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A UNIFORM LIMIT LAW FOR THE BRANCHING MEASURE ON A GALTON-WATSON TREE*

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Abstract. We prove a uniform asymptotic law for the branching measure on the boundary of a Galton–Watson tree, which is consistent with certain well-known uniform laws associated with Brownian motions. We also list a certain spectrum formula arising from this uniform law.

1. Introduction and Main Result. Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $\{p_n : n \in \mathbb{N}\}$ be a probability distribution on $\mathbb{N} = \{0, 1, ...\}$. For simplicity, we assume $p_0 = 0$. Put $\mathbb{N}^* = \{1, 2, ...\}$ and write $\mathbf{U} = \{\emptyset\} \cup \cup_{n=1}^{\infty} (\mathbb{N}^*)^n$ for the set of all finite sequences $u = u_1 \cdots u_n$ including the null sequence $\emptyset$. Let $\{N_u : u \in \mathbf{U}\}$ be a family of independent random variables defined on Ω, each distributed according to the law $\{p_n\}$. Let $\mathbf{T}(\omega)$ be the corresponding *Galton–Watson tree* with defining elements $\{N_u\}$: we have $\emptyset \in \mathbf{T}(\omega)$ and, if $u \in \mathbf{T}(\omega)$ and $i \in \mathbb{N}^*$, then $ui \in \mathbf{T}(\omega)$ if and only if $1 \le i \le N_u(\omega)$. For simplicity, when there is no confusion, we use $\mathbf{T}$ and $\mathbf{T}(\omega)$ interchangeably. If $u = u_1 ... u_n$ ($u_k \in \mathbb{N}, n \le \infty$), we write $|u| = n$ and $u|k = u_1 ... u_k, k \le n$. Let $\partial \mathbf{T} = \{u_1 u_2 ... : \forall n \in \mathbb{N}, u_1 ... u_n \in \mathbf{T}\}$ be the boundary of $\mathbf{T}$ endowed with the distance

$$d_e(u, v) = e^{-n}, \qquad \text{where } n = \max\{k \in \mathbb{N} : u|k = v|k\},\ u, v \in \partial \mathbf{T}.$$

For all $u \in \mathbf{U}$, let $\mathbf{T}_u$ be the *shifted tree* of $\mathbf{T}$ at u: we have $\emptyset \in \mathbf{T}_u$ and, if $v \in \mathbf{T}_u$ and $i \in \mathbb{N}^*$, then $vi \in \mathbf{T}_u$ if and only if $1 \le i \le N_{uv}$, where uv denotes the juxtaposition of u and v. Clearly $\mathbf{T} = \mathbf{T}_\emptyset$.

Write $N = N_\emptyset$ and assume $EN \log N < \infty$. Set $m = EN$ and put

$$Z = \lim_{n \to \infty} \frac{\text{card } \{v \in \mathbf{T} : |v| = n\}}{m^n};$$

the limit exists a.s. by the martingale convergence theorem. Then $EZ = 1$ and $Z > 0$ a.s. Similarly, for all $u \in \mathbf{U}$, we write

$$Z_u = \lim_{n \to \infty} \frac{\text{card } \{v \in \mathbf{T}_u : |v| = n\}}{m^n}.$$

Then $Z = Z_\emptyset$ and $\{Z_u : u \in \mathbf{U}\}$ is a family of identically distributed random variables. Since for all $u \in \mathbf{T}$, card $\{v \in \mathbf{T}_u : |v| = n+1\} = \sum_{i=1}^{N_u}$ card $\{v \in \mathbf{T}_{ui} : |v| = n\}$, it is easily seen that for all $u \in \mathbf{T}$,

$$m^{-|u|} Z_u = \sum_{i=1}^{N_u} m^{-|ui|} Z_{ui}.$$

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Therefore for almost all $\omega \in \Omega$, there is a unique Borel measure on $\partial\mathbf{T}(\omega)$, called henceforth by $\mu = \mu_\omega$, such that

$$\mu_\omega(B_u) = m^{-|u|} Z_u \qquad \forall u \in \mathbf{T}(\omega),$$

where

$$B_u = \{v \in \partial\mathbf{T} : u < v\}$$

is a ball in $\partial\mathbf{T}$ with diameter $|B_u| = e^{-|u|}$. Here for two sequences $u, v \in \mathbf{U}$, we write $u < v$ if $uu' = v$ for some $u' \in \mathbf{U}$. We can also normalize μ by putting $\bar{\mu} = \bar{\mu}_\omega = \mu_\omega / Z(\omega)$. Then $\bar{\mu}$ is a probability measure on $\partial\mathbf{T}$, such that for all $u \in \mathbf{T}$,

$$\bar{\mu}(B_u) = \lim_{n\to\infty} \frac{\text{card}\,\{v \in \mathbf{T} : u < v, |v| = n\}}{\text{card}\,\{v \in \mathbf{T} : |v| = n\}}.$$

