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HILBERT SCHEMES OF G-ORBITS IN DIMENSION THREE*

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Abstract. We study the precise structure of Hilbert scheme $\mathrm{Hilb}^G(\mathbf{C}^3)$ of G-orbits in the space $\mathbf{C}^3$ when the group G is a simple subgroup of $\mathrm{SL}(3, \mathbf{C})$ of either 60 or 168. These are the only possible non-abelian simple subgroups of $\mathrm{SL}(3, \mathbf{C})$.

0. Introduction. For a given finite subgroup G of $\mathrm{SL}(3, \mathbf{C})$ a somewhat complicated scheme, the Hilbert scheme $\mathrm{Hilb}^G(\mathbf{C}^3)$ of G-orbits has been introduced for the purpose of resolving singularities of the quotient space $\mathbf{C}^3/G$ as well as generalizing McKay correspondence in dimension two [McKay80]. It is defined to be the subscheme of $\mathrm{Hilb}^{|G|}(\mathbf{C}^3)$ parametrizing all the zero dimensional G-invariant subschemes with their structure sheaf isomorphic to the regular representation of the group G as G-modules. This scheme is now known by [BKR99] to be smooth and irreducible and it is a crepant resolution of the quotient space $\mathbf{C}^3/G$. See [Nakamura98] for smoothness and an algorithm of computation in the abelian case. See also [INakajima98].

On the other hand we know all the possibilities of finite subgroups of $\mathrm{SL}(3, \mathbf{C})$ by [Blichfeldt17], [BDM16] and [YY93]. There are exactly 4 infinite series labeled by A(belian), B, C, D, and 8 exceptional cases labeled by E though L. Among them there are only two non-abelian simple subgroups, which are of order either 60 or 168. We denote these subgroups of $\mathrm{SL}(3, \mathbf{C})$ simply by G_{60} and G_{168}. In this article we study $\mathrm{Hilb}^G(\mathbf{C}^3)$ when G is either G_{60} or G_{168}. The structure of $\mathrm{Hilb}^G(\mathbf{C}^3)$ is more or less easily understood over the quotient space $(\mathbf{C}^3/G) \setminus \{0\}$. Since smoothness of $\mathrm{Hilb}^G(\mathbf{C}^3)$ is already known by [BKR99], the remaining problem is to describe the fibre of $\mathrm{Hilb}^G(\mathbf{C}^3)$ over the origin precisely. However the computations we actually need in the cases $G = G_{60}$ and G_{168} are rather enormous. Therefore we wish to state the result precisely, indicating our ideas of the computations only.

We instead explain in detail the simplest non-abelian case - a trihedral group G_{12} of order 12. In this case we describe the precise structure of $\mathrm{Hilb}^G(\mathbf{C}^3)$. This gives a negative answer to a question posed by [INakamura99, p.227]. In fact, G_{12} has a normal subgroup N of order 4 by definition. The composite Hilbert scheme $\mathrm{Hilb}^{G/N}(\mathrm{Hilb}^N(\mathbf{C}^3))$ is a crepant resolution of $\mathbf{C}^3/G$ as well as $\mathrm{Hilb}^G(\mathbf{C}^3)$. Both of the fibres over the origin of $\mathbf{C}^3/G$ consist of 3 rational curves, but they have different configurations.

We note that crepant resolutions of $\mathbf{C}^3/G_{60}$ and $\mathbf{C}^3/G_{168}$ have been constructed by [M97] and [Roan96] by using the equations defining the quotient given by [YY93].

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It remains to compare the structures of their resolutions and ours.

1. Duality of the coinvariant algebra. The result of this section is more or less familiar to the specialists.

1.1. Complex reflection groups. Let V be a finite dimensional complex vector space, $V^\vee$ the dual of V and M a finite subgroup of $\mathrm{GL}(V)$ generated by complex reflections of V. Here we mean by a complex reflection an automorphism of V of finite order which has exactly one eigenvalue distinct from 1.

Let $S := S(V^\vee)$ be the symmetric algebra of $V^\vee$ over $\mathbf{C}$, $S = \oplus_{k=0}^{\infty} S_k$ be the homogeneous decomposition of S, $S_+ := \oplus_{k=1}^{\infty} S_k$. Since M acts on V, hence on $V^\vee$ contravariantly, it naturally acts on S by $g(F)(v) = F(g^{-1}v)$ for $F \in S$ and $v \in V$. Let S^M be the subalgebra of M-invariants of S, $S_+^M := S^M \cap S_+$, $\mathfrak{n}_M := SS_+^M$ and $S_M := S/\mathfrak{n}_M$. We call S_M the coinvariant algebra of M.

Let S^* be the symmetric algebra of V, and we identify it with the algebra of polynomial functions on $V^\vee$. Following [Steinberg64] we define an algebra homomorphism D of S into the endomorphism ring $\mathrm{End}\,(S^*)$ of S^* as follows. Let $v, w \in V^\vee$ and $F \in S^*$. Then we first define

$$(D_v F)(w) = \lim_{t \to 0} (F(w + tv) - F(w))/t$$

and extend it as an algebra homomorphism of S into $\mathrm{End}\,(S^*)$, in other words, we extend it to S by the conditions $D_{st} := D_s D_t$, $D_{s+t} = D_s + D_t$ for any $s, t \in S$.

We also define the G-invariant inner product

$$\alpha(s, F) := (D_s F)(0) \tag{1.1}$$

for $s \in S$, $F \in S^*$. In fact, since $(D_{\sigma s} \sigma F) = \sigma(D_s F)$ for $\sigma \in M$, we have $\alpha(\sigma s, \sigma F) = \sigma(D_s F)(0) = (D_s F)(0) = \alpha(s, F)$. This inner product extends the inner product between V and $V^\vee$. By the inner product we identify V with $V^\vee$, and S^* with S as well as the subalgebra $(S^*)^M$ of M-invariants in S^* with S^M, for instance in Theorem 1.2 (2) where we apply [Steinberg64, Lemma 3.1]. We use freely the identification in Theorem 1.2, in particular the notation (s, F) and $D_s F$ make sense for $s, F \in S$.

We recall a basic fact from [Bourbaki, Chapitre 5]

THEOREM 1.2. *Let V be an n-dimensional vector space and M a finite subgroup of $\mathrm{GL}(V)$ generated by complex reflections of V. Then*

1. *S^M is isomorphic to a polynomial ring of n variables, in other words, there are homogeneous polynomials $P_1, \cdots, P_n \in S$ such that*

$$S^M = \mathbf{C}[P_1, \cdots, P_n],$$

2. *There exists $P \in S$, unique up to constant multiples, such that*
   (2a) P is skew, that is, $g(P) = \det(g)P$ for any $g \in M$,
   (2b) if $P' \in S$ is skew, then P divides P'.

We note that P is given in two different ways. Let P_i be the set of homogeneous generators of S and x_i a basis of $V^\vee$. First P is given as $P = \mathrm{Jac}(P_1, \cdots, P_n) := \det(\partial P_i / \partial x_j)$.

Let Σ be the set of all reflections in G. For any $g \in \Sigma$ there is an element $e_g \in V^\vee$ unique up to constant multiples such that $g(x) = x + f_g(x)e_g$ ($\forall x \in V^\vee$) for some $f_g \in V$. Then $e_g = 0$ is a *linear* equation defining the reflection hyperplane in V of g. Then $P = c\prod_{g\in\Sigma} e_g \in S$ for some nonzero constant c. This is the same as [Steinberg64, Theorem 1.4 (a)] though the notation looks slightly different.

The basic degrees (or characteristic degrees) of M are by definition the set of integers $d_i := \deg P_i$ ($1 \leq i \leq n$), which is known to be independent of the choice of the generators P_i. It is easy to see $\deg P = \sum_{i=1}^n (d_i - 1) = |\Sigma|$.

The following follows from [Steinberg64] by identifying S with S^*.

THEOREM 1.3. *Let $m = \deg P$. Then*

1. *Let $U = \{D_s P; s \in S\}$. Then U is the orthogonal complement in S of $\mathfrak{n}_M$ with respect to α. It is a G-stable finite dimensional subspace of S such that $U \otimes_{\mathbf{C}} S^M \simeq S$ and $S_M \simeq U \simeq \mathbf{C}[M]$ as M-modules,*
2. *$s \in \mathfrak{n}_M$ if and only if $D_s P = 0$,*
3. *$S_k \subset \mathfrak{n}_M$ for $k > \deg P$.*

Proof. We note that (1) follows from [Steinberg64, Theorem 1.2 (c)], while (2) follows from [ibid.,Theorem 1.3 (b)]. (3) follows from (2). □

In particular by Theorem 1.3 (1) the maximum degree of elements in U is attained by $\deg(D_s P) = m$ with $s = 1$.

