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MIRROR PRINCIPLE III*

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Abstract. We generalize the theorems in *Mirror Principle I* and *II* to the case of general projective manifolds without the convexity assumption. We also apply the results to balloon manifolds, and generalize to higher genus.

1. Introduction.

The present paper is a sequel to *Mirror Principle I* and *II* [29][30]. For motivations and the main ideas of mirror principle, we refer the reader to the introductions of these two papers.

Let X be a projective manifold, and $d \in A_1(X)$. Let $M_{0,k}(d,X)$ denote the moduli space of k-pointed, genus 0, degree d, stable maps $(C, f, x_1, .., x_k)$ with target X [26]. *Note that our notation is without the bar.* By the construction of [27](see also [6][14]), each nonempty $M_{0,k}(d,X)$ admits a homology class $LT_{0,k}(d,X)$ of dimension $dim\ X + \langle c_1(X), d\rangle + k - 3$. This class plays the role of the fundamental class in topology, hence $LT_{0,k}(d,X)$ is called the virtual fundamental class. For background on this, we recommend [28].

Let V be a convex vector bundle on X. (ie. $H^1(\mathbf{P}^1, f^*V) = 0$ for every holomorphic map $f : \mathbf{P}^1 \to X$.) Then V induces on each $M_{0,k}(d,X)$ a vector bundle V_d, with fiber at $(C, f, x_1, .., x_k)$ given by the section space $H^0(C, f^*V)$. Let b be any multiplicative characteristic class [20]. (ie. if $0 \to E' \to E \to E'' \to 0$ is an exact sequence of vector bundles, then $b(E) = b(E')b(E'')$.) The problem we study here is to understand the intersection numbers

$$K_d := \int_{LT_{0,0}(d,X)} b(V_d)$$

and their generating function:

$$\Phi(t) := \sum K_d\ e^{d\cdot t}.$$

There is a similar and equally important problem if one starts from a concave vector bundle V [29]. (ie. $H^0(\mathbf{P}^1, f^*V) = 0$ for every holomorphic map $f : \mathbf{P}^1 \to X$.) More generally, V can be a direct sum of a convex and a concave bundle. Important progress made on these problems has come from mirror symmetry. All of it seems to point toward the following general phenomenon [9], which we call *the Mirror Principle.* Roughly, it says that there are functional identities which can be used to either constrain or to compute the K_d often in terms of certain explicit special functions, loosely called generalized hypergeometric functions. In this paper, we generalize this principle to include all projective manifolds. We apply this theory to compute the multiplicative classes $b(V_d)$ for vector bundles on balloon manifolds. The answer is in terms of certain universal virtual classes which are independent of V, b.

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When X is a toric manifold, b is the Euler class, and V is a sum of line bundles, there is a general formula derived in [21][23] based on mirror symmetry, giving $\Phi(t)$ in terms of generalized hypergeometric functions [15]. Similar functions were studied [16] in equivariant quantum cohomology theory based on a series of axioms. For further background, see introduction of [29].

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1.1. Outline. In section 2, we do the necessary preparation to set up the version of localization theorem we need. This is a (functorial localization) formula which translates a commutative square diagram into a relation between localizations on two T spaces related by an equivariant map.

We do basically three things in section 3. After we introduced the necessary notations, first we apply functorial localization to stable map moduli spaces. Second, we prove one of the main results of this paper: Theorem 3.6, which translate structure of fixed points on stable map moduli into an algebraic identity on the homology of a projective manifold (with or without T action). This motivates the notion of Euler data and Euler series. These are essentially solutions to the algebraic identity just mentioned. Third, we prove the main Theorems 3.12-3.13 which relate the generating functions $\Phi(t)$ with an Euler series $A(t)$ arising from induced bundles on stable map moduli.

In section 4, we specialize results in section 3 to balloon manifolds, and introduce the notion of linking. The main theorems here are 4.5 and 4.7. The first of these gives a description of an essential polar term of $A(t)$ upon localizing at a fixed point in X. The second theorem gives a sufficient condition for computing $A(t)$ in terms of certain universal virtual classes on stable map moduli. We then specialize this to the case when b_T is the Euler class or the Chern polynomial.

In section 5, we explain some other ways to compute $A(t)$, first by relaxing those sufficient conditions, then by finding an explicit closed formula for those universal virtual classes above by using an equivariant short exact sequence for the tangent bundle. This includes toric manifolds as a special case. We then formulate an inductive method for computing $A(t)$ in full generality for any balloon manifold. Next, we discuss a method in which functorial localization is used to study $A(t)$ via a resolution of the image of the collapsing map. In certain cases, this resolution can be described quite explicitly. Finally, we discuss a generalization of mirror principle to higher genus.

2. Set-up. Basic references: on intersection theory on algebraic schemes and stacks, we use [13][40]; on the virtual classes, we follow [27]; on their equivariant counterparts, see [1][2][7][25][12][17][41].

T denotes an algebraic torus. T-equivariant Chow groups (homology) with complex coefficients are denoted by $A_*^T(\cdot)$. T-equivariant operational Chow groups (cohomology) with complex coefficients are denoted by $A_T^*(\cdot)$. For $c \in A_T^p(X)$, and $\beta \in A_q^T(X)$, we denote by $c \cap \beta = \beta \cap c$ the image of $c \otimes \beta$ under the canonical homomorphisms

$$A_T^p(X) \otimes A_q^T(X) \to A_{q-p}^T(X).$$

The product on $A_T^*(X)$ is denoted by $a \cdot b$. The homomorphisms $\cap$ define an $A_T^*(X)$-module structure on the homology $A_*^T(X)$. When X is nonsingular, there is a compatible intersection product on $A_*^T(X)$ which we denote by $\beta \cdot \gamma$.

Given a T-equivariant (proper or flat) map $f : X \to Y$, we denote by

$$f_* : A_*^T(X) \to A_*^T(Y), \; f^* : A_*^T(Y) \to A_*^T(X)$$

the equivariant (proper) pushforward and (flat) pullback; the notations f^* and f_* are also used for pullback and (flat) pushforward on cohomology. All maps used here will be assumed proper. A formula often used is the projection formula:

$$f_*(f^*c \cap \beta) = c \cap f_*(\beta)$$

for cohomology class c on Y and homology class β on X. Note that both $A_*^T(X)$ and $A_T^*(X)$ are modules over the algebra $A_T^*(pt) = \mathbf{C}[\mathcal{T}^*]$, where $\mathcal{T}^*$ is the dual of the Lie algebra of T, and the homomorphisms f_*, f^* are module homomorphisms. We often extend these homomorphisms over the field $\mathbf{C}(\mathcal{T}^*)$ without explicitly saying so. Finally, suppose we have a fiber square

$$\begin{array}{ccc} F & \stackrel{i'}{\longrightarrow} & M \\ p\downarrow & & \downarrow q \\ X & \stackrel{i}{\longrightarrow} & Y \end{array}$$

where i is a regular embedding of codimension d, then we have

$$p_* i^! \beta = i^* q_* \beta$$

for any homology class β on M. Here $i^! : A_*^T(M) \to A_{*-d}^T(F)$ is the refined Gysin homomorphism.

2.1. Functorial localization. Let X be an algebraic stack with a T action and equipped with a suitable perfect obstruction theory (see [27][17]). Let F_r denote the fixed point components in X. Let $[X]^{vir}, [F_r]^{vir}$ be the equivariant virtual classes of X and the F_r. Then by [17],

$$[X]^{vir} = \sum_r i_{r*} \frac{[F_r]^{vir}}{e_T(F_r/X)}$$

where $i_r : F_r \to X$ are the inclusions, and $e_T(F_r/X)$ the equivariant Euler class of the virtual normal bundle of $F_r \subset X$. Then for any cohomology class c on X, we have

$$c \cap [X]^{vir} = \sum_r i_{r*} \frac{i_r^* c \cap [F_r]^{vir}}{e_T(F_r/X)}. \tag{2.1}$$

Throughout this subsection, let

$$f : \; X \to Y$$

be an equivariant map with Y smooth. Let E be a fixed point component in Y, and let F be the fixed points in $f^{-1}(E)$. Let g be the restriction of f to F, and $j_E : E \to Y$, $i_F : F \to X$ be the inclusion maps. Thus we have the commutative diagram:

$$\begin{array}{ccc} F & \stackrel{i_F}{\longrightarrow} & X \\ g\downarrow & & \downarrow f \\ E & \stackrel{j_E}{\longrightarrow} & Y. \end{array} \tag{2.2}$$

Then we have the following *functorial localization formula*.

LEMMA 2.1. *Given a cohomology class $\omega \in A_T^*(X)$, we have the equality on E:*

$$\frac{j_E^* f_*(\omega \cap [X]^{vir})}{e_T(E/Y)} = g_* \left(\frac{i_F^* \omega \cap [F]^{vir}}{e_T(F/X)} \right).$$

Proof. Applying 2.1 to the class $c = \omega \cdot f^* j_{E*} 1$ on X, we get

$$\omega \cdot f^* j_{E*} 1 \cap [X]^{vir} = i_{F*} \left(\frac{i_F^* (\omega \cdot f^* j_{E*} 1) \cap [F]^{vir}}{e_T(F/X)} \right).$$

Note that the contributions from fixed components other than F vanish. Applying f_* to both sides, we get

$$f_*(\omega \cap [X]^{vir}) \cap j_{E*} 1 = f_* i_{F*} \left(\frac{i_F^* (\omega \cdot f^* j_{E*} 1) \cap [F]^{vir}}{e_T(F/X)} \right).$$

Now $f \circ i_F = j_E \circ g$ which, implies

$$f_* i_{F*} = j_{E*} g_*, \quad i_F^* f^* = g^* j_E^*.$$

Thus we get

$$f_*(\omega \cap [X]^{vir}) \cap j_{E*} 1 = j_{E*} g_* \left(\frac{i_F^* \omega \cdot g^* e_T(E/Y) \cap [F]^{vir}}{e_T(F/X)} \right).$$

Applying j_E^* to both sides here, we get

$$\begin{aligned} j_E^* f_*(\omega \cap [X]^{vir}) \cap e_T(E/Y) &= e_T(E/Y) \cap g_* \left(\frac{i_F^* \omega \cdot g^* e_T(E/Y) \cap [F]^{vir}}{e_T(F/X)} \right) \\ &= e_T(E/Y)^2 \cap g_* \left(\frac{i_F^* \omega \cap [F]^{vir}}{e_T(F/X)} \right). \end{aligned}$$

Since $e_T(E/Y)$ is invertible, our assertion follows. □

Note that if F has more than one component, then the right hand side of the formula above becomes a sum over those components in an obvious way.

COROLLARY 2.2. *Let Y' be a T-invariant submanifold of Y, $f' : X' = f^{-1}(Y') \to Y'$ be the restriction of $f : X \to Y$ to the substack X', and $j : Y' \to Y$, $i : X' \to X$ be the inclusions. Then for any $\omega \in A_T^*(X)$, we have*

$$\frac{j^* f_*(\omega \cap [X]^{vir})}{e_T(Y'/Y)} = f'_* \left(\frac{i^* \omega \cap [X']^{vir}}{e_T(X'/X)} \right).$$

Proof. Let E be any fixed point component of Y contained in Y', and F be the fixed points in $f^{-1}(E)$, as in the preceding lemma. Then we have the commutative diagram

$$(2.3) \qquad \begin{array}{ccccc} F & \xrightarrow{i'_F} & X' & \xrightarrow{i} & X \\ g \downarrow & & f' \downarrow & & f \downarrow \\ E & \xrightarrow{j'_E} & Y' & \xrightarrow{j} & Y. \end{array}$$

We will show that

$$(*) \; {j'_E}^* \left(\frac{j^* f_*(\omega \cap [X]^{vir})}{e_T(Y'/Y)} \right) = {j'_E}^* f'_* \left(\frac{i_F^* \omega \cap [X']^{vir}}{e_T(X'/X)} \right).$$

Then our assertion follows from the localization theorem.

