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NASH RESIDUES OF SINGULAR HOLOMORPHIC FOLIATIONS*

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Interesting invariants such as the local Euler obstruction of a singular variety arise considering the Nash modification of the variety ([Br2], [M]). Also, taking a Nash type modification with respect to a coherent sheaf, we have further insight to the characteristic classes of coherent sheaves ([K], [Sc]). The purpose of this paper is to study a similar modification associated to a singular holomorphic foliation and to analyze related local invariants, namely, the Baum-Bott residues [BB] and the Nash residues, which will be introduced in this paper. As an application, we give a partial answer to the rationality conjecture of Baum and Bott (see 2.5 below).

Let $\mathcal{F}$ be a singular holomorphic foliation on a complex manifold M. The Baum-Bott residues are basically the characteristic classes of the normal sheaf $\mathcal{N}_{\mathcal{F}}$ of the foliation localized at the singular set $S(\mathcal{F})$ of $\mathcal{F}$. In section 2, we recall the Baum-Bott residues in the framework of Čech-de Rham cohomology, which is slightly different from the original one in [BB]. The method is suitable also for dealing with problems on singular varieties and is reviewed in section 1 (see also [Su2]). In a more precise way, for each compact connected component S of $S(\mathcal{F})$ and for a homogeneous symmetric polynomial of certain degree, we have the residue in the homology of S, with $\mathbb{C}$-coefficients in general. Despite their importance, not much is known about these residues. In order to analyze them, we consider, in section 3, the Nash type modification M^ν of M with respect to the foliation $\mathcal{F}$. The advantage of doing this is that the coherent sheaf $\mathcal{N}_{\mathcal{F}}$ is modified to a locally free sheaf $\mathcal{N}_{\mathcal{F}}^\nu$ on M^ν, although M^ν acquires singularity in general. For each compact connected component of the transform of $S(\mathcal{F})$, we define the residues as localized characteristic classes of $\mathcal{N}_{\mathcal{F}}^\nu$ and call them Nash residues. In the case M^ν is compact, we have the residue formula in the homology of M^ν (Theorem 3.4). This generalizes the work in [Se], where M^ν is assumed to be non-singular. We compare, in section 4, the Baum-Bott and Nash residues and prove that, for a polynomial with rational coefficients, the difference of the corresponding residues is a rational class (Theorem 4.1). This immediately implies the aforementioned answer to the rationality conjecture (Corollary 4.7). In some cases we can compute the differences of the two residues explicitly and can show how it is related to other familiar invariants.

1. Characteristic classes on singular varieties. As to the theory of characteristic classes, we use the Chern-Weil theory modified to fit in the framework of Čech-de Rham cohomology. For the Chern-Weil theory of characteristic classes of vector bundles, we refer to [BB], [Bo] and [MS]. For the background on the Čech-de Rham cohomology, we refer to [BT]. The integration and characteristic classes in this cohomology theory are first studied in [L1-4]. See also [Su2-3] for these material. We use the notation and facts there.

(A) Poincaré and Alexander homomorphisms. Let V be an analytic subvariety of pure dimension n in a complex manifold W of dimension $m = n + k$. We set

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$V' = V \setminus \mathrm{Sing}(V)$. First, suppose V is compact and let $\tilde{U}$ be a regular neighborhood of V in W. Also, let $\tilde{\mathcal{U}} = \{\tilde{U}_\alpha\}_{\alpha \in I}$ be an open covering of $\tilde{U}$. We denote by $(A^*(\tilde{\mathcal{U}}), D)$ the Čech-de Rham complex associated to $\tilde{\mathcal{U}}$ and by $H^*(A^*(\tilde{\mathcal{U}}))$ its cohomology, which is canonically isomorphic with the de Rham cohomology $H^*(\tilde{U}, \mathbb{C})$. Since V is deformation retract of $\tilde{U}$, the last cohomology is isomorphic with the cohomology $H^*(V, \mathbb{C})$ of V, e.g., in the singular cohomology theory. Furthermore, let $\{\tilde{R}_\alpha\}_{\alpha \in I}$ be a system of honey-comb cells adapted to $\tilde{\mathcal{U}}$ ([L1-4], see also [Su2, Ch.II, 3]) such that V is transverse to each $\tilde{R}_{\alpha_0 \cdots \alpha_p} = \tilde{R}_{\alpha_0} \cap \cdots \cap \tilde{R}_{\alpha_p}$. We set $R_{\alpha_0 \cdots \alpha_p} = \tilde{R}_{\alpha_0 \cdots \alpha_p} \cap V$. Then we may define the integration

$$\int_V : H^{2n}(A^*(\tilde{\mathcal{U}})) \to \mathbb{C}$$

as in [Su2, Ch.IV, 2]. Also the bilinear pairing

$$A^\ell(\tilde{\mathcal{U}}) \times A^{2n-\ell}(\tilde{\mathcal{U}}) \to A^{2n}(\tilde{\mathcal{U}}) \to \mathbb{C}$$

defined as the composition of the cup product and the integration induces the *Poincaré homomorphism*

$$P_V : H^\ell(V, \mathbb{C}) \simeq H^\ell(A^*(\tilde{\mathcal{U}})) \to H^{2n-\ell}(A^*(\tilde{\mathcal{U}}))^\vee \simeq H_{2n-\ell}(V, \mathbb{C}), \tag{1.1}$$

which is not an isomorphism in general. We refer to [Br1] for a combinatorial definition of the above Poincaré homomorphism and also that of the Alexander homomorphism described in (1.2) below. Thus these homomorphisms are defined in cohomology and homology with $\mathbb{Z}$ coefficients. The above homomorphism P_V sends the class $[\sigma]$ in $H^\ell(A^*(\tilde{\mathcal{U}}))$ to the class $[C]$ in $H_{2n-\ell}(V, \mathbb{C})$ such that

$$\int_V \sigma \smile \tau = \int_C \tau$$

for all τ in $A^{2n-\ell}(\tilde{\mathcal{U}})$ with $D\tau = 0$. Also we see that the fundamental class $[V]$ of V in $H_{2n}(V, \mathbb{C})$ is the image of $[1]$ in $H^0(V, \mathbb{C})$ by P_V.

Second, suppose V may not be compact. Let S be a compact set in V admitting a regular neighborhood in W such that there is an open set U in V with $S \subset U$ and $U \setminus S \subset V' = V \setminus \mathrm{Sing}(V)$. Letting $\tilde{U}_1$ be a regular neighborhood of S in W with $\tilde{U}_1 \cap V \subset U$ and $\tilde{U}_0$ a tubular neighborhood of $U_0 = U \setminus S$ in W with the projection $\rho : \tilde{U}_0 \to U_0$, we consider the covering $\tilde{\mathcal{U}} = \{\tilde{U}_0, \tilde{U}_1\}$ of $\tilde{U} = \tilde{U}_0 \cup \tilde{U}_1$. We may assume that U is deformation retract of $\tilde{U}$. We define the subcomplex $A^*(\tilde{\mathcal{U}}, \tilde{U}_0)$ of $A^*(\tilde{\mathcal{U}})$ by the exact sequence

$$0 \to A^*(\tilde{\mathcal{U}}, \tilde{U}_0) \to A^*(\tilde{\mathcal{U}}) \to A^*(\tilde{U}_0) \to 0.$$

Then we see that, by the five lemma,

$$H^*(A^*(\tilde{\mathcal{U}}, \tilde{U}_0)) \simeq H^*(U, U \setminus S; \mathbb{C}).$$

Let $\tilde{R}_1$ be a compact real $2m$-dimensional manifold with C^∞ boundary in $\tilde{U}_1$ such that S is in its interior and that $\partial \tilde{R}_1$ is transverse to U. We set $R_1 = \tilde{R}_1 \cap U$. Then $\partial R_1 = \partial \tilde{R}_1 \cap U$ is a $(2n-1)$-dimensional C^∞ submanifold of U_0. We set $R_{01} = -\partial R_1$ (∂R_1 with the opposite orientation). As in [Su2, Ch.IV, 2], we have the integration

$$\int_U : H^{2n}(A^*(\tilde{\mathcal{U}}, \tilde{U}_0)) \to \mathbb{C}.$$

The cup product induces the pairing $A^\ell(\tilde{\mathcal{U}},\tilde{U}_0) \times A^{2n-\ell}(\tilde{U}_1) \to A^{2n}(\tilde{\mathcal{U}},\tilde{U}_0)$, which, followed by the integration, gives a bilinear pairing

$$A^\ell(\tilde{\mathcal{U}},\tilde{U}_0) \times A^{2n-\ell}(\tilde{U}_1) \to \mathbb{C}.$$