We define a bilinear form $\beta : S_M \times S_M \to \mathbf{C}$ by

$$\beta(f,g) := (D_{fg}P)(0). \tag{1.2}$$

THEOREM 1.4. *Let S_M be the coinvariant algebra of M, $U_k := U \cap S_k$ and $(S_M)_k :=$ the image of U_k in S_M for $k \leq m$. Then $\beta : S_M \times S_M \to \mathbf{C}$ is a nondegenerate bilinear form such that*

1. *$\beta(f, gh) = \beta(fg, h)$,*
2. *$\beta(\sigma f, \sigma g) = \det(\sigma)^{-1}\beta(f,g)$,*
3. *$(S_M)_k$ and $(S_M)_{m-k}$ are dual to each other with respect to β.*

Proof. Nondegeneracy of β follows from Theorem 1.3 (2). (1) is clear from the definition of β. Next we prove (2). In fact, we see

$$\begin{aligned}\beta(\sigma f, \sigma g) &= D_{(\sigma f \sigma g)}P(0)\\ &= \sigma(D_{(fg)}(\sigma^{-1}P))(0)\\ &= \det(\sigma)^{-1}(D_{(fg)}P)(0)\\ &= \det(\sigma)^{-1}\beta(f,g)\end{aligned}$$

because $\sigma(D_s F) = D_{\sigma s}(\sigma F)$ for any $s, F \in S$. See [Steinberg64, p. 392]. □

1.5. Subgroups of complex reflection groups of index two. Let V be a finite dimensional complex vector space and G a finite subgroup of $\mathrm{SL}(V)$. Suppose that there is a finite subgroup M of $\mathrm{GL}(V)$ generated by complex reflections of V such that $G = M \cap \mathrm{SL}(V)$ and $[M : G] = 2$. For instance any finite subgroup of

SL(2, **C**) and the subgroups G_{12}, G_{60} and G_{168} of SL(3, **C**) satisfy the conditions as we see later. We will see that the facts observed in [INakamura99, Section 11] are easily derived from [Steinberg64] in a more general situation, though these have been observed already in [GSV83] and [Knörrer85].

Let $S := S(V^\vee)$ be the symmetric algebra of $V^\vee$ over $\mathbf{C}$, $S = \oplus_{k=0}^{\infty} S_k$ be the homogeneous decomposition of S, $S_+ := \oplus_{k=1}^{\infty} S_k$. Let S^G be the subalgebra of G-invariants of S, $S_+^G := S^G \cap S_+$, $\mathfrak{n}_G := SS_+^G$ and $S_G := S/\mathfrak{n}_G$.

In the rest of this section we compare S^G and S_G with S^M and S_M. First we note $S^M \subset S^G$ and $\mathfrak{n}_M \subset \mathfrak{n}_G$.

THEOREM 1.6. *Let U be the same as in Theorem 1.4, $U_k = U \cap S_k$ and $(S_G)_k$ the image of U_k in S_G. Then the following is true.*

1. $S^G = S^M[P]$, $\mathfrak{n}_G = \mathfrak{n}_M + \mathbf{C}P$.
2. $S_G \simeq \oplus_{k=1}^{m-1} U_k \simeq \mathbf{C}[G] + \mathbf{C}[G]/\mathbf{C}$, $U \cap S^G = \mathbf{C} + \mathbf{C}P$,
3. $(S_G)_k$ *and* $(S_G)_{m-k}$ *are dual to each other if* $1 \le k \le m-1$.

Proof. Since det is a nontrivial character of M/G by our definition of G, we see $1_G^M = 1_G + \det$, and $\mathbf{C}[M]_G = \mathbf{C}[G] + \det \otimes \mathbf{C}[G]$. It follows from Theorem 1.2 that $U \cap S^G = \mathbf{C} + \mathbf{C}P$. Therefore we infer the rest of the assertions from Theorem 1.3 and Theorem 1.4. □

We can make Theorem 1.6 (3) more precise as follows.

THEOREM 1.7. *Let the notation be the same as in Theorem 1.6. Let ρ (resp. ρ') be (an equivalence class of) an irreducible representation of G and $\bar\rho$ (resp. $\bar\rho'$) the complex conjugate of ρ. Let $(S_G)_k[\rho]$ be the sum of all G-submodules of $(S_G)_k$ isomorphic to ρ. Then*

1. $(S_G)_k[\rho]$ *and* $(S_G)_{m-k}[\bar\rho]$ *are dual with respect to* β,
2. *there is a G-submodule of S_G isomorphic to ρ' in $S_1((S_G)_k[\rho])$ if and only if there is a G-submodule of S_G isomorphic to $\bar\rho$ in $S_1((S_G)_{m-k-1}[\bar\rho'])$.*

Proof. Let W be an irreducible G-submodule of $(S_G)_k[\rho]$ isomorphic to ρ. Let W^c be a G-submodule in U_k complementary to W and W^* the orthogonal complement in U_{m-k} to W^c with respect to β. By Theorem 1.6 (3) W^* is dual to W with respect to β. It is clear that $\sigma(W^*) \subset W^*$ for any $\sigma \in G$. For $f \in W$ and $g \in W^*$, we have $\beta(\sigma f, g) = \beta(f, \sigma^{-1} g)$ by Theorem 1.4 (2), whence by $G \subset \mathrm{SL}(V)$

$$\chi_\rho(\sigma) = \mathrm{Tr}\,((\sigma)_W) = \mathrm{Tr}\,((\sigma^{-1})_{W^*}).$$

Hence $\mathrm{Tr}\,((\sigma)_{W^*}) = \overline{\chi_\rho(\sigma)} = \chi_{\bar\rho}(\sigma)$. It follows $W^* \simeq \bar\rho$. This proves (1).

Let W be a G-submodule of $(S_G)_k[\rho]$ and W' a G-submodule of $(S_1 W)[\rho']$ such that $W \simeq \rho$, $W' \simeq \rho'$. Let W^c (resp. $(W')^c$) be a G-submodule of $(S_G)_k$ (resp. $(S_G)_{k+1}$) complementary to W (resp. W'). Let W^* and $(W')^*$ be the orthogonal complement in U_{m-k} and U_{m-k-1} to W^c and $(W')^c$ respectively. Hence by (1) $W^* \simeq \bar\rho$ and $(W')^* \simeq \bar\rho'$. By assumption and by Theorem 1.6 (3) there exist $x \in S_1$, $f \in W$ and $g \in (W')^*$ such that

$$\beta(f, xg) = \beta(xf, g) \neq 0.$$

This implies that $S_1(W')^*$ contains a G-submodule dual to W with respect to β, hence isomorphic to $\bar{\rho}$ by (1). This completes the proof. □

1.8. Finite subgroups of SL(2,C). Let V be a vector space of two dimension. It is well known that for any any finite subgroup of $\mathrm{SL}(V)$ there is a finite complex reflection group M of $\mathrm{GL}(V)$ such that $G = M \cap \mathrm{SL}(V)$ and $[M : G] = 2$. In fact, for G a cyclic group of order n, M is a dihedral group $I(n)$ of order $2n$. For a binary dihedral group G, M is a subgroup of $GL(V)$ generated by G and a permutation matrix $(1,2)$. For G a binary tetra-, octa-, or icosa-hedral group respectively, M is a complex reflection group with Shephard-Todd number 12, 13 and 22 respectively [ST54, p. 301]. In particular, $M = \mu_4 \cdot G$ for G the binary octa-, or icosa-hedral group where μ_4 is the subgroup of scalar matrices of fourth roots of unity.

2. G_{12}. The purpose of this section is to provide an example which solves negatively the question in [INakamura99, Section 17].

2.1. The action of G on $V^\vee$. This Subsection is included just to explain our convention and notation in the subsequent sections.

Let $V = \mathbf{C}^3$ and $V^\vee$ the dual of V. We choose and fix a basis e_i of V once for all. The space $V^\vee$ is spanned by the dual basis $x_1 = x$, $x_2 = y$ and $x_3 = z$ with $x_i(e_j) = \delta_{ij}$. The matrix form of $g \in \mathrm{SL}(V)$ in Subsection 2.2 etc. is that of g with respect to e_i. Hereafter we call this matrix representation ρ and hence $\rho(g) = g$. Then as in Section 1, G acts on $V^\vee$ by the contragredient representation $\rho^\vee$ of ρ and we have $\rho^\vee(g)(v^\vee)(p) = v^\vee(\rho(g^{-1})p)$ for $p \in V$, $v^\vee \in V^\vee$ and $g \in G$. In terms of pull back by the automorphism $\rho(g^{-1})$ of V, this means that $\rho^\vee(g)(v^\vee) = \rho(g^{-1})^*(v^\vee)$, where $V^\vee$ is regarded as the space of linear functions on V. In particular we have

$$(\rho^\vee(g)(x), \rho^\vee(g)(y), \rho^\vee(g)(z)) = (x, y, z)^t g^{-1}.$$

This is equivalent to $(g^*(x), g^*(y), g^*(z)) = (x, y, z)^t g$ where $g^* = \rho(g)^*$ is the pull back of functions on V by the automorphism g of V.

The action of G on $V^\vee$ via $\rho^\vee$ can be extended to S and hence to S_G, since $\mathfrak{n}$ is G-invariant. We denote this representation of G on S_G by $S_G(\rho^\vee)$. We note also that the action of G on S is the same as the one given in [YY93, p. 38], so we can apply their results for G_{60} and G_{168}.

We use the same notation as above from now on.

2.2. A trihedral group G_{12}. Let N be an order 4 abelian subgroup of $\mathrm{SL}(V)$ consisting of diagonal matrices with diagonal coefficients ± 1 and $\tau := (\delta_{i,j+1})$. To be more precise

$$N = \left\{ \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix} \in \mathrm{SL}(V) \right\}, \quad \tau = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Let $G = G_{12}$ be the subgroup of $\mathrm{SL}(V)$ of order 12 generated by N and τ. The group G is called *a ternary trihedral group.* It is clear that there is an exact sequence $1 \to N \to G \to \mathbf{Z}/3\mathbf{Z} \to 0$. We note that the group G is a subgroup of index two of a complex reflection group M. In fact, we choose as $\tilde{N}$ a diagonal subgroup of $\mathrm{GL}(V)$ with diagonal coefficients ± 1 and M the extension of $\tilde{N}$ by $\{1, \tau, \tau^2\}$. Thus Section 1 can be applied for G and M. The basic degrees of M are equal to $2, 3$ and 4 so that $\deg P = m = (2-1) + (3-1) + (4-1) = 6$ as we have seen in Section 1.