We call μ(or $\bar{\mu}$) the *(uniform) branching measure* of the tree $\mathbf{T}$. This measure is well studied since the work of Hawkes [2], see for example [9] and [10]. It is known that with probability 1,

$$\lim_{n\to\infty} \frac{\log \mu(B_{u|n})}{n} = -\log m, \text{ for } \mu_\omega\text{-almost all } \ u \in \partial\mathbf{T}$$

(see [10]). Recently Liu [8] proved that the conclusion holds for *all* $u \in \partial\mathbf{T}$ under some additional conditions. The purpose of this note is to prove the asymptotic behavior of $\max_{u\in\mathbf{T},|u|=n} \mu(B_u)$ as $n \to \infty$. Write

$$\alpha = \log m, \quad \beta = 1 - \log m / \log \|N\|_\infty,$$

where $\|N\|_\infty =$ ess sup N. By convension, $\beta = 1$ if $\|N\|_\infty = \infty$. We also put

$$r = \sup\Big\{t > 0 : \quad E\exp(tZ^{1/\beta}) < \infty\Big\}.$$

By [6],

$$r = \liminf_{x\to\infty} \frac{-\log P\{Z > x\}}{x^{1/\beta}}; \tag{1.1}$$

by [5], $0 < r < \infty$ if

(1.2) either $\|N\|_\infty < \infty$ or $E\exp(tN) < \infty$ for some but not all $t > 0$.

We shall always assume (1.2) if the contrary is not specified.

THEOREM 1.1. *With probability 1,*

$$\limsup_{n\to\infty} \sup_{u\in\partial\mathbf{T}} \frac{m^n \mu(B_{u|n})}{n^\beta} = C,$$

where $C = (\alpha/r)^\beta$.

REMARKS. (i) If (1.2) fails, the statement of Theorem 1.1 also holds with C interpreted as 0 or ∞ according as $r = \infty$ or 0 respectively. This can be seen by the proof. (ii) It is interesting to observe that we may rewrite the result as

$$\limsup_{n\to\infty} \sup_{u\in\partial\mathbf{T}} \frac{\mu(B_{u|n})}{|B_{u|n}|^\alpha \Big(\log \frac{1}{|B_{u|n}|}\Big)^\beta} = C,$$

and in this form the result is consistent with some well–known uniform asymptotic laws associated with Brownian motions or stables processes, see for example [4] (Théorème 52,2, p.172), [1] (Theorem 2) and [13] (Lemma 2.3 and Corollary 5.2). (iii) A similar result for $\liminf$ of $\inf_{u\in\partial\mathbf{T}}\ \mu(B_{u|n})$ and other associated results are established in [7] and [8]; Yimin Xiao has kindly informed us that after reading our preprint, he also obtained some results on $\liminf_n \inf_{u\in\partial\mathbf{T}}\ \mu(B_{u|n})$ and that he is working on some related problems. (iv) We may replace lim sup by lim under some conditional conditions: for example, this is the case when N is of geometric distribution, as was shown by Hawkes [2].

The proof of Theorem 1.1 is give in §2. In §3, we list a certain spectrum formula arising from our uniform law; the formula has the same flavor as those in [12] and [13] for Brownian fast points and local times. About general definitions and properties, the reader is refereed to [11] on Galton–Watson trees and to [14] on fractals associated with stochastic processes.

2. Proof of Theorem 1.1.

Upper bound proof. For $\theta > 0$, let

$$E_\theta = \{\omega \in \Omega :\ \limsup_{n\to\infty} \sup_{u\in\partial\mathbf{T}} \frac{m^n \mu(B_{u|n})}{n^\beta} > \theta C\}.$$

We prove that $P(E_\theta) = 0$ for all $\theta > 1$. Since $E_\theta \subset \limsup A_n$, where $A_n = A_{n,\theta}$ is the event

$$A_n = \{\omega \in \Omega :\ \sup_{u\in\partial\mathbf{T}(\omega)} \frac{m^n \mu_\omega(B_{u|n})}{n^\beta} > \theta C\}. \tag{2.1}$$

