathrm{Cone}\,(P, Q) \in \Gamma^*(f)_2^+$. *Fix an $\ell = 1, 2, 3$.*

1) *If $q_\ell \leq 1$, then $\{q_{\ell,j}\}_{j=0}^{k+1}$ is monotone increasing in j i.e. $q_{\ell,j+1} \geq q_{\ell,j}$ for $0 \leq j \leq k$.*
2) *If $q_\ell \geq 2$, then either $\{q_{\ell,j}\}$ is monotone increasing or monotone decreasing in j or there exists a j_0 $(1 \leq j_0 \leq k)$ such that $q_{\ell,j_0} \geq 1$ and*

$$p_\ell = q_{\ell,k+1} \geq \cdots \geq q_{\ell,j_0+1} \geq q_{\ell,j_0} \leq q_{\ell,j_0-1} \leq \cdots \leq q_{\ell,0} = q_\ell.$$

Proof. We prove the assertion 2). If the assertion does not hold, there exists an index $j, 1 \leq j \leq k$ such that $q_{\ell,j-1} \leq q_{\ell,j} > q_{\ell,j+1}$. This implies that the self intersection number of each component of $E(Q_j)$ is $-(q_{\ell,j-1}+q_{\ell,j+1})/q_{\ell,j} > -2$, which is a contradiction (cf. [9, II(2.3) and III(6.3)]). The assertion 1) follows from 2) as $Q_j, j=1,\dots,k$ are strictly positive. □

Let Σ^* be any regular simplicial cone subdivision of $\Gamma^*(f)$ and let $\pi : \tilde{X} \to X$ be the corresponding toric modification. We denote the line index of π by $\rho(\Sigma^*)$. Take a two dimensional cone $\sigma = \mathrm{Cone}(P,Q) \in \Gamma^*(f)_2^+$. Let $Q_0 := Q, Q_1, \dots, Q_k, Q_{k+1} := P$ be the canonical subdivision of σ and let $S_0 := Q, S_1, \dots, S_\eta, S_{\eta+1} := P$ be the vertices of Σ^* on this cone. By [9, II(2.3)], $\{Q_0,\dots,Q_{k+1}\} \subset \{S_0,\dots,S_{\eta+1}\}$. We consider the condition:

($\sharp$): Σ^* has no vertex in the interior of $\mathrm{Cone}(Q,Q_1)$.

We say that Σ^* satisfies the ($\sharp$)-condition if it satisfies ($\sharp$)-condition for any Cone (P,Q) in $\Gamma^*(f)_2^+$ such that Q is not strictly positive. The inclusion $\mathcal{V}(\Sigma^*_{\mathrm{can}}) \subset \mathcal{V}(\Sigma^*)$ implies that the following statements.

THEOREM 6. *There exists a canonical morphism $\phi : X_{\Sigma^*} \to X_{\Sigma^*_{\mathrm{can}}}$. Furthermore ϕ is a composition of line-admissible blowing-ups if Σ^* satisfies ($\sharp$)-condition. In particular, the line index $\rho(\Sigma^*)$ does not depend on the choice of a toric resolution associated with any regular simplicial subdivision satisfying ($\sharp$)-condition and $\rho(\Sigma^*) = \rho(\Sigma^*_{\mathrm{can}})$.*

Proof. Take a two dimensional cone $\sigma = \mathrm{Cone}(P,Q) \in \Gamma^*(f)_2^+$ and assume that P is strictly positive. Let $Q_0 := Q, Q_1, \dots, Q_k, Q_{k+1} := P$ be the canonical subdivision of σ and let $S_0 := Q, S_1, \dots, S_\eta, S_{\eta+1} := P$ be the vertices of Σ^* on this cone. Write $S_i = {}^t(s_{1,j}, s_{2,j}, s_{3,j})$. Assume that $Q_{i_0} = S_\nu$ and $Q_{i_0+1} = S_\mu$ and $\mu - \nu > 1$. Take S_j with $\nu < j < \mu$ and put $\alpha_j = \det(Q_{i_0}, S_j)$ and $\beta_j = \det(S_j, Q_{i_0+1})$. Then α_j and β_j are positive integers and $S_j = \alpha_j Q_{i_0+1} + \beta_j Q_{i_0}$. This implies that $s_{1,j} > s_{1,\nu} + s_{1,\mu}$. Suppose that $s_1^{\max} = \max\{s_{1,j}; \nu < j < \mu\}$ and put $\gamma = \min\{\gamma; s_{1,\gamma} = s_1^{\max}\}$. Then by [9, II(2.3)] the intersection number of (each component of) $E(S_\gamma)$ is $-(s_{1,\gamma-1} + s_{1,\gamma+1})/s_{1,\gamma} > -2$. Then the negativity of the intersection number implies that $s_{1,\gamma-1} + s_{1,\gamma+1} = s_{1,\gamma}$. Thus each component of $E(S_\gamma)$ is a rational sphere of the first kind. This implies also that $S_\gamma = S_{\gamma-1} + S_{\gamma+1}$ and $\det(S_{\gamma-1}, S_{\gamma+1}) = 1$. Put $\mathcal{V}' = \mathcal{V}(\Sigma^*) - \{S_\gamma\}$. Then we can extend $\mathcal{V}'$ to get a regular simplicial subdivision $\Sigma^{*\prime}$ such that its restriction to $\Gamma^*(f)_2^+$ is defined by the vertices $\mathcal{V}'$. Thus we get a toric resolution $\pi' : X_{\Sigma^{*\prime}} \to X$. Changing Σ^* outside of $\Gamma^*(f)_2^+$ if necessary, we may assume by Proposition 4 that Σ^* is a subdivision of $\Sigma^{*\prime}$. Thus we get a canonical morphism $\psi : X_{\Sigma^*} \to X_{\Sigma^{*\prime}}$ which factors π by π'. By the definition, ψ is the composition of blowing-up at $r(S_\gamma) + 1$ intersection points of respective components of $E(S_{\gamma-1})$ and $E(S_{\gamma+1})$ in $X_{\Sigma^{*\prime}}$. Note that ψ is line-admissible unless Q is not strictly positive and $S_\nu = Q_0$ and $S_\mu = Q_1$. This is the situation where ψ is the blowing up at the intersection of $E(Q_1)$ and $E(Q)$. This does not occur if Σ^* satisfies ($\sharp$)-condition. Now the assertion follows by the induction on the cardinality of $\mathcal{V}(\Sigma^*) \setminus \mathcal{V}(\Sigma^*_{\mathrm{can}})$. □

3.3. Computation of $\rho(\Sigma^*_{\mathrm{can}})$. Let $\pi : X_{\Sigma^*} \to X$ be a toric resolution. We assume that Σ^* satisfies the ($\sharp$)-condition. We define $\mathcal{V}_{\mathrm{ns}}(\Sigma^*) := \{P \in \mathcal{V}(\Sigma^*) \mid P \text{ has 1 as a coordinate}\}$. We know that $E(P)$ is a normally smooth divisor if and only if $P \in \mathcal{V}_{\mathrm{ns}}(\Sigma^*)$. Thus for each $\mathrm{Cone}(P,Q) \in \Gamma^*(f)_2^+$, we define $\rho_{PQ} := \#\mathcal{V}_{\mathrm{ns}}(\Sigma^*) \cap \mathrm{Cone}\,(P,Q)^\circ$, where $\mathrm{Cone}\,(P,Q)^\circ$ is the interior of $\mathrm{Cone}\,(P,Q)$. This number is independent of Σ^* by Theorem 6. Recall that $r(P,Q)$ is the number of integral points in the interior of $\Delta(P;f) \cap \Delta(Q;f)$. By the definition we have

(3.1) $\rho(\Sigma^*) = \sharp\{P \in \mathcal{V}_{\rm ns}(\Sigma^*); \dim \Delta(P; f) = 2\} + \sum_{\mathrm{Cone}\,(P,Q)\in\Gamma^*(f)_2^+} (r(P,Q)+1)\rho_{PQ}$

Thus we need only to compute ρ_{PQ} for the calculation of $\rho(\Sigma^*)$. Take a cone $\sigma = \mathrm{Cone}\,(P,Q)$ in $\Gamma^*(f)_2^+$. The following gives a practical method to compute ρ_{PQ}.