2.3. Characters of G. Table 2.1 is the character table of G, where the first (resp. the second, the third) line gives a representative of (resp. the age of, the number of elements in) each conjugacy class and $\sigma_{12} = \mathrm{diag}(-1, -1, 1)$. For $G = G_{12}$, we have $\rho = \rho^\vee$ and the character of ρ is given by χ_3. For each irreducible character χ_k in Table 2.1, we choose and fix an irreducible representation ρ_k which affords χ_k.

c.c	1	σ_{12}	τ	τ^2
age	0	1	1	1
$\sharp$	1	3	4	4
χ_{1_1}	1	1	1	1
χ_{1_2}	1	1	ω	ω^2
χ_{1_3}	1	1	ω^2	ω
χ_3	3	-1	0	0

TABLE 2.1
Character table of G_{12}

2.4. The coinvariant algebra of G_{12}. The ring of all G-invariant polynomials is generated by the following four homogeneous polynomials

$$
\begin{aligned}
&f_2 = x^2 + y^2 + z^2, \quad f_3 = xyz, \\
&f_4 = x^2y^2 + y^2z^2 + z^2x^2, \\
&f_6 = (x^2 - y^2)(y^2 - z^2)(z^2 - x^2)
\end{aligned}
\tag{2.1}
$$

where

$$
f_6^2 + 4f_4^3 - f_2^2 f_4^2 - 18 f_2 f_4 f_3^2 + 4 f_2^3 f_3^2 + 27 f_3^4 = 0. \tag{2.2}
$$

We note that the ring of M-invariant polynomials is generated by f_2, f_3, f_4 and $P = f_6 = (1/4)\,\mathrm{Jac}(f_2, f_3, f_4)$. The variety defined by the equation (2.2) is an irreducible singular variety $\mathbf{C}^3/G$ with non-isolated singularities.

Let $\mathfrak{n}$ be the ideal generated by these four polynomials and $S_G = S/\mathfrak{n}$ the coinvariant algebra of G. Then S_G is decomposed into irreducible components as in Table 2.2. We denote the homogeneous component of S_G of degree d by $\bar{S}_d$. We denote the ρ_k-component of $\bar{S}_d$ by $\bar{S}_d[\rho_k]$, which is in view of Table 2.2 irreducible except for $(d, k) = (3, 3)$ if it is nonzero. We identify S_1 and $\bar{S}_1 = \bar{S}_1[\rho_3]$.

For the notation we define

$$
f = x^2 + \omega y^2 + \omega^2 z^2, \quad \bar{f} = x^2 + \omega^2 y^2 + \omega z^2. \tag{2.3}
$$

We remark that

$$f\bar{f} = f_2^2 - 3f_4,$$
$$f^3 - \bar{f}^3 = \prod_{i=0}^{2}(f - \omega^i \bar{f}) = 3(\omega^2 - \omega)f_6,$$
$$f^3 + \bar{f}^3 = \prod_{i=0}^{2}(f + \omega^i \bar{f}) = 8f_3^2 - 4f_2f_4 + f_2^3.$$

$\bar{S}_d$	1_2	1_3	3	$\dim \bar{S}_d$	$\bar{S}_d[\rho]$
$\bar{S}_1$	0	0	1	3	$\{x, y, z\}$
$\bar{S}_2$	1	1	1	5	$\{f\} + \{\bar{f}\} + \{yz, zx, xy\}$
$\bar{S}_3$	0	0	2	6	$\{xf, \omega^2 yf, \omega zf\} + \{x\bar{f}, \omega y\bar{f}, \omega^2 z\bar{f}\}$
$\bar{S}_4$	1	1	1	5	$\{\bar{f}^2\} + \{f^2\} + \{yzf, \omega zxf, \omega^2 xyf\}$
$\bar{S}_5$	0	0	1	3	$\{x\bar{f}^2, \omega^2 y\bar{f}^2, \omega z\bar{f}^2\}$

TABLE 2.2
The coinvariant algebra of G_{12}

2.5. The exceptional locus. We have a natural morphism from $\mathrm{Hilb}^G(\mathbf{C}^3)$ onto the quotient $\mathbf{C}^3/G$. It is called the Hilbert-Chow morphism π, which is an isomorphism over $(\mathbf{C}^3 \setminus \{\text{Fixed points of } G\})/G$. Now we study the structure of the fibre of π over the origin. We define

$$(2.4)\qquad \begin{aligned} I([a:b]_{1_2}) &:= S\cdot(af + b\bar{f}^2) + \mathfrak{n} \quad (a \neq 0),\\ I([a:b]_{1_3}) &:= S\cdot(a\bar{f} + bf^2) + \mathfrak{n} \quad (a \neq 0),\\ I([a:b]_3) &:= S[G](ax\bar{f} + bxf) + Sf^2 + S\bar{f}^2 + \mathfrak{n},\\ J &:= S\cdot(xf, yf, zf) + S\cdot\bar{f}^2 + \mathfrak{n} = I([0:1]_3),\\ J' &:= S\cdot(x\bar{f}, y\bar{f}, z\bar{f}) + S\cdot f^2 + \mathfrak{n} = I([1:0]_3). \end{aligned}$$

THEOREM 2.6. *Let* $G = G_{12}$. *Then the fibre of the Hilbert Chow morphism* π *over the origin is one of the following*

$$I([a:b]_{1_2})\ (a \neq 0),\ I([a:b]_{1_3})\ (a \neq 0),\ I([a:b]_3).$$

Proof. Let $S = \mathbf{C}[x, y, z]$ and $\mathfrak{m}$ the maximal ideal of S defining the origin. Let I be any ideal in $\mathrm{Hilb}^G(\mathbf{C}^3)$ over the origin. Then $\mathfrak{n} \subset I \subset \mathfrak{m}$ because $S/I \simeq \mathbf{C}[G]$. Let $\bar{I} := I/\mathfrak{n}$. Since $S/\mathfrak{n} + S(f + b\bar{f}^2) \simeq \mathbf{C}[G]$ by Table 2.2, we have $I = \mathfrak{n} + S(f + b\bar{f}^2)$ if $f + b\bar{f}^2 \in I$. Similarly if $\bar{f} + bf^2 \in I$, then $I = \mathfrak{n} + S(\bar{f} + bf^2)$. Suppose $f + b\bar{f}^2 \notin I$ and $\bar{f} + bf^2 \notin I$ for any $b \in \mathbf{C}$. Then $f^2 \in I$ and $\bar{f}^2 \in I$, $\bar{S}_5[\rho_3] \subset \bar{I}$. If $\bar{S}_2[\rho_3] \subset \bar{I}$, then $\bar{S}_4[\rho_3] \subset I$ and $\bar{S}_3[\rho_3] \cap I \neq \{0\}$, which is absurd. Hence $\bar{S}_2[\rho_3] \not\subset \bar{I}$. Similarly we see that if $I \not\subset \mathfrak{m}^3 + \mathfrak{n}$, then we have a contradiction. Hence $I \subset \mathfrak{m}^3 + \mathfrak{n}$ and $\bar{I} \cap \bar{S}_3[\rho_3] \neq \{0\}$. It follows that $I = I([a:b]_3)$ for some $[a:b] \in \mathbf{P}^1$. This completes the proof. □

COROLLARY 2.7. *The fibre* $\pi^{-1}(0)$ *is a chain of three smooth rational curves intersecting transversally.*

Proof. We see that in $\mathrm{Hilb}^G(\mathbf{C}^3)$

$$\lim_{a\to 0} I([a:b]_{1_2}) = J, \quad \lim_{a\to 0} I([a:b]_{1_3}) = J'.$$

Let $C_{1_2} := \{I([a:b]_{1_2}), J; a \neq 0\} \simeq \mathbf{P}^1$, $C_{1_3} := \{I([a:b]_{1_3}), J'; a \neq 0\} \simeq \mathbf{P}^1$ and $C_3 := \{I([a:b]_3); [a:b] \in \mathbf{P}^1\} \simeq \mathbf{P}^1$. Then $C_{1_2} \cap C_{1_3} = \emptyset$, $C_{1_2} \cap C_3 = \{J\}$ and $C_{1_3} \cap C_3 = \{J'\}$. The intersection of C_k is transversal. In fact, the tangent space of $\mathrm{Hilb}^G(\mathbf{C}^3)$ is the direct sum of $\mathrm{Hom}(I[\rho_i], S/I[\rho_i])$ for ρ_i distinct, hence for instance at J, C_{1_2} and C_3 are transversal because so are $\mathrm{Hom}(I[\rho_{1_2}], S/I[\rho_{1_2}])$ ($\simeq \mathbf{C}$) and $\mathrm{Hom}(I[\rho_3], S/I[\rho_3])$ ($\simeq \mathbf{C}$). This completes the proof. See also the proof of Corollary 4.6. □

2.8. $\mathrm{Hilb}^{G/N}(\mathrm{Hilb}^N(\mathbf{C}^3))$. By [Nakamura98] $\mathrm{Hilb}^N(\mathbf{C}^3)$ is a crepant resolution of $\mathbf{C}^3/N$, on which G/N acts naturally. Hence $\mathrm{Hilb}^{G/N}(\mathrm{Hilb}^N(\mathbf{C}^3))$ is a crepant resolution of $\mathrm{Hilb}^N(\mathbf{C}^3)/(G/N)$, hence it is a crepant resolution of $\mathbf{C}^3/G$. Let $\phi : \mathrm{Hilb}^{G/N}(\mathrm{Hilb}^N(\mathbf{C}^3)) \to \mathbf{C}^3/G$ be the natural morphism, and $\psi : \mathrm{Hilb}^N(\mathbf{C}^3) \to \mathbf{C}^3/N$ the Hilbert-Chow morphism.