This induces the *Alexander homomorphism*

$$A_{U_S} : H^\ell(U, U \setminus S; \mathbb{C}) \simeq H^\ell(A^*(\tilde{\mathcal{U}},\tilde{U}_0)) \to H^{2n-\ell}(\tilde{U}_1,\mathbb{C})^\vee \simeq H_{2n-\ell}(S,\mathbb{C}), \tag{1.2}$$

which is not an isomorphism in general. Similarly we have a homomorphism

$$H^{2n-\ell}(S,\mathbb{C}) \simeq H^{2n-\ell}(\tilde{U}_1,\mathbb{C}) \to H^\ell(A^*(\tilde{\mathcal{U}},\tilde{U}_0))^\vee \simeq H_\ell(U, U \setminus S; \mathbb{C}). \tag{1.3}$$

The homomorphism A_{U_S} sends the class $[\sigma]$ in $H^\ell(A^*(\tilde{\mathcal{U}},\tilde{U}_0))$ to the class $[C]$ in $H_{2n-\ell}(S,\mathbb{C})$ such that

$$\int_U \sigma \smile \tau = \int_C \tau,$$

for all $\tau = \tau_1$ in $A^{2n-\ell}(\tilde{U}_1)$ with $d\tau_1 = 0$. We denote by $[U_S]$ the class in the homology $H_{2n}(U, U \setminus S; \mathbb{C})$ assigned to $[1]$ in $H^0(S,\mathbb{C})$ by the homomorphism (1.3). We may also define the cap product

$$H_r(U, U \setminus S; \mathbb{C}) \times H^s(U, U \setminus S; \mathbb{C}) \to H_{r-s}(S,\mathbb{C})$$

as in [Su2, Ch.II, 1]. Then we may write

$$A_{U_S}([\sigma]) = [\sigma] \frown [U_S].$$

The following is proved similarly as [Su2, Ch.II, Proposition 3.11].

PROPOSITION 1.4. *Let V be a compact subvariety of dimension n in a complex manifold W and S a compact set in V which admits a regular neighborhood in W and contains* $\mathrm{Sing}(V)$. *Then the following diagram is commutative :*

$$\begin{array}{ccc} H^\ell(V, V \setminus S; \mathbb{C}) & \xrightarrow{j^*} & H^\ell(V,\mathbb{C}) \\ \downarrow{\scriptstyle A_{V_S}} & & \downarrow{\scriptstyle P_V} \\ H_{2n-\ell}(S,\mathbb{C}) & \xrightarrow{i_*} & H_{2n-\ell}(V,\mathbb{C}), \end{array}$$

where i and j denote, respectively, the inclusions $S \hookrightarrow V$ and $(V,\emptyset) \hookrightarrow (V, V \setminus S)$.

Remarks 1.5. 1. If V is non-singular, P_V and A_{V_S} are the usual Poincaré and Alexander isomorphisms. In this case, we may set $W = V$.

2. In the above, the assumption that $U \setminus S$ is in the regular part $V' = V \setminus \mathrm{Sing}(V)$ is not necessary. However, with this condition, to define a cochain $\sigma = (\sigma_0, \sigma_1, \sigma_{01})$ in $A^*(\tilde{\mathcal{U}})$ we only need to define σ_0 on $U_0 = U \setminus S$, since there is a C^∞ retraction $\rho : \tilde{U}_0 \to U_0$.

(B) Characteristic classes in the Čech-de Rham cohomology. Again, let V be a subvariety of dimension n in a complex manifold W. First, suppose V is compact and let $\tilde{U}$ and $\tilde{\mathcal{U}}$ be as in the first paragraph of (A) above. Let $\tilde{E}$ be a complex C^∞ vector bundle over $\tilde{U}$ and ∇ a connection for $\tilde{E}$. For a homogeneous symmetric polynomial φ of degree d, we may define a closed $2d$-form $\varphi(\nabla)$ whose class in $H^{2d}(\tilde{U},\mathbb{C})$ is the characteristic class $\varphi(\tilde{E})$ of $\tilde{E}$ with respect to φ [Su2, II, 7]. Also, taking a connection ∇_α for $\tilde{E}$ on each $\tilde{U}_\alpha$, we may define the characteristic class $\varphi(\tilde{E})$ in $H^{2d}(A^*(\tilde{\mathcal{U}}))$, which corresponds to the previous one under the isomorphism

$H^{2d}(A^*(\tilde{\mathcal{U}})) \simeq H^{2d}(\tilde{U},\mathbb{C})$ [Su2, Ch.II, 8. D]. The corresponding class in $H^{2d}(V,\mathbb{C}) \simeq H^{2d}(A^*(\tilde{\mathcal{U}}))$ is denoted by $\varphi(E)$ with $E = \tilde{E}|_V$.

Similarly, for a virtual bundle $\tilde{\xi} = \sum_{i=0}^q (-1)^i \tilde{E}_i$ over $\tilde{U}$, taking a family of connections $\nabla_\alpha^\bullet = (\nabla_\alpha^{(q)},\dots,\nabla_\alpha^{(0)})$, each $\nabla_\alpha^{(i)}$ being a connection for $\tilde{E}_i$, on each $\tilde{U}_\alpha$, we may define the characteristic class $\varphi(\tilde{\xi})$ in $H^{2d}(A^*(\tilde{\mathcal{U}}))$ and the corresponding class $\varphi(\xi)$ in $H^{2d}(V,\mathbb{C}) \simeq H^{2d}(A^*(\tilde{\mathcal{U}}))$ with $\xi = \tilde{\xi}|_V$. We also have the class $P_V(\varphi(\xi)) = \varphi(\xi) \frown [V]$ in $H_{2n-2d}(V,\mathbb{C})$.

Second, suppose V may not be compact. Let S be a compact set in V admitting a regular neighborhood in W such that there is an open set U in V with $S \subset U$ and $U \setminus S \subset V'$. Let $\tilde{U}_1$, $\tilde{U}_0$, $\tilde{\mathcal{U}} = \{\tilde{U}_0, \tilde{U}_1\}$ and $\tilde{U} = \tilde{U}_0 \cup \tilde{U}_1$ be as in the second paragraph in (A) above. For a virtual bundle $\tilde{\xi} = \sum_{i=0}^q (-1)^i \tilde{E}_i$ over $\tilde{U}$ and a homogeneous symmetric polynomial φ of degree d, the characteristic class $\varphi(\tilde{\xi})$ in $H^{2d}(A^*(\tilde{\mathcal{U}})) \simeq H^{2d}(\tilde{U},\mathbb{C})$ is represented by the cocycle $\varphi(\nabla_\star^\bullet)$ in $A^{2d}(\tilde{\mathcal{U}})$ given by

$$\varphi(\nabla_\star^\bullet) = (\varphi(\nabla_0^\bullet), \varphi(\nabla_1^\bullet), \varphi(\nabla_0^\bullet, \nabla_1^\bullet)), \tag{1.6}$$

where $\nabla_0^\bullet = (\nabla_0^{(q)},\dots,\nabla_0^{(0)})$ and $\nabla_1^\bullet = (\nabla_1^{(q)},\dots,\nabla_1^{(0)})$ denote families of connections for $\tilde{\xi}$ on $\tilde{U}_0$ and $\tilde{U}_1$, respectively. Note that it is sufficient if $\nabla_0^\bullet$ is defined only on $U_0 = U \setminus S$ (see Remark 1.5.2). We have the corresponding class $\varphi(\xi)$ in $H^{2d}(U,\mathbb{C}) \simeq H^{2d}(\tilde{U},\mathbb{C})$ with $\xi = \tilde{\xi}|_U$.

Here we recall the construction of the "difference form" $\varphi(\nabla_0^\bullet, \nabla_1^\bullet)$ for later use ([Bo, p.65], see also [Su2, Ch.II, (8.2)]). It is a $(2d-1)$-form on $\tilde{U}_{01} = \tilde{U}_0 \cap \tilde{U}_1$ with the property

$$d\varphi(\nabla_0^\bullet, \nabla_1^\bullet) = \varphi(\nabla_1^\bullet) - \varphi(\nabla_0^\bullet).$$

To define $\varphi(\nabla_0^\bullet, \nabla_1^\bullet)$, we consider the vector bundle $\tilde{E}_i \times \mathbb{R} \to \tilde{U}_{01} \times \mathbb{R}$ and let $\tilde{\nabla}^{(i)}$ be the connection for it given by $\tilde{\nabla}^{(i)} = (1-t)\nabla_0^{(i)} + t\nabla_1^{(i)}$, for $i = 0,\dots,q$. We set $\tilde{\nabla}^\bullet = (\tilde{\nabla}^{(q)},\dots,\tilde{\nabla}^{(0)})$. Denoting by ϖ_* the integration along the fibers of the projection $\varpi : \tilde{U}_{01} \times [0,1] \to \tilde{U}_{01}$, we define $\varphi(\nabla_0^\bullet, \nabla_1^\bullet) = \varpi_* \varphi(\tilde{\nabla}^\bullet)$.