Put $j_E := j \circ j'_E$, $i_F := i \circ i'_F$. The left hand side of (*) is

$$\frac{{j'_E}^* j^* f_*(\omega \cap [X]^{vir})}{{j'_E}^* e_T(Y'/Y)} = e_T(E/Y') \cap \frac{j_E^* f_*(\omega \cap [X]^{vir})}{e_T(E/Y)}$$
$$= e_T(E/Y') \cap g_* \left(\frac{i_F^*\omega \cap [F]^{vir}}{e_T(F/X)}\right) \text{ (preceding lemma).}$$

Now apply the left hand square in 2.3 and the preceding lemma again to the class $\frac{i^*\omega}{e_T(X'/X)}$ on X'. Then the right hand side of (*) becomes

$${j'_E}^* f'_* \left(\frac{i_F^*\omega}{e_T(X'/X)} \cap [X']^{vir}\right) = e_T(E/Y') \cap g_* \left(\frac{{i'_F}^* \frac{i^*\omega}{e_T(X'/X)} \cap [F]^{vir}}{e_T(F/X')}\right)$$
$$= e_T(E/Y') \cap g_* \left(\frac{i_F^*\omega \cap [F]^{vir}}{e_T(F/X)}\right).$$

This proves (*). □

3. General Projective T-manifolds. Let X be a projective T-manifold. Let $M_d(X)$ be the degree $(1,d)$, arithmetic genus zero, 0-pointed, stable map moduli stack with target $\mathbf{P}^1 \times X$. The standard $\mathbf{C}^\times$ action on $\mathbf{P}^1$ together with the T action on X induces a $G = \mathbf{C}^\times \times T$ action on $M_d(X)$. Let $LT_d(X) \in A_*^G(M_d(X))$ be the virtual class of this moduli stack. This is an equivariant homology class of dimension $\langle c_1(X), d\rangle + dim\ X$.

The $\mathbf{C}^\times$ fixed point components F_r, labelled by $0 \preceq r \preceq d$, in $M_d(X)$ can be described as follows (see [30]). Let F_r be the substack

$$F_r := M_{0,1}(r,X) \times_X M_{0,1}(d-r,X)$$

obtained from gluing the two one pointed moduli stacks. More precisely, consider the map

$$e_r^X \times e_{d-r}^X : M_{0,1}(r,X) \times M_{0,1}(d-r,X) \to X \times X$$

given by evaluations at the corresponding marked points; and

$$\Delta : \ X \to X \times X$$

the diagonal map. Then we have

$$F_r = (e_r^X \times e_{d-r}^X)^{-1}\Delta(X).$$

Note that $F_d = M_{0,1}(d,X) = F_0$ by convention, but F_0 and F_d will be embedded into $M_d(X)$ in two different ways. The F_r can be identified with a $\mathbf{C}^\times$ fixed point component of $M_d(X)$ as follows. Consider the case $r \neq 0, d$ first. Given a pair $(C_1, f_1, x_1) \times (C_2, f_2, x_2)$ in F_r, we get a new curve C by gluing C_1, C_2 to $\mathbf{P}^1$ with x_1, x_2 glued to $0, \infty \in \mathbf{P}^1$ respectively. The new curve C is mapped into $\mathbf{P}^1 \times X$ as follows. Map $\mathbf{P}^1 \subset C$ identically onto $\mathbf{P}^1$, and collapse C_1, C_2 to $0, \infty$ respectively; then map C_1, C_2 into X with f_1, f_2 respectively, and collapse the $\mathbf{P}^1$ to $f(x_1) = f(x_2)$. This defines a stable map (C, f) in $M_d(X)$. For $r = d$, we glue (C_1, f_1, x_1) to $\mathbf{P}^1$ at x_1 and 0. For $r = 0$, we glue (C_2, f_2, x_2) to $\mathbf{P}^1$ at x_2 and ∞.

Notations:

(i) We identify F_r as a substack of $M_d(X)$ as above, and let

$$i_r : F_r \to M_d(X)$$

denote the inclusion map.

(ii) We have evaluation maps

$$e^X : F_r \to X,$$

which sends a pair in F_r to the value at the common marked point. While the notation e^X doesn't reflect the dependence on r, the domain F_r that e^X operates on will be clear.

(iii) We have the obvious inclusion

$$\Delta' : F_r \subset M_{0,1}(r, X) \times M_{0,1}(d - r, X),$$

and projections

$$p_0 : F_r \to M_{0,1}(r, X), \ p_\infty : F_r \to M_{0,1}(d - r, X).$$

(iv) Let L_r denote the universal line bundle on $M_{0,1}(r, X)$.

(v) We have the natural forgetting, evaluation, and projection maps:

$$\begin{aligned} \rho &: M_{0,1}(d, X) \to M_{0,0}(d, X) \\ e_d^X &: M_{0,1}(d, X) \to X \\ \pi &: M_d(X) \to M_{0,0}(d, X). \end{aligned}$$

We also have the obvious commutative diagrams

$$\begin{array}{ccc} M_d(X) & & \\ \pi \downarrow & \nwarrow i_0 & \\ M_{0,0}(d, X) & \overset{\rho}{\longleftarrow} & M_{0,1}(d, X) \end{array}$$

$$\begin{array}{ccc} F_r & \overset{\Delta'}{\longrightarrow} & M_{0,1}(r, X) \times M_{0,1}(d - r, X) \\ e^X \downarrow & & \downarrow e_r^X \times e_{d-r}^X \\ X & \overset{\Delta}{\longrightarrow} & X \times X \end{array} \tag{3.1}$$

where Δ is the diagonal map. Note that we have a diagram similar to 3.1 but with X replaced by Y in the bottom row. From the fiber square 3.1, we have a refined Gysin homomorphism

$$\Delta^! : A_*^T(M_{0,1}(r, X) \times M_{0,1}(d - r, X)) \to A_{*-dim\ X}^T(F_r).$$

We refer the reader to section 6 [27] for the following

LEMMA 3.1. *[27] For $r \neq 0, d$, $[F_r]^{vir} = \Delta^!(LT_{0,1}(r, X) \times LT_{0,1}(d - r, X))$.*

(vi) Let α be the weight of the standard $\mathbf{C}^\times$ action on $\mathbf{P}^1$. We denote by $A_*^T(X)(\alpha)$ the algebra obtained from $A_*^T(X)[\alpha]$ by inverting the classes ω such that $(i_F^*\omega)^{-1}$ is well-defined in $A_*(F) \otimes \mathbf{C}(\mathcal{T}^*)(\alpha)$, for every fixed point component F. If β is an element in $A_*^T(X)(\alpha)$, we let $\overline{\beta}$ be the class obtained from β by replacing $\alpha \to -\alpha$. We also introduce formal variables $\zeta = (\zeta_1, ..., \zeta_m)$ such that $\bar{\zeta}_a = -\zeta_a$. Denote $\mathcal{R} = \mathbf{C}(\mathcal{T}^*)[\alpha]$. When a multiplicative class b_T, such as the Chern polynomial $c_T = x^r + xc_1 + \cdots + c_r$, is considered, we must replace the ground field $\mathbf{C}$ by $\mathbf{C}(x)$, so that c_T takes value in Chow groups with appropriate coefficients. This change of ground field will be implicit whenever necessary.

(vii) For each d, let $\varphi : M_d(X) \to W_d$ be a G-equivariant map into smooth manifold (or orbifold) W_d with the property *that the $\mathbf{C}^\times$ fixed point components in W_d are G-invariant submanifolds Y_r such that $\varphi^{-1}(Y_r) = F_r$.*

The spaces W_d exist but are not unique. Two specific kinds will be used here. First, choose an equivariant projective embedding

$$\tau : X \to Y = \mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_m}$$

which induces an isomorphism $A^1(X) \cong A^1(Y)$. Then we have a G-equivariant embedding

$$M_d(X) \to M_d(Y).$$

There is a G-equivariant map (see [29] and references there)

$$M_d(Y) \to W_d := N_{d_1} \times \cdots \times N_{d_m}$$

where the $N_{d_a} := \mathbf{P}H^0(\mathbf{P}^1, \mathcal{O}(d_a))^{n_a+1} \cong \mathbf{P}^{(n_a+1)d_a+n_a}$, which are the linear sigma model for the $\mathbf{P}^{n_i}$. Thus composing the two maps above, we get a G-equivariant map $\varphi : M_d(X) \to W_d$. It is also easy to check that the $\mathbf{C}^\times$ fixed point components in W_d have the desired property. Second, if X is a toric variety, then there exist toric varieties W_d [31] where Y_r are submanifolds of X. We postpone the discussion of this till section 5 when we discuss the case of toric manifolds. From now on, unless specified otherwise, W_d will be the first kind as defined above.

(viii) We denote the equivariant hyperplane classes on W_d by κ_a (which are pullbacked from the each of the N_{d_a} to W_d). We denote the equivariant hyperplane classes on Y by H_a (which are pullbacked from each of the $\mathbf{P}^{n_a}$ to Y). We use the same notations for their restrictions to X. We write $\kappa \cdot \zeta = \sum_a \kappa_a \zeta_a$, $H \cdot t = \sum_a H_a t_a$, $d \cdot t = \sum_a d_a t_a$, where the t and ζ are formal variables.

3.1. Localization on stable map moduli. Clearly we have the commutative diagram:

$$\begin{array}{ccc} F_r & \xrightarrow{i_r} & M_d(X) \\ e^Y \downarrow & & \downarrow \varphi \\ Y_r & \xrightarrow{j_r} & W_d. \end{array} \tag{3.2}$$

Let $\varphi : M_d(X) \to W_d$, $e^Y : F_r \to Y_r$ play the respective roles of $f : X \to Y$, $f' : X' \to Y'$ in functorial localization. Then it follows that

LEMMA 3.2. *Given a cohomology class ω on $M_d(X)$, we have the following equality on $Y_r \cong Y$ for $0 \preceq r \preceq d$:*

$$\frac{j_r^* \varphi_*(\omega \cap LT_d(X))}{e_G(Y_r/W_d)} = e_*^Y \left(\frac{i_r^* \omega \cap [F_r]^{vir}}{e_G(F_r/M_d(X))} \right).$$

Following [29], one can easily compute the Euler classes $e_G(Y_r/W_d)$, and they are given as follows. For $d = (d_1, .., d_m)$, $r = (r_1, ..., r_m) \preceq d$, we have

$$e_G(Y_r/W_d) = \prod_{a=1}^{m} \prod_{i=0}^{n_a} \prod_{k=0 k\neq r_a}^{d_a} (H_a - \lambda_{a,i} - (k - r_a)\alpha)$$

where the $\lambda_{a,i}$ are the T weights of $\mathbf{P}^{n_a}$. Note that e^Y is the composition of $e^X : F_r \to X$ with $\tau : X \to Y = Y_r$. Thus

$$e_*^Y = \tau_* e_*^X.$$

It follows that

LEMMA 3.3. *Given a cohomology class ω on $M_d(X)$, we have the following equality on X for $0 \preceq r \preceq d$:*

$$\tau^* \left(\frac{j_r^* \varphi_*(\omega \cap LT_d(X))}{e_G(Y_r/W_d)} \right) = e_T(X/Y) \cap e_*^X \left(\frac{i_r^* \omega \cap [F_r]^{vir}}{e_G(F_r/M_d(X))} \right).$$

Now if ψ is a cohomology class on $M_{0,0}(d, X)$, then for $\omega = \pi^* \psi$, we get $i_0^* \omega = i_0^* \pi^* \psi = \rho^* \psi$. It follows that

LEMMA 3.4. *Given a cohomology class ψ on $M_{0,0}(d, X)$, we have the following equality on X:*

$$\tau^* \left(\frac{j_0^* \varphi_*(\pi^* \psi \cap LT_d(X))}{e_G(Y_0/W_d)} \right) = e_T(X/Y) \cap e_*^X \left(\frac{\rho^* \psi \cap LT_{0,1}(d, X)}{e_G(F_0/M_d(X))} \right).$$

LEMMA 3.5. *For $r \neq 0, d$,*

$$e_G(F_r/M_d(X)) = \alpha(\alpha + p_0^* c_1(L_r)) \cdot \alpha(\alpha - p_\infty^* c_1(L_{d-r})).$$

For $r = 0, d$,

$$e_G(F_0/M_d(X)) = \alpha(\alpha - c_1(L_d)),\ e_G(F_d/M_d(X)) = \alpha(\alpha + c_1(L_d)).$$

The computation done in section 2.3 of [29] and in section 3 of [30] (see also references there), for the normal bundles $N_{F_r/M_d(X)}$, makes no use of the convexity assumption on TX. Therefore it carries over here with essentially no change.

3.2. From gluing identity to Euler data. Fix a T-equivariant multiplicative class b_T. Fix a T-equivariant bundle of the form $V = V^+ \oplus V^-$, where $V^\pm$ are respectively the convex/concave bundles on X. We assume that

$$\Omega := \frac{b_T(V^+)}{b_T(V^-)}$$

is a well-defined invertible class on X. By convention, if $V = V^\pm$ is purely convex/concave, then $\Omega = b_T(V^\pm)^{\pm 1}$. Recall that the bundle $V \to X$ induces the bundles

$$V_d \to M_{0,0}(d, X),\ U_d \to M_{0,1}(d, X),\ \mathcal{U}_d \to M_d(X).$$

Moreover, they are related by $U_d = \rho^* V_d$, $\mathcal{U}_d = \pi^* V_d$, Define linear maps

$$i_r^{vir} : A_G^*(M_d(X)) \to A_*^T(X)(\alpha),\ i_r^{vir} \omega := e_*^X \left(\frac{i_r^* \omega \cap [F_r]^{vir}}{e_G(F_r/M_d(X))} \right).$$

THEOREM 3.6. *For $0 \preceq r \preceq d$, we have the following identity in $A_*^T(X)(\alpha)$:*

$$\Omega \cap i_r^{vir} \pi^* b_T(V_d) = \overline{i_0^{vir} \pi^* b_T(V_r)} \cdot i_0^{vir} \pi^* b_T(V_{d-r}).$$

Proof. For simplicity, let's consider the case $V = V^+$. The general case is entirely analogous. The proof here is the one in [29][30], but slightly modified to take into account the new ingredient coming from the virtual class.