By [Nakamura98] the fan which describes the torus embedding $\mathrm{Hilb}^N(\mathbf{C}^3)$ is given by a decomposition of the triangle $\langle e_1, e_2, e_3\rangle$ in $\mathbf{R}^3$ by junior elements $\sigma_{12}, \sigma_{23}, \sigma_{31}$ of N:

$$\begin{aligned} \Delta_0 &= \langle \sigma_{12}, \sigma_{23}, \sigma_{31}\rangle, \\ \Delta_1 &= \langle \sigma_{23}, e_2, e_3\rangle, \\ \Delta_2 &= \langle \sigma_{31}, e_3, e_1\rangle, \\ \Delta_3 &= \langle \sigma_{12}, e_1, e_2\rangle. \end{aligned} \tag{2.5}$$

Let x, y, z be the standard coordinate of $\mathbf{C}^3$. The one-dimensional strata $\langle \sigma_{12}, \sigma_{23}\rangle$, $\langle \sigma_{23}, \sigma_{31}\rangle$, $\langle \sigma_{31}, \sigma_{12}\rangle$ corresponds to torus orbit rational curves C_1, C_2, C_3. The fibre of ψ over the origin is the union of C_i. The curves C_i meet at a unique point where they intersect as three axes in the affine space $\mathbf{C}^3$. We also note that the fixed locus of the action of N consists of three coordinate axes $\ell_x : x = 0$, $\ell_y : y = 0$ and $\ell_z : z = 0$ where the action of N reduces to a cyclic group of order two, say A_1. This implies that the structure of $\mathrm{Hilb}^N(\mathbf{C}^3)$ over the coordinate axis is a $\mathbf{P}^1$-bundle. Let $D_i := \psi^{-1}(\ell_i)$ $(i = x, y, z)$. The group $G/N \simeq \mathbf{Z}/3\mathbf{Z}$ permutes D_x, D_y and D_z cyclically.

We write the chart of $\mathrm{Hilb}^N(\mathbf{C}^3)$ corresponding to Δ_0 as

$$U_0 := \mathrm{Spec}\ \mathbf{C}[p, q, r]$$

where $p = yz/x, q = zx/y, r = xy/z$. The action of τ turns out to be $\tau^*(p) = q, \tau^*(q) = r, \tau^*(r) = p$. It follows that the fixed point locus of the (induced) action of τ on $\mathrm{Hilb}^N(\mathbf{C}^3)$ is $\ell : p = q = r$, along which the action of τ is A_2. Therefore the structure of $\mathrm{Hilb}^{G/N}(\mathrm{Hilb}^N(\mathbf{C}^3))$ over ℓ is a union of two $\mathbf{P}^1$-bundles E_1, E_2 meeting transversally along a section over ℓ. In particular the fibre of $(p, q, r) = (0, 0, 0)$ is the union of two rational curves m_1, m_2. G/N permutes the curves $n_i := D_i \cap \psi^{-1}(0)$ $(i = x, y, z)$, whence it yields a unique rational curve n on $\mathrm{Hilb}^{G/N}(\mathrm{Hilb}^N(\mathbf{C}^3))$. It follows that the fibre of ϕ over the origin is the union of three rational curves m_1, m_2 and n. Taking it into account that the geometry about m_i and n_j is G/N-symmetric, the three rational curves meet at a unique point. By a calculation we see that they meet as three coordinate axes of $\mathbf{C}^3$ at the intersection. We also see that the normal bundle of n in $\mathrm{Hilb}^{G/N}(\mathrm{Hilb}^N(\mathbf{C}^3))$ is $O_n(-1)^{\oplus 2}$. We also see that the exceptional divisors of ϕ are E_1,E_2 and D where D is the image of D_i mod G/N. The divisor D is a $\mathbf{P}^1$-bundle over $\ell \setminus \{0\}$ with $D_0 = \phi^{-1}(0) \cap D = \phi^{-1}(0)$. $\mathrm{Hilb}^G(\mathbf{C}^3)$ is obtained from $\mathrm{Hilb}^{G/N}(\mathrm{Hilb}^N(\mathbf{C}^3))$ by a flop with center n.

3. G_{60}.

3.1. Characters of G_{60}. Let $G = G_{60}$. The group G is isomorphic to the alternating group of degree 5 and is the normal subgroup of index 2 of the Coxeter group H_3. We note $H_3 = G \times \{\pm 1\}$.

Let $V = \mathbf{C}^3$, and $V^\vee$ the dual of V. The space $V^\vee$ is spanned by the dual basis x, y and z as before. By [YY93, p.72] G is realized as a subgroup of $\mathrm{SL}(V)$ generated by

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \epsilon^4 & 0 \\ 0 & 0 & \epsilon \end{pmatrix}, \; v = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \; \tau = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 & 1 \\ 2 & s & t \\ 2 & t & s \end{pmatrix}$$

where $\epsilon = \exp(2\pi i/5)$, $s = \epsilon^2 + \epsilon^3$ and $t = \epsilon + \epsilon^4$.

c. c.	1	$\sigma\tau$	v	σ	σ^2
age	0	1	1	1	1
$\sharp$	1	20	15	12	12
χ_1	1	1	1	1	1
χ_{3_1}	3	0	-1	$-s$	$-t$
χ_{3_2}	3	0	-1	$-t$	$-s$
χ_4	4	1	0	-1	-1
χ_5	5	-1	1	0	0

TABLE 3.1
Character table of G_{60}

Table 3.1 is the character table of G. The first (resp. the second, the third) line of Table 3.1 gives a representative of (resp. the age of, the number of elements in) each conjugacy class. For each irreducible character χ_k in Table 3.1, we choose and fix an irreducible representation ρ_k which affords χ_k. We note that $\bar{\rho}_k \simeq \rho_k$ for any k and $\rho = \rho_{3_1} \simeq \rho^\vee$. We also note that $\rho_{3_1}(g)$ has an eigenvalue one for g belonging to any junior conjugacy class, whence by [IR96], there is no compact irreducible divisor in the fibre of the Hilbert-Chow morphism over the origin. See Corollary 3.6.

3.2. The coinvariant algebra of G_{60}. The ring of invariant polynomials of G is generated by the four polynomials h_i $(i = 1, 2, 3, 4)$ of degrees 2, 6, 10 and 15 respectively ([YY93, pp. 72-74])

$$\begin{aligned} &h_1 = x^2 + yz, \; h_2 = 8x^4yz - 2x^2y^2z^2 - x(y^5 + z^5) + y^3z^3, \\ &h_3 = (1/25)(-256h_1^5 + \mathrm{BH}(h_2, h_1) + 480h_1^2h_2), \\ &h_4 = (1/10)\,\mathrm{Jac}(h_2, h_1, h_3), \end{aligned} \tag{3.1}$$

where $\mathrm{BH}(h_2, h_1)$ is the bordered Hessian of h_2, and h_1. See [YY93, p. 71].

Let $\mathfrak{n}$ be the ideal generated by these four polynomials and $S_G = S/\mathfrak{n}$ the coinvariant algebra of G. We can apply Section 1 to H_3 and G. Let $\bar{S}_d$ be the homogeneous component of S_G of degree d. Then $\bar{S}_d$ is decomposed into irreducible components as in Table 3.2.

$\bar{S}_d$	1	3_1	3_2	4	5	$\dim \bar{S}_d$
$\bar{S}_0$	1	0	0	0	0	1
$\bar{S}_1$	0	1	0	0	0	3
$\bar{S}_2$	0	0	0	0	1	5
$\bar{S}_3$	0	0	1	1	0	7
$\bar{S}_4$	0	0	0	1	1	9
$\bar{S}_5$	0	1	1	0	1	11
$\bar{S}_6$	0	1	0	1	1	12
$\bar{S}_7$	0	0	1	1	1	12
$\bar{S}_8$	0	0	1	1	1	12
$\bar{S}_9$	0	1	0	1	1	12
$\bar{S}_{10}$	0	1	1	0	1	11
$\bar{S}_{11}$	0	0	0	1	1	9
$\bar{S}_{12}$	0	0	1	1	0	7
$\bar{S}_{13}$	0	0	0	0	1	5
$\bar{S}_{14}$	0	1	0	0	0	3

TABLE 3.2
The coinvariant algebra of G_{60}

We will denote the ρ_k component of $\bar{S}_d$ by $\bar{S}_d[\rho_k]$, which is irreducible in view of Table 3.2 if it is nonzero. We identify S_1 and $\bar{S}_1$ and we decompose $S_1\bar{S}_d[\rho_k]$ for each d and k. The result is summarized in Diagram G_{60} in Subsection 4.7. Omitting the details we just mention how we calculated Diagram G_{60}.

1. Choose a monomial basis $x^iy^jz^k$ of $\bar{S}_d$ for each d.
2. Calculate the projections of each basis element to each component $\bar{S}_d[\rho_k]$, and thus we get a basis for every $\bar{S}_d[\rho_k]$.
3. Multiply the basis elements of $\bar{S}_d[\rho_k]$ by x, y, z and calculate again their projections to the components $\bar{S}_{d+1}[\rho_{k'}]$ for all k'.