If, for some reason as discussed in (C) below, we may choose $\nabla_0^\bullet$ so that $\varphi(\nabla_0^\bullet) = 0$, then the cocycle $\varphi(\nabla_\star^\bullet)$ is in $A^{2d}(\tilde{\mathcal{U}}, \tilde{U}_0)$ and defines a class $\varphi_S(\xi)$ in $H^{2d}(A^*(\tilde{\mathcal{U}}, \tilde{U}_0)) \simeq H^{2d}(U, U \setminus S; \mathbb{C})$, which is mapped to $\varphi(\xi)$ by the canonical homomorphism $j^* : H^{2d}(U, U \setminus S; \mathbb{C}) \to H^{2d}(U,\mathbb{C})$. We also have the class $A_{U_S}(\varphi_S(\xi)) = \varphi_S(\xi) \frown [U_S]$ in $H_{2n-2d}(S,\mathbb{C})$.

Note that the cup product induces a $H^*(U,\mathbb{C})$-module structure on the relative cohomology $H^*(U, U \setminus S; \mathbb{C})$.

(C) Localization. In this article, we consider the following two types of localizations :

(I) Localization of certain characteristic classes by the existence of a "basic connection" on the non-singular part of a holomorphic foliation, and

(II) localization of characteristic classes by exactness of a complex of vector bundles.

The type (I) is used to define the Baum-Bott residues and the Nash residues and will be explained in Sections 2 and 3 below.

The type (II) is used to compare these two types of residues in Section 4 (see [Su3, 1. (C)] for this type of localization when the ambient space is non-singular). For this, consider the second situation in (A) and (B) above, and let $\tilde{U}_1$, $\tilde{U}_0$, $\tilde{\mathcal{U}} = \{\tilde{U}_0, \tilde{U}_1\}$ and $\tilde{U} = \tilde{U}_0 \cup \tilde{U}_1$ be as before. Let $\tilde{\xi} = \sum_{i=0}^q (-1)^i \tilde{E}_i$ be a virtual bundle over $\tilde{U}$ with $\tilde{E}_i$

C^∞ vector bundles. Suppose there is a complex of continuous bundle homomorphisms

$$0 \to E_q \xrightarrow{h_q} \cdots \xrightarrow{h_1} E_0 \to 0 \tag{1.7}$$

on U, $E_i = \tilde{E}_i|_U$, such that, on U_0, the h_i's are C^∞ and the sequence is exact. If we take the family of connections $\nabla_0^\bullet = (\nabla_0^{(q)}, \dots, \nabla_0^{(0)})$ so that it is compatible with (1.7) on U_0, for a symmetric polynomial φ without constant term, we have $\varphi(\nabla_0^\bullet) = 0$ [BB, Lemma (4.22)]. For $\nabla_1^\bullet = (\nabla_1^{(q)}, \dots, \nabla_1^{(0)})$, we take arbitrary family of connections for $\tilde{\xi}$ on $\tilde{U}_1$. Then the cocycle (1.6) is in $A^*(\tilde{\mathcal{U}}, \tilde{U}_0)$ and it defines a class in $H^*(U, U \setminus S; \mathbb{C})$, which we denote $\varphi_S(\xi)$ as in (B) above. It is sent to $\varphi(\xi)$ by the canonical homomorphism $j^* : H^*(U, U \setminus S; \mathbb{C}) \to H^*(U, \mathbb{C})$. In particular, we have the local Chern classes $c_S^i(\xi)$ in $H^{2i}(U, U \setminus S; \mathbb{C})$, $i = 1, 2, \dots$, and the local Chern character $\mathrm{ch}_S^*(\xi)$ in $H^*(U, U \setminus S; \mathbb{C})$. Note that the local Chern characters defined as above have all the necessary properties and coincide with the ones in [I] (cf. [Su3, 1. (C)]). Hence they are in the cohomology $H^*(U, U \setminus S; \mathbb{Q})$ with $\mathbb{Q}$ coefficients. Also, the local Chern classes above are in the image of the canonical homomorphism $H^*(U, U \setminus S; \mathbb{Z}) \to H^*(U, U \setminus S; \mathbb{C})$, i.e., integral classes. Hence, if φ is a symmetric polynomial with integral coefficient and without constant term, then $\varphi_S(\xi)$ is also an integral class.

2. Baum-Bott residues. Let M be a (connected) complex manifold of dimension n and TM its holomorphic tangent bundle. Also, denoting by $\mathcal{O}_M$ the structure sheaf of M, let $\Theta_M = \mathcal{O}_M(TM)$ be the tangent sheaf of M. (The tangent sheaf of) a singular holomorphic foliation on M is defined to be an involutive coherent subsheaf $\mathcal{F}$ of Θ_M. Let $\mathcal{N}_\mathcal{F}$ denote the quotient sheaf $\Theta_M/\mathcal{F}$ (the normal sheaf of the foliation) so that we have the exact sequence

$$0 \to \mathcal{F} \to \Theta_M \to \mathcal{N}_\mathcal{F} \to 0. \tag{2.1}$$

In general, for a coherent $\mathcal{O}_M$-module $\mathcal{S}$, we define its singular set $\mathrm{Sing}(\mathcal{S})$ to be the set of points x in M where the stalk $\mathcal{S}_x$ is not $\mathcal{O}_{M,x}$-free. We define the singular set $S(\mathcal{F})$ of the foliation $\mathcal{F}$ to be the singular set $\mathrm{Sing}(\mathcal{N}_\mathcal{F})$ of the coherent sheaf $\mathcal{N}_\mathcal{F}$. Note that $S(\mathcal{F}) \supset \mathrm{Sing}(\mathcal{F})$. If p is the rank of (the locally free part of) $\mathcal{F}$, $\mathcal{F}$ defines a non-singular foliation of dimension p on $M \setminus S(\mathcal{F})$. If we set, for each point x in M,

$$F(x) = \{\, v(x) \mid v \in \mathcal{F}_x \,\},$$

then $F(x)$ is a subspace of the tangent space T_xM. In general, $\dim F(x) \le p$, where the equality holds if and only if $x \notin S(\mathcal{F})$.

We briefly recall how we define the Baum-Bott residues. On $M \setminus \mathrm{Sing}(\mathcal{F})$, there is a vector bundle F such that $\mathcal{F} = \mathcal{O}(F)$. If we set $M_0 = M \setminus S(\mathcal{F})$, then it is a subset of $M \setminus \mathrm{Sing}(\mathcal{F})$ and on M_0, $F_0 = F|_{M_0}$ may be identified with a subbundle of TM_0, whose fiber over $x \in M_0$ is $F(x)$. If we set $N_{F_0} = TM_0/F_0$, we have $\mathcal{N}_\mathcal{F} = \mathcal{O}(N_{F_0})$ on M_0. Recall that the bundle N_{F_0} admits a basic connection [BB, Definition (3.24)].

We try to compute the characteristic class $\varphi(\mathcal{N}_\mathcal{F})$ for a homogeneous symmetric polynomial φ of degree d and will see that, if $d > n - p$, the class $\varphi(\mathcal{N}_\mathcal{F})$ is localized at $S(\mathcal{F})$.