Recall that a point (f, C) in $F_r \subset M_d$ comes from gluing together a pair of stable maps $(f_1, C_1, x_1), (f_2, C_2, x_2)$ with $f_1(x_1) = f_2(x_2) = p \in X$. From this, we get an exact sequence over C:

$$0 \to f^*V \to f_1^*V \oplus f_2^*V \to V|_p \to 0.$$

Passing to cohomology, we have

$$0 \to H^0(C, f^*V) \to H^0(C_1, f_1^*V) \oplus H^0(C_2, f_2^*V) \to V|_p \to 0.$$

Hence we obtain an exact sequence of bundles on F_r:

$$0 \to i_r^*\mathcal{U}_d \to U_r' \oplus U_{d-r}' \to e^{X*}V \to 0.$$

Here $i_r^*\mathcal{U}_d$ is the restriction to F_r of the bundle $\mathcal{U}_d \to M_d(X)$. And U_r' is the pullback of the bundle $U_r \to M_{0,1}(d, X)$, and similarly for U_{d-r}'. Taking the multiplicative class b_T, we get the identity on F_r:

$$e^{X*}b_T(V) \cdot b_T(i_r^*\mathcal{U}_d) = b_T(U_r') \cdot b_T(U_{d-r}').$$

We refer to this as the *gluing identity.*

Now put

$$\omega = \frac{b_T(U_r)}{e_G(F_r/M_r(X))} \times \frac{b_T(U_{d-r})}{e_G(F_0/M_{d-r}(X))} \cap LT_{0,1}(r, X) \times LT_{0,1}(d-r, X)$$

From the commutative diagram 3.1, we have the identity:

$$e_*^X \Delta^!(\omega) = \Delta^*(e_r^X \times e_{d-r}^X)_*(\omega).$$

On the one hand is

$$\begin{aligned}
&\Delta^*(e_r^X \times e_{d-r}^X)_*(\omega) \\
&= (e_r^X)_* \frac{b_T(U_r) \cap LT_{0,1}(r, X)}{e_G(F_r/M_r(X))} \cdot (e_{d-r}^X)_* \frac{b_T(U_{d-r}) \cap LT_{0,1}(d-r, X)}{e_G(F_0/M_{d-r}(X))} \\
&= (e_r^X)_* \frac{\rho^* b_T(V_r) \cap LT_{0,1}(r, X)}{e_G(F_r/M_r(X))} \cdot (e_{d-r}^X)_* \frac{\rho^* b_T(V_{d-r}) \cap LT_{0,1}(d-r, X)}{e_G(F_0/M_{d-r}(X))} \\
&= \overline{i_0^{vir}\pi^* b_T(V_r)} \cdot i_0^{vir}\pi^* b_T(V_{d-r}).
\end{aligned}$$

On the other hand, applying the gluing identity, we have

$$\begin{aligned}
e_*^X \Delta^!(\omega) &= e_*^X \left(\frac{b_T(U_r')}{\alpha(\alpha + p_0^* c_1(L_r))} \cdot \frac{b_T(U_{d-r}')}{\alpha(\alpha - p_\infty^* c_1(L_{d-r}))} \cap [F_r]^{vir} \right) \\
&= e_*^X \left(\frac{e^{X*} b_T(V) \cdot i_r^* b_T(\mathcal{U}_d) \cap [F_r]^{vir}}{e_G(F_r/M_d(X))} \right) \\
&= b_T(V) \cap e_*^X \left(\frac{i_r^* b_T(\mathcal{U}_d) \cap [F_r]^{vir}}{e_G(F_r/M_d(X))} \right) \\
&= b_T(V) \cap i_r^{vir}\pi^* b_T(V_d).
\end{aligned}$$

This proves our assertion. □

Specializing the theorem to $b_T \equiv 1$, we get

COROLLARY 3.7. *$i_r^{vir}1_d = \overline{i_0^{vir}1_r} \cdot i_0^{vir}1_{d-r}$ where 1_d is the identity class in on $M_d(X)$.*

For a given convex/concave bundle V on X, and multiplicative class b_T, we put

$$A^{V,b_T}(t) = A(t) := e^{-H\cdot t/\alpha} \sum_d A_d \; e^{d\cdot t}$$

$$A_d := i_0^{vir}\pi^* b_T(V) = e_*^X \left(\frac{\rho^* b_T(V_d) \cap LT_{0,1}(d,X)}{e_G(F_0/M_d(X))} \right).$$

Here we will use the convention that $A_0 = \Omega$, and the sum is over all $d = (d_1, ..., d_m) \in \mathbf{Z}_+^m$. When the reference to V, b_T is clear, we'll drop them from the notations. The special case in the corollary will play an important role. So we introduce the notation:

$$\mathbf{1}(t) := e^{-H\cdot t/\alpha} \sum_d \mathbf{1}_d e^{d\cdot t}, \; \mathbf{1}_d = i_0^{vir} 1_d.$$

By the preceding theorem and Lemma 3.2, it follows immediately that for $\omega = \varphi_*(\pi^* b_T(V_d) \cap LT_d(X))$, we have

$$\begin{aligned}
\int_{W_d} \omega \cap e^{\kappa\cdot\zeta} &= \sum_{0 \preceq r \preceq d} \int_{Y_r} \frac{j_r^* \omega}{e_G(Y_r/W_d)} \; e^{(H+r\alpha)\cdot\zeta} \\
&= \sum_r \int_{Y_r} \tau_* i_r^{vir} \pi^* b_T(V_d) \; e^{(H+r\alpha)\cdot\zeta} \\
&= \sum_r \int_X i_r^{vir} \pi^* b_T(V_d) \; e^{(H+r\alpha)\cdot\zeta} \\
&= \sum_r \int_X \Omega^{-1} \cap \bar{A}_r \cdot A_{d-r} \; e^{(H+r\alpha)\cdot\zeta} \; (\mathit{Theorem}\ 3.6).
\end{aligned}$$

Since $\omega \in A_*^G(W_d)$, hence $\int_{W_d} \omega \cap c \in A_*^G(pt) = \mathbf{C}[\mathcal{T}^*, \alpha]$ for all $c \in A_G^*(W_d)$, it follows that both sides of the eqn. above lie in $\mathcal{R}[[\zeta]]$. This motivates the following (cf. [16])

DEFINITION 3.8. *Let $\Omega \in A_T^*(X)$, invertible. We call a power series of the form*

$$B(t) := e^{-H\cdot t/\alpha} \sum_d B_d \; e^{d\cdot t}, \; B_d \in A_*^T(X)(\alpha)$$

an Ω-Euler series if $\sum_{0 \preceq r \preceq d} \int_X \Omega^{-1} \cap \bar{B}_r \cdot B_{d-r} \; e^{(H+r\alpha)\cdot\zeta} \in \mathcal{R}[[\zeta]]$ for all d.

Thus we have seen above that an elementary consequence of the gluing identity in Theorem 3.6 is that

COROLLARY 3.9. *$A^{V,b_T}(t) = e^{-H\cdot t/\alpha} \sum_d i_0^{vir} \pi^* b_T(V_d) \; e^{d\cdot t}$ is an Euler series.*

DEFINITION 3.10. *[29] Let $\Lambda \in A_T^*(Y)$. We call a sequence $P: \; P_d \in A_G^*(W_d)$ an Λ-Euler data if*

$$\Lambda \cdot j_r^* P_d = \overline{j_0^* P_r} \cdot j_0^* P_{d-r}, \; 0 \preceq r \preceq d.$$

Let P be an Λ-Euler data such that $\tau^*\Lambda$ is invertible. Then we have

$$\tau^*\Lambda \cdot \tau^* j_r^* P_d \cap i_r^{vir} 1_d = \tau^* \overline{j_0^* P_r} \cdot \tau^* j_0^* P_{d-r} \cap \overline{i_0^{vir} 1_r} \cdot i_0^{vir} 1_{d-r}. \quad (3.3)$$

By Lemma 3.3,

$$\begin{aligned}\tau^* j_r^* P_d \cap i_r^{vir} 1_d &= \tau^* j_r^* P_d \cdot e_T(X/Y)^{-1} \cap \tau^* \left(\frac{j_r^* \varphi_* LT_d(X)}{e_G(Y_r/W_d)} \right) \\ &= e_T(X/Y)^{-1} \cap \tau^* \left(\frac{j_r^* \varphi_*(\varphi^* P_d \cap LT_d(X))}{e_G(Y_r/W_d)} \right) \\ &= i_r^{vir} \varphi^* P_d.\end{aligned}$$

Thus 3.3 becomes

$$\tau^* \Lambda \cap i_r^{vir} \varphi^* P_d = \overline{i_0^{vir} \varphi^* P_r} \cdot i_0^{vir} \varphi^* P_{d-r}.$$

(cf. Theorem 3.6.) From this we get, as before,

$$\int_{W_d} \varphi_* LT_d(X) \cap P_d \cdot e^{\kappa \cdot \zeta} = \int_X \tau^* \Lambda^{-1} \cap \overline{i_0^{vir} \varphi^* P_r} \cdot i_0^{vir} \varphi^* P_{d-r}\ e^{(H + r\alpha)\cdot \zeta} \in \mathcal{R}[[\zeta]].$$

Therefore, that

$$B(t) = e^{-H \cdot t/\alpha} \sum_d i_0^{vir} \varphi^* P_d\ e^{d \cdot t}$$

is an Euler series, is just an elementary consequence of the Euler data identity. More generally, we have

THEOREM 3.11. *Let P be an Λ-Euler data as before, and let $\mathbf{O}(t) = e^{-H \cdot t/\alpha} \sum_d O_d\ e^{d \cdot t}$ be any Ω-Euler series. Then*

$$B(t) = e^{-H \cdot t/\alpha} \sum_d \tau^* j_0^* P_d \cap O_d\ e^{d \cdot t}$$

is an $\Omega \cdot \tau^* \Lambda$-Euler series.

Proof. Define P_d' on W_d by setting

$$j_r^* P_d' := \tau_*(\Omega^{-1} \bar{O}_r \cdot O_{d-r}) \cap e_G(Y_r/W_d).$$

By the localization theorem, this defines a class on W_d. Moreover, we have

$$\int_{W_d} P_d' \cap e^{\kappa \cdot \zeta} = \sum_r \int_X \Omega^{-1} \bar{O}_r \cdot O_{d-r}\ e^{(H + r\alpha)\cdot \zeta} \in \mathcal{R}.$$

It follows that $P_d' \in A_*^G(W_d) \otimes \mathcal{R}$ (see proof of Lemma 2.15 [29]). Now

$$\int_{W_d} P_d \cap P_d'\ e^{\kappa \cdot \zeta} = \sum_r \int_X \Omega^{-1} \tau^* \Lambda^{-1}\ \overline{(\tau^* j_0^* P_r \cap O_r)} \cdot (\tau^* j_0^* P_{d-r} \cap O_{d-r})\ e^{(H + r\alpha)\cdot \zeta},$$

which lies in $\mathcal{R}$ because $P_d \cap P_d'$ lies in $A_*^G(W_d) \otimes \mathcal{R}$. □

Note that if $O_d = \mathbf{1}_d$, then P_d' in the proof above is just $\varphi_* LT_d(X)$. For explicit examples of Euler data, see [29][30].

3.3. From Euler data to intersection numbers. Again, fix the data V, b_T

as before. From now on we write e^X simply as e. We recall the notations

$$A^{V,b_T}(t) = A(t) = e^{-H\cdot t/\alpha} \sum_d A_d\, e^{d\cdot t},$$

$$A_d = i_0^{vir} \pi^* b_T(V_d) = e_*^X \left(\frac{\rho^* b_T(V_d) \cap LT_{0,1}(d,X)}{e_G(F_0/M_d(X))} \right)$$

$$K_d^{V,b} = K_d = \int_{LT_{0,0}(d,X)} b(V_d)$$

$$\Phi^{V,b} = \Phi = \sum K_d\, e^{d\cdot t}.$$

THEOREM 3.12. *(i)* $deg_\alpha A_d \le -2$.