In view of Theorem 1.6 (3), we only need to calculate $S_1\bar{S}_d$ up to $d = 7$ to complete Diagram G_{60}.

LEMMA 3.3. *Let I be a G-invariant ideal of S containing $\mathfrak{n}$ with $S/I \simeq \mathbf{C}[G]$ as G-modules. Then $I_b \subset I \subset I_t$, where I_b and I_t are the ideals of S containing $\mathfrak{n}$ such that*

$$(3.2)\qquad \begin{aligned} I_b/\mathfrak{n} &= \sum_{d=10}^{14} \bar{S}_d + \bar{S}_9[\rho_4] + \bar{S}_9[\rho_5], \\ I_t/\mathfrak{n} &= \sum_{d=7}^{14} \bar{S}_d + \bar{S}_6[\rho_{3_1}]. \end{aligned}$$

Proof. Let I be an ideal satisfying the conditions in the lemma and put $\bar{I} = I/\mathfrak{n}$.

Let $\rho = \rho_{3_1}$. We first consider the ρ component $\bar{I}[\rho]$ of $\bar{I}$. Since S/I is isomorphic to $\mathbf{C}[G]$ as a G-module, $\bar{I} \simeq \mathbf{C}[G]/\mathbf{C}$ and $\bar{I}[\rho] \simeq \rho^{\oplus 3}$ in view of Theorem 1.6. Take an irreducible (not necessarily homogeneous) submodule W of $\bar{I}[\rho]$ and let d_0 be the largest number such that $W \subset \bar{S}_{\geq d_0} := \sum_{d=d_0}^{14} \bar{S}_d$. Then by Diagram G_{60}, we see that $S_{14-d_0}W = \bar{S}_{14}$ and thus $\bar{S}_{14} \subset \bar{I}$. We may assume that $W \neq \bar{S}_{14}$, since $\bar{I}[\rho] \simeq \rho^{\oplus 3}$. Then again by Diagram G_{60} we see that $pr_{[3_1]}S_{10-d_0}W = \bar{S}_{10}[\rho] \mod \bar{S}_{14}$, where $pr_{[3_1]}$ is the projection onto the ρ_{3_1}-component, and thus $\bar{S}_{10}[\rho] + \bar{S}_{14}[\rho] \subset \bar{I}$.

Next we consider the ρ_{3_2} component of $\bar{I}$. Again by Diagram G_{60}, we see $\bar{S}_{12}[\rho_{3_2}] \subset \bar{I}$. By $\bar{I}[\rho_{3_2}] \simeq \rho_{3_2}^{\oplus 3}$, we can choose an irreducible submodule W of $\bar{I}[\rho_{3_2}]$ such that $W \neq \bar{S}_{12}[\rho_{3_2}]$. Then $pr_{[3_2]}S_{10-d_0}W = \bar{S}_{10}[\rho_{3_2}] \mod \bar{S}_{12}$ by a similar argument and by a similar definition of d_0. Hence $\bar{S}_{10}[\rho_{3_2}] + \bar{S}_{12}[\rho_{3_2}] \subset \bar{I}$.

Similar arguments can be applied to ρ_4 and ρ_5 to conclude $I_b \subset I$.

Next we prove $I \subset I_t$. Suppose that I is not contained in I_t. Then there is an irreducible G-module W of $\bar{I}$ which is not contained in $\bar{I}_t := I_t/\mathfrak{n}$. For instance assume $W \simeq \rho_{3_1}$. Since W is not contained in $\bar{I}_t[\rho_{3_1}] = \oplus_{d\geq 6}\bar{S}_d[\rho_{3_1}]$, by Diagram G_{60} we see $\bar{S}_9[\rho_{3_1}] \subset \bar{I}$, whence $\rho_{3_1}^{\oplus 4} \subset \bar{I}$, which is a contradiction. In the other cases we can proceed in the same way to derive a contradiction. This completes the proof. □

3.4. The eigenvectors $v_d(k)$. Let $1 \leq d \leq 14$, and $k \in \{3_1, 4, 5\}$ (or resp. $k = 3_2$). Suppose $\bar{S}_d[\rho_k] \neq 0$. Then we define $v_d[k]$ to be an eigenvector of $S_G(\rho^\vee)(\sigma^{-1})$ with eigenvalue ϵ (or resp. ϵ^2) in $\bar{S}_d[\rho_k]$. The eigenvector $v_d[k]$ is unique up to constant multiples.

Now we define

$$\begin{aligned} I([a:b]_{3_1}) &= S[G](av_6[3_1] + bv_9[3_1]) + \mathfrak{n}, \ (a \neq 0) \\ I([a:b]_k) &= S[G](av_7[k] + bv_8[k]) + \mathfrak{n}, \ (a \neq 0, k = 3_2, 4) \\ I([a:b]_5) &= S[G](av_7[5] + bv_8[5]) + \mathfrak{n}, \ (ab \neq 0) \\ I_1 &= S[G]v_7[5] + S[G]v_9[3_1] + \mathfrak{n}, \\ I_0 &= S[G]v_8[3_2] + S[G]v_8[4] + S[G]v_8[5] + \mathfrak{n} \end{aligned} \tag{3.3}$$

where $S[G] = S \otimes_{\mathbf{C}} \mathbf{C}[G]$. It is easy to check that all $I([a:b]_k)$ $(k = 3_1, 3_2, 4, 5)$ and I_i $(i = 0, 1)$ belong to $\mathrm{Hilb}^G(\mathbf{C}^3)$.

THEOREM 3.5. *Let $G = G_{60}$. Then the fiber of the Hilbert-Chow morphism π over the origin consists of the following ideals:*

$$I([a:b]_{3_1})\ (a \neq 0),\ I([a:b]_{3_2})\ (a \neq 0),$$
$$I([a:b]_4)\ (a \neq 0),\ I([a:b]_5)\ (ab \neq 0),\ I_1,\ I_0.$$

Proof. Let I be an ideal in $\mathrm{Hilb}^G(\mathbf{C}^3)$ over the origin and $\bar{I} = I/\mathfrak{n}$. Then since $\bar{I}_b := I_b/\mathfrak{n} \subset \bar{I} \subset \bar{I}_t$, by Lemma 3.3, $\bar{I}$ contains $W := \mathbf{C}[G](av_6[3_1] + bv_9[3_1])$ for some $[a:b] \in \mathbf{P}_1$. If $a \neq 0$, then $SW + \mathfrak{n}$ is G-stable and $S/(SW + \mathfrak{n})$ is isomorphic to $\mathbf{C}[G]$. It follows that $I = SW + \mathfrak{n} = I([a:b]_{3_1})$.

Next we assume $a = 0$. Hence $v_9[3_1] \in I$ and $\bar{I}[\rho_{3_1}] = \oplus_{d\geq 9}\bar{S}_d[\rho_{3_1}] \simeq \rho_{3_1}^{\oplus 3}$.

By Lemma 3.3 $(\bar{S}_{10}+\bar{S}_{12})[\rho_{3_2}] \subset \bar{I}$. Hence $\bar{I}$ contains $W := \mathbf{C}[G](av_7[3_2]+bv_8[3_2])$ for some $[a:b] \in \mathbf{P}_1$. If $a \neq 0$, we see that $I = SW + \mathfrak{n} = I([a:b]_{3_2})$. Now we assume

$a = 0$. Then $v_8[3_2] \in \bar{I}$ and $\bar{I}[\rho_{3_2}] = \oplus_{d\geq 8}\bar{S}_d[\rho_{3_2}] \simeq \rho_{3_2}^{\oplus 3}$. Hence by Diagram G_{60} we see $\bar{S}_d[\rho_4] \subset \bar{I}$ for $d \geq 9$. Hence there exists $[a:b] \in \mathbf{P}^1$ such that $av_7[4] + bv_8[4] \in \bar{I}$. If $a \neq 0$, then $I = I([a:b]_4)$. If $a = 0$, then $\bar{S}_d[\rho_4] \subset \bar{I}$ for $d \geq 8$ while $\bar{S}_d[\rho_5] \subset \bar{I}$ for $d \geq 9$. Hence there is $[a:b] \in \mathbf{P}^1$ such that $av_7[5] + bv_8[5] \in \bar{I}$. If moreover $ab \neq 0$, then $I = I([a:b]_5)$. If $b = 0$, then $\bar{S}_d[\rho_4] \subset \bar{I}$ for $d \geq 8$ and $\bar{S}_d[\rho_4] \subset \bar{I}$ for $d \geq 9$ and $k = 3_1, 3_2, 5$. Hence $I = I_1$. If $a = 0$, then $\bar{S}_d[\rho_k] \subset \bar{I}$ for $d \geq 8$ and $k = 4, 5$. Thus we see $I = I_0 = \oplus_{d\geq 8}\bar{S}_d + \mathfrak{n}$. This completes the proof. □

COROLLARY 3.6. *The fibre $\pi^{-1}(0)$ is a connected curve consisting of four smooth rational curves. Three of the four meet at a point as three coordinate axes of $\mathbf{C}^3$, while two of the four intersect transversally at another point.*

Proof. Let $C_{3_1} = \{I_1, I([a:b]_{3_1}), a \neq 0\}$, $C_k = \{I_0, I([a:b]_k), a \neq 0\}$ $(k = 3_2, 4)$ and $C_5 = \{I_0, I_1, I([a:b]_5), ab \neq 0\}$. Then C_k is a smooth rational curve. In fact, we easily see

$$\lim_{a\to 0} I([a:b]_{3_1}) = \lim_{b\to 0} I([a:b]_5) = I_1,$$
$$\lim_{a\to 0} I([a:b]_k)) = I_0 \quad (k = 3_2, 4, 5).$$