THEOREM 7. *Let $P = {}^t(p_1,p_2,p_3)$ be strictly positive and let $Q = {}^t(q_1,q_2,q_3)$ and assume that $d := \det(P,Q) > 1$. Let $Q_i = {}^t(q_{1,i},q_{2,i},q_{3,i}), i = 0,\dots,k+1$ be the vertices defining the canonical subdivision from Q with $Q_0 = Q$ and $Q_{k+1} = P$. Fix an $\ell \in \{1,2,3\}$. Then*

1. *For each $1 \le i \le k$, there exists positive integers $0 < \alpha_i, \beta_i < d$ such that $Q_i = (\beta_i P + \alpha_i Q)/d$. Putting $\alpha_0 = \beta_{k+1} = d$, $\alpha_{k+1} = \beta_0 = 0$, they satisfy the inequality:*

$$\alpha_i > \alpha_{i+1}, \quad \beta_i < \beta_{i+1}, \quad i = 0,\dots,k$$

2. *Let $\mathcal{V}_{\rm ns}^{(\ell)}(P,Q)$ be the set of integral covectors R expressed as $R = (\beta P + \alpha Q)/d$ where α, β are positive integers satisfying*

$$\text{(3.2)} \qquad \begin{cases} \alpha q_\ell + \beta p_\ell = d, \ 0 < \alpha, \beta < d \\ \alpha q_k + \beta p_k \equiv 0 \mod d \quad (k \ne \ell) \end{cases}$$

and let $\mathcal{V}_{\rm ns}^{(\ell)}(P,Q;\Sigma_{\rm can}^*)$ be the set of covectors Q_i, $1 \le i \le k$ such that $q_{\ell,i} = 1$. Then $\mathcal{V}_{\rm ns}^{(\ell)}(P,Q) = \mathcal{V}_{\rm ns}^{(\ell)}(P,Q;\Sigma_{\rm can}^*)$. Note that the inequality $\alpha, \beta < d$ follows automatically from the positivity if both p_ℓ and q_ℓ are positive.

Proof. The first assertion follows by an inductive argument. Write $Q_i = (\beta_i P + \alpha_i Q)/d$ with positive rational numbers α_i, β_i. As $\det(P, Q_i) = \alpha_i$ and $\det(Q_i, Q) = \beta_i$, α_i, β_i are positive integers. By the definition of Q_1, we can write $Q_1 = (P + \alpha_1 Q)/d$ for some $0 < \alpha_1 < d$. The assertion for Q_1 holds and $\det(P, Q_1) = \alpha_1$. Assume that $Q_j = (\beta_j P + \alpha_j Q)/d$ with $0 < \alpha_j < d$. As $\det(P, Q_j) = \alpha_j$ and $\{Q_j, \dots, Q_{k+1}\}$ is the vertices of the canonical subdivision of Cone (P, Q_j), there exists α', $0 < \alpha' < \alpha_i$, such that

$$Q_{j+1} = \frac{1}{\alpha_j}P + \frac{\alpha'}{\alpha_j}Q_j = \frac{1}{\alpha_j}P + \frac{\alpha'}{\alpha_j}\frac{(\beta_j P + \alpha_j Q)}{d} = (\frac{1}{\alpha_j} + \frac{\alpha'\beta_j}{\alpha_j d})P + \frac{\alpha'}{d}Q$$

Thus $\alpha_{j+1} = \alpha' < \alpha_j < d$. The inequality $\beta_{j+1} > \beta_j$ can be proved similarly by using the fact that $\{P, Q_k, \dots, Q_1, Q\}$ is the vertices of the canonical subdivision of the cone Cone (P,Q) from P (cf. [9, II(2.3)]). Now we show the second assertion. The inclusion $\mathcal{V}_{\rm ns}^{(\ell)}(P,Q;\Sigma_{\rm can}^*) \subset \mathcal{V}_{\rm ns}^{(\ell)}(P,Q)$ is obvious. Suppose that $R = (\beta P + \alpha Q)/d \in \mathcal{V}_{\rm ns}^{(\ell)}(P,Q)$ is not contained in $\mathcal{V}_{\rm ns}^{(\ell)}(P,Q;\Sigma_{\rm can}^*)$. Suppose that $R \in \mathrm{Cone}\,(Q_i, Q_{i+1})^\circ$. Then we can write $R = mQ_i + nQ_{i+1}$ for some positive integers m, n. If $i \ge 1$, this gives a contradiction by comparing the ℓ-th coefficient: $1 = mq_{\ell,i} + nq_{\ell,i+1} \ge m + n$. Suppose that $i = 0$. Write $Q_1 = (P + \alpha_1 Q)/d$ as above. Then $R = mQ + (P + \alpha_1 Q)n/d = nP/d + (md + n\alpha_1)Q/d$. Thus we get $\alpha = md + \alpha_1 n \ge d$ which contradicts to the assumption. □

REMARK 8. The computation of $\mathcal{V}_{\rm ns}(P,Q)$ is most difficult for the case $p_\ell, q_\ell > 1$. Assume that $p_\ell, q_\ell > 0$. If we have a solution (α_0, β_0), the other solutions are reduce to the following equation. Put $\alpha = \alpha_0 + \alpha', \beta = \beta_0 + \beta'$. Then

$$\begin{cases} \alpha' q_\ell + \beta' p_\ell = 0 \\ \alpha' q_k + \beta' p_k \equiv 0 \mod d \quad (k \neq \ell) \end{cases} \tag{3.3}$$

Let $\Delta := \Delta(P; f) \cap \Delta(Q; f)$. Let $T = {}^t(t_1, t_2, t_3)$ be a covector in $\mathcal{V}_{\rm ns}^{(\ell)}(P, Q)$ (thus $t_\ell = 1$). Geometrically this implies that $\Delta(T; f) = \Delta$. In particular, $\Gamma_+(f) \subset \{(\nu_1, \nu_2, \nu_3); t_1\nu_1 + t_2\nu_2 + t_3\nu_3 \geq d(T; f)\}$. This gives a practical way to find such a T.

The case $q_\ell = 0$ or 1, the computation is much easier. See Corollary 11.

The canonical subdivision of Cone (P, Q) takes sometimes a lot of computations (see Example 9). Theorem 7 gives us a criterion on the existence or non-existence of normally smooth divisors, without computing the whole subdivision $Q_i, i = 1, \dots, k$.

EXAMPLE 9. For simplicity, we write $x = z_1, y = z_2, z = z_3$. Let us consider $f(x, y, z) = x^m + y^n + x^r y^r + z^2$. We assume that $m, n > 2r$. Put $n = n_1 r + n_0, m = m_1 r + m_0$ with $0 \leq m_0, n_0 \leq r - 1$. Then $\Gamma(f)$ has two compact faces whose covectors are $P = {}^t(2(n-r), 2r, nr)/\delta_1$ and $Q = {}^t(2r, 2(m-r), mr)/\delta_2$ where $\delta_1 = \gcd(2(n-r), 2r, nr)$ and $\delta_2 = \gcd(2r, 2(m-r), mr)$ and the corresponding dual Newton diagram is as in Figure 3.1. Note that $d := \det(P, Q)$ is given by $d = 2(mn - mr - nr)/(\delta_1\delta_2)$. We consider $\mathcal{V}_{\rm ns}^{(1)}(P, Q)$. First we consider the covector $T_0 = {}^t(1, 1, r)$, which is a weight vector of $x^r y^r + z^2$. As $m, n > 2r$, T_0 must be on Cone (P, Q). To proceed the further computation, let us assume that n, m, r are odd and $\gcd(m, r) = \gcd(n, r) = 1$. This implies $\delta_1 = \delta_2 = 1$. By Theorem 7, we have

$$\begin{cases} 2\beta(n-r) + 2\alpha r = d \\ 2\beta r + 2\alpha(m-r) \equiv 0 \mod d \\ \beta nr + \alpha mr \equiv 0 \mod d \end{cases}$$

First we have a canonical solution $(\alpha_0, \beta_0) = (n - 2r, m - 2r)$ which corresponds to the covector $T_0 = {}^t(1, 1, r)$. Thus putting $\alpha = \alpha_0 + a$ and $\beta = \beta_0 + b$, we can reduce the equation as

$$\begin{cases} 2b(n-r) + 2ar = 0 \\ 2br + 2a(m-r) \equiv 0 \mod d \\ bnr + amr \equiv 0 \mod d \end{cases}$$

Taking the positivity of α, β into account, we have the solution

$$\{(\alpha, \beta)\} = \left\{((n-2r) + 2j(n-r), (m-2r) - 2jr)\, ; 0 \leq j \leq \left[\frac{m_1 - 2}{2}\right]\right\}$$

For example, consider the easiest case $m = n$. This has a unique solution $(\alpha, \beta) = (n - 2r, n - 2r)$ and $\mathcal{V}_{\rm ns}^{(1)}(P, Q) = \{B\}$ where $B = {}^t(1, 1, r)$. By symmetry, we have $\mathcal{V}_{\rm ns}^{(2)} = \{B\}$. Note $r(P, Q) = 1$. By writing down the equation described by Theorem 7, we can show $\mathcal{V}_{\rm ns}^{(3)}(P, Q) = \emptyset$.