(ii) If for each d the class $b_T(V_d)$ has homogeneous degree the same as the dimension of $LT_{0,0}(d,X)$, then in the nonequivariant limit we have

$$\int_X e^{-H\cdot t/\alpha} A_d = \alpha^{-3}(2 - d\cdot t) K_d$$

$$\int_X \left(A(t) - e^{-H\cdot t/\alpha}\Omega \right) = \alpha^{-3}(2\Phi - \sum t_i \frac{\partial \Phi}{\partial t_i}).$$

Proof. By definition,

$$A_d = e_* \left(\frac{\rho^* b(V_d) \cap LT_{0,1}(d,X)}{e_G(F_0/M_d(X))} \right).$$

So assertion (i) follows immediately from this formula Lemma 3.5.

The second equality in assertion (ii) follows from the first equality in (ii). Now consider

$$\begin{aligned} I &:= \int_X e^{-H\cdot t/\alpha} A_d \\ &= \int_{LT_{0,1}(d,X)} e^{-e^* H\cdot t/\alpha} \frac{\rho^* b(V_d)}{e_{\mathbf{C}^\times}(F_0/M_d(X))} \\ &= \int_{LT_{0,0}(d,X)} b(V_d)\ \rho_* \left(\frac{e^{-e^* H\cdot t/\alpha}}{e_{\mathbf{C}^\times}(F_0/M_d(X))} \right). \end{aligned}$$

Now $b(V_d)$ has homogeneous degree the same as the dimension of $LT_{0,0}(d,X)$. The second factor in the last integrand contributes a scalar factor given by integration over a fiber E of ρ. By Lemma 3.5, the degree 1 term in the second factor is $\frac{-e^* H\cdot t}{\alpha^3} + \frac{c}{\alpha^3}$ where $c = c_1(L_d)$.

Now the line bundle L_d on $M_{0,1}(d,X)$ is the restriction of the universal bundle L'_d on $M_{0,1}(d,Y)$ ($Y = \mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_m}$), and the map $\rho : M_{0,1}(d,X) \to M_{0,0}(d,X)$, is the restriction of the forgetting map $\rho' : M_{0,1}(d,Y) \to M_{0,0}(d,Y)$. For the latter, the general fiber of ρ' is smooth $E' \cong \mathbf{P}^1$ so that

$$\int_{E'} i_{E'}^* c_1(L'_d) = \int_{E'} c_1(TE') = 2.$$

Since ρ' is flat,

$$\int_E i_E^* c_1(L_d) = \int_{E'} i_{E'}^* c_1(L'_d) = 2.$$

Restricting to a fiber E say over $(C,f) \in M_{0,0}(d,X)$, the evaluation map e is equal to f, which is a degree d map $E \to X$. It follows that

$$\int_E e^* H = d.$$

So we have

$$I = (-\frac{d \cdot t}{\alpha^3} + \frac{2}{\alpha^3}) K_d. \square$$

THEOREM 3.13. *More generally suppose b_T is an equivariant multiplicative class of the form*

$$b_T(V) = x^r + x^{r-1} b_1(V) + \cdots + b_r(V), \; rk\ V = r$$

where x is a formal variable, b_i is a class of degree i. Suppose

$$s := rk\ V_d - exp.\ dim\ M_{0,0}(d,X) \geq 0$$

is independent of $d \succ 0$. Then in the nonequivariant limit,

$$\frac{1}{s!}\left(\frac{d}{dx}\right)^s |_{x=0} \int_X e^{-H \cdot t/\alpha} A_d = \alpha^{-3} x^{-s} (2 - d \cdot t) K_d$$

$$\frac{1}{s!}\left(\frac{d}{dx}\right)^s |_{x=0} \int_X \left(A(t) - e^{-H \cdot t/\alpha} \Omega\right) = \alpha^{-3} x^{-s} (2\Phi - \sum t_i \frac{\partial \Phi}{\partial t_i}).$$

Proof. The proof is entirely analogous to (ii) above. $\square$

In the case of $b_T(V) = 1$, one can improve the α degree estimates for $A_d = \mathbf{1}_d$ given by Theorem 3.12 (i).

LEMMA 3.14. *For all d,*

$$deg_\alpha\ \mathbf{1}_d \leq min(-2, -\langle c_1(X), d\rangle).$$

Proof. If $\langle c_1(X), d\rangle \leq 2$, then the assertion is a special case Theorem 3.12 (i). So suppose that $\langle c_1(X), d\rangle > 2$. The class $LT_{0,1}(d,X)$ is of dimension

$$s = exp.dim\ M_{0,1}(d,X) = \langle c_1(X), d\rangle + dim\ X - 2.$$

Let $c = c_1(L_d)$. Then $c^k \cap LT_{0,1}(d,X)$ is of dimension $s-k$, and so $e_*(c^k \cap LT_{0,1}(d,X))$ lies in the group $A^T_{s-k}(X)$. But this group is zero unless $s - k \leq dim\ X$ or $k \geq s - dim\ X = \langle c_1(X), d\rangle - 2$. Now by Lemma 3.5, it follows that

$$\mathbf{1}_d = i_0^{vir} 1_d = \sum_{k \geq \langle c_1(X), d\rangle - 2} \frac{1}{\alpha^{k+2}}\ e_*(c^k \cap LT_{0,1}(d,X)).$$

This completes the proof. $\square$

REMARK 3.15. *The entire theory discussed in this section obviously specializes to the case $T = 1$, hence applies to any projective manifold X.*

4. Linking.

DEFINITION 4.1. *A projective T-manifold X is called a balloon manifold if X^T is finite, and if for $p \in X^T$, the weights of the isotropic representation T_pX are pairwise linearly independent.*

The second condition in the definition is known as the GKM condition [18]. We will assume that our balloon manifold has the property that if $p, q \in X^T$ such that $c(p) = c(q)$ for all $c \in A_T^1(X)$, then $p = q$. *From now on, unless stated otherwise, X will be a balloon manifold with this property.* If two fixed points p, q in X are connected by a T-invariant 2-sphere, then we call that 2-sphere a balloon and denote it by pq. For examples and the basic facts we need to use about these manifolds, see [30] and references there. All the results in sections 5-6 in [30] are proved for balloon manifolds without any convexity assumption, and are therefore also applicable here. We will quote the ones we need here without proof, but with only slight change in notations and terminology.

DEFINITION 4.2. *Two Euler series A, B are linked if for every balloon pq in X and every $d = \delta[pq] \succ 0$, the function $(A_d - B_d)|_p \in \mathbf{C}(\mathcal{T}^*)(\alpha)$ is regular at $\alpha = \frac{\lambda}{\delta}$ where λ is the weight on the tangent line $T_p(pq) \subset T_pX$.*

THEOREM 4.3. *(Theorem 5.4 [30]) Suppose A, B are linked Euler series satisfying the following properties: for $d \succ 0$,*

(i) For $p \in X^T$, every possible pole of $(A_d - B_d)|_p$ is a scalar multiple of a weight on T_pX.
(ii) $deg_\alpha(A_d - B_d) \leq -2$.
Then we have $A = B$.

THEOREM 4.4. *(Theorem 6.6 [30]) Suppose that A, B are two linked Euler series having property (i) of the preceding theorem. Suppose that $deg_\alpha A_d \leq -2$ for all $d \succ 0$, and that there exists power series $f \in \mathcal{R}[[e^{t_1}, .., e^{t_m}]]$, $g = (g_1, .., g_m)$, $g_j \in \mathcal{R}[[e^{t_1}, .., e^{t_m}]]$, without constant terms, such that*

$$e^{f/\alpha} B(t) = \Omega - \Omega \frac{H \cdot (t+g)}{\alpha} + O(\alpha^{-2}) \tag{4.1}$$

when expanded in powers of α^{-1}. Then

$$A(t+g) = e^{f/\alpha}\ B(t).$$

The change of variables effected by f, g above is an abstraction of what's known as mirror transformations [9].

THEOREM 4.5. *Let $p \in X^T$, $\omega \in A_T^*\ (M_{0,1}\ (d, X))\ [\alpha]$, and consider $i_p^* e_* \left(\frac{\omega \cap LT_{0,1}(d,X)}{e_G(F_0/M_d(X))}\right) \in \mathbf{C}(\mathcal{T}^*)(\alpha)$ as a function of α. Then*

(i) Every possible pole of the function is a scalar multiple of a weight on T_pX.
(ii) Let pq be a balloon in X, and λ be the weight on the tangent line $T_p(pq)$. If $d = \delta[pq] \succ 0$, then the pole of the function at $\alpha = \lambda/\delta$ is of the form

$$e_T(p/X)\ \frac{1}{\delta} \frac{1}{\alpha(\alpha - \lambda/\delta)} \frac{i_F^* \omega}{e_T(F/M_{0,1}(d, X))}$$

where F is the (isolated) fixed point $(\mathbf{P}^1, f_\delta, 0) \in M_{0,1}(d, X)$ with $f_\delta(0) = p$, and $f_\delta : \mathbf{P}^1 \to X$ maps by a δ-fold cover of pq.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} \{F\} & \xrightarrow{i_F} & M_{0,1}(d,X) \\ e' \downarrow & & e \downarrow \\ p & \xrightarrow{i_p} & X \end{array}$$

where e is the evaluation map, $\{F\}$ are the fixed point components in $e^{-1}(p)$, e' is the restriction of e to $\{F\}$, and i_F, i_p are the usual inclusions. By functorial localization we have, for any $\beta \in A_T^*(M_{0,1}(d,X))(\alpha)$,

$$\begin{aligned} i_p^* e_*(\beta \cap LT_{0,1}(d,X)) &= e_T(p/X) \sum_F e'_* \left(\frac{i_F^*\beta \cap [F]^{vir}}{e_T(F/M_{0,1}(d,X))} \right) \\ &= e_T(p/X) \sum_F \int_{[F]^{vir}} \frac{i_F^*\beta}{e_T(F/M_{0,1}(d,X))}. \end{aligned} \tag{4.2}$$

We apply this to the class

$$\beta = \frac{\omega}{e_G(F_0/M_d(X))} = \frac{\omega}{\alpha(\alpha - c)}$$

where $c = c_1(L_d)$ (cf. Lemma 3.5). For (i), we will show that a pole of the sum 4.2 is at either $\alpha = 0$ or $\alpha = \lambda'/\delta'$ for some tangent weight λ' on T_pX. For (ii), we will show that only one F in the sum 4.2 contributes to the pole at $\alpha = \lambda/\delta$, that the contributing F is the isolated fixed point $(\mathbf{P}^1, f_\delta, 0)$ as asserted in (ii), and that the contribution has the desired form.

A fixed point (C, f, x) in $e^{-1}(p)$ is such that $f(x) = p$, and that the image curve $f(C)$ lies in the union of the T-invariant balloons in X. The restriction of the first Chern class c to an F must be of the form

$$i_F^* c = c_F + w_F$$

where $c_F \in A^1(F)$, and $w_F \in \mathcal{T}^*$ is the weight of the representation on the line T_xC induced by the linear map $df_x : T_xC \to T_pX$ (cf. [26]). The image of df_x is either 0 or a tangent line $T_p(pr)$ of a balloon pr. Thus w_F is either zero (when the branch $C_1 \subset C$ containing x is contracted), or $w_F = \lambda'/\delta'$ (when $C_1 \xrightarrow{f} X$ maps by a δ'-fold cover of a balloon pr with tangent weight λ'). The class $e_T(F/M_{0,1}(d,X))$ is obviously independent of α. Since c_F is nilpotent, a pole of the sum 4.2 is either at $\alpha = 0$ or $\alpha = w_F$ for some F. This proves (i).

Now, an F in the sum 4.2 contributes to the pole at $\alpha = \lambda/\delta$ only if $w_F = \lambda/\delta$. Since the weights on T_pX are pairwise linearly independent, that $\lambda/\delta = \lambda'/\delta'$ implies that $\lambda = \lambda'$ and $\delta = \delta'$. Since $d = \delta[pq]$, it follows that the only fixed point contributing to the pole at $\alpha = \lambda/\delta$ is (C, f, x) where $C \xrightarrow{f} X$ maps by a δ-fold cover of the balloon pq with $C \cong \mathbf{P}^1$ and $f(x) = 0$. This is an isolated fixed point, which we denote by $F = (\mathbf{P}^1, f_\delta, 0)$. It contributes to the sum 4.2 the term

$$\int_F \frac{i_F^*\beta}{e_T(F/M_{0,1}(d,X))} = \frac{1}{\delta} \frac{i_F^*\omega}{\alpha(\alpha - \lambda/\delta)} \frac{1}{e_T(F/M_{0,1}(d,X))}.$$

Here F is an orbifold point of order δ, and hence the integration contributes the factor $1/\delta$. This proves (ii). □

Fix the data V, b_T and a Λ-Euler data $P: P_d$ such that

$$\tau^*\Lambda = \Omega := b_T(V^+)/b_T(V^-).$$

We now discuss the interplay between four Euler series: $A^{V,b_T}(t)$, $\mathbf{1}(t)$, and two others

$$\mathbf{O}(t) := e^{-H\cdot t/\alpha} \sum O_d \; e^{d\cdot t}$$
$$B(t) := e^{-H\cdot t/\alpha} \sum_d \tau^* j_0^* P_d \cap O_d \; e^{d\cdot t}$$

where $\mathbf{O}(t)$ *denote some unspecified Euler series linked to* $\mathbf{1}(t)$. (In particular $\mathbf{O}(t)$ may be specialized to $\mathbf{1}(t)$ itself.) That $B(t)$ is an Euler series follows from Theorem 3.11.