This completes the proof. See also the proof of Corollary 4.6. □

4. G_{168}.

4.1. Characters of G_{168}. Let $G = G_{168}$. The group G is isomorphic to the simple group PSL(2, 7). Let $V = \mathbf{C}^3$ and $V^\vee$ the dual of V. The space $V^\vee$ is spanned by x, y and z as before. The group G is realized as a subgroup of SL(V) generated by the elements given below :

$$\sigma = \begin{pmatrix} \beta & 0 & 0 \\ 0 & \beta^2 & 0 \\ 0 & 0 & \beta^4 \end{pmatrix}, \tau = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$v = \frac{-1}{\sqrt{-7}} \begin{pmatrix} \beta^4 - \beta^3 & \beta^2 - \beta^5 & \beta - \beta^6 \\ \beta^2 - \beta^5 & \beta - \beta^6 & \beta^4 - \beta^3 \\ \beta - \beta^6 & \beta^4 - \beta^3 & \beta^2 - \beta^5 \end{pmatrix},$$

where $\beta = \exp(2\pi i/7)$. See [YY93, p. 74]. Let $M = G \times \{\pm 1\}$. Then $[M : G] = 2$ and M is a complex reflection group with Shephard-Todd number 24 [ST54, p. 301]. Hence we can apply Section 1 to G.

Table 4.1 is the character table of G, where $a = \beta + \beta^2 + \beta^4$. We choose β as a primitive 7-th root of unity to define a conjugacy class to be junior or senior in Table 4.1. For each irreducible character χ_k in Table 4.1, we choose and fix an irreducible representation ρ_k which affords χ_k, in particular $\rho_{3_1} = \rho^\vee$ and $\rho_{3_2} = \rho$. We note $\bar{\rho}_{3_1} \simeq \rho_{3_2}$. There is a unique junior conjugacy class with all eigenvalues of ρ_{3_1} distinct from one. Hence by [IR96], there is a unique compact irreducible divisor in the fibre of the Hilbert-Chow morphism over the origin. See Corollary 4.6.

c. c.	1	v	τ	σv	σ	σ^3
age	0	1	1	1	1	2
$\sharp$	1	21	56	42	24	24
χ_1	1	1	1	1	1	1
χ_{3_1}	3	-1	0	1	-1-a	a
χ_{3_2}	3	-1	0	1	a	-1-a
χ_6	6	2	0	0	-1	-1
χ_7	7	-1	1	-1	0	0
χ_8	8	0	-1	0	1	1

TABLE 4.1
Character table of G_{168}

4.2. The coinvariant algebra of G. By [YY93] the ring of G-invariant polynomials is generated by the four polynomials h_i ($i = 1, 2, 3, 4$) of degrees 4, 6, 14 and 21 respectively:

$$\begin{aligned} h_1 &= xy^3 + yz^3 + zx^3, \quad h_2 = 5x^2y^2z^2 - x^5y - y^5z - z^5x, \\ h_3 &= (1/9)\,\mathrm{BH}(h_1, h_2), \; h_4 = (1/4)\,\mathrm{Jac}(h_1, h_2, h_3). \end{aligned} \tag{4.1}$$

$\bar{S}_d$	1	3_1	3_2	6	7	8	$\dim \bar{S}_d$
$\bar{S}_0$	1	0	0	0	0	0	1
$\bar{S}_1$	0	1	0	0	0	0	3
$\bar{S}_2$	0	0	0	1	0	0	6
$\bar{S}_3$	0	0	1	0	1	0	10
$\bar{S}_4$	0	0	0	1	0	1	14
$\bar{S}_5$	0	0	1	0	1	1	18
$\bar{S}_6$	0	0	0	1	1	1	21
$\bar{S}_7$	0	0	0	0	1	2	23
$\bar{S}_8$	0	1	0	1	1	1	24
$\bar{S}_9$	0	1	0	1	1	1	24
$\bar{S}_{10}$	0	0	1	1	1	1	24
$\bar{S}_{11}$	0	1	0	1	1	1	24
$\bar{S}_{12}$	0	0	1	1	1	1	24
$\bar{S}_{13}$	0	0	1	1	1	1	24
$\bar{S}_{14}$	0	0	0	0	1	2	23
$\bar{S}_{15}$	0	0	0	1	1	1	21
$\bar{S}_{16}$	0	1	0	0	1	1	18
$\bar{S}_{17}$	0	0	0	1	0	1	14
$\bar{S}_{18}$	0	1	0	0	1	0	10
$\bar{S}_{19}$	0	0	0	1	0	0	6
$\bar{S}_{20}$	0	0	1	0	0	0	3

TABLE 4.2
The coinvariant algebra of G_{168}

Let S_G be the coinvariant algebra of G, and $\bar{S}_d$ the homogeneous component of S_G of degree d. $\bar{S}_d$ is decomposed into irreducible components $\bar{S}_d[\rho_k]$ as in Table 4.2. In view of Table 4.2, $\bar{S}_d[\rho_k]$ is irreducible except when it is zero, or $(d, k) = (7, 8)$, $(14, 8)$.

We give the irreducible decomposition of $S_1\bar{S}_d[\rho_k]$ for each (d,k) in Diagram G_{168} in Subsection 4.7.

For $(d,k) = (7,8)$ or $(14,8)$, the decomposition of $S_1\bar{S}_d[\rho_k]$ is given more precisely as follows. $W_{7,\ell} := pr_{[8]}S_1\bar{S}_6[\rho_\ell]$ is irreducible and isomorphic to ρ_8 for $\ell = 6,7,8$ and they are however all distinct. Moreover

$$S_1W_{7,6} = \bar{S}_8[\rho_6] + \bar{S}_8[\rho_7] + \bar{S}_8[\rho_8],$$
$$S_1W_{7,\ell} = \bar{S}_8, \quad \text{for} \quad \ell = 7,8.$$

Similarly $W_{14,\ell} := pr_{[8]}S_1\bar{S}_{13}[\rho_\ell]$ is irreducible and isomorphic to ρ_8 for $\ell = 3_2, 6, 7, 8$ and they are all distinct. Moreover

$$S_1W_{14,3_2} = \bar{S}_{15}[\rho_7] + \bar{S}_{15}[\rho_8],$$
$$S_1W_{14,\ell} = \bar{S}_{15}, \quad \text{for} \quad \ell = 6,7,8.$$

In a manner similar to the case of G_{60}, we can prove the following by chasing Diagram G_{168}.

LEMMA 4.3. *Let I be a G-invariant ideal of S containing $\mathfrak{n}$ with $S/I \simeq \mathbf{C}[G]$ as G-modules. Then $I_b \subset I \subset I_t$, where I_b and I_t are ideals of S containing $\mathfrak{n}$ such that*

$$I_b/\mathfrak{n} = \bar{S}_{\geq 14} + \sum_{k=6,7,8} \bar{S}_{13}[\rho_k] + \sum_{k=7,8} \bar{S}_{12}[\rho_k], \tag{4.2}$$
$$I_t/\mathfrak{n} = \bar{S}_{\geq 10} + \bar{S}_9[\rho_6] + \bar{S}_9[\rho_{3_1}] + \bar{S}_8[\rho_{3_1}].$$

4.4. The eigenvectors $v_d(k)$. Let $1 \leq d \leq 20$ and $k = 3_1, 3_2, 6, 7, 8$. Suppose $(d,k) \neq (7,8), (14,8)$ and $\bar{S}_d[\rho_k] \neq 0$. Then we define $v_d[k]$ to be an eigenvector of $S_G(\rho^\vee)(\sigma^{-1})$ with eigenvalue β (resp. β^3) in $\bar{S}_d[\rho_k]$ for $k \neq 3_2$ (resp. $k = 3_2$). The eigenvector $v_d[k]$ is unique up to constant multiples. Also we define $v'_d[6]$ (resp. $v''_d[7]$) to be an eigenvector of $S_G(\rho^\vee)(\sigma^{-1})$ with eigenvalue β^2 (resp. β^0) in $\bar{S}_d[\rho_6]$ (resp. $\bar{S}_d[\rho_7]$).