Thus, by Borel–Cantelli Lemma, it suffices to prove that $\sum P(A_n) < \infty$. We have

$$\begin{aligned} P(A_n) &\le E \sum_{u:|u|=n} 1\{\ \frac{m^n \mu_\omega(B_{u|n})}{n^\beta} > \theta C\} \\ &= E \sum_{u:|u|=n} 1\{Z_u > N^\beta \theta C\} \\ &= e^{n\alpha} P\{Z > n^\beta \theta C\}, \end{aligned}$$

in the above, the notation $1\{\cdot\}$ denotes the indicator of the event $\{\cdot\}$. Note that we have used the branching property that for all $u \in \mathbf{U}$ with $|u| = n$, the random variables Z_u are independent and have the same distribution as Z. By the definition of C and the assumption $\theta > 1$, we can find an $\epsilon > 0$ such that

$$(r-\epsilon)(\theta C)^{1/\beta} > \alpha.$$

By (1.1), we have, for all n large enough,

$$P\{Z > N^\beta \theta C\} \le \exp\big(-(r-\epsilon)(\theta C)^{1/\beta} n\},$$

from which it follows that

$$P(A_n) \le \exp\{-[(r-\epsilon)(\theta C)^{1/\beta} - \alpha]n\}.$$

Thus, $\sum P(A_n) < \infty$; this ends the upper bound proof.

Lower bound proof. Consider again the event A_n defined by (2.1). We shall prove that, for each $\theta : 0 < \theta < 1$, $P(\limsup A_n) = 1$. From which it follows that, with probability 1,

$$\limsup_n \sup_{u \in \partial \mathbf{T}} \frac{m^n \mu_\omega(B_{u|n})}{n^\beta} \geq \theta C.$$

Thus, the lower bound proof is obtained by letting $\theta \uparrow 1$ through rational numbers. It suffices to prove that $\liminf P(A_n^c) = 0$. Observe that

$$\begin{aligned} P(A_n^c) &\leq P\Big\{ \cap_{u:T,|u|=n} \{\frac{m^n \mu(B_{u|n})}{n^\beta} < \theta C\}\Big\} \\ &= E \prod_{u:|u|=n} 1\Big\{\frac{m^n \mu(B_{u|n})}{n^\beta} < \theta C\Big\} \\ &= E \prod_{u \in \mathbf{T},|u|=n} 1\Big\{Z_u < n^\beta \theta C\Big\} \\ &= E\Big[\big(P\{Z < n^\beta \theta C\}\big)^{Z^{(n)}}\Big], \end{aligned}$$

where $Z^{(n)} = \mathrm{card}\{u : |u| = n\}$. In the above we have again used the branching property. Since $\theta < 1$, we can find a small $\epsilon > 0$ such that

$$\lambda := \frac{(r+\epsilon)\theta^{1/\beta}}{r} < 1.$$

Since (1.1) also holds with x replaced by $n^\beta \theta C$, there exists a sequence $n' \uparrow \infty$ such the following holds for $n = n'$:

$$P(Z \geq n^\beta \theta C) \geq e^{-(r+\epsilon)(\theta C)^{1/\beta} n} = e^{-\lambda \alpha n}.$$

Thus, using $1 - x \leq e^{-x}$ $\forall x \in (0,1)$, we see that for $n = n'$,

$$P(A_n^c) \leq E\Big[(1 - e^{-\lambda \alpha n})^{Z^{(n)}}\Big] \leq E \exp\Big[- e^{-\lambda \alpha n} Z^{(n)}\Big].$$

We note that, with probability 1,

$$e^{-\lambda \alpha n} Z^{(n)} = m^{-\lambda n} Z^{(n)} = m^{(1-\lambda)n} \cdot \frac{Z^{(n)}}{m^n}.$$

Since $Z^{(n)}/m^n \to Z > 0$ a.s. and $\lambda < 1$, the quantity in the above display tends to ∞. Applying this result to $n = n'$, we see that $\liminf P(A_n^c) = 0$. This completes the proof. □

3. A spectrum formula. Write, for $\theta > 0$,

$$F_\theta = \Big\{u \in \partial \mathbf{T} : \limsup_{n \to \infty} \frac{m^n \mu(B_{u|n})}{n^\beta} = \theta C\Big\},$$

where C is the constant in Theorem 1.1. By Theorem 1.1, $F_\theta = \emptyset$ if $\theta > 1$. It is interesting to calculate $\dim F_\theta$, the Hausdorff dimension of F_θ, for $0 \leq \theta \leq 1$. We can modify the technique in §2 to obtain an upper bound.

PROPOSITION 3.1. *With probability 1,*

$$\dim F_\theta \le \alpha(1-\theta^{1/\beta}), \quad 0 \le \theta \le 1. \tag{3.1}$$

Proof. The assertion is evident if $\theta = 0$, because $\dim F_\theta \le \dim \partial\mathbf{T} = \alpha$. Assume $0 < \theta \le 1$. We search for the smallest $b > 0$ so that $\dim F_\theta \le b$. We observe that, for $\epsilon : 0 < \epsilon < \theta$ and positive integer k,

$$F_\theta \subset \cup_{n\ge k}\Big\{u \in \partial\mathbf{T} : \frac{m^n\mu(B_{u|n})}{n^\beta} > (\theta-\epsilon)C\Big\}.$$