COROLLARY 4.6. *Suppose that at* $\alpha = \lambda/\delta$ *and* $F = (\mathbf{P}^1, f_\delta, 0)$, *we have* $i_p^* j_0^* P_d = i_F^* \rho^* b_T(V_d)$ *for all* $d = \delta[pq]$. *Then* $B(t)$ *is linked to* $A^{V,b_T}(t)$.

Proof. Since $\mathbf{O}(t)$ is linked to $\mathbf{1}(t)$ by assumption, it follows trivially that

$$B(t) = e^{-H\cdot t/\alpha} \sum_d \tau^* j_0^* P_d \cap O_d \; e^{d\cdot t}$$
$$C(t) = e^{-H\cdot t/\alpha} \sum_d \tau^* j_0^* P_d \cap \mathbf{1}_d \; e^{d\cdot t}$$

are linked. So it suffices to show that $A(t)$ and $C(t)$ are linked. Denote their respective Fourier coefficients by A_d, C_d. Then
(4.3)
$$i_p^* C_d - i_p^* A_d = i_p^* j_0^* P_d \cdot i_p^* e_* \left(\frac{LT_{0,1}(d,X)}{e_G(F_0/M_d(X))} \right) - i_p^* e_* \left(\frac{\rho^* b_T(V_d) \cap LT_{0,1}(d,X)}{e_G(F_0/M_d(X))} \right).$$

By Theorem 4.5 (ii), this difference is regular because the zero of the function $i_p^* j_0^* P_d - i_F^* \rho^* b_T(V_d)$ cancels the *simple* pole of each term in 4.3 at $\alpha = \lambda/\delta$. □

We now formulate one of the main theorems of this paper. It'll also give a more directly applicable form of Theorem 4.4. Given the data $V, b_T, \mathbf{O}(t), P$, and

$$B(t) := e^{-H\cdot t/\alpha} \sum_d \tau^* j_0^* P_d \cap O_d \; e^{d\cdot t},$$

assume that the preceding corollary holds. Suppose in addition, that

(*) For each d, we have the form

$$\tau^* j_0^* P_d = \Omega \alpha^{\langle c_1(X), d\rangle} \left(a + (a' + a'' \cdot H)\alpha^{-1} + \cdots\right),$$

for some $a, a', a_i'' \in \mathbf{C}(\mathcal{T}^*)$ (depending on d).

(**) For each d, we have the form (written in cohomology $A_T^*(X)$):

$$O_d = \alpha^{-\langle c_1(X), d\rangle} \left(b + (b' + b'' \cdot H)\alpha^{-1} + \cdots\right),$$

for some $b, b', b_i'' \in \mathbf{C}(\mathcal{T}^*)$ (depending on d).

THEOREM 4.7. *Suppose that* $A^{V,b_T}(t), B(t)$ *are as in the preceding corollary. Under the assumptions (*)-(**), there exist power series* $f \in \mathcal{R}[[e^{t_1}, .., e^{t_m}]]$, $g = (g_1, .., g_m)$, $g_j \in \mathcal{R}[[e^{t_1}, .., e^{t_m}]]$, *without constant terms, such that*

$$A^{V,b_T}(t+g) = e^{f/\alpha} B(t).$$

Proof. Recall that

$$B(t) := e^{-H\cdot t/\alpha} \sum_d \tau^* j_0^* P_d \cap O_d \; e^{d\cdot t}.$$

By the preceding corollary, $B(t)$ is linked to $A(t)$. We will use the asymptotic forms (*)-(**) to explicitly construct f, g satisfying the condition 4.1. Our assertion then follows from Theorem 4.4.

By (*)-(**), the Fourier coefficient B_d, $d \succ 0$, of $B(t)$ has the form

$$B_d = \Omega \left(ab + (ab' + a'b)\alpha^{-1} + (ab'' + a''b)\cdot H\alpha^{-1} + \cdots \right)$$

(and $B_0 = \Omega$). Multiplying this by $e^{-H\cdot t/\alpha} = 1 - H \cdot t\alpha^{-1} + \cdots$, and $e^{d\cdot t}$, and then sum over d, we get the form

$$B(t) = \Omega \left(C + (C' + C'' \cdot H - C\; H \cdot t)\alpha^{-1} + \cdots \right)$$

where $C, C', C_i'' \in \mathbf{C}(\mathcal{T}^*)[[e^{t_1}, .., e^{t_m}]]$ having constant terms $1, 0, 0$ respectively. It follows that

$$\frac{e^{-C'/C\alpha}}{C} B(t) = \Omega \left(1 - (t - \frac{C''}{C}) \cdot H\alpha^{-1} + \cdots \right)$$

So putting $f = -\alpha log\; C - \frac{C'}{C}$ and $g = -\frac{C''}{C}$ yields the eqn. 4.1. This completes the proof. □

COROLLARY 4.8. *The preceding theorem holds if we specialize the choice of* $O(t)$ *to* $\mathbf{1}(t)$, *ie.*

$$B(t) = e^{-H\cdot t/\alpha} \sum_d \tau^* j_0^* P_d \cap \mathbf{1}_d \; e^{d\cdot t}.$$

Proof. The preceding theorem holds for any Euler series $\mathbf{O}(t)$ satisfying the condition (**) linked to $\mathbf{1}(t)$. Now by Lemma 3.14, $\mathbf{1}(t)$ satisfies condition (**); and obviously it is also linked to itself. □

4.1. Linking values. In this subsection, we continue using the notations $V, b_T, \mathbf{1}(t), O(t), A(t)$, introduced above, where $O(t)$ is linked to $\mathbf{1}(t)$. We will apply Theorem 4.7 to the case when b_T is the Euler class or the Chern polynomial.

For simplicity, we will assume that V has the following property: that there exist nontrivial T-equivariant line bundles $L_1^+, .., L_{N^+}^+; L_1^-, .., L_{N^-}^-$ on X with $c_1(L_i^+) \geq 0$ and $c_1(L_j^-) < 0$, such that for any balloon $pq \cong \mathbf{P}^1$ in X we have

$$V^\pm|_{pq} = \oplus_{i=1}^{N^\pm} L_i^\pm|_{pq}.$$

Note that $N^\pm = rk\; V^\pm$. We also require that

$$\Omega := b_T(V^+)/b_T(V^-) = \prod_i b_T(L_i^+)/\prod_j b_T(L_j^-). \tag{4.4}$$

In this case we call the list $(L_1^+, .., L_{N^+}^+; L_1^-, .., L_{N^-}^-)$ the splitting type of V. Note that V is not assumed to split over X.

THEOREM 4.9. *Let* $b_T = e_T$ *be the equivariant Euler class. Let* pq *be a balloon,* $d = \delta[pq] \succ 0$, *and* λ *be the weight on the tangent line* $T_q(pq)$. *Let* $F = (\mathbf{P}^1, f_\delta, 0)$ *be*

the fixed point, as in Theorem 4.5(ii). Then
(4.5)

$$i_F^*\rho^* b_T(V_d) = \prod_i \prod_{k=0}^{\langle c_1(L_i^+),d\rangle} (c_1(L_i^+)|_p - k\lambda/\delta) \times \prod_j \prod_{k=1}^{-\langle c_1(L_j^-),d\rangle-1} (c_1(L_j^-)|_p + k\lambda/\delta)\,.$$

In particular, $A^{V,e_T}(t)$ *is linked to the Euler series* $B(t) = e^{-H\cdot t/\alpha}\sum_d B_d\, e^{d\cdot t}$ *where*

$$B_d = O_d \cap \prod_i \prod_{k=0}^{\langle c_1(L_i^+),d\rangle} (c_1(L_i^+) - k\alpha) \times \prod_j \prod_{k=1}^{-\langle c_1(L_j^-),d\rangle-1} (c_1(L_j^-) + k\alpha).$$

Proof. Define $P:\ P_d \in A_G^*(W_d)$ by

$$P_d := \prod_i \prod_{k=0}^{\langle c_1(L_i^+),d\rangle} (\hat{L}_i^+ - k\alpha) \times \prod_j \prod_{k=1}^{-\langle c_1(L_j^-),d\rangle-1} (\hat{L}_j^- + k\alpha),$$

where $\hat{L}_i^\pm \in A_G^*(W_d)$ denotes the canonical lifting of $c_1(L_i^\pm) \in A_T^*(Y)$. Then P is an Ω-Euler data (see section 2.2 [29]). By Theorem 3.11, it follows that $B_d = \tau^* j_0^* P_d \cap O_d$ is an Euler series. By (corollary to) Theorem 4.5, $A(t)$ is linked to $B(t)$, provided that eqn. 4.5 holds. We now prove eqn. 4.5.

We first consider a single convex line bundle $V = L$. As before, the fixed point $F = (\mathbf{P}^1, f_\delta, 0)$ in $M_{0,1}(d,X)$ is a δ-fold cover of the balloon $pq \cong \mathbf{P}^1$. We can write it as

$$f_\delta : \mathbf{P}^1 \to pq \cong \mathbf{P}^1,\ [w_0, w_1] \mapsto [w_0^\delta, w_1^\delta].$$

Note that the T-action on X induces the standard rotation on $pq \cong \mathbf{P}^1$ with weight λ. Clearly, we have

$$i_F^*\rho^* e_T(V_d) = i_{\rho(F)}^* e_T(V_d) = e_T(i_{\rho(F)}^* V_d).$$

The right hand side is the product the weights of the T representation on the vector space

$$i_{\rho(F)}^* V_d = H^0(\mathbf{P}^1, f_\delta^* L) = H^0(\mathbf{P}^1, f_\delta^* \mathcal{O}(l))$$

where $l = \langle c_1(L), [pq]\rangle$. Thus we get (cf. section 2.4 [29])

$$e_T(i_{\rho(F)}^* V_d) = \prod_{k=0}^{l\delta} (c_1(L)|_p - k\frac{\lambda}{\delta}).$$

This proves 4.5 for a single convex line bundle.

Similarly for a concave line bundle $V = L$, if its restriction to the balloon pq is $\mathcal{O}(-l)$ with $-l = \langle c_1(L), [pq]\rangle$, then

$$e_T(i_{\rho(F)}^* V_d) = \prod_{k=1}^{l\delta-1} (c_1(L)|_p + k\frac{\lambda}{\delta}).$$

This is 4.5 for a single concave line bundle. The general case can clearly be obtained by taking products. □

A parallel argument for $b_T =$ the Chern polynomial yields

THEOREM 4.10. *Let $b_T = c_T$ be the equivariant Chern polynomial, with the rest of the notations as in the preceding theorem. Then*

$$i_F^*\rho^* c_T(V) = \prod_i \prod_{k=0}^{\langle c_1(L_i^+),d\rangle} (x + c_1(L_i^+)|_p - k\lambda/\delta) \times \prod_j \prod_{k=1}^{-\langle c_1(L_j^-),d\rangle - 1} (x + c_1(L_j^-)|_p + k\lambda/\delta)\,.$$

In particular, $A^{V,e_T}(t)$ is linked to the Euler series $B(t) = e^{-H\cdot t/\alpha}\sum_d B_d\, e^{d\cdot t}$ where

$$B_d = O_d \cap \prod_i \prod_{k=0}^{\langle c_1(L_i^+),d\rangle} (x + c_1(L_i^+) - k\alpha) \times \prod_j \prod_{k=1}^{-\langle c_1(L_j^-),d\rangle - 1} (x - c_1(L_j^-) + k\alpha).$$

By Theorem 4.7, we can therefore compute $A(t) = A^{V,b_T}(t)$ in terms of the Euler series $B(t)$ given above, provided that the Euler data P and the Euler series $O(t)$ both have the appropriate asymptotic forms (*)-(**) required by Theorem 4.7.

COROLLARY 4.11. *Let b_T be either e_T or c_T. Suppose that*

$$c_1(V^+) - c_1(V^-) \le c_1(X). \tag{4.6}$$

Then the condition (*) holds for the Euler data P in the two preceding theorems. In this case, if $O(t)$ is any Euler series linked to $\mathbf{1}(t)$ and satisfies condition (**), then Theorem 4.7 applies to compute $A^{V,b_T}(t)$ in terms of $O(t)$ and P.