Here is a list of some of these polynomials used later. Notice that these polynomials are in $\bar{S}$, i.e. taken modulo $\mathfrak{n}$.

$$\begin{aligned}
v_8[3_1] &= 3x^8 - 29xz^7 - 112y^5z^3 - 149xy^7,\\
v_9[3_1] &= 48y^7z^2 - 6xy^2z^6 + z^9 - 198x^2y^4z^3,\\
v_{11}[3_1] &= -198x^2yz^8 - 990xy^6z^4 + y^{11} - 748y^4z^7,\\
v_{10}[3_2] &= -264y^8z^2 + x^{10} - 116xy^3z^6 + 161yz^9,\\
v_{12}[3_2] &= 3y^{12} - 11693xy^7z^4 - 344xz^{11} - 9988y^5z^7,\\
v_{13}[3_2] &= 2z^{13} + 3315xy^9z^3 + 559xy^2z^{10} + 3198y^7z^6,\\
v'_9[6] &= x^9 + 40x^2z^7 + 360xy^5z^3 + 298y^3z^6,\\
v'_{10}[6] &= 4y^5z^5 + 5xy^7z^2,\\
v'_{11}[6] &= 538xy^9z - 68xy^2z^8 + 338y^7z^4 + 3z^{11},\\
v'_{12}[6] &= 49xy^4z^7 + 10y^9z^3 - 6y^2z^{10},\\
v''_{10}[7] &= 19xy^8z + 4xyz^8 + 14y^6z^4,\\
v''_{11}[7] &= 34xy^3z^7 + 7y^8z^3 - 5yz^{10}.
\end{aligned} \tag{4.3}$$

Now we define G-invariant ideals of S by

$$
(4.4)\quad
\begin{aligned}
I([a:b:c]_{3_1}) &:= S[G](av_8[3_1] + bv_9[3_1] + cv_{11}[3_1]) + \mathfrak{n},\\
I([a:b]_6) &:= S[G](av'_{10}[6] + bv'_{11}[6]) + S[G]v_{11}[3_1] + \mathfrak{n},\\
I([a:b]_7) &:= S[G](av''_{10}[7] + bv''_{11}[7]) + S[G]v_{11}[3_1] + \mathfrak{n},\\
I([a:b]_8) &:= S[G](av_{10}[8] + bv_{11}[8]) + \mathfrak{n},\\
I_0 &:= \sum_{k=3_1,6,7,8} S[G]v_{11}[k] + \mathfrak{n}
\end{aligned}
$$

where $S[G] = S \otimes_{\mathbf{C}} \mathbf{C}[G]$, $[a:b:c]_{3_1} \neq [0:0:1]$ and $[a:b]_k \neq [0:1]$. It is not difficult to check by using Diagram G_{168} that these ideals belong to $\mathrm{Hilb}^G(\mathbf{C}^3)$. Then we have

THEOREM 4.5. *Let $G = G_{168}$. Then the fiber of the Hilbert-Chow morphism π over the origin consists of the following ideals:*

$$
\begin{aligned}
&I([a:b:c]_{3_1}) \quad (a,b) \neq (0,0),\\
&I_0,\ I([a:b]_k) \quad a \neq 0,\ k = 6,7,8.
\end{aligned}
$$

Proof. Let I be an ideal in $\mathrm{Hilb}^G(\mathbf{C}^3)$ over the origin and $\bar{I} = I/\mathfrak{n}$. In view of Lemma 4.3 $\bar{I}$ contains $v := av_8[3_1] + bv_9[3_1] + cv_{11}[3_1]$ for some $[a:b:c] \in \mathbf{P}_2$. Let $W = \mathbf{C}[G]v$ and $W^* = \sum_{\ell=1}^4 S_\ell W$. We have $[SW : \rho_{3_2}] \leq 3$, because $\bar{I} \simeq \mathbf{C}[G]/\mathbf{C}$. Since $\bar{S}_{20}[\rho_{3_1}] \subset SW$, the multiplicity $[W^* : \rho_{3_2}]$ is at most two. To calculate the multiplicity, we consider the eigenspace of $S_G(\rho^\vee)(\sigma^{-1})$ in $pr_{[3_2]}W^*$ with eigenvalue β^3. It is easy to see that this space is spanned by the images by $pr_{[3_2]}$ of the four vectors yv, x^2v, xy^2zv and z^4v. By calculation it turns out that

$$
\begin{aligned}
pr_{[3_2]}(yv) &= -28bv_{10}[3_2] + (28/3)cv_{12}[3_2],\\
pr_{[3_2]}(x^2v) &= 42av_{10}[3_2] + 7cv_{13}[3_2],\\
pr_{[3_2]}(xy^2zv) &= -2av_{12}[3_2] - bv_{13}[3_2],\\
pr_{[3_2]}(z^4v) &= -3pr_{[3_2]}(xy^2zv).
\end{aligned}
$$

We easily see

$$
pr_{[3_2]}(3ayv + 2bx^2v + 14cxy^2zv) = 0,
$$

whence the β^3-eigenspace of $S_G(\rho^\vee)(\sigma^{-1})$ in $pr_{[3_2]}W^*$ is exactly 2-dimensional. Since W^* is a G-module, $pr_{[3_2]}$ is an isomorphism on $W^*[\rho_{3_2}]$. Hence $[SW : \rho_{3_2}] = 3$ for any $[a:b:c] \in \mathbf{P}_2$. Similarly it can be verified that $pr_{[6]}S_{\leq 3}W$ is generated by

$$
\begin{aligned}
(1/84)pr_{[6]}(xv) &= 3av'_9[6] + 49bv'_{10}[6] + 49cv'_{12}[6],\\
(-1/4)pr_{[6]}(y^2zv) &= av'_{11}[6] + 49bv'_{12}[6]
\end{aligned}
$$

as an $S[G]$-module. Hence $[SW : \rho_6] = 6$ if $(a,b) \neq (0,0)$. Also we see $[SW : \rho_k] = k$ for $k = 7$ and 8 if $(a,b) \neq (0,0)$. Thus if $(a,b) \neq (0,0)$, then $SW \simeq \mathbf{C}[G]/\mathbf{C}$ and $I = SW + \mathfrak{n} = I([a:b:c]_{3_1})$.

Now we assume $(a,b) = (0,0)$. Then $\bar{I}[\rho_{3_1}] = \oplus_{k\geq 11}\bar{S}_k[\rho_{3_1}]$. It follows $\bar{I}[\rho_{3_2}] = \oplus_{k\geq 12}\bar{S}_k[\rho_{3_2}]$, and $\bar{S}_k[\rho_6] \subset \bar{I}[\rho_6]$ for $k \geq 12$. Since $\bar{I} \simeq \mathbf{C}[G]/\mathbf{C}$, there exists $av'_{10}[6]+$

$bv'_{11}[6] \in \bar{I}$ for some $[a:b] \in \mathbf{P}^1$ by Lemma 4.3. If $a \neq 0$, then we see $I = I([a:b]_6)$. If $a = 0$, then $\bar{I}[\rho_6] = \oplus_{k \geq 11} \bar{S}_k[\rho_6]$ and $\bar{S}_k[\rho_7] \subset \bar{I}[\rho_7]$ for $k \geq 12$.

It follows from Lemma 4.3 that there exists $av''_{10}[7] + bv''_{11}[7] \in \bar{I}$ for some $[a : b] \in \mathbf{P}^1$. If $a \neq 0$, then we see $I = I([a:b]_7)$. If $a = 0$, then $\bar{I}[\rho_7] = \oplus_{k \geq 11} \bar{S}_k[\rho_7]$ and $\bar{S}_k[\rho_8] \subset \bar{I}[\rho_8]$ for $k \geq 12$. Therefore there exists $av_{10}[8] + bv_{11}[8] \in \bar{I}$ for some $[a:b] \in \mathbf{P}^1$ by Lemma 4.3. If $a \neq 0$, then we see $I = I([a:b]_8)$. If $a = 0$, we see $I = I_0$. This completes the proof. □

COROLLARY 4.6. *The fibre $\pi^{-1}(0)$ is the union of a smooth rational curve and a doubly blown-up projective plane with infinitely near centers, both intersecting transversally at a unique point.*

Proof. Let $C_8 := \{I_0, I([a:b]_8)\ (a \neq 0)\}$. Then $C_8 \simeq \mathbf{P}^1$ because $I_0 = \lim_{a \to 0} I([a:b]_8)$. To be more precise we construct a G-invariant zero-dimensional subscheme Z of $C_8 \times \mathbf{C}^3$ flat over C_8 such that the fibre of Z over $I \in C_8$ is Spec S/I. For the purpose we define a G-invariant ideal $\mathcal{I}$ of $O_{C_8} \otimes_{\mathbf{C}} S$ by

$$\mathcal{I} = O_{C_8}(-1)I([a:b]_8) + O_{C_8}\Big(\sum_{k=3_1,6,7} S[G]v_{11}[k] + \mathfrak{n}\Big).$$

We note that $\mathcal{I}_{[a:b]\times\mathbf{C}^3} = I([a:b]_8)$ if $a \neq 0$, while $\mathcal{I}_{[0:1]\times\mathbf{C}^3} = I_0$. Since $\dim S/\mathcal{I}_{[a:b]}$ is constant ($= 168$) on C_8, the subscheme Z of $C_8 \times \mathbf{C}^3$ defined by $\mathcal{I}$ is flat over C_8.

Next let T be a doubly blown-up projective plane with infinitely near centers and we construct a G-invariant zero-dimensional subscheme Z of $T \times \mathbf{C}^3$ flat over T such that

(i) any fibre of Z over T is one of the subschemes Spec S/I defined by the ideals I among $I([a:b:c]_{3_1})$, $I([a:b]_k)$ $(k = 6, 7)$ and I_0,
(ii) the natural morphism ϕ of T into $\mathrm{Hilb}^G(\mathbf{C}^3)$ is a closed immersion,
in other words,
(ii′) ϕ is an injection and for any $t \in T$ the Kodaira-Spencer map of ϕ is a monomorphism of the tangent space $T_t(T)$ of T at t into the subspace $\mathrm{Hom}_S(I/\mathfrak{n}, S/I)$ of $\mathrm{Hom}_S(I, S/I)$ where I is the unique ideal of S such that the fibre of Z over t is Spec S/I.

In fact, (ii′) is easy to check by the construction of Z below.

For the purpose let $U = \mathbf{P}^2 \setminus \{p_0\}$ and we first define a G-invariant ideal $\mathcal{I}_U$ on $U \times \mathbf{C}^3$ by

$$\mathcal{I}_U := O_U(-1)I([a:b:c]_{3_1}) + O_U\mathfrak{n}$$

where $O_U(-1) = O_{\mathbf{P}^2}(-1)_{|U}$ and $[a:b:c] \in \mathbf{P}^2$. Then we extend it to T so that $(O_T \otimes_{\mathbf{C}} S)/\mathcal{I}$ may be O_T-flat.

