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A DESCRIPTION OF THE DISCRETE SPECTRUM OF $(SL(2), E_{7(-25)})^*$

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1. Introduction. Let $E_{8,4}$ be the simply connected quarternionic E_8. It is the unique simply connected simple Lie group of exceptional type E_8 and real rank 4. In [GrW] Gross and Wallach constructed the minimal representation π_{min} of $E_{8,4}$. It is an irreducible unitary representation with minimal Gelfand-Kirillov dimension, and its annihilator in the (complexified) universal enveloping algebra is the Joseph ideal.

In this paper we will give a description of the discrete spectrum of the restriction of π_{min} to the symmetric subgroup $E_{7,3} \times SU(1,1)$. Here $E_{7,3}$ is a connected simple Lie group of type E_7 that gives rise to the hermitian tube domain of Cartan type EVII. For an integer $k \geq 2$ let π_k be the holomorphic discrete series representation of $SU(1,1) = SL(2)$ of lowest weight k. Let π_{-k} be the contragredient of π_k. Write the unitary decomposition of the restriction of π_{min} as

$$\pi_{min}|_{E_{7,3}\times SU(1,1)} = (\bigoplus_{|k|\geq 2} \theta_k \otimes \pi_k) \bigoplus (\text{continuous spectrum}) \tag{1.1}$$

Since π_{min} is self-contragredient, we see θ_{-k} is contragredient to θ_k. So it suffices to describe θ_k for $k \geq 2$. It turns out that

$$\theta_k = \sigma_k \oplus \sigma'_k$$

is the sum of two representations, where σ_k is an irreducible highest weight unitary representation which belongs to the discrete series when $k \geq 10$. The representation σ'_k is admissible (in fact it has multiplicity free K-types) and non-tempered. If $k \geq 4$ the K-type structure of σ'_k is identical to that of a derived functor module. For $k \geq 10$, σ'_k is an irreducible unitary representation with non-zero cohomology at bi-degree $(10,1)$ (and so σ'_{-k} has cohomology at bi-degree $(1,10)$). Thus when $k \geq 10$, θ_k is the sum of two irreducible representations, one of them belongs to the discrete series while the other is very far from being tempered. This is the rough description of the discrete spectrum. For details see §5-6.

The determination of θ_k depends heavily on the fact that the groups $E_{7,3}$ and $SU(1,1)$ (essentially) form a reductive dual pair in $E_{8,4}$. As such they fit into the following seesaw diagram

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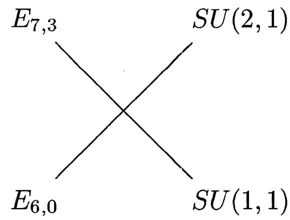
(1.2)

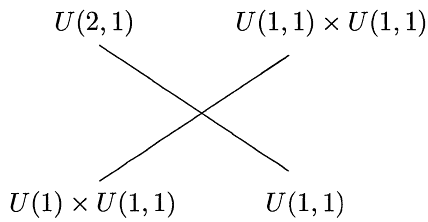


Here $E_{6,0}$ is the semi-simple part of the maximal compact subgroup of $E_{7,3}$, a compact simple Lie group of type E_6. With this diagram in mind, we first determine the decomposition of π_{min} restricted to $E_{6,0} \times SU(2,1)$ in §3. This is not difficult since $E_{6,0}$ is compact. The decomposition takes the form

$$\pi_{min}|_{E_{6,0}\times SU(2,1)} = \bigoplus_{p,q\geq 0} \pi(p\omega_6 + q\omega_1) \otimes \sigma_{p,q}$$

where ω_j denotes the j-th fundamental weight for $E_{6,0}$ and $\pi(\lambda)$ is the irreducible finite dimensional representation with highest weight λ. It turns out that each representation $\sigma_{p,q}$ is irreducible and belongs to the generic discrete series of $SU(2,1)$.

Next in §4 we study the restriction of $\sigma_{p,q}$ to $SU(1,1)$. We consider $U(2,1)$ and $U(1,1)$ instead, extending $\sigma_{p,q}$ to $U(2,1)$. Then there is another seesaw diagram for dual pairs in the rank 6 symplectic group $Sp_{12}(\mathbb{R})$:



This diagram is analized using our earlier results in [Lib] and results of Repka [Rep] on tensor products of holomorphic and anti-holomorphic representations. This leads to the explicit determination of the K-type structure of θ_k. Upon inspection of this structure we find that θ_k contains a highest weight module, namely σ_k. Finally, we know *a priori* that θ_k is quasi-simple with infinitesimal character given by Rallis and Schiffimann [RaS] (see also [Lia]). Together we have enough information to give the description of θ_k outlined above.