Proof. The Euler data P in either of the preceding theorems has the form: for each $d \succ 0$,

$$\tau^* j_0^* P_d = \Omega \alpha^{\langle c_1(V^+) - c_1(V^-),d\rangle - N_-} \left(a + (a' + a''\cdot H)\alpha^{-1} + \cdots\right),$$

for some $a, a', a_i'' \in \mathbf{C}(cT^*)$ (depending on d). By assumption,

$$\langle c_1(V^+) - c_1(V^-), d\rangle \le \langle c_1(X), d\rangle.$$

This implies that P satisfies the condition (*). □

This result shows that if $\Omega = b_T(V^+)/b_T(V^-)$ has a certain factorized form 4.4, and if there is a suitable bound 4.6 on first Chern classes, then $A(t) = A^{V,b_T}(t)$ is computable in terms of the $\mathbf{1}(t)$ (or a suitable Euler series $O(t)$ linked to it). Note that even though $\mathbf{1}(t)$ is not known explicitly in closed form in general, it is *universal* in the sense that it is natural and is independent of any choice of V or b_T. Its Fourier coefficients also happen to be related to the universal line bundle on $M_{0,1}(d,X)$. In the next section, we specialize $O(t)$ to something quite explicit. We also discuss some other ways to compute $A(t)$. We consider situations in which the first Chern class bound and the factorization condition on Ω can be removed.

5. Applications and Generalizations. Throughout this section, we continue to use the same notations: $V, b_T, \Omega, A(t), \ldots$.

5.1. Inverting $\mathbf{1}_d$. Suppose $\mathbf{1}_d$ is invertible for all d. Then obviously, there exist unique $B_d \in A_T^*(X)(\alpha)$ such that

$$A(t) = e^{-H\cdot t/\alpha} \sum B_d \cap \mathbf{1}_d\, e^{d\cdot t}.$$

In particular this says that for $d = \delta[pq]$, $F = (\mathbf{P}^1, f_\delta, 0)$, we must have

$$i_p^* B_d = i_F^* \rho^* b_T(V_d) \tag{5.1}$$

at $\alpha = \lambda/\delta$. By Theorem 4.3, the B_d are the unique classes in $A_T^*(X)(\alpha)$ such that
(i) eqn. 5.1 holds.
(ii) $deg_\alpha\ B_d \cap \mathbf{1}_d \leq -2$.
(iii) $e^{-H\cdot t/\alpha} \sum B_d \cap \mathbf{1}_d\ e^{d\cdot t}$ is an Ω-Euler series.

In other words these algebraic conditions completely determine the B_d. Thus in principle the B_d can be computed in terms of the classes $\mathbf{1}_d$ and the linking values 5.1. The point is that *this is true whether or not the bound 4.6 or the factorized form Ω 4.4 holds.* Here are a few examples.

Example 1. $X = Y$ is a product of projective space with the maximal torus action. In this case,

$$\mathbf{1}_d = \frac{1}{e_G(Y_0/W_d)}$$

which is given explicitly in section 2. We also have $B_d = j_0^* \varphi_*(\pi^* b_T(V_d) \cap LT_d(X)) \in A_T^*(X)[\alpha]$ (cf. Lemma 3.3). Finding the B_d explicitly amounts to finding polynomials in H_a, α with the prescribed values 5.1, and the degree bound (ii). This is a linear problem! This approach is particularly useful for computing $b_T(V_d)$ for nonsplit bundles V (e.g. $V = TX$), or for bundles where the bound 4.6 fails (e.g. $\mathcal{O}(k)$ on $\mathbf{P}^n$ with $k > n+1$).

Example 2. Suppose X is a balloon manifold such that every balloon pq generates the integral classes in $A_1(X)$. Then every integral class $d \in A_1(X)$ is of the form $\delta[pq]$ (e.g. Grassmannians). We claim that, in this case, $\mathbf{1}_d$ is invertible for all d. It suffices to show that $i_p^* \mathbf{1}_d$ is nonzero for every fixed point p in X. Given p, we know that there are $n = dim\ X$ other fixed points q joint to p by balloons pq. Pick such a q. Then $d = \delta[pq]$ for some δ. It follows from Theorem 4.5 that the function $i_p^* \mathbf{1}_d$ has a *nontrivial* simple pole at $\alpha = \lambda/\delta$ where λ is the weight on the tangent line $T_p(pq)$. This completes the proof.

Obviously, we can take product of these examples and still get invertible $\mathbf{1}_d$ for the product manifold.

5.2. Toric manifolds. Let X be a toric manifold of dimension n. Denote by D_a, $a = 1,..,N$, the T-invariant divisors in X. We denote by the same notations the equivariant homology classes they represent. Recall that [3][11][32] X can be represented as an orbit space

$$X = (\Gamma - Z)/K$$

where K is an algebraic torus of dimension $N - n$, $\Gamma = \mathbf{C}^N$ is a linear representation of K, and Z is a K-invariant monomial variety of $\mathbf{C}^N$, all determined by the fan of X. The T action on the orbit space is induced by $(\mathbf{C}^\times)^N$ acting on Γ by the usual scaling. Define

$$O(t) = e^{-H\cdot t/\alpha} \sum_d O_d\ e^{d\cdot t},\ O_d := \frac{\prod_{\langle D_a, d\rangle < 0} \prod_{k=0}^{-\langle D_a, d\rangle - 1} (D_a + k\alpha)}{\prod_{\langle D_a, d\rangle \geq 0} \prod_{k=1}^{\langle D_a, d\rangle} (D_a - k\alpha)}. \tag{5.2}$$

We will prove that $O(t)$ is a 1-Euler series.

First we recall a construction in [31][42]. Given an integral class $d \in A_1(X)$, let $\Gamma_d = \oplus_a H^0(\mathbf{P}^1, \mathcal{O}(\langle D_a, d\rangle))$. Let K act on Γ_d by $\phi_a \mapsto t^{\lambda_a} \phi_a$ where the λ_a are the same weights with which K acts on Γ. Let

$$Z_d = \{\phi \in \Gamma_d | \phi(z, w) \in Z,\ \forall (z, w) \in \mathbf{C}^2\}.$$

(Note that ϕ here is viewed as a polynomial map $\mathbf{C}^2 \to \mathbf{C}^N$.) It is obvious that Z_d is K-invariant. Define the orbit space

$$\mathcal{W}_d := (\Gamma_d - Z_d)/K.$$

(i) If not empty, $\mathcal{W}_d$ is a toric manifold of dimension

$$dim\ \mathcal{W}_d = {\sum_a}'(\langle D_a, d\rangle + 1) - dim\ K$$

where $\sum_a'$ means summing only those terms which are positive.

(ii) T acts on $\mathcal{W}_d$ in an obvious way. There is also a $\mathbf{C}^\times$ action on $\mathcal{W}_d$ induced by the standard action on $\mathbf{P}^1$ with weight α. Each $\mathbf{C}^\times$ fixed point component in $\mathcal{W}_d$ is (consisting of K-orbits of)

$$\mathcal{Y}_r = \{\phi = (x_1 w_0^{\langle D_1, r\rangle} w_1^{\langle D_1, d-r\rangle}, ..., x_N w_0^{\langle D_N, r\rangle} w_1^{\langle D_N, d-r\rangle}) | (x_1, .., x_N) \in \mathbf{C}^N,$$
$$x_b = 0\ if\ the\ corresp.\ monomial\ has\ negative\ exponent\}.$$

Let $j_r : \mathcal{Y}_r \to \mathcal{W}_d$ be the inclusion maps. If nonempty, $\mathcal{Y}_r$ is canonically isomorphic to a T-invariant submanifold in X given by intersecting those divisors $x_b = 0$ corresponding to negative exponents above. Denote the canonical inclusions by $\tau_r : \mathcal{Y}_r \to X$. Then $\tau_{r*}(1) = \prod_{\langle D_a, r\rangle < 0\ or\ \langle D_a, d-r\rangle < 0} D_a$. We will denote the class of $D_a \cap \mathcal{Y}_r$ in $\mathcal{Y}_r$ simply by D_a.

(iii) The $G = \mathbf{C}^\times \times T$-equivariant Euler class of the normal bundle of $\mathcal{Y}_r$ in $\mathcal{W}_d$ is

$$e_G(\mathcal{Y}_r/\mathcal{W}_d) = \prod_{\langle D_a, d\rangle \geq 0} \prod_{k=0\ \ k \neq \langle D_a, r\rangle}^{\langle D_a, d\rangle} (D_a + \langle D_a, r\rangle\alpha - k\alpha).$$

(iv) Corresponding canonically to every T-divisor class D_a on X is a G-divisor class $\hat{D}_a$ on $\mathcal{W}_d$. It is determined by

$$j_r^* \hat{D}_a = D_a + \langle D_a, r\rangle\alpha.$$

Similarly, every linear combination D of the D_a corresponds to some $\hat{D}$ on $\mathcal{W}_d$.

LEMMA 5.1. *$O(t)$ introduced above is a 1-Euler series.*

Proof. Let

$$\omega_d = \prod_{\langle D_a, d\rangle < 0} \prod_{k=1}^{-\langle D_a, d\rangle - 1} (\hat{D}_a + k\alpha) \in A_*^G(\mathcal{W}_d). \tag{5.3}$$

By the localization theorem,

$$\int_{\mathcal{W}_d} \omega_d \cdot e^{\hat{H}\cdot\zeta} = \sum_r \int_{\mathcal{Y}_r} \frac{j_r^* \omega_d}{e_G(\mathcal{Y}_r/\mathcal{W}_d)}\ e^{(H + r\alpha)\cdot\zeta}.$$

Obviously, the left hand side lies in $A_G^*(pt)[[\zeta]] \subset \mathcal{R}[[\zeta]]$. Now observe that the right hand side is

$$\sum_r \int_{\mathcal{Y}_r} \frac{\prod_{\langle D_a,d\rangle<0}\prod_{k=1}^{-\langle D_a,d\rangle-1}(D_a+\langle D_a,r\rangle\alpha+k\alpha)}{\prod_{\langle D_a,d\rangle\geq 0}\prod_{k=0}^{\langle D_a,d\rangle}{}_{k\neq\langle D_a,r\rangle}(D_a+\langle D_a,r\rangle\alpha-k\alpha)}\, e^{(H+r\alpha)\cdot\zeta}$$

$$=\sum_r \int_X \frac{\prod_{\langle D_a,r\rangle<0}\prod_{k=0}^{-\langle D_a,r\rangle-1}(D_a-k\alpha)}{\prod_{\langle D_a,r\rangle\geq 0}\prod_{k=1}^{\langle D_a,r\rangle}(D_a+k\alpha)}$$

$$\times \frac{\prod_{\langle D_a,d-r\rangle<0}\prod_{k=0}^{-\langle D_a,d-r\rangle-1}(D_a+k\alpha)}{\prod_{\langle D_a,d-r\rangle\geq 0}\prod_{k=1}^{\langle D_a,d-r\rangle}(D_a-k\alpha)}\cdot e^{(H+r\alpha)\cdot\zeta}$$

$$=\sum_r \int_X \bar{O}_r\cdot O_{d-r}\; e^{(H+r\alpha)\cdot\zeta}.$$

This shows that $O(t)$ is a 1-Euler series. □

REMARK 5.2. *One can define the notion of Euler data on the basis of $\mathcal{W}_d$ in a way similar to Definition 3.10. The classes 5.3 in fact give an example of Euler data for $\mathcal{W}_d$. One can also construct the whole parallel theory of mirror principle for toric manifolds using $\mathcal{W}_d$.*

LEMMA 5.3. *The two Euler series $O(t)$ and $\mathbf{1}(t)$ are linked.*

Proof. Let $p\in X^T$, pq be a balloon in X, $d=\delta[pq]\succ 0$, and λ be the weight on the tangent line on $T_p(pq)$. Let $F=(\mathbf{P}^1,f_\delta,0)$ be the fixed point in $M_{0,1}(d,X)$, as given in Theorem 4.5, which says that the function $i_p^*\mathbf{1}_d$ has the polar term, at $\alpha=\lambda/\delta$,

$$e_T(p/X)\,\frac{1}{\lambda}\frac{1}{\alpha-\lambda/\delta}\frac{1}{e_T(F/M_{0,1}(d,X))}. \tag{5.4}$$

We now compute the contribution from $e_T(F/M_{0,1}(d,X))$ for a toric manifold X. The virtual normal bundle of the point $F=(C=\mathbf{P}^1,f_\delta,0)$ in $M_{0,1}(d,X)$ is

$$N_{F/M_{0,1}(d,X)}=[H^0(C,f_\delta^*TX)]-[H^1(C,f_\delta^*TX)]-A_C$$

(notation as in section 2.3 [29]). From the Euler sequence of X [24], we get an equivariant exact sequence for every balloon pq in X,

$$0\to\mathcal{O}^{N-n}\to\oplus_a\mathcal{O}(D_a)|_{pq}\to TX|_{pq}\to 0$$

where $\mathcal{O}$ is the trivial line bundle. At p, there are exactly n nonzero $D_a(p):=i_p^*D_a$ giving the weights for the isotropic representation T_pX, and $N-n$ zero $D_a(p)$ corresponding to the trivial representation $\mathcal{O}^{N-n}|_p$. As usual we ignore the zero weights below, which must drop out at the end.