This is done as follows. Let $p_0 := [0:0:1] \in \mathbf{P}^2$ and let ℓ_0 be the (-1)-curve on $Q_{p_0}(\mathbf{P}^2)$. Let p_1 be a point of ℓ_0 and $T = Q_{p_1}Q_{p_0}(\mathbf{P}^2)$. Since T is a torus embedding, we may assume that we can choose an open covering $\{U, U_1, U_2, U_3\}$ of T such that $U = \mathbf{P}^2 \setminus \{p_0\}$, $U_k = \mathrm{Spec}\ \mathbf{C}[s_k, t_k]$ where

$$p = a/c,\ q = b/c,$$
$$s_1 = p,\ t_1 = q/p,\ s_2 = p/q,\ t_2 = q^2/p,\ s_3 = p/q^2,\ t_3 = q.$$

Let $v = s_1v_8[3_1] + s_1t_1v_9[3_1] + v_{11}[3_1]$. We define on U_1

$$\mathcal{I} = O_T \otimes S[G]v + O_T\mathfrak{n} + O_T \otimes S[G](3v''_{10}[7] + 7t_1v''_{11}[7]).$$

We see $\mathcal{I} = \mathcal{I}_U$ on $(U \cap U_1) \times \mathbf{C}^3$ because if $s_1 \neq 0$, then

$$(-1/56s_1)pr_{[7]}(yzv) = 3v''_{10}[7] + 7t_1v''_{11}[7].$$

We note that $\mathcal{I}_{\tau\times\mathbf{C}^3} = I([3 : 7t_1]_7)$ where $\tau := (s_1, t_1) = (0, t_1) \in U_1$. It is evident that $\mathcal{I} = O_T \otimes S[G]I([s_1, s_1t_1, 1]_{3_1})$ if $(s_1, t_1) \in U_1$ and $s_1 \neq 0$.

Let $v = s_2^2t_2v_8[3_1] + s_2t_2v_9[3_1] + v_{11}[3_1]$. Next we define on U_2

$$\begin{aligned}\mathcal{I} = O_T \otimes S[G]v + O_T\mathfrak{n} + O_T \otimes S[G](3s_2t_2v'_9[6] + 49t_2v'_{10}[6] - v'_{11}[6])\\ + O_T \otimes S[G](3s_2v''_{10}[7] + 7v''_{11}[7]) + O_T \otimes S[G]v_{11}[8].\end{aligned}$$

We see $\mathcal{I} = \mathcal{I}_U$ on $(U \cap U_2) \times \mathbf{C}^3$ because if $s_2t_2 \neq 0$, then

$$\begin{aligned}-(1/56s_2t_2)pr_{[7]}(yzv) &= 3s_2v''_{10}[7] + 7v''_{11}[7],\\ (1/84)pr_{[6]}(xv) &= 3s_2^2t_2v'_9[6] + 49s_2t_2v'_{10}[6] + 49v'_{12}[6],\\ -(1/4s_2t_2)pr_{[6]}(y^2zv) &= s_2v'_{11}[6] + 49v'_{12}[6],\end{aligned}$$

whence

$$\begin{aligned}(1/84s_2)pr_{[6]}(xv) + (1/4s_2^2t_2)pr_{[6]}(y^2zv)\\ = 3s_2t_2v'_9[6] + 49t_2v'_{10}[6] - v'_{11}[6].\end{aligned}$$

We note that $\mathcal{I}_{\tau\times\mathbf{C}^3} = I_0$ if $\tau := (s_2, t_2) = (0, 0)$ while $\mathcal{I}_{\tau\times\mathbf{C}^3} = I([3s_2 : 7]_7)$ (resp. $I([49t_2 : (-1)]_6)$ if $\tau = (s_2, 0), s_2 \neq 0$ resp. if $\tau = (0, t_2)$, $t_2 \neq 0$.

Finally let $v = s_3t_3^2v_8[3_1] + t_3v_9[3_1] + v_{11}[3_1]$. We define on U_3

$$\begin{aligned}\mathcal{I} = O_T \otimes S[G]v + O_T\mathfrak{n}\\ + O_T \otimes S[G](3s_3t_3v'_9[6] + 49v'_{10}[6] - s_3v'_{11}[6]).\end{aligned}$$

It is easy to see that $\mathcal{I} = \mathcal{I}_U$ on $(U \cap U_3) \times \mathbf{C}^3$. We note that $\mathcal{I}_{\tau\times\mathbf{C}^3} = I([49 : (-s_3)]_6)$ for $\tau = (s_3, 0) \in U_3$.

Let Z be a subscheme of $T\times\mathbf{C}^3$ defined by $\mathcal{I}$. Then Z is T-flat because $(O_T\otimes S)/\mathcal{I}$ is locally O_T-free of rank 168. This completes the proof. □

4.7. Diagrams. Diagram G_{60} and Diagram G_{168} express the decomposition of $S_1\bar{S}_i[j]$. The rows are indexed by degrees and the columns by irreducible representations. Each vertex corresponds to $\bar{S}_i[j]$ and we join $\bar{S}_i[j]$ and $\bar{S}_{i+1}[k]$ when $\bar{S}_{i+1}[k]$ appears in $S_1\bar{S}_i[j]$. Two double circles in Diagram G_{168} mean that the multiplicities of ρ_8 in $\bar{S}_7$ and $\bar{S}_{14}$ are equal to two. We note that Diagram G_{60} is symmetric with center at degrees 7 and 8 because $\bar{\rho}_k \simeq \rho_k$ for any irreducible representation ρ_k. However Diagram G_{168} loses apparent symmetry with center at degrees 10 and 11 because $\bar{\rho}_{3_1} \simeq \rho_{3_2}$.

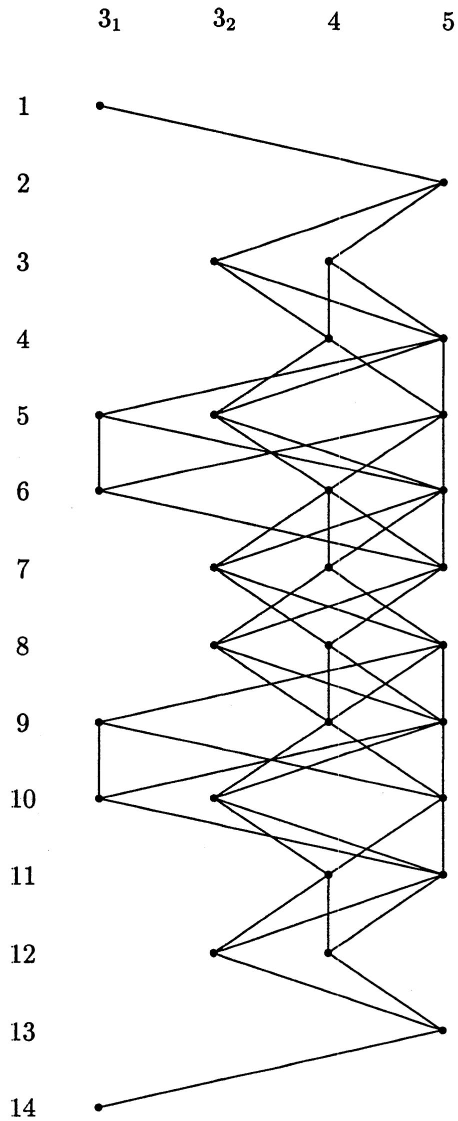

Diagram G_{60}

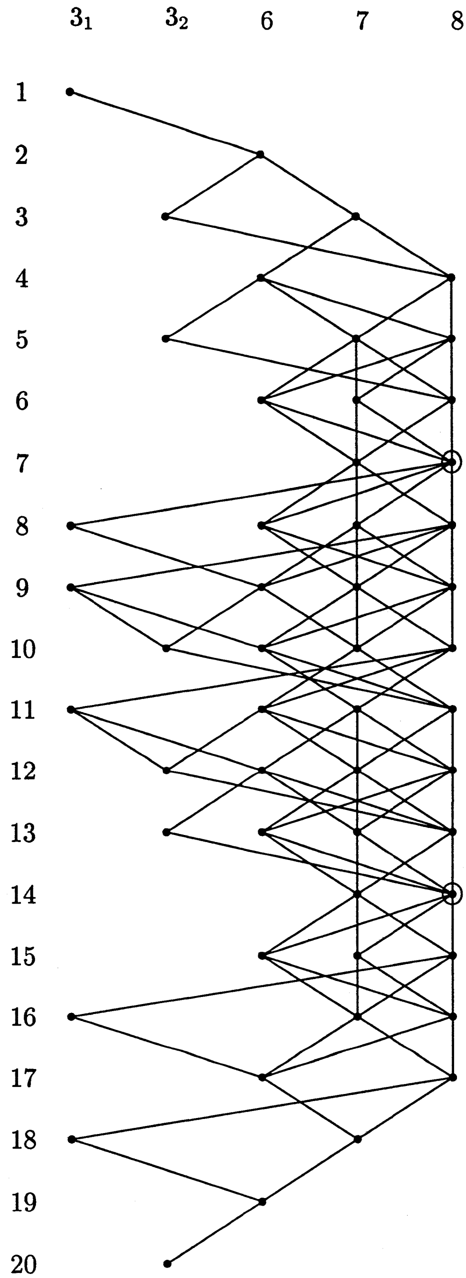

Diagram G_{168}

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