Now we look at Cone (P, E_1) and Cone (P, E_3). Note that $\det(P, E_1) = r$ and $\det(P, E_3) = 2$. It is easy to see that there are no normally smooth divisor on these cones. Observe that the computation of canonical subdivision of Cone (P, Q) is not so

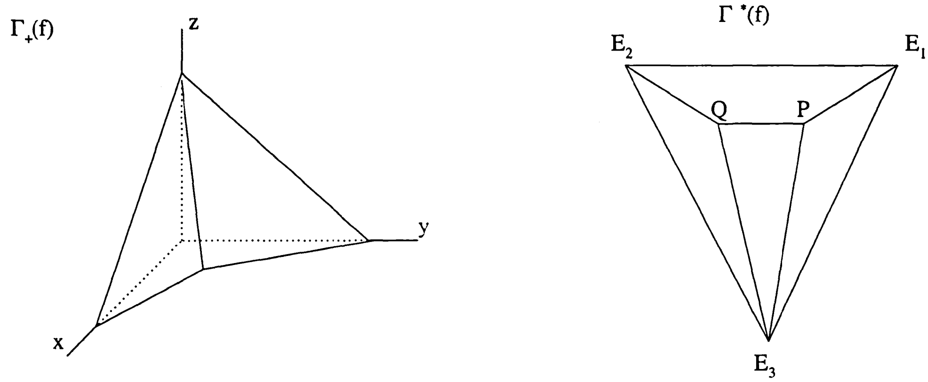

FIG. 3.1. *The Newton polyhedron and the dual Newton diagram*

easy. For example, if $r = 15, n = 37$, then $B = {}^t(1,1,15)$ and first covector B_1 (from Q) is given by $(P + 223Q)/518 = {}^t(13,19,240)$ and $518/223 = [3,2,2,12,2,2,3]$ and it takes some computation to complete the subdivision.

The following lemma describes the covectors corresponding to the non-compact faces.

LEMMA 10. *Assume that* $X = \{f(z_1, z_2, z_3) = 0\}$ *and assume that* f *is non-degenerate and* $\Gamma(f)$ *has at least one compact two dimensional face for simplicity. Suppose that* $z_2 = z_3 = 0$ *is a line in* X. *(So* f *is not convenient.) Then there is a unique covector* $Q = {}^t(q_1, q_2, q_3) \in \mathrm{Vertex}(\Gamma^*(f))$ *such that* $q_1 = 0$. *Furthermore* Q *takes the form* ${}^t(0,1,q_3)$ *or* ${}^t(0,q_2,1)$.

There exists a unique covector $P = {}^t(p_1,p_2,p_3)$ *which corresponds to a compact divisor and adjacent to* Q *in* $\Gamma^*(f)_2^+$. *Then we have* $\det(P,Q) = p_1$.

Proof. As X has an isolated singularity, f must contain a monomial of type $z_1^a z_2$ or $z_1^a z_3$. Suppose that $B := (a,1,0) \in \Gamma(f)$. Let $C = (b,0,c)$ be the vertex of $\Gamma(f) \cap \{z_2 = 0\}$ adjacent to B by an edge. It is clear that the non-compact face Ξ which has $\overline{BC}$ as a face and is unbounded to the direction of the z_1-axis has covector $Q = {}^t(0,c,1)$. One can see that there exists no other non-compact face which is unbounded to the z_1-axis direction and bounded to z_2, z_3-direction. Let Δ be the compact face which has $\overline{BC}$ as a boundary and let $P = {}^t(p_1,p_2,p_3)$ be the corresponding covector. As $\Delta(P;f)$ contains B, C, we need to have $p_1 a + p_2 = b p_1 + c p_3$. Now the last assertion follows from $\det(P,Q) = \gcd(p_1, p_2 - cp_3) = \gcd(p_1, p_1(b-a)) = p_1$. □

The following corollary describes explicitly $\mathcal{V}_{\mathrm{ns}}^{(1)}(P,Q)$ in the case $q_1 = 0$ or 1.

COROLLARY 11. *With the assumptions of Theorem 7, we have the following.*

1) *Assume* $q_1 = 0$. *Then* $\mathcal{V}_{\mathrm{ns}}^{(1)}(P,Q) \neq \emptyset$ *if and only if* $d := \det(P,Q) > 1$ *and* $d = p_1$. *In this cases,* $\mathcal{V}_{\mathrm{ns}}^{(1)}(P,Q) = \{Q_1\}$. *If* $Q \neq E_2, E_3$, *then* $\{y = z = 0\} \subset X$ *and* $d = \det(P,Q) = p_1$.
2) *Assume* $q_1 = 1$. *Then* $\mathcal{V}_{\mathrm{ns}}^{(1)}(P,Q) \neq \emptyset$ *if and only if* $d > p_1$. *In this case, we have* $Q_i = (iP + (d - ip_1)Q)/d$ *for* $i = 1, \ldots, [d/p_1]$ *and* $\mathcal{V}_{\mathrm{ns}}^{(1)}(P,Q) = \{Q_i; i = 1, \ldots, [d/p_1]\}$.

Proof. Assume that $Q' = (\beta P + \alpha Q)/d \in \mathcal{V}_{\mathrm{ns}}^{(1)}(P,Q)$ with $0 < \alpha, \beta < d$.

1) If $q_1 = 0$, we have $\gcd(q_2, q_3) = 1$. As $d = \gcd(p_1q_2, p_1q_3, p_2q_3 - p_3q_2) = \gcd(p_1, p_2q_3 - p_3q_2)$, d divides p_1. Thus $Q' \in \mathcal{V}_{\rm ns}^{(1)}(P,Q)$ if and only if $d = p_1$ and $\beta = 1$. In this case, $Q' = Q_1$ and $\mathcal{V}_{\rm ns}^{(1}(P,Q) = \{Q_1\}$. Assume that $Q \neq E_2, E_3$. By the definition of $\Gamma^*(f)_2^+$, $\Delta(Q; f)$ is a non-compact face with dimension 2. In particular, $\{y = z = 0\} \subset X$. By Lemma 10, we have $d = p_1$.

2) Suppose that $q_1 = 1$. Then $\beta p_1 + \alpha = d$. This implies $d > p_1$. Put $d = rp_1 + d'$ with $0 \leq d' < p_1$ and $r = [d/p_1]$. Then by the above equality, we have $(\alpha, \beta) = (d - jp_1, j)$, $j = 1, \dots, [d/p_1]$. Put $Q'_j := (jP + (d - jp_1)Q)/d$. By the definition, d divides the minors of (P,Q) which are $p_1q_2 - p_2, p_1q_3 - p_3, p_2q_3 - p_3q_2$. Thus $\beta p_j + \alpha q_j = \beta p_j + (d - \beta p_1)q_j \equiv \beta(p_j - p_1q_j) \equiv 0 \mod d$ for $j = 2, 3$. Thus Q'_j is an integral covector for $\beta = 1, \dots, r$. It is clear that $Q'_1 = Q_1$. Assume that $Q'_r = Q_\iota$ for some ι. By the monotonity of the coefficients (Lemma 5), we have $Q_j \in \mathcal{V}_{\rm ns}^{(1)}(P,Q)$ for $j \leq \iota$. Thus $\iota = r$ and $Q'_j = Q_j$ for $j \leq r$. □

REMARK 12. In the case of non-convenient surface with $q_1 = 0$, the divisor $E(Q_1)$ corresponds to the deformations of the line $z_2 = z_3 = 0$. In fact, $E(Q)$ is a non-compact divisor which is the strict transform of z_1-axis and $E(Q)$ intersects transversely with $E(Q_1)$.

For $R \in \mathcal{V}_{\rm ns}^{(\ell)}$, write $R = (\beta P + \alpha Q)/d$. We call β/d *the P-coefficient* of R.

COROLLARY 13. *With the assumptions of Theorem 7, suppose that* $q_1 > 1$. *Let* $\bar{Q} = (\bar{\beta}P + \bar{\alpha}Q)/d \in \mathcal{V}_{\rm ns}^{(\ell)}$ *and* $\underline{Q} = (\underline{\beta}P + \underline{\alpha}Q)/d \in \mathcal{V}_{\rm ns}^{(\ell)}$ *be the covectors with maximal and minimal P-coefficients in* $\mathcal{V}_{\rm ns}^{(\ell)}$. *Then*

$$\rho_{PQ}^{(\ell)} = 1 + |\det(\bar{Q}, \underline{Q})| = 1 + \frac{|\bar{\beta}\underline{\alpha} - \bar{\alpha}\underline{\beta}|}{d} \tag{3.4}$$

Proof. Denote by $d' := |\det(\bar{Q}, \underline{Q})|$. Suppose that $\underline{Q} = Q_i$ and $\bar{Q} = Q_{i+j}$. Then $\mathcal{V}_{\rm ns}^{(\ell)} = \{Q_i, \dots, Q_{i+j}\}$ by Lemma 5 and $\rho_{PQ}^{(\ell)} = j + 1$. By the assumption, we have $Q_{i+1} = (Q_{i+j} + (d' - 1)Q_i)/d'$. As the continuous fraction $d'/(d' - 1)$ is given by $[2, \dots, 2]$ ($(d' - 1)$ copies of 2), we get $j - 1 = d' - 1$ and the assertion follows immediately. □

4. Applications.

4.1. Weighted homogeneous surfaces. In this section we study lines on weighted homogeneous surface singularities, which are classified as follows ([12, 9]):

$X_{\rm I}$: $h_{\rm I} = x^a + y^b + z^c = 0$,
$X_{\rm II}$: $h_{\rm II} = x^a y + y^b + z^c = 0$,
$X_{\rm III}$: $h_{\rm III} = x^a y + xy^b + z^c = 0$,
$X_{\rm IV}$: $h_{\rm IV} = x^a y + y^b z + z^c = 0$,
$X_{\rm V}$: $h_{\rm V} = x^a y + y^b z + z^c x = 0$,
$X_{\rm VI}$: $h_{\rm VI} = xy + z^c = 0$,
$X_{\rm VII}$: $h_{\rm VII} = x^a z + y^b z + z^c + tx^{c_1}y^{c_2} = 0, \quad t \neq 0$
$X_{\rm VIII}$: $h_{\rm VIII} = x^a y + xy^b + xz^c + ty^{c_1}z^{c_2} = 0, \quad t \neq 0$.