Let $\lambda=D_b(p)$. Note that $\langle D_b,d\rangle=1$ (section 2.3 [30]). The bundle $\mathcal{O}(D_b)$ contributes to $e_T([H^0(C,f_\delta^*TX)])$ the term

$$\prod_{k=0}^{\delta-1}(D_b(p)-k\lambda/\delta).$$

For each $a\neq b$ with $\langle D_a,d\rangle\geq 0$, the bundle $\mathcal{O}(D_a)$ contributes to $e_T([H^0(C,f_\delta^*TX)])$

the term

$$\prod_{k=0}^{\langle D_a,d\rangle} (D_a(p) - k\lambda/\delta) \ if \ D_a(p) \neq 0,$$

$$\prod_{k=1}^{\langle D_a,d\rangle} (D_a(p) - k\lambda/\delta) \ if \ D_a(p) = 0.$$

For each a with $\langle D_a, d\rangle < 0$, the bundle $\mathcal{O}(D_a)$ contributes to $e_T([H^1(C, f_\delta^* TX)])$ the term

$$\prod_{k=1}^{-\langle D_a,d\rangle-1} (D_a(p) + k\lambda/\delta).$$

The automorphism group A_C contributes $e_T(A_C) = -\lambda/\delta$. Finally, we have

$$e_T(p/X) = \prod_{D_a(p)\neq 0} D_a(p).$$

Combining all the contributions, we see that 5.4 becomes $\frac{1}{\alpha-\lambda/\delta}$ times

$$\frac{-1}{\delta} \frac{\prod_{\langle D_a,d\rangle<0} \prod_{k=0}^{-\langle D_a,d\rangle-1}(D_a(p) + k\lambda/\delta)}{\prod_{\langle D_a,d\rangle\geq 0 \, a\neq b} \prod_{k=1}^{\langle D_a,d\rangle}(D_a(p) - k\lambda/\delta) \times \prod_{k=1}^{\delta-1}(D_b(p) - k\lambda/\delta)}.$$

But this coincides with

$$lim_{\alpha\to\lambda/\delta} \ (\alpha - \lambda/\delta) i_p^* O_d.$$

This shows that $i_p^* O_d - i_p^* \mathbf{1}_d$ is regular at $\alpha = \lambda/\delta$. □

Note that $O_d = \alpha^{\langle c_1(X), d\rangle} +$ lower order terms, because $\sum D_a = c_1(X)$. Thus $O(t)$ is an Euler series linked to $\mathbf{1}(t)$ and meets the condition (**) of Theorem 4.7. In particular to apply to the case $b_T = e_T$ or c_T, all we need is the form 4.4 for Ω and the bound 4.6. For then Corollary 4.11 holds.

Example. Take $b_T = c_T$. Take V to be any direct sum of convex equivariant line bundles L_i, so that 4.6 holds. Note that in this case 4.4 holds automatically. Then Theorem 4.7 yields an explicit formula for $A^{V,b_T}(t)$ in terms of the O_d 5.2 and the P_d in Theorem 4.10. For $b_T = e_T$, and V a direct sum of convex line bundles L_i with $\sum_i c_1(L_i) = c_1(X)$, we get a similar explicit formula for $A(t)$. Plugging this formula into Theorem 3.12 in the nonequivariant limit, we get

COROLLARY 5.4. *Let*

$$B(t) = e^{-H\cdot t/\alpha} \sum_d \prod_i \prod_{k=0}^{\langle c_1(L_i),d\rangle} (c_1(L_i) - k\alpha) \cap \frac{\prod_{\langle D_a,d\rangle<0} \prod_{k=0}^{-\langle D_a,d\rangle-1}(D_a + k\alpha)}{\prod_{\langle D_a,d\rangle\geq 0} \prod_{k=1}^{\langle D_a,d\rangle}(D_a - k\alpha)} e^{d\cdot t},$$

as in Theorem 4.7. Then we have

$$\int_X \left(e^{f/\alpha} B(t) - e^{-H\cdot T/\alpha}\Omega\right) = \alpha^{-3}(2\Phi - \sum T_i \frac{\partial\Phi}{\partial T_i})$$

where $T = t + g$, and f, g are the power series computed in Theorem 4.7.

This is the general mirror formula in [21][22] (see also references there), formulated in the context of mirror symmetry and reflexive polytopes [4][5].

5.3. A generalization. We have now seen several ways to compute $A(t) = A^{V,b_T}(t)$ under various assumptions on either V, b_T, or TX, or the classes $\mathbf{1}_d$, or some combinations of these assumptions. We now combine these approaches to formulate an algorithm for computing $A(t)$ in full generality on any balloon manifold X, for arbitrary V, b_T. The result will be in terms of certain (computable) T representations.

(i) By Lemma 3.4, the A_d is of the form

$$A_d = \frac{\tau^* \phi_d}{e_G(X/W_d)},$$

where $\phi_d \in A_*^T(Y)[\alpha]$, hence represented by a polynomial $\mathbf{C}[\mathcal{T}^*][H_1, ..., H_m, \alpha]$. Note that the denominator of A_d is $e_G(X/W_d) = e_T(X/Y) \cdot \tau^* e_G(Y_0/W_d)$. Thus the goal is to compute the class $\tau^*\phi_d$ for all d. We shall set up a (over-determined) system of linear equations with a solution (unique up to $ker\ \tau^*$) given by the ϕ_d.

(ii) By Theorem 3.12, the degree of ϕ_d is bounded according to

$$deg_\alpha A_d \leq -2.$$

(iii) By Theorem 4.5 (i), at any fixed point p, the function $i_p^* A_d$ is regular away from $\alpha = 0$ or λ/δ, where λ is a weight on T_pX. In other words,

$$Res_{\alpha=\gamma}(\alpha - \gamma)^k\ i_p^* A_d = 0$$

for all $\gamma \neq 0, \lambda/\delta$ and $k \geq 0$. Note that these are all linear conditions on ϕ_d.

(iv) By Theorem 4.5 (ii) (see notations there), for any balloon pq in X and $d = \delta[pq] \succ 0$, we have

$$\begin{aligned} lim_{\alpha\to\lambda/\delta}(\alpha - \lambda/\delta)\ i_p^* A_d &= \frac{e_T(p/X)}{\lambda\ e_T(F/M_{0,1}(d,X))} i_F^* \rho^* b_T(V_d) \\ &= \frac{-e_T(p/X)}{\delta} \frac{e_T[H^1(\mathbf{P}^1, f_\delta^* TX)]'}{e_T[H^0(\mathbf{P}^1, f_\delta^* TX)]'}\ b_T(i_{\rho(F)}^* V_d). \end{aligned} \tag{5.5}$$

Here we have used the fact that $N_{F/M_{0,1}(d,X)} = [H^0(\mathbf{P}^1, f_\delta^* TX)] - [H^1(\mathbf{P}^1, f_\delta^* TX)] - A_C$ (cf. section 5.2). The prime in the Euler classes above means that we drop the zero weights in the T representations $[H^i(\mathbf{P}^1, f_\delta^* TX)]$. Now if $V = V^+ \oplus V^-$ is a convex/concave bundle on X, then we have the T representation

$$i_{\rho(F)}^* V_d = H^0(\mathbf{P}^1, f_\delta^* V^+) \oplus H^1(\mathbf{P}^1, f_\delta^* V^-).$$

Thus

$$b_T(i_{\rho(F)}^* V_d)) = b_T[H^0(\mathbf{P}^1, f_\delta^* V^+)]\ b_T[H^1(\mathbf{P}^1, f_\delta^* V^-)],$$

which is just the value of b_T for a trivial bundle over a point. Note that if U is any T representation with weight decomposition $U = \oplus_i \mathbf{C}_{\nu_i}$, then $b_T(U) = \prod_i b_T(\mathbf{C}_{\nu_i})$ by the multiplicativity of b_T. Hence once the T representations $H^i(\mathbf{P}^1, f_\delta^* V^\pm)$ are given, eqn. 5.5 becomes a linear condition on the ϕ_d, where the right hand side is some known element in $\mathbf{C}(\mathcal{T}^*)$.

(v) Finally, we know that $A(t)$ is an Ω-Euler series. This is (inductively) a linear condition on the ϕ_d.

(vi) By Theorem 4.3, any solution to the linear conditions in (ii)-(v) necessarily represents the class $\tau^*\phi_d$ we seek.

Of course, this algorithm relies on knowing the T representations $[H^i(\mathbf{P}^1, f_\delta^* TX)]$, $[H^i(\mathbf{P}^1, f_\delta^* V)]$ induced by the T-equivariant bundles $TX|_{pq}$ and $V|_{pq}$ on each balloon

$pq \cong \mathbf{P}^1$. But describing them for any given X and V is clearly a classical question. We have seen that these representations are easily computable in many cases. We now discuss a general situation in which these representations can also be computed similarly.

Let V be any T-equivariant vector bundle on X and let

$$0 \to V_N \to \cdots \to V_1 \to V \to 0$$

be an equivariant resolution. Then by the Euler-Poincare Principle, we have

$$[H^0(\mathbf{P}^1, f_\delta^* V)] - [H^1(\mathbf{P}^1, f_\delta^* V)] = \sum_a (-1)^{a+1}([H^0(\mathbf{P}^1, f_\delta^* V_a)] - [H^1(\mathbf{P}^1, f_\delta^* V_a)]).$$

Note that there is a similar equality of representations whenever V is a term in any given exact sequence

$$0 \to V_N \to \cdots \to V_i \to V \to V_{i-1} \to \cdots \to V_1 \to 0.$$

Now suppose that each V_a is a direct sum of T-equivariant line bundles. Then each summand L will contribute to $[H^0(\mathbf{P}^1, f_\delta^* V_a)] - [H^1(\mathbf{P}^1, f_\delta^* V_a)]$ the representations

$$c_1(L)|_p - k\lambda/\delta, \ k = 0, 1, ..., l\delta,$$

if $l = \langle c_1(L), [pq] \rangle \geq 0$; and

$$c_1(L)|_p + k\lambda/\delta, \ k = 1, ..., l\delta - 1,$$

if $-l = \langle c_1(L), [pq] \rangle < 0$ (cf. proof of Theorem 4.9). In this case, $[H^0(\mathbf{P}^1, f_\delta^* V)] - [H^1(\mathbf{P}^1, f_\delta^* V)]$ are then determined completely. Thus whenever a T-equivariant resolution by line bundles is known for TX and the convex/concave bundle $V^{\pm}$, the right hand side of eqn. 5.5 becomes known.

Example. Consider the case $X = \mathbf{P}^n$, $V = TX$, and b_T the Chern polynomial. This will be an example where V has no splitting type, but $A(t)$ can be computed via a T-equivariant resolution nevertheless. Recall the T equivariant Euler sequence

$$0 \to \mathcal{O} \to \oplus_{i=0}^n \mathcal{O}(H - \lambda_i) \to TX \to 0.$$

For $F = (\mathbf{P}^1, f_\delta, 0)$, where f_δ is the δ-fold cover of the balloon pq, this gives

$$b_T(i_{\rho(F)}^* V_d) = \frac{1}{x} \prod_i \prod_{k=0}^{d} (x + \lambda_j - \lambda_i - k\lambda/\delta).$$

Here p, q are the jth and the lth fixed points in $\mathbf{P}^n$, so that $\lambda = \lambda_j - \lambda_l$. We can use this to set up a linear system to solve for $A(t)$ inductively. However, there is an easier way to compute $A(t)$ in this case. Observe that $\Omega = b_T(V) = \frac{1}{x}\prod_i (x + H - \lambda_i)$, and that

$$P: \ P_d := \frac{1}{x} \prod_i \prod_{k=0}^{d} (x + \kappa - \lambda_i - k\alpha)$$

defines an Ω-Euler data (see section 2.2 [29]). Since $j_0^* \kappa = H$ and $i_p^* H = \lambda_j$, it follows that

$$b_T(i_{\rho(F)}^* V_d) = i_p^* j_0^* P_d$$

at $\alpha = \lambda/\delta$. By the corollary to Theorem 4.5, the Euler series

$$B(t) := e^{-H \cdot t/\alpha} \sum j_0^* P_d \cap \mathbf{1}_d$$

is linked to $A(t)$. Obviously, we have $deg_\alpha j_0^* P_d = (n+1)d$, hence P meets condition (*) of Theorem 4.7 (τ is the identity map here). For $O_d = \mathbf{1}_d$, condition (**) there is also automatic. It follows that

$$A(t+g) = e^{f/\alpha} B(t)$$

where f, g are explicitly computable functions from Theorem 4.7. Note that $rank\ V_d = (n+1)d + n$, and so Theorem 3.13 yields immediately the codimension $s = 3$ Chern class of V_d.