Let S be a compact connected component of $S(\mathcal{F})$ and U a relatively compact open neighborhood of S in M disjoint from the other components of $S(\mathcal{F})$. We set $U_0 = U \setminus S$ and $U_1 = U$ (or any neighborhood of S in U) and consider the covering

$\mathcal{U} = \{U_0, U_1\}$ of U. We take a resolution

$$0 \to \mathcal{A}_U(E_q) \to \cdots \to \mathcal{A}_U(E_0) \to \mathcal{A}_U \otimes_{\mathcal{O}_U} \mathcal{N}_{\mathcal{F}} \to 0$$

of the normal sheaf $\mathcal{N}_{\mathcal{F}}$ by real analytic (complex) vector bundles E_i on U [AH]. In the above sequence, we denote by $\mathcal{A}_U$ the sheaf of germs of real analytic functions on U. By definition, the characteristic class $\varphi(\mathcal{N}_{\mathcal{F}})$ on U is the characteristic class $\varphi(\xi)$ of the virtual bundle $\xi = \sum_{i=0}^{q}(-1)^i E_i$. On U_0 we have the exact sequence of vector bundles

$$0 \to E_q \to \cdots \to E_0 \to N_{F_0} \to 0.$$

Let ∇ be a basic connection for N_{F_0} on U_0. There is a connection $\nabla_0^{(i)}$ on U_0 for each E_i so that the family of connections $(\nabla_0^{(q)}, \dots, \nabla_0^{(0)}, \nabla)$ is compatible with the above sequence. If we denote by $\nabla_0^\bullet$ the family of connections $(\nabla_0^{(q)}, \dots, \nabla_0^{(0)})$ on U_0, we have

$$\varphi(\nabla_0^\bullet) = \varphi(\nabla). \tag{2.2}$$

On U_1, we take an arbitrary family $\nabla_1^\bullet = (\nabla_1^{(q)}, \dots, \nabla_1^{(0)})$ of connections, each $\nabla_1^{(i)}$ being a connection for E_i on U_1. Then the class $\varphi(\mathcal{N}_{\mathcal{F}}) = \varphi(\xi)$ in $H^{2d}(U, \mathbb{C})$ is represented by the cocycle $\varphi(\nabla_*^\bullet)$ in $A^{2d}(\mathcal{U})$ given by

$$\varphi(\nabla_*^\bullet) = (\varphi(\nabla_0^\bullet), \varphi(\nabla_1^\bullet), \varphi(\nabla_0^\bullet, \nabla_1^\bullet)).$$

If $d > n - p$, then by (2.2) and the Bott vanishing theorem [BB, Proposition (3.27)], we have $\varphi(\nabla_0^\bullet) \equiv 0$. Hence $\varphi(\nabla_*^\bullet)$ is in $A^{2d}(\mathcal{U}, U_0)$. We denote its class $[\varphi(\nabla_*^\bullet)]$ in $H^{2d}(U, U \setminus S; \mathbb{C})$ by $\varphi_S(\mathcal{N}_{\mathcal{F}}, \mathcal{F})$ and call it the localization of $\varphi(\mathcal{N}_{\mathcal{F}})$ by $\mathcal{F}$ at S. It does not depend on the choices of various connections and is sent to $\varphi(\mathcal{N}_{\mathcal{F}})$ by the canonical homomorphism $j^* : H^{2d}(U, U \setminus S; \mathbb{C}) \to H^{2d}(U, \mathbb{C})$. We have the residue $\mathrm{Res}_\varphi(\mathcal{N}_{\mathcal{F}}, \mathcal{F}; S)$ in $H_{2n-2d}(S, \mathbb{C})$ as the image of $\varphi_S(\mathcal{N}_{\mathcal{F}}, \mathcal{F})$ by the Alexander isomorphism

$$H^{2d}(U, U \setminus S; \mathbb{C}) \xrightarrow{\sim} H_{2n-2d}(S, \mathbb{C}).$$

Let R_1 be a $2n$-dimensional manifold with C^∞ boundary in U_1 containing S in its interior and set $R_0 = U \setminus \mathrm{Int}\, R_1$ so that $\{R_0, R_1\}$ is a system of honey-comb cells adapted to $\mathcal{U}$. Then $\mathrm{Res}_\varphi(\mathcal{N}_{\mathcal{F}}, \mathcal{F}; S)$ is represented by a $2(n-d)$-cycle C in S such that

$$\int_C \tau_1 = \int_{R_1} \varphi(\nabla_1^\bullet) \wedge \tau_1 + \int_{R_{01}} \varphi(\nabla_0^\bullet, \nabla_1^\bullet) \wedge \tau_1 \tag{2.3}$$

for any closed $2(n-d)$-form τ_1 on U_1. In particular, if $d = n$, the residue is a complex number given by

$$\mathrm{Res}_\varphi(\mathcal{N}_{\mathcal{F}}, \mathcal{F}; S) = \int_{R_1} \varphi(\nabla_1^\bullet) + \int_{R_{01}} \varphi(\nabla_0^\bullet, \nabla_1^\bullet). \tag{2.4}$$

Thus for such a polynomial φ and a compact connected component S of $S(\mathcal{F})$, we have the Baum-Bott residue $\mathrm{Res}_\varphi(\mathcal{N}_{\mathcal{F}}, \mathcal{F}; S)$ in $H_{2n-2d}(S, \mathbb{C})$. Moreover, if M is compact, from Proposition 1.4, we have the residue formula [BB, Theorem 2]

$$\sum_S i_* \mathrm{Res}_\varphi(\mathcal{N}_{\mathcal{F}}, \mathcal{F}; S) = \varphi(\mathcal{N}_{\mathcal{F}}) \frown [M] \qquad \text{in} \qquad H_{2n-2d}(M, \mathbb{C}),$$

where i denotes the embedding $S \hookrightarrow M$ and the sum is taken over the components of $S(\mathcal{F})$.

We quote the following conjecture in [BB] :

2.5. Rationality conjecture. Let $\mathcal{F}$ be a singular foliation of dimension p on a complex manifold M of dimension n. Also let S be a compact connected component of the singular set $S(\mathcal{F})$ and φ a homogeneous symmetric polynomial of degree d. If $n-p+1<d\le n$ and if φ is with rational coefficients, then

$$\mathrm{Res}_{\varphi}(\mathcal{N}_{\mathcal{F}},\mathcal{F};S)\in H_{2n-2d}(S,\mathbb{Q}).$$

3. Nash residues. Let M be a complex manifold of dimension n and $\mathcal{F}$ a singular foliation of dimension p on M. As in Section 2, we set, for each point x in M, $F(x)=\{\,v(x)\mid v\in\mathcal{F}_x\,\}$. Recall that $F(x)$ is a subspace of the tangent space T_xM and it is p-dimensional if and only if $x\notin S(\mathcal{F})$. If we let $\tilde{\pi}:G_p(TM)\to M$ be the Grassmann bundle of p-planes in TM, we have a section $\sigma:M\setminus S(\mathcal{F})\to G_p(TM)$ defined by $\sigma(x)=F(x)$. We denote by M^ν the closure in $G_p(TM)$ of the image of σ and call it the tangential (or Nash) modification of M with respect to $\mathcal{F}$. It is an analytic subvariety in $G_p(TM)$. If we denote by π the restriction of $\tilde{\pi}$ to M^ν and set $S(\mathcal{F})^\nu=\pi^{-1}(S(\mathcal{F}))$, $\pi:M^\nu\to M$ is a holomorphic map which is biholomorphic from $M_0^\nu=M^\nu\setminus S(\mathcal{F})^\nu$ onto $M_0=M\setminus S(\mathcal{F})$. We denote by $\tilde{T}^\nu$ and $\tilde{N}^\nu$, respectively, the tautological bundle and the tautological quotient bundle over $G_p(TM)$, which are defined as follows. We have

$$\tilde{\pi}^*TM=\{\,(P,v)\in G_p(TM)\times TM\mid P\subset T_xM,\ v\in T_xM,\ x=\tilde{\pi}(P)\,\}.$$

The tautological bundle $\tilde{T}^\nu$ is the subbundle of $\tilde{\pi}^*TM$ given by

$$\tilde{T}^\nu=\{\,(P,v)\in G_p(TM)\times TM\mid v\in P\,\}.$$

Thus the fiber of $\tilde{T}^\nu$ over $P\in G_p(TM)$ is P. We set $\tilde{N}^\nu=\tilde{\pi}^*TM/\tilde{T}^\nu$. Let T^ν and N^ν be their restrictions to M^ν so that we have an exact sequence of vector bundles over M^ν :

$$0\to T^\nu\to\pi^*TM\to N^\nu\to 0,\tag{3.1}$$

which is equivalent to (2.1) away from the singular sets. In fact, $T^\nu|_{M_0^\nu}=\pi^*F_0$ and $N^\nu|_{M_0^\nu}=\pi^*N_{F_0}$.

We try to compute the characteristic class $\varphi(N^\nu)$ of the vector bundle N^ν over M^ν for a homogeneous symmetric polynomial φ of degree d and will see that, if $d>n-p$, the class $\varphi(N^\nu)$ is localized at $S(\mathcal{F})^\nu$.