Wee Teck Gan [Gan] has shown that the minimal representation of $E_{8,4}$ is automorphic. It follows that the non-tempered representations σ'_k decribed here are also automorphic representations.

NOTATIONS. We use $E_{n,r}$ to denote a connected simple Lie group of exceptional type E_n and split rank r. This will be made more precise when the group is actually introduced.Let $\mathfrak{e}_{n,r}$ be the corresponding complexified Lie algebra. Up to isomorphisms the later is of course independent of the second subscript r. But we keep it in our notations to reminder ourselves which real Lie group the complex Lie algebra comes from. The root subgroup corresponding to a compact root is denoted $SU(2)$ and one corresponding to a non-compact root will be $SU(1,1)$. Their complexified Lie algebras will be $\mathfrak{su}(2)$ and $\mathfrak{su}(1,1)$ respectively. We will denote by V_μ or $V(\mu)$ the irreducible module with highest weight μ, of whatever Lie group or Lie algebra in question.

2. Subalgebras of $\mathfrak{e}_{8,4}$. Consider $S = E_{8,4}$ (simply connected). The maximal compact subgroup of S is $SU(2) \times E_{7,0}$ which contains a Cartan subgroup H. We introduce coordinates so that the complexified Lie algebra of H is $\mathfrak{h} \simeq \mathbb{C}^8$, and that the restriction of the Cartan-Killing form is the standard inner product given by

$$< x, y >= x_1 y_1 + \cdots + x_8 y_8$$

Let e_j denotes evaluation on the j-th coordinate. We may assume that the roots of $\mathfrak{h}$ in $\mathfrak{e}_{8,4}$ are as enumerated in [Hel, Ch. X], namely

$$\pm e_i \pm e_j \qquad (1 \leq i < j \leq 8)$$

and

$$\frac{1}{2}(\pm e_1 \pm \cdots \pm e_8)$$

with an even number of minus signs. The simple roots are

$$\alpha_1 = \frac{1}{2}(e_1 + e_8 - e_2 - \cdots - e_7), \qquad \alpha_2 = e_1 + e_2$$

and

$$\alpha_j = e_{j-1} - e_{j-2}, \qquad (3 \leq j \leq 8)$$

We assume that $SU(2)$ is the root subgroup corresponding to the root $e_7 + e_8$. Consequently roots in $\mathfrak{e}_{7,0}$ are precisely those perpendicular to $e_7 + e_8$. Write the Cartan decomposition of $\mathfrak{e}_{8,4}$ as

$$\mathfrak{e}_{8,4} = \mathfrak{su}(2) \oplus \mathfrak{e}_{7,0} \oplus \mathfrak{p} \tag{2.1}$$

As a module for $SU(2) \times E_{7,0}$, $\mathfrak{p}$ has highest weight $e_6 + e_8$. It can also be written as

$$\mathfrak{p} = \mathbb{C}^2 \otimes U(\lambda) \tag{2.2}$$

Here $U(\lambda)$ is the miniscule module of $E_{7,0}$ of dimension 56, and λ is the highest weight which is also the 7-th fundamental weight for $E_{7,0}$.

Next let $\mathfrak{e}_{6,0} \subset \mathfrak{e}_{7,0}$ be the (simple) subalgebra generated by all the roots which are orthogonal to both $e_6 + e_8$ and $e_7 + e_8$. The centralizer of $\mathfrak{e}_{6,0}$ in $\mathfrak{e}_{7,0}$ is the one-dimensional torus $\mathbb{C} \cdot h$ where

$$h = (0, 0, 0, 0, 0, 2, -1, 1) \in \mathfrak{h}$$

Let $\mathbb{C}(k)$ be the one-dimensional space on which the element h acts via the scalar k. As a module for $\mathfrak{e}_{6,0} + \mathbb{C}h$ one has

$$U(\lambda) = \mathbb{C}(3) + \mathbb{C}(-3) + V \otimes \mathbb{C}(1) + V^* \otimes \mathbb{C}(-1) \tag{2.3}$$

where V is an irreducible representation of $\mathfrak{e}_6$ of dimension 27. More precisely we take the first six simple roots of $\mathfrak{e}_{8,4}$ as simple roots for $\mathfrak{e}_{6,0}$ and let ω_j be the j-th fundamental weight. Then the highest weights of V and V^* are ω_6 and ω_1 respectively.