The surface $X_{\rm I}$ is called a Pham-Brieskorn surface. This type of surfaces have been studied in the previous paper [4]. The surface $X_{\rm VI}$ is an A_{c-1} type singularity. There are exact $c-1$ families of lines on this surface (see [1, 2, 4, 5]). On surface $X_{\rm VII}$ and $X_{\rm VIII}$, the term $y^{c_1}z^{c_2}$ must be on the supporting plane of the previous three monomials. Thus a, b, c are not arbitrary. The Newton boundaries of the surfaces other than $X_{\rm VI}, X_{\rm VII}$ and $X_{\rm VIII}$ are triangles. Note that for a weighted homogeneous surface, the Newton boundary has only one compact 2-dimensional face. Let $P = {}^t(p_1, p_2, p_3)$ be the corresponding covector. The formula (3.1) in §2 reduces to

$$\rho(\Sigma^*_{\rm can}) = \varepsilon + \sum_{{\rm Cone}\,(P,Q)\in\Gamma^*(f)^+_2} (r(P,Q)+1)\rho_{PQ}(\Sigma^*_{\rm can}). \tag{4.1}$$

where $\varepsilon = 1$ if $P \in \mathcal{V}_{\rm ns}(\Sigma^*_{\rm can})$ and $\varepsilon = 0$ otherwise.

For each type of surfaces, one can calculate $\rho_{PQ}(\Sigma^*_{\rm can})$ for each Cone (P, Q) in the dual Newton diagram by using the method described in the previous sections.

LEMMA 14. *Assume that* Cone (P, E_i) *be a cone in* $\Gamma^*(f)^+_2$. *Then* $\det(P, E_i)$ *is given by* $\delta_i := \gcd(p_j, p_k)$ *with* $\{i, j, k\} = \{1, 2, 3\}$. *Assume that* $\delta_i > 1$.

1) $\mathcal{V}^{(i)}_{\rm ns}(P, E_i) \neq \emptyset$ *if and only if* $\delta_i > p_i$ *and* $\rho^{(i)}_{PE_i} = \left[\frac{\delta_i}{p_i}\right]$.

2) $\mathcal{V}^{(j)}_{\rm ns}(P, E_i) \neq \emptyset$ *if and only if* $p_j | p_k$. *In this case,* $\rho^{(j)}_{PE_i} = 1$.

3)

$$\rho_{PE_i} = \begin{cases} 0, & \text{if } \left[\frac{\delta_i}{p_i}\right] = 0 \text{ and } \delta_i < \min\{p_j, p_k\} \\ \max\{1, \left[\frac{\delta_i}{p_i}\right]\}, & \text{otherwise} \end{cases}$$

Proof. This follows from Corollary 11. □

LEMMA 15. *Let* Cone (P, Q) *be a cone in* $\Gamma^*(f)^+_2$ *with* $Q = {}^t(0, c, 1)$. *Suppose that* $\det(P, Q) = p_1 > 1$. *Then*

$$\rho_{PQ} = \begin{cases} \max\{1, \left[\frac{p_1}{p_2}\right], \left[\frac{p_1}{p_3}\right]\}, & c = 1 \\ \rho^{(2)}_{PQ} + \max\{1, \left[\frac{p_1}{p_3}\right]\} - \varepsilon, & c > 1 \end{cases}$$

where $\varepsilon = 1$ *if either* $Q_1 \in \mathcal{V}^{(2)}_{\rm ns}(P, Q)$ *or* $Q_{j_1} \in \mathcal{V}^{(2)}_{\rm ns}(P, Q)$ *with* $j_1 := \left[\frac{p_1}{p_3}\right] \geq 1$ *and* $\varepsilon = 0$ *otherwise.*

Proof. Let $Q_1, \dots, Q_k$ be the primitive covectors in Cone (P, Q) inserted by the canonical subdivision from Q. If $c = 1$, the assertion is immediate from Corollary 11, as $q_{1,1} = 1$. We assume that $c > 1$. If $[p_1/p_3] = 0$, the assertion is obvious. Assume that $[p_1/p_3] \geq 1$. By Corollary 11, Q_j is given by $(jP + (p_1 - jp_3)Q)/p_1$ for $1 \leq j \leq j_1$. Thus $q_{2,j} = c - j(cp_3 - p_2)/p_1$. If $cp_3 - p_2 < 0$, $q_{2,j}$ is monotone increasing by Lemma 5 and we see that $\mathcal{V}^{(2)}_{\rm ns}(P, Q) = \emptyset$ and the assertion follows immediately. Assume that $cp_3 - p_2 \geq 0$. Then $q_{2,j}$ is monotone decreasing for $0 \leq j \leq j_1$. Thus $\mathcal{V}^{(2)}_{\rm ns}(P, Q) \cap \mathcal{V}^{(3)}_{\rm ns}(P, Q) \neq \emptyset$ if and only if $q_{2,j_1} = 1$. If this is the case, Q_{j_1} is the unique covector in common. Thus the assertion follows from these observations. □

4.2. Normally smooth divisors on $X_{\rm II}$. By using Lemmas 14 and 15, we can compute the number $\rho(\Sigma^*_{\rm can})$. We show this by considering the surface $X_{\rm II}$. One can do the same consideration for the other types of surfaces. Let $X_{\rm II} : h_{\rm II}(x, y, z) = x^a y + y^b + z^c = 0$. Put $\hat{a} := \gcd(a, b-1), e := \gcd(b, c)$ and $d := \gcd(c(b-1), ac, ab) = e\gcd(a, c(b-1)/e)$. The dual Newton diagram $\Gamma^*(h_{\rm II})^+_2$ consists of three cones: Cone (P, Q), Cone (P, E_1) and Cone (P, E_3) where $P := {}^t(c(b-1)/d, ac/d, ab/d)$ and $Q := {}^t(0, c, 1)$.

The following three propositions are special cases of Lemmas 14 and 15.

PROPOSITION 16. Cone (P, E_1) *is regular if and only if* a *divides* $c(b-1)/e$. *Assume that* $a \nmid (c(b-1)/e)$. *Then*

1) $\mathcal{V}_{\rm ns}^{(1)}(P,E_1) \neq \emptyset$ *if and only if* $ae > (b-1)c$. *And in this case* $\rho_{PE_1}^{(1)} = \left[\frac{ae}{(b-1)c}\right]$.
2) $\mathcal{V}_{\rm ns}^{(2)}(P,E_1) \neq \emptyset$ *if and only if* $c|b$.
3) $\mathcal{V}_{\rm ns}^{(3)}(P,E_1) \neq \emptyset$ *if and only if* $b|c$.
4) $\rho_{PE_1} = \max\{\rho_{PE_1}^{(2)}, \rho_{PE_1}^{(3)}, \left[\frac{ae}{(b-1)c}\right]\}$. □

PROPOSITION 17. *As* $\det(P,E_3) = c\hat{a}/d$, Cone (P, E_3) *is regular if and only if* $d = c\hat{a}$. *Assume that* $c\hat{a} > d$. *Then*

1) $\mathcal{V}_{\rm ns}^{(1)}(P,E_3) \neq \emptyset$ *if and only if* $(b-1)|a$.
2) $\mathcal{V}_{\rm ns}^{(2)}(P,E_3) \neq \emptyset$ *if and only if* $a|(b-1)$.
3) $\mathcal{V}_{\rm ns}^{(3)}(P,E_3) \neq \emptyset$ *if and only if* $c\hat{a} > ab$ *and* $\rho_{PE_3}^{(3)} = [\frac{c\hat{a}}{ab}]$.
4) $\rho_{PE_3} = \max\{\rho_{PE_3}^{(1)}, \rho_{PE_3}^{(2)}, [\frac{c\hat{a}}{ab}]\}$.

Recall that $\rho_{P,E_i}^{(j)} \leq 1$ for $i = 1, 3$ and $j \neq i$ by Lemma 5.