For $A \subset \partial\mathbf{T}$, write $\mathcal{H}^b(A) = \lim_{k\to\infty}\mathcal{H}^b_k(A)$, where

$$\mathcal{H}^b_k(A) = \inf\{\sum_v |B_v|^b : \ A \subset \cup B_v, \ |v| \ge k, \ \forall v\}, \qquad k \in \mathbb{N}.$$

Then

$$\mathcal{H}^b_k(F_\theta) \le \sum_{n\ge k}\sum_{|v|=n}|B_v|^b 1\Big\{\frac{m^n\mu(B_v)}{n^\beta} > (\theta-\epsilon)C\Big\}.$$

Let I_k denote the random variable defined by the right hand side of the above display; then, by the same reasoning as the first part of §2, we have

$$\begin{aligned} EI_k &= \sum_{n\ge k} e^{-nb}m^nP\big[Z > (\theta-\epsilon)Cn^\beta\big] \\ &\le \sum_{n\ge k} e^{-(b-\alpha)n}e^{-\tau(\theta-\epsilon)^{1/\beta}C^{1/\beta}n}, \end{aligned}$$

where $\tau = r - \epsilon$, and $k = k(\epsilon)$ is large enough. The series in the above display is convergent, so that I_k tends to 0 a.s., whenever

$$b > \alpha - \tau(\theta-\epsilon)^{1/\beta}C^{1/\beta}.$$

Since ϵ is arbitrarily chosen, we conclude that $\mathcal{H}^b(F_\theta) = 0$, whenever $b > \alpha - r\theta^{1/\beta}C^{1/\beta} = \alpha(1-\theta^{1/\beta})$. This implies the assertion. □

REMARK. In view of the results in [12] and [13] and the formula (1.1), we could expect the equality in (3.1). Clearly, this is the case if $\theta = 0$ or 1.

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Note added in proof. Recently, (a) Q.S. Liu has proved that the conclusion in Theorem 1.1 remains valid when the lim sup is replaced by lim if (1.1) holds with lim inf replaced by lim, and that a similar result also holds for $\inf_{u\in\partial\mathbf{T}}\mu(B_{u|n})$; (b) N.R. Shieh and S.J.Taylor have shown that we do have the equality in (3.1).

REFERENCES

[1] J. HAWKES, *A lower Lipschitz condition for the stable subordinator*, Z. Wahr. verw. Geb., 17 (1971), pp. 23–32.

[2] J. HAWKES, *Trees generated by a simple branching process*, J. London Math. Soc., 24 (1981), pp. 373–384.

[3] A. JOFFE, *Remarks on the structure of trees with applications to supercritical Galton-Watson processes*, in Advances in Prob., 5, A. Joffe and P. Ney, ed., Dekker, New-York, 1978, pp. 263–268.

[4] P. LÉVY, *Théorie De L'addition Des Variables Aléatoires*, 2ème éd., Gautier-Villars, Paris, 1954.

[5] Q. S. LIU, *The exact Hausdorff dimension of a branching set*, Prob. Th. Rel. Fields, 104 (1996), pp. 515–538.

[6] Q. S. LIU, *The growth of an entire characteristic function and the tail probabilities of the limit of a tree martingale*, in "Trees", B. Chauvin, S. Cohen, and A. Rouault, ed., Progress in Probability, 40, Birkhöuser, Verlag Basel, 1996, pp. 51–80.

[7] Q. S. LIU, *Exact packing measure of the boundary of a Galton-Watson tree*, To appear in Stoch. Processes Appl.

[8] Q. S. LIU, *Local properties of the branching measure on the Galton-Watson tree*, Preprint, Université Rennes 1, France, 1998.

[9] Q. S. LIU AND A. ROUAULT, *On two measures defined on the boundary of a branching tree*, in Classical and modern branching processes, K. B. Athreya and P. Jagers, ed., IMA Volumes in Mathematics and its applications, 84, Springer-Verlag, 1996, pp. 187–202.

[10] R. LYONS, R. PEMANTLE, AND Y. PERES, *Ergodic theory on Galton-Watson trees, Speed of random walk and dimension of harmonic measure*, Ergodic Theory Dyn. Systems, 15, 1995, pp. 593–619.

[11] J. NEVEU, *Arbre et processus de Galton-Watson*, Ann. Inst. Henri Poincaré, 22 (1986), pp. 199–207.

[12] S. OREY AND S. J. TAYLOR, *How often on a Brownian path does the law of iterated logarithm fail?*, Proc. London Math. Soc., 3 (1974), pp. 174–192.

[13] N. R. SHIEH AND S. J. TAYLOR, *Logarithmic multifractal spectrum of stable occupation measure*, Stoch. Processes Appl., 75 (1998), pp. 249-261.

[14] S. J. TAYLOR *The measure theory of random fractals*, Math. Proc. Camb. Phi Soc., 100 (1986), pp. 383–406.