Let $E_{6,0}$ and T_h be the compact connected subgroups of $E_{8,4}$ corresponding to the Lie algebras $\mathfrak{e}_{6,0}$ and $\mathbb{C}h$ respectively. The centralizer of $E_{6,0}$ in $E_{8,4}$ is of type A_2 (see [Rub]) with maximal compact subgroup $SU(2) \times T_h$. Thus the centralizer is just $SU(2,1)$, which contains the root subgroup $SU(1,1)$ corresponding to the root $e_6 + e_8$. In this way T_h is identified with the center of the maximal compact subgroup of $SU(2,1)$. We write

$$T(SU(2,1)) = T_h = E_{7,0} \cap SU(2,1)$$

Note that the roots in $\mathfrak{su}(2,1)$ are

$$\pm(e_7 + e_8), \quad \pm(e_6 + e_8), \quad \pm(e_6 - e_7)$$

We take $-e_7 - e_8, e_6 + e_8$ as the simple roots.

On the other hand we can realize $\mathfrak{su}(2,1)$ as follows. Let I be the 3×3 diagonal matrix with $1, 1, -1$ on the diagonal. Then $SU(2,1)$ can be identified with the group of all complex matrices A with determinant 1, such that $\overline{A}^t I A = I$. We take the space of diagonal matrices in $\mathfrak{su}(2,1)$ as a Cartan subalgebra. Let ε_j denote evaluation on the j-th diagonal element and take $\varepsilon_1 - \varepsilon_2$, $\varepsilon_2 - \varepsilon_3$ as the simple roots. Thus on the Cartan alegebra of $\mathfrak{su}(2,1)$ we have

$$-e_7 - e_8 = \varepsilon_1 - \varepsilon_2, \qquad e_6 + e_8 = \varepsilon_2 - \varepsilon_3 \tag{2.4}$$

The element h is then identified with

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \in \mathfrak{su}(2,1)$$

Next let $\mathfrak{e}_{7,3}$ be the centralizer of $\mathfrak{su}(1,1)$ in $\mathfrak{e}_{8,4}$. Let $E_{7,3} \subset E_{8,4}$ be the corresponding connected subgroup. The group $E_{7,3}$ has maximal compact subgroup

$$K = E_{6,0} \times T(E_{7,3}),$$

where $T(E_{7,3})$ is a one dimensional torus which is the same as the centralizer of $SU(1,1)$ in $SU(2,1)$. Thus

$$T(E_{7,3}) = \{t(a) = \begin{pmatrix} a^{-2} & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \mid a \in U(1)\} \tag{2.5}$$

For an integer μ let $(\mu)_{7,3}$ denote the character of $T(E_{7,3})$ taking $t(a)$ to a^μ. We fix an orientation of the circle $T(E_{7,3})$ by choosing the element

$$t_{7,3} = (0,0,0,0,0,-1,2,1) \in \mathfrak{t}(\mathfrak{e}_{7,3}) \subseteq \mathfrak{h} \tag{2.6}$$

In view of (2.4), we see that $t_{7,3}$ is identified with the matrix

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathfrak{su}(2,1)$$

The Cartan decomposition of $\mathfrak{e}_{7,3}$ is

$$\mathfrak{e}_{7,3} = (\mathfrak{e}_{6,0} \oplus \mathfrak{t}(\mathfrak{e}_{7,3})) \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$$

where as $(\mathfrak{e}_{6,0} \oplus \mathfrak{t}(\mathfrak{e}_{7,3}))$-modules

$$\mathfrak{p}^+ = V(\omega_1) \otimes (2)_{7,3}, \quad \mathfrak{p}^- = V(\omega_6) \otimes (-2)_{7,3} \tag{2.7}$$

Thus $\mathfrak{p}^{\pm}$ are the ± 2 eigenspaces for $ad(t_{7,3})$. The highest weights for $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are $-e_6 + e_8$ and $e_5 - e_7$ respectively.

3. Restriction to $E_{6,0} \times SU(2,1)$. Let π_{min} be the minimal representation of $E_{8,4}$ constructed by Gross and Wallach [GrW]. A great deal of information about π_{min} is available from that paper. But here all we need is its $(SU(2) \times E_{7,0})$-type structure given by

$$\pi_{min}|_{SU(2)\times E_{7,0}} = \bigoplus_{n=0}^{\infty} S^{n+8}(\mathbb{C}^2) \otimes U(n\lambda) \tag{3.1}$$

Here $S^k(\mathbb{C})$ denotes the k-th symmetric power of the standard action of $SU(2)$ on $\mathbb{C}^2$. The weight λ is as in §2, and $U(n\lambda)$ is the irreducible module with highest weight $n\lambda$. The reader will realize that the following lemma is contained in Proposition 3.1 of [HPS].