PROPOSITION 18. Cone (P, Q) *is regular if and only if* $(b-1)c$ *divides* ae, *or equivalently* $(b-1)|a$ *and* $c|b\frac{a}{b-1}$. *Assume that* Cone (P,Q) *is not regular. Then we have*

1) $\mathcal{V}_{\rm ns}^{(1)}(P,Q) = \{Q_1\}$.
2) $\mathcal{V}_{\rm ns}^{(3)}(P,Q) \neq \emptyset$ *if and only if* $c(b-1) > ab$. *And in this case* $\rho_{PQ}^{(3)} = \left[\frac{c(b-1)}{ab}\right]$.
3) $\mathcal{V}_{\rm ns}^{(2)}(P,Q) \neq \emptyset$ *if and only if there exist positive integers* α *and* β *such that*

$$a\beta + d\alpha = b - 1, \tag{4.2}$$

$$ab\beta + d\alpha \equiv 0 \mod c(b-1). \tag{4.3}$$

The second condition can be replaced by $a\beta + 1 \equiv 0$ modulo c.

Proof. The last assertion follows from by (4.2) as $ab\beta + d\alpha = (b-1)(a\beta + 1)$. □

The non-trivial computation is required only for $\mathcal{V}_{\rm ns}^{(2)}(P,Q)$ which we will explain more in detail. Write $b = eb_1$ and $c = ec_1$.

COROLLARY 19. I. *For* $\mathcal{V}_{\rm ns}^{(2)}(P,Q) \neq \emptyset$, *it is necessary that*

$$\gcd(a,c) = 1, \quad b > a, c \tag{4.4}$$

In this case, we have $d = e\hat{a}$ *and* $\mathcal{V}_{\rm ns}^{(2)}(P,Q)$ *is the set of covectors* $T = (\alpha Q + \beta P)/d$ *which satisfies*

$$a\beta + e\hat{a}\alpha = b - 1 \tag{4.5}$$

$$0 < \alpha, \beta \tag{4.6}$$

$$b - e\hat{a}\alpha \equiv 0 \ \text{ modulo c} \tag{4.7}$$

II. *Furthermore* $\mathcal{V}_{\rm ns}^{(2)}(P,Q)$ *is non-empty if* $[b/c] \geq a + \hat{a}$.

Proof. From the congruence $a\beta + 1 \equiv 0$ modulo c, it is clear that $\gcd(a,c) = 1$. Thus $d = e\gcd(a, c_1(b-1)) = e\hat{a}$. The equality (4.7) results from

$$a\beta + 1 = b - d\alpha = e(b_1 - \hat{a}\alpha) \equiv 0 \ \text{ modulo c}$$

Thus $b > a\beta \geq a$ and $b > c$. The last congruence equation is equivalent to $b_1 - \hat{a}\alpha \equiv 0$ modulo c_1.

Assume that $[b/c] - a - \hat{a} \geq 0$. As $\gcd(\hat{a}, b_1) = 1$, there exists positive integer α_0, $0 < \alpha_0 < c_1$, such that $b_1 - \hat{a}\alpha_0 \equiv 0$ modulo c_1. Put $b_1 - \alpha_0\hat{a} = j_0c_1$. We see that $j_0 = b_1/c_1 - \alpha_0\hat{a}/c_1 > [b/c] - \hat{a}$. Take α which satisfies the congruence $a\beta + 1 \equiv 0$ modulo c. Then α takes the form $\alpha = \alpha_0 + jc_1$ with $j \in \mathbf{N}$ and thus $b_1 - \hat{a}\alpha = (j_0 - j\hat{a})c_1$. For the positivity of β, we need to have $0 \leq j < j_0/\hat{a}$. The integrity of T implies

$$e(b_1 - \hat{a}\alpha) - 1 = ec_1(j_0 - j\hat{a}) - 1 \equiv 0 \quad \text{modulo a}$$

As j can move $0 \leq j < j_0/\hat{a}$ and $j_0 > [b/c] - \hat{a} \geq a$ or $j_0/\hat{a} > a/\hat{a}$, this congruence equation has a positive solution j_1, $0 \leq j_1 \leq j_0/\hat{a}$. Then put $\beta = (ec_1(j_0 - j_1\hat{a}) - 1)/a$ for such a solution j_1. This gives a covector $T = (\alpha Q + \beta P) \in \mathcal{V}_{ns}^{(2)}(P, Q)$. □

EXAMPLE 20. Consider $X_{II} : x^9y + y^b + z^8 = 0$ with $b = 22 + 36k$. Then $e = 2, \hat{a} = 3$ and the equation is

$$9\beta + 6\alpha = 21 + 36k, \quad 9\beta + 1 \equiv 0 \text{ modulo } 8$$

In this case, $[b/c] - a - \hat{a} = (22+36k)/8 - 12 \geq 0$ if $k \geq 37/18$. For $k \geq 3$ (in fact, for $k \geq 2$), we have a solution $(\alpha, \beta) = (6k-7, 7)$. In this case, $P = {}^t(28+48k, 12, 33+54k)$ and $Q = {}^t(0, 8, 1)$ and $T := (\alpha Q + \beta Q)/(28 + 48k) = {}^t(7, 1, 8)$. We leave the computation of the other covectors in $\mathcal{V}_{ns}^{(2)}(P, Q)$ to the reader.

4.3. The minimality of the canonical toric resolutions. We study when the canonical toric resolution of a weighted homogeneous surface is minimal. Though the canonical toric resolution is not always minimal (see Example 28), we can expect that the minimality hold for almost all classes of non-degenerate surfaces. By [9, III(6.3)], for each weighted homogeneous surface the resolution graph associated with the canonical toric resolution is star-shaped. Hence, when the resolution graph has at least three arms, the canonical resolution is minimal.

We have the following general statement which is very helpful to see if a given toric modification is minimal.

LEMMA 21. *Let $X := f^{-1}(0)$ be a non-degenerate surface. Suppose that $P \in \Gamma^*(f)$ is the strictly positive covector corresponding to a compact face Δ of the Newton boundary $\Gamma(f)$.*

1) *Let $\Delta_1, \ldots, \Delta_\ell$ be the boundary edges of Δ. The exceptional divisor $E(P)$ is rational if and only if*

$$-\frac{6\mathrm{Vol}(\mathrm{Cone}\,\Delta)}{d(P; f)} + \sum_{i=1}^{l}(r(\Delta_i) + 1) = 2$$

where Cone Δ *is the cone over Δ with vertex O and $r(\Delta_i)$ is the number of integral points in the interior of Δ_i.*

2) *The canonical toric resolution $\pi : \tilde{X} \longrightarrow (X, 0)$ is not minimal if and only if there exists a compact face Δ of $\Gamma(f)$ such that $E(P)$ is rational, $E(P)^2 = -1$ and $E(P)$ intersects at most two other exceptional divisors where P is the covector corresponding to Δ.*

Proof. The first statement is a conclusion of [9, III(6.4)]. The assertion 2) follows from the Castelnuovo-Enriques criterion and [9, III §4(A) and §6]. □

THEOREM 22. *Let X be one of the surfaces of type X_{II}, X_{III}, X_{IV}, X_{V}, X_{VII} or X_{VIII}. We assume that $a, b, c > 1$ in 4.1. Then the canonical toric resolution of X is minimal. In particular, $\rho(X, 0) = \rho(\Sigma^*_{\mathrm{can}})$.*

Proof. We first check when the central exceptional divisor $E(P)$ is rational by using Lemma 21 (see also [9, III(6.9)]). If this is the case, we compute the number of arms from $E(P)$. If this number is less than 3, we show that $E(P)^2 \le -2$. Recall that the number of arms in the resolution graph is the sum of $r(P,Q)+1$ for non-regular cones $\mathrm{Cone}\,(P,Q) \in \Gamma^*(f)_2^+$.

(II). Let $X = X_{\mathrm{II}} : x^a y + y^b + z^c = 0$. Put $e = \gcd(b,c), \hat{a} = \gcd(a, b-1)$. Then $P = {}^t(c(b-1), ac, ab)/d$ with $d = e\gcd(a, c(b-1)/e)$. Note that $r(P,Q)+1 = 1$, $r(P,E_1)+1 = e$ and $r(P,E_3)+1 = \hat{a}$. By loc. cit. $E(P)$ is rational if and only if 1) $e = \gcd(c, a/\hat{a}) = 1$ or 2) $\hat{a} = \gcd(a, c/e) = 1$. If 1) holds, then $d = \hat{a}$. We have $\det(P,Q) = c(b-1)/\hat{a} > 1$, $\det(P,E_3) = c > 1$ and $\det(P,E_1) = a/\hat{a}$. If $\hat{a} = a$, $\mathrm{Cone}\,(P,E_3)$ gives $\hat{a} = a$ arms. Hence, in any case the resolution graph of X_{II} has at least three arms centered at $E(P)$.