Let S be a compact connected component of $S(\mathcal{F})$ and set $S^\nu=\pi^{-1}(S)$. Also, let U^ν be a neighborhood of S^ν in M^ν disjoint from the other components of $S(\mathcal{F})^\nu$. Letting $\tilde{U}_1^\nu$ be a regular neighborhood of S^ν in $G_p(TM)$ with $\tilde{U}_1^\nu\cap M^\nu\subset U^\nu$ and $\tilde{U}_0^\nu$ a tubular neighborhood of $U_0^\nu=U^\nu\setminus S^\nu$ in $G_p(TM)$ with the projection $\rho:\tilde{U}_0^\nu\to U_0^\nu$, we consider the covering $\tilde{\mathcal{U}}^\nu=\{\tilde{U}_0^\nu,\tilde{U}_1^\nu\}$ of $\tilde{U}^\nu=\tilde{U}_0^\nu\cup\tilde{U}_1^\nu$. We may assume that U^ν is deformation retract of $\tilde{U}^\nu$. The characteristic class $\varphi(\tilde{N}^\nu)$ on $\tilde{U}^\nu$ is represented by the cocycle $\varphi(\nabla_\star^\nu)$ in $A^{2d}(\tilde{\mathcal{U}}^\nu)$ given by

$$\varphi(\nabla_\star^\nu)=(\varphi(\nabla_0^\nu),\varphi(\nabla_1^\nu),\varphi(\nabla_0^\nu,\nabla_1^\nu)),$$

where ∇_0^ν and ∇_1^ν denote connections for $\tilde{N}^\nu$ on $\tilde{U}_0^\nu$ and $\tilde{U}_1^\nu$, respectively. Note that it is sufficient if ∇_0^ν is defined only on U_0^ν (see Remark 1.5.2). If we set $U=\pi(U^\nu)$, π induces a biholomorphic map from U_0^ν onto $U_0=U\setminus S$. On U_0, we have a basic connection ∇ for N_{F_0}. We take as ∇_0^ν the connection for N^ν given by $\nabla_0^\nu=\pi^*\nabla$. Then $\varphi(\nabla_0^\nu)\equiv 0$ and the cocycle $\varphi(\nabla_\star^\nu)$ is in $A^{2d}(\tilde{\mathcal{U}}^\nu,\tilde{U}_0^\nu)$ and it defines a class $\varphi_{S^\nu}(N^\nu,\mathcal{F})$

in $H^{2d}(U^\nu, U^\nu \setminus S^\nu; \mathbb{C})$. It is not difficult to show that the class $\varphi_{S^\nu}(N^\nu, \mathcal{F})$ does not depend on the choice of the basic connection ∇ or the connection ∇_1^ν (see, e.g., [Su2, Ch.III, Lemma 3.1]). We denote its image by the Alexander homomorphism

$$H^{2d}(U^\nu, U^\nu \setminus S^\nu; \mathbb{C}) \to H_{2n-2d}(S^\nu, \mathbb{C})$$

by $\mathrm{Res}_\varphi(N^\nu, \mathcal{F}; S^\nu)$ and call it the Nash residue of $\mathcal{F}$ with respect to φ at S^ν.

Let $\tilde{R}_1^\nu$, R_1^ν and R_{01}^ν be as in Section 1. Then the residue $\mathrm{Res}_\varphi(N^\nu, \mathcal{F}; S^\nu)$ is represented by a $2(n-d)$-cycle C in S^ν such that

$$\int_C \tau_1 = \int_{R_1^\nu} \varphi(\nabla_1^\nu) \wedge \tau_1 + \int_{R_{01}^\nu} \varphi(\nabla_0^\nu, \nabla_1^\nu) \wedge \tau_1 \tag{3.2}$$

for any closed $2(n-d)$-form τ_1 on $\tilde{U}_1^\nu$. In particular, if $d = n$, the residue is a complex number given by

$$\mathrm{Res}_\varphi(N^\nu, \mathcal{F}; S^\nu) = \int_{R_1^\nu} \varphi(\nabla_1^\nu) + \int_{R_{01}^\nu} \varphi(\nabla_0^\nu, \nabla_1^\nu). \tag{3.3}$$

Summarizing the above, we have the following theorem, the second statement of which follows from Proposition 1.4.

THEOREM 3.4. *Let $\mathcal{F}$ be a singular foliation of dimension p on a complex manifold M of dimension n and φ a homogeneous symmetric polynomial of degree $d > n-p$.*
(1) *For each compact connected component S of the singular set $S(\mathcal{F})$, we have the residue $\mathrm{Res}_\varphi(N^\nu, \mathcal{F}; S^\nu)$ in $H_{2n-2d}(S^\nu, \mathbb{C})$, which is given by (3.2).*
(2) *If M is compact,*

$$\sum_S i_* \mathrm{Res}_\varphi(N^\nu, \mathcal{F}; S^\nu) = \varphi(N^\nu) \frown [M^\nu] \quad \text{in} \quad H_{2n-2d}(M^\nu, \mathbb{C}),$$

where i denotes the embedding $S^\nu \hookrightarrow M^\nu$ and the sum is taken over all the connected components S of $S(\mathcal{F})$.

4. Comparison of Baum-Bott and Nash residues. Let M be a complex manifold of dimension n and $\mathcal{F}$ a singular foliation of dimension p on M. Also let S be a compact connected component of $\mathrm{Sing}(\mathcal{F})$ and set $S^\nu = \pi^{-1}(S)$, as in Section 3. Then there is a canonical homomorphism

$$\pi_* : H_{2n-2d}(S^\nu, \mathbb{C}) \to H_{2n-2d}(S, \mathbb{C}).$$

THEOREM 4.1. *Let φ be a homogeneous symmetric polynomial of degree $d > n-p$. If φ is with integral coefficients, then the difference*

$$\mathrm{Res}_\varphi(\mathcal{N}_\mathcal{F}, \mathcal{F}; S) - \pi_* \mathrm{Res}_\varphi(N^\nu, \mathcal{F}; S^\nu)$$

is in the image of the canonical homomorphism $H_{2n-2d}(S, \mathbb{Z}) \to H_{2n-2d}(S, \mathbb{C})$.

Proof. Let U^ν, $\tilde{U}_0^\nu$, $\tilde{U}_1^\nu$, $\tilde{U}^\nu$ and U be as in Section 3. We also set $U_0 = U \setminus S$, $U_1 = \tilde{\pi}(\tilde{U}_1^\nu)$ and $\mathcal{U} = \{U_0, U_1\}$. We may assume that U is relatively compact. Let

$$0 \to \mathcal{A}_U(E_q) \to \cdots \to \mathcal{A}_U(E_1) \to \mathcal{A}_U \otimes_{\mathcal{O}_U} \mathcal{F} \to 0 \tag{4.2}$$

be a resolutin of $\mathcal{F}$ by real analytic vector bundles E_i on U. Since $\mathcal{A}_U \otimes_{\mathcal{O}_U} \cdot$ is a right exact functor, this gives a resolution of the sheaf $\mathcal{N}_\mathcal{F}$ on U :

$$0 \to \mathcal{A}_U(E_q) \xrightarrow{\eta_q} \cdots \to \mathcal{A}_U(E_1) \xrightarrow{\eta_1} \mathcal{A}_U(TM) \to \mathcal{A}_U \otimes_{\mathcal{O}_U} \mathcal{N}_\mathcal{F} \to 0$$

and $\mathcal{N}_{\mathcal{F}}$ is equivalent to the virtual bundle $\xi = \sum_{i=0}^{q}(-1)^i E_i$, where we set $E_0 = TM$. We have an exact sequence on $U_0 = U \setminus S$:

$$(4.3) \qquad 0 \to E_q \to \cdots \to E_1 \to TM \to N_{F_0} \to 0.$$

Note that the sheaf homomorphism η_i, $i = 1, \ldots, q$, induces a bundle homomorphism $h_i : E_i \to E_{i-1}$ on U and, in turn, $\pi^* h_i : \pi^* E_i \to \pi^* E_{i-1}$ on U^ν. We claim that the image of $\pi^* h_1$ is in T^ν. In fact, the image coincides with T^ν on $U_0^\nu = U^\nu \setminus S^\nu$, which is dense in U^ν. Hence by the continuity of $\pi^* h_1$, the image is in T^ν on U^ν. Thus we have a complex of vector bundles on U^ν

$$(4.4) \qquad 0 \to \pi^* E_q \to \cdots \to \pi^* E_1 \to \pi^* TM \to N^\nu \to 0,$$

which is exact on U_0^ν (cf. (3.1)). Consider the virtual bundle $\tilde{\varepsilon} = \tilde{\pi}^*\xi - \tilde{N}^\nu$ over $\tilde{U}^\nu$ so that we have

$$\tilde{\pi}^*\xi = \tilde{N}^\nu + \tilde{\varepsilon}.$$

We set $\varepsilon = \tilde{\varepsilon}|_{U^\nu} = \pi^*\xi - N^\nu$. We may write

$$\varphi(\tilde{\pi}^*\xi) = \varphi(\tilde{N}^\nu) + \sum_{i=1}^{r} \varphi^{(i)}(\tilde{N}^\nu) \cdot \psi^{(i)}(\tilde{\varepsilon}),$$

where the $\varphi^{(i)}$'s are symmetric polynomials with integral coefficients and the $\psi^{(i)}$'s are symmetric polynomials with integral coefficients without constant term. Recall that the Baum-Bott residue is the image of the localized class of $\varphi(\xi)$ by the Alexander isomorphism and the Nash residue is the image of the localized class of $\varphi(N^\nu)$ by the Alexander homomorphism. We compare them using the characteristic classes of ε which are all localized.