5.4. Blowing up the image. In this section, we discuss another approach to compute $A(t)$. For clarity, we restrict to the case of a convex T-manifold X (T may be trivial), and $b_T \equiv 1$, so that $A(t) = \mathbf{1}(t)$. Thus we will study the classes

$$\mathbf{1}_d = e_*^X \left(\frac{LT_{0,1}(d,X)}{e_G(F_0/M_d(X))} \right).$$

We will actually be interested in the integrals $\int_X \tau^* e^{H \cdot \zeta} \cap \mathbf{1}_d$, where $\tau : X \to Y$ is a given projective embedding. For the purpose of studying the intersection numbers in section 3.3, this is adequate. Since X is assumed convex, $LT_{0,1}(d,X)$ is represented by $M_{0,1}(d,X)$. Likewise for $M_d(X)$.

Suppose that we have a commutative diagram

$$\begin{array}{ccccc} F_0 & \xrightarrow{e^Y} & Y_0 & \xleftarrow{g} & E_0 \\ \downarrow i & & \downarrow j & & \downarrow k \\ M_d & \xrightarrow{\varphi} & W_d & \xleftarrow{\psi} & \mathcal{Q}_d. \end{array} \tag{5.6}$$

Here the left hand square is as in 3.2 ($i_0, j_0, M_d(X)$ there are written as i, j, M_d here for clarity). We assume that $\mathcal{Q}_d$ is a G-manifold, that $\psi : \mathcal{Q}_d \to W_d$ is a G-equivariant resolution of singularities of $\varphi(M_d)$, and that E_0 is the fixed points in $\psi^{-1}(Y_0)$. Here g denotes the restriction of ψ, and k the inclusion. Recall that φ is an isomorphism into its image away from the singular locus of M_d. The singularities in $\varphi(M_d)$ is the image of the compactifying divisor in M_d, which has codimension at least 2. Then we have the equality in $A_*^G(W_d)$:

$$\varphi_*[M_d] = \psi_*[\mathcal{Q}_d].$$

Applying functorial localization to the left hand square in 5.6 as in section 3.1, we get

$$\frac{j^* \varphi_*[M_d]}{e_G(Y_0/W_d)} = e_*^Y \left(\frac{[F_0]}{e_G(F_0/M_d)} \right).$$

Doing the same for the right hand square, we get

$$\frac{j^* \psi_*[\mathcal{Q}_d]}{e_G(Y_0/W_d)} = g_* \left(\frac{[E_0]}{e_G(E_0/\mathcal{Q}_d)} \right).$$

It follows that

LEMMA 5.5. *In $A_*^T(Y)$, we have the equality*

$$\tau_* \mathbf{1}_d = e_*^Y \left(\frac{[F_0]}{e_G(F_0/M_d)} \right) = g_* \left(\frac{[E_0]}{e_G(E_0/\mathcal{Q}_d)} \right).$$

It follows that

$$\begin{aligned}\int_X \tau^* e^{H\cdot\zeta} \cap \mathbf{1}_d &= \int_{Y_0} e^{H\cdot\zeta} \cap g_* \left(\frac{[E_0]}{e_G(E_0/\mathcal{Q}_d)}\right) \\ &= \int_{E_0} \frac{g^* e^{H\cdot\zeta}}{e_G(E_0/\mathcal{Q}_d)} \\ &= \int_{E_0} \frac{g^* j^* e^{\kappa\cdot\zeta}}{e_G(E_0/\mathcal{Q}_d)} \\ &= \int_{E_0} \frac{k^* \psi^* e^{\kappa\cdot\zeta}}{e_G(E_0/\mathcal{Q}_d)}.\end{aligned}$$

In many cases, the spaces $\mathcal{Q}_d$ can be explicitly described, and the classes $\psi^*\kappa$ on $\mathcal{Q}_d$ can be expressed in terms of certain universal classes. For example, when X is a flag variety, then the $\mathcal{Q}_d$ can be chosen to be the Grothendieck Quot scheme (cf. [10][8]). Integration on the Quot scheme can be done by explicit localization (cf. [39]). When X is a Grassmannian and $\tau : X \to Y = \mathbf{P}^N$ is the Plucker embedding, then $\psi^*\kappa = -c_1(S)$, where S is a universal subbundle on the Quot scheme. In this case, the image $\psi(\mathcal{Q}_d)$ has been studied extensively in [37][38].

When X is not convex, a similar method still works if we can find an explicit cycles Z_d in $\mathcal{Q}$ such that

$$\varphi_* LT_d(X) = \psi_*[Z_d]$$

in $A_*^G(W_d)$. This approach deserves further investigation.

5.5. Higher genus. In this section, we discuss a generalization of mirror principle to higher genus. More details will appear elsewhere. As before X will be a projective T-manifold, and $\tau : X \to Y$ a given T-equivariant projective embedding. (Again, T may be the trivial group.)

Let $M_{g,k}(d, X)$ denote the k-pointed, arithmetic genus g, degree d, stable map moduli stack of X. Let M_d^g denote $M_{g,0}((d,1), X \times \mathbf{P}^1)$. Note that for each stable map $(C, f) \in M_d^g$ there is a unique branch $C_0 \cong \mathbf{P}^1$ in C such that f composed with the projection $X \times \mathbf{P}^1 \to \mathbf{P}^1$ maps $C_0 \to \mathbf{P}^1$ isomorphically. Moreover, C is a union of C_0 with some disjoint curves $C_1, .., C_N$, where each C_i intersects C_0 at a point $x_i \in C_0$. The map f composed with $X \times \mathbf{P}^1 \to \mathbf{P}^1$ collapses all $C_1, .., C_N$.

The standard $\mathbf{C}^\times$ action on $\mathbf{P}^1$ induces an action on M_d^g. The fixed point components are labelled by $F_{d_1,d_2}^{g_1,g_2}$ with $d_1 + d_2 = d$, $g_1 + g_2 = g$. As in the genus zero case, a stable map (C, f) in this component is given by gluing two 1-pointed stable maps $(f_1, C_1, x_1) \in M_{g_1,1}(d_1, X)$, $(f_2, C_2, x_2) \in M_{g_2,1}(d_2, X)$ with $f_1(x_1) = f_2(x_2)$, to a $\mathbf{P}^1$ at 0 and ∞ at the marked points (cf. section 3). We can therefore identify $F_{d_1,d_2}^{g_1,g_2}$ with $M_{g_1,1}(d_1, X) \times_X M_{g_2,1}(d_2, X)$. We denote by

$$F_{d_1,d_2}^g := \coprod_{g_1+g_2=g} F_{d_1,d_2}^{g_1,g_2}, \; i_{d_1,d_2} : F_{d_1,d_2}^g \to M_d^g,$$

the disjoint union and inclusions. There are two obvious projection maps

$$p_0 : F_{d_1,d_2}^g \to \coprod_{g_1=0}^{g} M_{g_1,1}(d_1, X), \; p_\infty : F_{d_1,d_2}^g \to \coprod_{g_2=0}^{g} M_{g_2,1}(d_2, X).$$

The map p_0 strips away the stable maps (f_2, C_2, x_2) glued to ∞ and forgets the $\mathbf{P}^1$; p_∞ strips away the stable map (f_1, C_1, x_1) glued to 0 and forgets the $\mathbf{P}^1$. We also have the usual evaluation maps, and the forgetting map:

$$e_{d_1,d_2} : F^g_{d_1,d_2} \to X,\ e_d : M_{g,1}(d,X) \to X,\ \rho : M_{g,1}(d,X) \to M_{g,0}(d,X).$$

Relating and summarizing the natural maps above is the following diagram:

$$(5.7)\qquad \begin{array}{ccccccc} X & \xleftarrow{e_{d_1,d_2}} & F^{g_1,g_2}_{d_1,d_2} & \xrightarrow{i_{d_1,d_2}} & M^g_d & \xrightarrow{\pi} & M_{g,0}(d,X) \\ e_{d_1}\uparrow & p_0 \swarrow & & \searrow p_\infty & & & \\ M_{g_1,1}(d_1,X) & & & & M_{g_2,1}(d_2,X) & & \\ \rho\downarrow & & & & \downarrow\rho & & \\ M_{g_1,0}(d_1,X) & & & & M_{g_2,0}(d_2,X) & & \end{array}$$

Fix a class $\Omega \in A^*_T(X)$. We call a list $b^g_d \in A^*_T(M_{g,0}(d,X))$ an Ω-gluing sequence if we have the identities on the $F^g_{d_1,d_2}$:

$$e^*_{d_1,d_2}\Omega \cdot i^*_{d_1,d_2}\pi^* b^g_d = \sum_{g_1+g_2=g} p^*_0\rho^* b^{g_1}_{d_1} \cdot p^*_\infty \rho^* b^{g_2}_{d_2}.$$

It is easy to verify that $b^g_d \equiv 1$ is an example of a 1-gluing sequence. Restricted to $g = 0$, the identity above is precisely the gluing identity in section 3.2. There we have found that the gluing identity results in an Euler series. It turns out that a gluing sequence too leads to an Euler series. For $\omega \in A^*_G(M^g_d)$ and $d = d_1 + d_2$, define (cf. section 3.2)

$$i^{vir}_{d_1,d_2}\omega := e_{d_1,d_2*}\left(\frac{i^*_{d_1,d_2}\omega \cap [F^g_{d_1,d_2}]^{vir}}{e_G(F^g_{d_1,d_2}/M^g_d)}\right) \in A^T_*(X)(\alpha).$$

Then for a given gluing sequence $b^g_d \in A^*_T(M_{g,0}(d,X))$, we have the identities

$$\Omega \cap i^{vir}_{d_1,d_2}\pi^* b^g_d = \sum_{g_1+g_2=g} \overline{i^{vir}_{0,d_1}\pi^* b^{g_1}_{d_1}} \cdot i^{vir}_{0,d_2}\pi^* b^{g_2}_{d_2}.$$

Again, putting $g = 0$, we get the identities in Theorem 3.6. The argument in the higher genus case is essentially the same as the genus zero case. Here, one chases through a fiber diagram analogous to 3.1 using the associated refined Gysin homomorphism, together with the diagram 5.7.

Now given a gluing sequence, we put

$$A^g_d := i^{vir}_{0,d}\pi^* b^g_d,\ A_d := \sum_g A^g_d\, \mu^g,\ A(t) := e^{-H\cdot t/\alpha} \sum_d A_d\, e^{d\cdot t}.$$

Here μ is a formal variable. Then $A(t)$ is an Euler series. (We must, of course, replace the ring $\mathcal{R}$ by $\mathcal{R}[[\mu]]$.) The argument is also similar to the genus zero case: one applies functorial localization to the diagram

$$\begin{array}{ccc} F^g_{d_1,d_2} & \xrightarrow{i_{d_1,d_2}} & M^g_d \\ e_{d_1,d_2}\downarrow & & \downarrow\varphi, \\ X \subset Y & \xrightarrow{j_{d_1,d_2}} & W_d \end{array}$$

the same way we have done to diagram 3.2 in section 3.2.

We can proceed further in a way parallel to the genus zero case. Namely, to find further constraints to a gluing sequence, we should compute the linking values of the Euler series $A(t)$. For this, let's assume that X is a balloon manifold, as in sections 4.1 and 5.3. In genus zero, the linking values of an Euler series, say coming from $b_T(V_d)$, are determined by the restrictions $i_F^* b_T(\rho^* V_d)$ to the isolated fixed point $F = (\mathbf{P}^1, f_\delta, 0) \in M_{0,1}(d, X)$, which is a δ-fold cover of a balloon pq in X (see Theorem 4.5). In higher genus, this is replaced by a component in $M_{g,1}(d, X)$ consisting of the following stable maps (C, f, x). Here C is a union of two curves C_1 and $C_0 \cong \mathbf{P}^1$ such that $C_0 \xrightarrow{f} pq$ is a δ-fold cover with $f(x) = p$, and $f(C_1) = q$. Therefore this fixed point component can be identified canonically with $\bar{M}_{g,1}$, the moduli space of genus g, 1-pointed, stable curves. Let's abbreviate it F. The linking values of $A(t)$ for this component is then a power series summing over integrals on $\bar{M}_{g,1}$ of classes given in terms of $i_F^* \rho^* b_d^g$ and $e_T(F/M_{g,1}(d, X))$ (cf. Theorem 4.5).

The entire discussion in this section can be generalized to the case of multiple marked points, ie. $M_{g,k}(d, X)$. The details will appear elsewhere.

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