In case 2), we have $\det(P,Q) = c(b-1)/e > 1$, $\det(P,E_1) = a > 1$ and $\det(P,E_3) = c/e$. If $e < c$, we have at least three arms in the resolution graph. Suppose that $e = c$. Then the number of arms at $E(P)$ is $e+1 \ge 3$, unless $b = 2$ and $e = c = 2$. In this case, the resolution graph has two similar arms and $E(P)$ is normally smooth with $E(P)^2 \le -2$.

Outline of other cases:

(III) Let $X_{\mathrm{III}} : x^a y + xy^b + z^c = 0$. Then $P = {}^t(c(b-1), c(a-1), ab-1)/d$ with $d = e\gcd(c, (ab-1)/e)$ and $e = \gcd(a-1, b-1)$. The dual Newton diagram $\Gamma^*(f)_2^+$ has 3 arms $\mathrm{Cone}\,(P,E_3)$, $\mathrm{Cone}\,(P,Q)$, $\mathrm{Cone}\,(P,R)$ where $Q = {}^t(0,c,1)$ and $R = {}^t(c,0,1)$. The central divisor $E(P)$ is rational if and only if $\gcd(c,(ab-1)/e) = 1$. If $E(P)$ is rational, then $d = e$ and $\det(P,Q) = c(b-1)/e > 1, \det(P,R) = c(a-1)/e > 1$, and $\det(P,E_3) = c > 1$. Hence, the resolution graph has at least three arms.

(IV) Let $X_{\mathrm{IV}} : x^a y + y^b z + z^c = 0$. Then $P := {}^t(bc-c+1, a(c-1), ab)/d$ with $d = e\gcd(a, (bc-c+1)/e)$ and $e := \gcd(b, c-1)$. The dual Newton diagram $\Gamma^*(f)_2^+$ has 3 arms $\mathrm{Cone}\,(P,E_1)$, $\mathrm{Cone}\,(P,Q)$, $\mathrm{Cone}\,(P,S)$ where $Q = {}^t(0,c,1)$ and $S = {}^t(1,0,a)$. The divisor $E(P)$ is rational if and only if $\gcd(a,(bc-c+1)/e) = 1$ which is equivalent to $d = e$. We have $\det(P,E_1) = a > 1$, $\det(P,S) = a(c-1)/e > 1$ and $\det(P,Q) = (bc-c+1)/e$. As $\mathrm{Cone}\,(P,E_1)$ has e-copies of arms, $E(P)$ has at least three arms.

(V) Let $X_{\mathrm{V}} : x^a y + y^b z + z^c x = 0$. Then $P := {}^t(bc-c+1, ca-a+1, ab-b+1)/d$ with $d = \gcd(bc-c+1, ca-a+1, ab-b+1)$. The dual Newton diagram $\Gamma^*(f)_2^+$ has 3 arms $\mathrm{Cone}\,(P,Q)$, $\mathrm{Cone}\,(P,S)$, $\mathrm{Cone}\,(P,T)$ where $Q = {}^t(0,c,1)$, $S = {}^t(1,0,a)$ and $T := {}^t(b,1,0)$. The divisor $E(P)$ is rational if and only if $d = 1$. In this case, we have $\det(P,Q) = bc-c+1 > 1$, $\det(P,S) = ca-a+1 > 1$ and $\det(P,T) = ab-b+1 > 1$. Thus $E(P)$ has three arms.

(VII) Let $X_{\mathrm{VII}} : x^a z + y^b z + z^c + tx^{c_1}y^{c_2} = 0$. Then $P = {}^t(b(c-1), a(c-1), ab)/\delta$ with $\delta = \gcd(b(c-1), a(c-1), ab)$. The dual Newton diagram $\Gamma^*(f)_2^+$ has 4 arms $\mathrm{Cone}\,(P,Q)$, $\mathrm{Cone}\,(P,S)$, $\mathrm{Cone}\,(P,E_1)$, $\mathrm{Cone}\,(P,E_2)$ where $Q = {}^t(0,1,c_2)$ and $S = {}^t(1,0,c_1)$. By the weighted homogenuity, we have the equality $b(c-1)c_1 + a(c-1)c_2 = abc$ which implies that $(c-1)|ab$. Hence $\delta = (c-1)\gcd(a, b, ab/(c-1))$. By loc. cit., $E(P)$ is rational if and only if either (i) $\gcd(a,b) = \gcd(a, c-1) = 1$, or (ii)

$\gcd(a,b) = \gcd(b,c-1) = 1$. By symmetry, we may assume that the first case (i). Then $\delta = c-1$, $\det(P,Q) = b > 1$, $\det(P,S) = a > 1$, $\det(P,E_1) = a > 1$. Thus the resolution graph has at least three arms.

(VIII) Let $X_{\rm VIII} : x^a y + xy^b + xz^c + ty^{c_1}z^{c_2} = 0$. Then $P = {}^t(c(b-1), c(a-1), b(a-1))/\delta$ with $\delta = \gcd(c(b-1), c(a-1), b(a-1))$. By the weighted homogenuity, we must have $c(a-1)c_1 + b(a-1)c_2 = c(ab-1)$ which implies that $(a-1)|c(ab-1)$ and $cc_1 + bc_2 = bc + c(b-1)/a - 1$. Thus $\delta = (a-1)\gcd(b,c,c(b-1)/(a-1))$. The dual Newton diagram $\Gamma^*(f)_2^+$ has 4 arms Cone(P,E_3), Cone(P,Q), Cone(P,S) and Cone(P,T) where $Q = {}^t(0,c,1)$, $S = {}^t(c_2,0,1)$ and $T = {}^t(c_1,1,0)$. The divisor $E(P)$ is rational if and only if $(b-1) = k(a-1)$ for some $k \in \mathbf{N}$ and $\gcd(b,c) = 1$. Then $d = a-1$ and $\det(P,Q) = ck > 1$, $\det(P,S) = c > 1$, $\det(P,T) = b > 1$ and $\det(P,E_3) = c$. Thus the $E(P)$ has at least 3 arms. □

4.4. Normally smooth divisors on $T_{p,q,r}$-surfaces. Let $T_{p,q,r} : x^p + y^q + z^r + xyz = 0$ with $1/p + 1/q + 1/r < 1$.

(1) Suppose that p,q,r are pairwisely coprime and $p < q < r$. The diagram $\Gamma^*(f)_2^+$ has three strictly positive vertices $P := {}^t(rq-r-q,r,q)$, $Q := {}^t(r,pr-p-r,p)$, and $R := {}^t(q,p,pq-q-p)$. The cones Cone(P,E_1), Cone(Q,E_2) and Cone(R,E_3) are regular. Put $\delta := pqr - pr - qr - pq$. Then $\det(P,Q) = \det(Q,R) = \det(P,R) = \delta$.

PROPOSITION 23. *Under the above assumption, we have*

$$\rho(X_{p,q,r},O) = \rho_{QR}^{(1)} + \rho_{QR}^{(2)} + \rho_{QR}^{(3)} + \rho_{PR}^{(2)} + \rho_{PR}^{(3)} + \rho_{PQ}^{(3)} - 2 - \epsilon,$$

where $\varepsilon = 1$ if $p = 3$, and $\epsilon = 0$ if $p \neq 3$.

Proof. This is a summary of the following three lemmas. □

LEMMA 24.
1) $\mathcal{V}_{\rm ns}^{(1)}(Q,R) = \{P_k = {}^t(1,k,p-k-1) \mid p/q < k < (rp-r-p)/r\}$.
2) $\mathcal{V}_{\rm ns}^{(2)}(Q,R) = \{P_k' = {}^t(k,1,pk-k-1) \mid r/(pr-p-r) < k < q/p\}$.
3) $\mathcal{V}_{\rm ns}^{(3)}(Q,R) = \{P_k'' = {}^t(k,pk-k-1,1) \mid q/(pq-p-q) < k < r/p\}$.
4) $\mathcal{V}_{\rm ns}^{(1)}(Q,R) \cap \mathcal{V}_{\rm ns}^{(2)}(Q,R) \cap \mathcal{V}_{\rm ns}^{(3)}(Q,R) \neq \emptyset$ *if and only if* $p = 3$.
5) $\rho_{QR} = \rho_{QR}^{(1)} + \rho_{QR}^{(2)} + \rho_{QR}^{(3)} - 1 - \epsilon$, *where $\epsilon = 1$ if $p = 3$, and $\epsilon = 0$ if $p \neq 3$.*

Proof. We mainly use Theorem 7. Let $P' := (\beta Q + \alpha R)/\delta = {}^t(p_1,p_2,p_3)$. The equation is

$$\begin{cases} \beta r + \alpha q = p_1\delta \\ \beta(pr-p-r) + \alpha p = p_2\delta \\ \beta p + \alpha(pq-p-q) = p_3\delta \end{cases} \quad \text{this implies} \quad \begin{cases} \alpha = (pr-p-r)p_1 - rp_2 \\ \beta = qp_2 - pp_1 \\ p_2 + p_3 = (p-1)p_1 \end{cases}$$

Hence, we have the following conclusions.