Let ∇ be a basic connection for N_{F_0} on U_0. We choose a connection $\nabla_0^{(i)}$ for each E_i, $i = 0, \ldots, q$, on U_0 so that $(\nabla_0^{(q)}, \ldots, \nabla_0^{(0)}, \nabla)$ is compatible with (4.3) and set $\nabla_0^\bullet = (\nabla_0^{(q)}, \ldots, \nabla_0^{(0)})$. Also, choose a connection $\nabla_1^{(i)}$ for each E_i on U_1 and set $\nabla_1^\bullet = (\nabla_1^{(q)}, \ldots, \nabla_1^{(0)})$. The Baum-Bott residue $\mathrm{Res}_\varphi(\mathcal{N}_{\mathcal{F}}, \mathcal{F}; S)$ is the class in $H_{2n-2d}(S, \mathbb{C})$ which corresponds, under the Alexander isomorphism, to the localized class $\varphi_S(\mathcal{N}_{\mathcal{F}}, \mathcal{F})$ of $\varphi(\xi)$ in $H^{2d}(U, U \setminus S; \mathbb{C}) \simeq H^{2d}(A^*(\mathcal{U}, U_0))$ represented by the cocycle

$$\varphi(\nabla_\star^\bullet) = (0, \varphi(\nabla_1^\bullet), \varphi(\nabla_0^\bullet, \nabla_1^\bullet)).$$

Let ∇_0^ν be a connection for N^ν on U_0^ν given by $\nabla_0^\nu = \pi^*\nabla$ and let ∇_1^ν be a connection for $\tilde{N}^\nu$ on $\tilde{U}_1^\nu$. Then the Nash residue $\mathrm{Res}_\varphi(N^\nu, \mathcal{F}; S^\nu)$ is the class in $H_{2n-2d}(S^\nu, \mathbb{C})$ which is the image by the Alexander homomorphism of the localized class $\varphi_S(N^\nu, \mathcal{F})$ of $\varphi(N^\nu)$ in $H^{2d}(U^\nu, U^\nu \setminus S^\nu; \mathbb{C}) \simeq H^{2d}(A^*(\tilde{\mathcal{U}}^\nu, \tilde{U}_0^\nu))$ represented by the cocycle

$$\varphi(\nabla_\star^\nu) = (0, \varphi(\nabla_1^\nu), \varphi(\nabla_0^\nu, \nabla_1^\nu)).$$

We have a commutative diagram

$$\begin{array}{ccc}
H^{2d}(U, U \setminus S; \mathbb{C}) & \xrightarrow{\sim} & H^{2d}(A^*(\mathcal{U}, U_0)) \\
\Big\downarrow{\scriptstyle \pi^*} & & \Big\downarrow{\scriptstyle \tilde{\pi}^*} \\
H^{2d}(U^\nu, U^\nu \setminus S^\nu; \mathbb{C}) & \xrightarrow{\sim} & H^{2d}(A^*(\tilde{\mathcal{U}}^\nu, \tilde{U}_0^\nu)).
\end{array}$$

If we consider the families of connections $\pi^*\nabla_0^\bullet$ for $\pi^*\xi$ on U_0^ν and $\tilde\pi^*\nabla_1^\bullet$ for $\tilde\pi^*\xi$ on $\tilde U_1^\nu$, the class $\pi^*\varphi_S(\mathcal{N}_{\mathcal{F}},\mathcal{F})$ is represented by the cocycle

$$\varphi(\tilde\pi^*\nabla_\star^\bullet)=(0,\varphi(\tilde\pi^*\nabla_1^\bullet),\varphi(\pi^*\nabla_0^\bullet,\tilde\pi^*\nabla_1^\bullet))$$

in $A^*(\tilde{\mathcal{U}}^\nu,\tilde U_0^\nu)$. The class $\varphi^{(i)}(\tilde N^\nu)$ on $\tilde U^\nu$ is represented by the cocycle $\varphi^{(i)}(\nabla_\star^\nu)$ in $A^*(\tilde{\mathcal{U}}^\nu)$ given by

$$\varphi^{(i)}(\nabla_\star^\nu)=(\varphi^{(i)}(\nabla_0^\nu),\varphi^{(i)}(\nabla_1^\nu),\varphi^{(i)}(\nabla_0^\nu,\nabla_1^\nu)).$$

If we denote by ∇_0^ε the family of connections $(\pi^*\nabla_0^\bullet,\nabla_0^\nu)=(\pi^*\nabla_0^{(q)},\dots,\pi^*\nabla_0^{(0)},\nabla_0^\nu)$ on U_0^ν and by ∇_1^ε the family of connections $(\tilde\pi^*\nabla_1^\bullet,\nabla_1^\nu)=(\tilde\pi^*\nabla_1^{(q)},\dots,\tilde\pi^*\nabla_1^{(0)},\nabla_1^\nu)$ on $\tilde U_1^\nu$, then the family ∇_0^ε is compatible with (4.4) on U_0^ν and the class $\psi^{(i)}(\tilde\varepsilon)$ on $\tilde U^\nu$ is represented by the cocycle $\psi^{(i)}(\nabla_\star^\varepsilon)$ in $A^*(\tilde{\mathcal{U}}^\nu,\tilde U_0^\nu)$ given by

$$\psi^{(i)}(\nabla_\star^\varepsilon)=(0,\psi^{(i)}(\nabla_1^\varepsilon),\psi^{(i)}(\nabla_0^\varepsilon,\nabla_1^\varepsilon)),$$

whose class in $H^*(U^\nu,U^\nu\setminus S^\nu;\mathbb{C})$ is the localised class $\psi_S^{(i)}(\varepsilon)$. Recall that (Section 1 (C)) they are in the image of the canonical homomorphism $H^*(U^\nu,U^\nu\setminus S^\nu;\mathbb{Z})\to H^*(U^\nu,U^\nu\setminus S^\nu;\mathbb{C})$.

We recall the construction of the "difference forms" in Section 1, (B). Again, it suffices to construct them on $U_{01}^\nu=U_0^\nu\cap\tilde U_1^\nu$. Let $\tilde\nabla^\nu$ be the connection for the vector bundle $N^\nu\times\mathbb{R}$ over $U_{01}^\nu\times\mathbb{R}$ given by $\tilde\nabla^\nu=(1-t)\nabla_0^\nu+t\nabla_1^\nu$. We denote by ϖ_* the integration along the fibers of the projection $\varpi:U_{01}^\nu\times[0,1]\to U_{01}^\nu$, as before. We claim that, in $A^*(\tilde{\mathcal{U}}^\nu,\tilde U_0^\nu)$, we have

$$\varphi(\tilde\pi^*\nabla_\star^\bullet)=\varphi(\nabla_\star^\nu)+\sum_{i=1}^r\varphi^{(i)}(\nabla_\star^\nu)\smile\psi^{(i)}(\nabla_\star^\varepsilon)+D\,\tau,\tag{4.5}$$

where

$$\tau=(0,0,\tau_{01}),\qquad \tau_{01}=\sum_{i=1}^r\varpi_*(\varphi^{(i)}(\varpi^*\nabla_0^\nu,\tilde\nabla^\nu)\cdot\psi^{(i)}(\tilde\nabla^\varepsilon)).$$

In fact, we have, from the definition of the cup product (see, e.g., [Su2, II, 3]),

$$\begin{aligned}&\varphi^{(i)}(\nabla_\star^\nu)\smile\psi^{(i)}(\nabla_\star^\varepsilon)\\&\quad=(0,\varphi^{(i)}(\nabla_1^\nu)\cdot\psi^{(i)}(\nabla_1^\varepsilon),\varphi^{(i)}(\nabla_0^\nu)\cdot\psi^{(i)}(\nabla_0^\varepsilon,\nabla_1^\varepsilon)+\varphi^{(i)}(\nabla_0^\nu,\nabla_1^\nu)\cdot\psi^{(i)}(\nabla_1^\varepsilon)).\end{aligned}$$

First, we have

$$\varphi(\tilde\pi^*\nabla_\star^\bullet)=\varphi(\nabla_1^\nu)+\sum_{i=1}^r\varphi^{(i)}(\nabla_1^\nu)\cdot\psi^{(i)}(\nabla_1^\varepsilon).$$