1) $p_1 = 1$ if and only if there exists an integer $p_2 > 0$ such that $\alpha > 0$ and $\beta > 0$. This is equivalent to $p/q < p_2 < (pr-p-r)/r$. And in this case $P' = (1,p_2,p-1-p_2)$.

2) $p_2 = 1$ if and only if there exists an integer $p_1 > 0$ such that $r/(pr-p-r) < p_1 < q/p$. And in this case $P' = (p_1,1,(p-1)p_1 - 1)$.

3) $p_3 = 1$ if and only if there exists an integer $p_1 > 0$ such that $q/(pq-p-q) < p_1 < r/p$. And in this case $P' = {}^t(p_1, pp_1 - p_1 - 1, 1)$.

4) is obvious now.

5) One can see this by comparing the three sets $\mathcal{V}_{\rm ns}^{(i)}(Q,R)$. In case $p = 2$, we have $\mathcal{V}_{\rm ns}^{(1)}(Q,R) = \emptyset$ and $\mathcal{V}_{\rm ns}^{(2)}(Q,R) \cap \mathcal{V}_{\rm ns}^{(3)}(Q,R) = \{{}^t(2,1,1)\}$. Hence, $\rho_{QR} = \rho_{QR}^{(2)} + \rho_{QR}^{(3)} - 1$.

In case $p = 3$, we have $\mathcal{V}_{\rm ns}^{(i)}(Q,R) \cap \mathcal{V}_{\rm ns}^{(j)}(Q,R) = \mathcal{V}_{\rm ns}^{(1)}(Q,R) \cap \mathcal{V}_{\rm ns}^{(2)}(Q,R) \cap \mathcal{V}_{\rm ns}^{(3)}(Q,R) = \{{}^t(1,1,1)\}$ for $i \neq j$. Hence, $\rho_{QP} = \rho_{QR}^{(1)} + \rho_{QR}^{(2)} + \rho_{QR}^{(3)} - 2$.

In case $p > 3$, we have $\mathcal{V}_{\rm ns}^{(1)}(Q,R) \cap \mathcal{V}_{\rm ns}^{(2)}(Q,R) = \{{}^t(1,1,p-2)\}$ and $\mathcal{V}_{\rm ns}^{(1)}(Q,R) \cap \mathcal{V}_{\rm ns}^{(3)}(Q,R) = \mathcal{V}_{\rm ns}^{(2)}(Q,R) \cap \mathcal{V}_{\rm ns}^{(3)}(Q,R) = \emptyset$. Hence, $\rho_{QP} = \rho_{QR}^{(1)} + \rho_{QR}^{(2)} + \rho_{QR}^{(3)} - 1$. □

Similarly, one can prove the following two lemmas.

LEMMA 25.
1) $\mathcal{V}_{\rm ns}^{(1)}(P,R) = \emptyset$.
2) $\mathcal{V}_{\rm ns}^{(2)}(P,R) = \{Q'_\ell = {}^t(q-\ell-1,1,\ell) \mid q/r < \ell < (pq-p-q)/p\}$.
3) $\mathcal{V}_{\rm ns}^{(3)}(P,R) = \{Q''_\ell = {}^t(q\ell-\ell-1,\ell,1) \mid p/(pq-p-q) < \ell < r/q\}$.
4) Let $Q' = {}^t(q_1,q_2,q_3) = (\beta P + \alpha R)/\delta$. *Then* $(q-1)q_2 = q_1 + q_3$.
5) $\rho_{PR} = \rho_{PR}^{(2)} + \rho_{PR}^{(3)} - 1$.

LEMMA 26.
1) $\mathcal{V}_{\rm ns}^{(1)}(P,Q) = \mathcal{V}_{\rm ns}^{(2)}(P,Q) = \emptyset$.
2) $\mathcal{V}_{\rm ns}^{(3)}(P,Q) = \{R'_\ell = {}^t(r-\ell-1,\ell,1) \mid r/q < \ell < (pr-p-r)/p\}$ *and* $\rho_{PQ} = \rho_{PQ}^{(3)}$.

EXAMPLE 27. (1) Let $p = 2, q = 3$ and $r \geq 7$. By the canonical subdivisions of the three cones, we see that $\rho_{QR} = [\frac{r-6}{2}] \geq 1$, $\rho_{PR} = [\frac{r-6}{3}] \geq 1$, and $\rho_{PQ} = [\frac{r-3}{6}]$.
(2) Let $p = 3, q = 4$ and $r > 4$. By the canonical subdivisions of the three cones, we see that $\rho_{QR} = [\frac{r}{3}] \geq 1$, $\rho_{PR} = [\frac{r}{4}] \geq 1$ and $\rho_{PQ} = [\frac{2r}{3}] - [\frac{r}{4}] - 1$.

(2) Another case. Let $f(x,y,z) = x^n + y^n + z^n + xyz$ $(n \geq 4)$. The dual Newton diagram has three covectors $P_i, i = 1,2,3$ corresponding to the three compact faces. They are given by ${}^t(n-2,1,1), {}^t(1,n-2,1), {}^t(1,1,n-2)$. And for $i \neq j$, $\det(P_i,P_j) = n-3$. Let $B_1,\dots,B_k$ be the vertices of the canonical subdivision of Cone(P_1,P_2) from P_1. Then $B_1 = (P_2 + (n-4)P_1)/(n-3) = {}^t(n-3,2,1)$. Thus $(n-3)/(n-4) = [2,\dots,2]$ with $(n-4)$-copies of 2. This implies $k = n-4$ and $B_j = {}^t(n-2-j,1+j,1), j = 1,\dots,n-4$. In fact, by Lemma 5 the third coordinate of B_j is always 1 as both of P_1,P_2 have 1 as the third coordinate. Hence $\rho_{P_1P_2} = n-4$. The branch Cone(P_i,E_i) is regular. Thus $\rho(V,O) = \rho(\Sigma^*_{\rm can}) = 3n-9$ and *every exceptional divisor is normally smooth.*

5. Remarks.

5.1. Example of the inequality $\rho(\Sigma^*_{\rm can}) > \rho(X,O)$. Let us consider A_{2c-1}-singularity, $X = \{x^2+y^2+z^{2c} = 0\}$. The resolution graph has two arms and the central divisor $E(P)$ is a rational curve with $E(P)^2 = -1$. Thus we have to blow-down the central divisor once (Example (6.7.1) in [9, III]). However in this example, the central exceptional divisor is not normally smooth, i.e., the extra blowing-up is line-admissible. So $\rho(\Sigma^*_{\rm can}) = \rho(X,O)$. The following gives an example of $\rho(\Sigma^*_{\rm can}) > \rho(X,O)$.

EXAMPLE 28. Let X be defined by $h = xy + y^{bc} + z^c$ with $b, c \geq 2$. This is an A_{c-1}-singularity and a special case of $X_{\rm II}$ with $P := {}^t(bc-1,1,b)$ and $Q := {}^t(0,c,1)$.

Since $\det(P,E_1) = \det(P,E_3) = 1$ and $\det(P,Q) = bc-1$, we make the canonical subdivision of Cone(P,Q). The first covector T_1 from P is given by

$$T_1 = (Q + (bc-c-1)P)/(bc-1) = {}^t(bc-c-1,1,b-1)$$

We have the continuous fraction expansion $(bc-1)/(bc-c-1) = [2,\dots,2,3,2,\dots,2]$ where the number of 2 in the first 2-series (respectively in the second 2-series) is

$(b-2)$ (resp. $c-2$). Thus we have $c+b-3$ covectors $T_1, \dots, T_{b+c-3}$. The exceptional divisor $E(P)$ is rational with $E(P)^2 = -1$ and $E(T_j)$ with self intersection number $E(T_j)^2 = -2$ for $j \neq b-1$ and -3 for $j = b-1$ (see Theorem (6.3), Chapter III, [9]). In fact first $b-2$ covectors are given by

$$Q_j = {}^{t}(cb - jc - 1, 1, b - j), \quad j = 1, \dots, b-1$$
$$Q_{b-1+j} = {}^{t}(c - j - 1, j + 1, 1), \quad j = 1, \dots, c-2$$

and we see that they are normally minimal. To get a minimal reslution, we need to blow down $b-1$ divisors $E(P), E(T_1), \dots, E(T_{b-2})$ in this order. Then the self-intersection number of $E(T_{b-1})$ changes to -2 and we get A_{c-1} graph. In this example, we have $\rho(X, O) = c-1$ and $\rho(\Sigma^*_{\mathrm{can}}) = b + c - 2$.

5.2. Parametrization of lines. The normally smooth divisors on a surface X correspond to the lines on X. By using a toric resolution, one can give the exact parameterizations of the lines on X. This was done already for the Pham-Brieskorn surfaces in [4].

PROPOSITION 29. *Suppose that we have a line L in a non-degenerate surface $X : f(x, y, z) = 0$ and assume that L is parametrized as*

$$x(t) = \alpha t^a + \alpha_1 t^{a+1} \dots, \quad y(t) = \beta t^b + \beta_1 t^{b+1} + \dots, \quad z(t) = \gamma t^c + \gamma_1 t^{c+1} + \dots$$

with $\alpha, \beta, \gamma \neq 0$ and $\min(a, b, c) = 1$. Let $P = {}^{t}(a, b, c)$. Then the pull back of L intersects $E(P)$ transversally and $f_P(\alpha, \beta, \gamma) = 0$. Conversely any curve in $\mathcal{L}_{E(P)}$ has such a parametrization.

EXAMPLE 30. (1) Let X be defined by $h = x^a y + y^b - z^b = 0$ with $a = a_1(b-1)$ and $a_1 > 1$. This is a special case of X_{II}. We use the notations in §4.2. Note that $P = {}^{t}(1, a_1, a_1)$, $Q = {}^{t}(0, b, 1)$, $\det(P, Q) = \det(P, E_3) = 1$ and $\det(P, E_1) = a_1$. By canonical subdivision of Cone (P, E_1) we have $R_i := {}^{t}(1, i, i)$ with $i = 0, 1, \dots, i_1 = a_1$, where $R_0 := E_1$ and $R_{i_1} := P$. Hence $\rho_{PE_1} = a_1 - 1$. Since $r(P, E_1) + 1 = b$, each $E(R_i)$ has b components. By [9, III(6.3)], $E(P)^2 = -b < -1$. Hence π is minimal and $\rho(X, 0) = b(a_1 - 1) + 1$. The restriction of π on the toric chart associated with $\sigma_i :=$ Cone (R_i, R_{i-1}, E_2) is given by

$$\pi_{\sigma_i}: \quad x = uv, \quad y = u^i v^{i-1} w, \quad z = u^i v^{i-1}.$$

and the pull-back of h is given by

$$h \circ \pi_{\sigma_i} = u^{ib} v^{(i-1)b} \left(u^{(a_1-i)(b-1)} v^{(a_1-i+1)(b-1)} w + w^b - 1 \right)$$

The divisor $E(R_i)$ is defined by $u = 0$ and $w^b - 1 = 0$, hence $E(R_i)$ has b components. On this toric chart, the resolution $\tilde{X}$ of X is defined by

$$\tilde{h}_i(u, v, w) := u^{(a_1-i)(b-1)} v^{(a_1-i+1)(b-1)} w + w^b - 1 = 0$$

and in a neighborhood of $q \in E(R_i)$ we take u, v to be the local coordinates of $\tilde{X}$. Let $q = (0, s)$ in this coordinates. We consider the lines C_s defined by $t \mapsto (t, s)$. The image of C_s by π_{σ_i} is given by

$$\pi_{\sigma_i}(C_s): \quad x = st, \quad y = s^{i-1} w_k(t, s) t, \quad z = s^{i-1} t^i,$$

where $w_k(t,s)$ is the solution of $\tilde{h}_i(t,s,w)=0$ with $w_k(0)=\exp(2\pi ki/b)$. As a special case, take $i=1$. Then C_s is a normal line on $E(Q_1)$. When we moves $s\to 0$, this line approaches to $E(E_1)$ and $w_k(t)\equiv \exp(2k\pi i/b)$ and the image is the obvious line $t\to (x,y,z)=(0,w_kt,t)$.

(2) Let $X=T_{2,3,7}: x^2+y^3+z^7+xyz=0$. We have three covectors

$$P={}^t(11,7,3),\ Q={}^t(7,5,2),\ R={}^t(3,2,1)$$

and we do not need any other covector. Consider the toric chart $\sigma:=(Q,R,E_3)$ with coordinates (u,v,w). Then the line $u=1, v=t$ produces a line parametrized as $t\mapsto (t^3,t^2,-2t+128t^2+\dots)$.

5.3. Obvious lines on surfaces. We consider a surface $X=\{f(x,y,z)=0\}$ where f has a non-degenerate Newton boundary. There are surfaces having obvious lines which can be read off from the polynomial defining the surface.

(1) Assume that $f(x,y,z)$ is not convenient and assume for example $\{y=z=0\}\subset X$. Then as we have seen in Lemma 10, there is a unique non-compact face, different from the coordinate planes, which has the covector of the type $Q={}^t(0,c,1)$ or ${}^t(0,1,c)$ and a unique covector P such that Cone (P,Q) is in $\Gamma^*(f)_2^+$ and P corresponds to a compact face. Let $Q_1,\dots,Q_k$ be the covectors defining the canonical regular subdivision from Q. Then Q_1 is a normally smooth divisor and $\mathcal{L}_{Q_1}$ contains the canonical line $\{y=z=0\}$.

(2) Assume that $h(x,y):=f(x,y,0)$ (the section of f with $z=0$) is a non-monomial homogeneous polynomial of degree d. Then we can factor $h(x,y)=cx^ay^b\prod_{i=1}^k(y-\alpha_ix)$. Thus X has the lines $z=0, y=\alpha_ix$ for $i=1,\dots,k$. Combinatorially this says the following. There exists a compact face Δ such that $\Delta\supset\Delta(h)$. The corresponding covector takes the form $P={}^t(p,p,r)$ with $\gcd(p,r)=1$. Then the first covector Q_1 from E_3 in the canonical regular subdivision of Cone (P,E_3) takes the form $Q_1={}^t(1,1,s)$ with $s=1+[r/p]$. So we can see that $Q_1\in\mathcal{V}_{\mathrm{ns}}(P,E_3)$. A typical example is $T_{n,n,n}: x^n+y^n+z^n-xyz=0$. Another example is (1) of Example 30.

(3) Assume that the monomial x^A in f such that $(A,0,0)\in\Gamma(f)$. We say that x^A is negligibly truncatable if $f_t(x,y,z)=(f(x,y,z)-f(x,0,0))+tf(x,0,0)$ defines a μ-constant family for $0\le t\le 1$ (cf. [11]). Assume for example, the monomials x^ay and x^bz^c are on the non-compact face of $\Gamma(f_0)$. Let $Q':={}^t(c/d,c(A-a)/d,(A-b)/d)$ with $d=\gcd(c,A-b)$. The covector Q' corresponds to the negligible compact face of f_1 containing $(a,1,0),(b,0,c)$ and $(A,0,0)$. Then there is a normally smooth divisor on Cone (Q,E_3). In fact, $\det(Q',E_3)=c/d$. If $c=d$, Q gives normally smooth divisor. If $c>d$, the first covector Q_1' of the canonical regular subdivision of Cone (Q',E_3) is normally smooth. An example is given by $f(x,y,z)=x^2y+y^2+z^5+x^5$. Then x^5 is negligibly truncatable.

(4) Assume that $\Gamma(f)$ has a compact face whose covector P has 1 in its coefficients. Then $E(P)$ is a normally smooth divisor. This is the case, for example, if $P={}^t(1,1,1)$ and $f_P(x,y,z)$ has a two-dimensional support. We can see easily that $E(P)$ is isomorphic to the projective curve $f_P(x,y,z)=0$ in $\mathbb{P}^2$. The tangent cone of X at O is given by the cone of $f_P=0$.

5.4. Normally smooth divisors on complete intersections. In this paper we mainly considered normally smooth divisors on two dimensional hypersurface singularities. However every assertion can be generalized to non-degenerate complete intersections. We give an example. Consider the surface given by $X=\{f_1(x,y,z,w)=$

$f_2(x,y,z,w) = 0\}$ where f_1 and f_2 has the same Newton boundary. Assume that f_1, f_2 are Pham-Brieskorn polynomials of the same type, with generic coefficients:

$$f_i = a_i x^{p_1} + b_i y^{p_2} + c_i z^{p_3} + d_i w^{p_4}, i = 1,2$$

We assume that $p_1,\dots,p_4 \geq 2$ and mutually coprime. Then the dual Newton diagram $\Gamma^*(f_1, f_2)$ is the same with $\Gamma^*(f_i)$ and $\Gamma^*(f)_2^+$ is star-shaped with the center $P = {}^t(p_2p_3p_4, p_1p_3p_4, p_1p_2p_4, , p_1p_2p_3)$ and four arms $\mathrm{Cone}\,(P, E_i), i = 1,\dots,4$. We consider the $\mathrm{Cone}\,(P, E_1)$. First $\det(P, E_1) = p_1$. By Lemma 11, $\mathcal{V}_{ns}^{(i)}(P, E_1) = \emptyset$ for $2 \leq i \leq 4$. As for $\mathcal{V}_{ns}^{(1)}(P, E_1) \neq \emptyset$ if and only if $p_2p_3p_4 < p_1$ and putting $r = [p_1/p_2p_3p_4]$, $\mathcal{V}_{ns}^{(1)}(P, E_1) = \{Q_j = (jP + (p_1 - jp_2p_3p_4)E_1)/p_1; j = 1,\dots,r\}$.

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