Hence we have the identity of the terms of forms on $\tilde U_1^\nu$. To compare the terms of forms on U_{01}^ν, we set $\tilde\nabla^\bullet=(\tilde\nabla^{(q)},\dots,\tilde\nabla^{(0)})$ with $\tilde\nabla^{(i)}=(1-t)\pi^*\nabla_0^{(i)}+t\tilde\pi^*\nabla_1^{(i)}$ and $\tilde\nabla^\varepsilon=(\tilde\nabla^\bullet,\tilde\nabla^\nu)$. Then, by definition,

$$\varphi(\pi^*\nabla_0^\bullet,\tilde\pi^*\nabla_1^\bullet)=\varpi_*\varphi(\tilde\nabla^\bullet)=\varpi_*\varphi(\tilde\nabla^\nu)+\sum_{i=1}^r\varpi_*(\varphi^{(i)}(\tilde\nabla^\nu)\cdot\psi^{(i)}(\tilde\nabla^\varepsilon)).$$

We have $\varpi_*\varphi(\tilde{\nabla}^\nu) = \varphi(\nabla_0^\nu, \nabla_1^\nu)$. On the other hand, we have

$$\begin{aligned}\varphi^{(i)}(\nabla_0^\nu)\cdot\psi^{(i)}(\nabla_0^\varepsilon,\nabla_1^\varepsilon) &+ \varphi^{(i)}(\nabla_0^\nu,\nabla_1^\nu)\cdot\psi^{(i)}(\nabla_1^\varepsilon)\\ &= \varpi_*(\varpi^*\varphi^{(i)}(\nabla_0^\nu)\cdot\psi^{(i)}(\tilde{\nabla}^\varepsilon)) + \varphi^{(i)}(\nabla_0^\nu,\nabla_1^\nu)\cdot\psi^{(i)}(\nabla_1^\varepsilon).\end{aligned}$$

We compute

$$\begin{aligned}\varpi_*(\varphi^{(i)}(\tilde{\nabla}^\nu)\cdot\psi^{(i)}(\tilde{\nabla}^\varepsilon) &- \varpi^*\varphi^{(i)}(\nabla_0^\nu)\cdot\psi^{(i)}(\tilde{\nabla}^\varepsilon))\\ &= -d\,\varpi_*(\varphi^{(i)}(\varpi^*\nabla_0^\nu,\tilde{\nabla}^\nu)\cdot\psi^{(i)}(\tilde{\nabla}^\varepsilon)) + \varphi^{(i)}(\nabla_0^\nu,\nabla_1^\nu)\cdot\psi^{(i)}(\nabla_1^\varepsilon).\end{aligned}$$

This proves the identity (4.5), which shows that, we have, in $H^{2d}(U^\nu, U^\nu \setminus S^\nu; \mathbb{C})$,

$$\pi^*\varphi_S(\mathcal{N}_\mathcal{F},\mathcal{F}) = \varphi_S(N^\nu,\mathcal{F}) + \sum_{i=1}^r \varphi^{(i)}(\mathcal{N}_\mathcal{F})\cdot\psi_S^{(i)}(\varepsilon). \tag{4.6}$$

In view of the commutative diagram

$$\begin{array}{ccc} H^{2d}(U^\nu, U^\nu\setminus S^\nu;\mathbb{C}) & \xleftarrow{\ \pi^*\ } & H^{2d}(U, U\setminus S;\mathbb{C}) \\ \downarrow{\scriptstyle A_{U^\nu S^\nu}} & & \wr\downarrow{\scriptstyle A_{U_S}} \\ H_{2n-2d}(S^\nu,\mathbb{C}) & \xrightarrow{\ \pi_*\ } & H_{2n-2d}(S,\mathbb{C}), \end{array}$$

we have the theorem. □

Since N^ν is a vector bundle of rank $n-p$, if $\varphi = c^{i_1}\cdots c^{i_r}$ with $i_\nu > n-p$ for some ν, then $\mathrm{Res}_\varphi(N^\nu, \mathcal{F}; S) = 0$. Hence we have

COROLLARY 4.7. *If $\varphi = c^{i_1}\cdots c^{i_r}$ with $i_\nu > n-p$ for some ν, then $\mathrm{Res}_\varphi(\mathcal{N}_\mathcal{F}, \mathcal{F}; S)$ comes from an integral class, in particular it is a rational class, where c^i denotes the i-th Chern polynomial.*

Remarks 4.8. 1. The above is proved in [Su1] under the assumption that the annihilator of $\mathcal{F}$ is locally free.

2. The equality (4.6) gives an explicit expression of the difference in [K] Théorème 8.1 for the sheaf $\mathcal{N}_\mathcal{F}$.

Example 4.9. Let M be an open disk U about the origin 0 in $\mathbb{C}^2 = \{(z_1, z_2)\}$ and $\mathcal{F}$ the foliation generated by a holomorphic vector field $v = a_1\frac{\partial}{\partial z_1} + a_2\frac{\partial}{\partial z_2}$, where a_1 and a_2 are holomorphic functions on U with the set of common zeros $\{0\}$. We have

$$G_1(TM) = U\times\mathbb{CP}^1 \supset M^\nu = \{\,(z_1,z_2;\zeta_0,\zeta_1)\in U\times\mathbb{CP}^1 \mid a_1\zeta_1 - a_2\zeta_0 = 0\,\},$$

$S = \{0\}$ and $S^\nu = 0\times\mathbb{CP}^1$. In this case we have residues essentially for $\varphi = c_2$ and c_1^2 and they are complex numbers. If we set

$$A = \begin{pmatrix} \frac{\partial a_1}{\partial z_1} & \frac{\partial a_1}{\partial z_2} \\ \frac{\partial a_2}{\partial z_1} & \frac{\partial a_2}{\partial z_2} \end{pmatrix},$$

we have

$$\mathrm{Res}_{c_2}(\mathcal{N}_\mathcal{F},\mathcal{F};S) = \left(\frac{1}{2\pi\sqrt{-1}}\right)^2 \int_\Gamma \frac{\det(A)dz_1\wedge dz_2}{a_1 a_2},$$

where Γ is a 2-cycle in U given by $\Gamma = \{\,(z_1,z_2) \mid |a_1| = |a_2| = \varepsilon\,\}$, for a sufficiently small positive number ε, and is oriented so that $d(\arg a_1)\wedge d(\arg a_2)$ is a positive form [BB]. The above integral is equal to the intersection number $(a_1, a_2)_0$ of two curves

$a_1 = 0$ and $a_2 = 0$ in U at 0, which is equal to $\dim_{\mathbb{C}} \mathcal{O}/(a_1, a_2)$, where (a_1, a_2) is the ideal generated by a_1 and a_2 in $\mathcal{O} = \mathcal{O}_{\mathbb{C}^2,0}$. Hence we have

$$\mathrm{Res}_{c_2}(\mathcal{N}_{\mathcal{F}}, \mathcal{F}; S) = (a_1, a_2)_0.$$

On the other hand, since the rank of N^ν is one,

$$\mathrm{Res}_{c_2}(N^\nu, \mathcal{F}; S^\nu) = 0.$$

We also have

$$\mathrm{Res}_{c_1^2}(\mathcal{N}_{\mathcal{F}}, \mathcal{F}; S) = \left(\frac{1}{2\pi\sqrt{-1}}\right)^2 \int_\Gamma \frac{\mathrm{tr}(A) dz_1 \wedge dz_2}{a_1 a_2}.$$

Now we compute the difference $\mathrm{Res}_{c_1^2}(\mathcal{N}_{\mathcal{F}}, \mathcal{F}; S) - \mathrm{Res}_{c_1^2}(N^\nu, \mathcal{F}; S^\nu)$ of the Baum-Bott and Nash residues. Since, in this case, $\mathcal{F}$ is a free $\mathcal{O}$-module of rank one, in (4.2), we may set $q = 1$ and may take the trivial bundle as E_1 ; $E_1 = F = U \times \mathbb{C}$. Thus the exact sequence (4.3) becomes

$$0 \to F_0 \xrightarrow{h} TU_0 \xrightarrow{g} N_{F_0} \to 0, \tag{4.10}$$

where $F_0 = F|_{U_0}$, $U_0 = U \setminus \{0\}$. Using also the notation in the proof of Theorem 4.1, from $c_1(\tilde{\pi}^*\xi) = c_1(\tilde{N}^\nu) + c_1(\tilde{\varepsilon})$, we have

$$c_1^2(\tilde{\pi}^*\xi) = c_1^2(\tilde{N}^\nu) + 2c_1(\tilde{N}^\nu) \cdot c_1(\tilde{\varepsilon}) + c_1^2(\tilde{\varepsilon}).$$

Hence (4.5) becomes

$$c_1^2(\tilde{\pi}^*\nabla_\star^\bullet) = c_1^2(\nabla_\star^\nu) + 2c_1(\nabla_\star^\nu) \cdot c_1(\nabla_\star^\varepsilon) + c_1^2(\nabla_\star^\varepsilon) + D\tau.$$

The left hand side corresponds to the Baum-Bott residue and the first term in the right hand side corresponds to the Nash residue. We recall that the cocycles in the right hand side are given by

$$\begin{aligned} c_1(\nabla_\star^\nu) &= (c_1(\nabla_0^\nu), c_1(\nabla_1^\nu), c_1(\nabla_0^\nu, \nabla_1^\nu)) \qquad \text{and} \\ c_1(\nabla_\star^\varepsilon) &= (0, c_1(\nabla_1^\varepsilon), c_1(\nabla_0^\varepsilon, \nabla_1^\varepsilon)). \end{aligned}$$

We have

$$c_1(\nabla_1^\varepsilon) = -c_1(\tilde{\pi}^*\nabla_1^{(1)}) + c_1(\tilde{\pi}^*\nabla_1^{(0)}) - c_1(\nabla_1^\nu).$$

We denote by 1 the frame for $E_1 = F = U \times \mathbb{C}$ which assigns $(z, 1)$ to each point z in U and take as $\nabla_1^{(1)}$ the 1-trivial connection for F. Note that the frame 1 is sent to v by the map g in (4.10). We also take as $\nabla_1^{(0)}$ the connection for TM on U which is $\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right)$-trivial. Then we have $c_1(\nabla_1^\varepsilon) = -c_1(\nabla_1^\nu)$. Recalling that $c_1(\nabla_\star^\nu) \cdot c_1(\nabla_\star^\varepsilon)$ is cohomologous to $c_1(\nabla_\star^\varepsilon) \cdot c_1(\nabla_\star^\nu)$ and that $c_1^2(\nabla_\star^\varepsilon)$ is cohomologous to $c_1(\nabla_\star^\varepsilon)^2$ [Su3, Proposition 1.6], we compute and see that the cocycle $2c_1(\nabla_\star^\nu) \cdot c_1(\nabla_\star^\varepsilon) + c_1^2(\nabla_\star^\varepsilon)$ is cohomologous to

$$(0, 0, c_1(\nabla_0^\varepsilon, \nabla_1^\varepsilon) \cdot c_1(\nabla_1^\nu)).$$

Hence the difference of the Baum-Bott and Nash residues is given by the integral

$$\int_{R_{01}^\nu} c_1(\nabla_0^\varepsilon, \nabla_1^\varepsilon) \cdot c_1(\nabla_1^\nu). \tag{4.11}$$

In our case, we may take as $\tilde{U}_1^\nu$ a tubular neighborhood of S^ν with projection $\tilde{p} : \tilde{U}_1^\nu \to S^\nu$, and as $\tilde{R}_1^\nu$ a closed tubular neighborhood of S^ν in $\tilde{U}_1^\nu$. Recall that

$R_{01}^\nu = -\partial R_1^\nu$, $R_1^\nu = \tilde{R}_1^\nu \cap M^\nu$. Since $\tilde{N}^\nu$ is isomorphic with $\tilde{p}^*\tilde{N}^\nu|_{S^\nu}$, we may take as ∇_1^ν the connection equivalent to the pull-back $\tilde{p}^*\nabla_S^\nu$, where ∇_S^ν denotes a connection for $\tilde{N}^\nu|_{S^\nu}$. If we denote by p the restriction of $\tilde{p}$ to R_{01}^ν, we see that, by the projection formula, the integral (4.11) is equal to

$$-\int_{S^\nu} p_* c_1(\nabla_0^\varepsilon, \nabla_1^\varepsilon) \cdot c_1(\nabla_S^\nu),$$

where p_* denotes the integration along the fibers of p. To compute $c_1(\nabla_0^\varepsilon, \nabla_1^\varepsilon)$, we need to fix frames of the bundles involved. First we cover U_0 by two open sets W_i, $i = 1, 2$, given by $W_i = \{\, z \in U \mid a_i(z) \neq 0 \,\}$. We identify F_0 with the image $h(F_0)$ in TU_0. Then, on W_1, we may choose v, $\left(v, \frac{\partial}{\partial z_2}\right)$ and $g\left(\frac{\partial}{\partial z_2}\right)$, respectively, as frames for F_0, TU_0 and N_{F_0}. Let $\theta_0^{(1)}$, $\theta_1^{(1)}$, $\theta_0^{(0)}$, $\theta_1^{(0)}$ and θ be the connection matrices of $\nabla_0^{(1)}$, $\nabla_1^{(1)}$, $\nabla_0^{(0)}$, $\nabla_1^{(0)}$ and ∇, respectively, with respect to these frames. Also, let θ_S^ν be the connection matrix (form) of ∇_S^ν with respect to some frame of $N^\nu|_{S^\nu}$. Then we compute

$$\begin{aligned} c_1(\nabla_0^\varepsilon, \nabla_1^\varepsilon) &= -c_1(\pi^*\nabla_0^{(1)}, \pi^*\nabla_1^{(1)}) + c_1(\pi^*\nabla_0^{(0)}, \pi^*\nabla_1^{(0)}) - c_1(\nabla_0^\nu, \nabla_1^\nu) \\ &= \frac{\sqrt{-1}}{2\pi}\left(-\pi^*\theta_1^{(1)} + \pi^*\theta_0^{(1)} + \operatorname{tr}(\pi^*\theta_1^{(0)} - \pi^*\theta_0^{(0)}) - p^*\theta_S^\nu + \pi^*\theta\right). \end{aligned}$$

Since $\nabla_1^{(1)}$ is v-trivial, we have $\theta_1^{(1)} = 0$. Also, since $(\nabla_0^{(1)}, \nabla_0^{(0)}, \nabla)$ is compatible with (4.10), we have $\operatorname{tr}\theta_0^{(0)} = \theta_0^{(1)} + \theta$. We compute $\operatorname{tr}\theta_1^{(0)} = \frac{da_1}{a_1}$. Hence we have, on $\pi^{-1}W_1$,

$$c_1(\nabla_0^\varepsilon, \nabla_1^\varepsilon) = \frac{\sqrt{-1}}{2\pi}\left(\pi^*\left(\frac{da_1}{a_1}\right) - p^*\theta_S^\nu\right).$$

Similarly, on $\pi^{-1}W_2$, we have

$$c_1(\nabla_0^\varepsilon, \nabla_1^\varepsilon) = \frac{\sqrt{-1}}{2\pi}\left(\pi^*\left(\frac{da_2}{a_2}\right) - p^*\theta_S^\nu\right).$$

Note that, since M^ν is given by $a_1\zeta_1 - a_2\zeta_0 = 0$, we have $\pi^*\left(\frac{da_1}{a_1}\right) = \pi^*\left(\frac{da_2}{a_2}\right)$ on $\pi^{-1}(W_1 \cap W_2)$. We identify $S^\nu = 0 \times \mathbb{CP}^1$ with $\mathbb{CP}^1$ and take a point $(\zeta_0 : \zeta_1)$ in S^ν with $\zeta_0 \neq 0$ and set $\zeta = \zeta_1/\zeta_0$. Then $p_*c_1(\nabla_0^\varepsilon, \nabla_1^\varepsilon)$ is a function on S^ν whose value at ζ is

$$\frac{\sqrt{-1}}{2\pi}\int_{p^{-1}(\zeta)} \pi^*\left(\frac{da_1}{a_1}\right) = \frac{\sqrt{-1}}{2\pi}\int_{\pi_* p^{-1}(\zeta)} \frac{da_1}{a_1}.$$

Since $\pi_* p^{-1}(\zeta)$ is the link of the curve in U defined by $a_2 - \zeta a_1 = 0$, the last integral is equal to

$$-(a_2 - \zeta a_1, a_1)_0 = -(a_1, a_2)_0.$$

Finally, noting that $\int_{S^\nu} c_1(\nabla_S^\nu) = 1$, we get

$$\operatorname{Res}_{c_1^2}(\mathcal{N}_\mathcal{F}, \mathcal{F}; S) = \operatorname{Res}_{c_1^2}(N^\nu, \mathcal{F}; S^\nu) + (a_1, a_2)_0,$$

which is announced in [Su2, Ch.VI, Example 5.5] (note the sign correction).

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