LEMMA 3.1. *When restricted to $\mathfrak{e}_{6,0} \oplus \mathbb{C}h$ the irreducible $\mathfrak{e}_{7,0}$ module $U(n\lambda)$ decomposes as*

$$U(n\lambda) = \bigoplus V(p\omega_6 + q\omega_1) \otimes \mathbb{C}(p - q + 3r - 3s) \tag{3.2}$$

where the sum is over all non-negative integers p, q, r, s with

$$p + q + r + s = n$$

Proof. See [HPS], section 3.

The spectrum decomposition of π_{min} restricted to $E_{6,0} \times SU(2,1)$ can now be described as

PROPOSITION 3.2. *We have*

$$\pi_{min}|_{E_6\times SU(2,1)} = \bigoplus \pi(p\omega_6 + q\omega_1) \otimes \sigma_{p,q} \tag{3.3}$$

where $\sigma_{p,q}$ is a generic discrete series representation. More precisely, take

$$\epsilon_1 - \epsilon_3,\ \epsilon_3 - \epsilon_2,\ \epsilon_1 - \epsilon_2$$

to be the positive roots, with simple roots $\alpha_1 = \epsilon_1 - \epsilon_3$, $\alpha_2 = \epsilon_3 - \epsilon_2$. Let ω_1', ω_2' be the corresponding fundamental weights. Then $\sigma_{p,q}$ has Harish-Chandra parameter $\lambda = (p+4)\omega_1' + (q+4)\omega_2'$.

Proof. From (3.1) and (3.2) we immediately conclude

$$\pi_{min}|_{E_6\times SU(2,1)} = \bigoplus V(p\omega_6 + q\omega_1) \otimes \sigma_{p,q}$$

with

$$\sigma_{p,q}|_{\mathfrak{su}(2)+\mathbb{C}h} = \bigoplus_{r,s\geq 0} S^{p+q+r+s+8}(\mathbb{C}^2) \otimes \mathbb{C}(p - q + 3r - 3s) \tag{3.4}$$

Let $\sigma'_{p,q}$ be the discrete series representation of $SU(2,1)$ with Harish-Chandra parameter $(p+4)\omega'_1 + (q+4)\omega'_2$. Up to equivalence there is only one representation with the same $(\mathfrak{su}(2) \times \mathbb{C}h)$-type structure as $\sigma'_{p,q}$. But the $(\mathfrak{su}(2) \times \mathbb{C}h)$-types of $\sigma'_{p,q}$ are given by the Blattner formula [HeS] and those of $\sigma_{p,q}$ are given by (3.4). One checks that $\sigma_{p,q}$ and $\sigma'_{p,q}$ have exactly the same $(\mathfrak{su}(2) \times \mathbb{C}h)$-types. Therefore $\sigma_{p,q} \simeq \sigma'_{p,q}$ as claimed.

4. Restriction from $SU(2,1)$ to $S(U(1) \times U(1,1))$. We need to understand the restriction of $\sigma_{p,q}$ to the subgroup $SU(1,1)$. This is more or less the same as understanding restriction from $U(2,1)$ to the symmetric subgroup $U(1) \times U(1,1)$, and will be done using suitable seesaw dual pairs in the symplectic group $Sp_{12}(\mathbb{R})$.

Let $\tilde{U}(1,1)$ be the two-fold cover of $U(1,1)$ determined by $det(\cdot)^{1/2}$. Local theta correspondence gives rise to a bijection between certain discrete series of $\tilde{U}(1,1)$ and of $U(2,1)$. We shall describe the correspondence for the cases we need here. The relevant Harish-Chandra parameters of $\tilde{U}(1,1)$ will be of the form $\lambda = (a,b)$, where a,b are integers, and either $a > b > 0$, or $0 > a > b$. Let $\pi(\lambda)$ be the corresponding discrete series representation of $\tilde{U}(1,1)$. To each such λ we associate a Harish-Chandra parameter $\Lambda = \theta(\lambda)$ by the formula

$$\theta(\lambda) = \begin{cases} (a,0;b) & \text{if} \quad a > b > 0 \\ (0,b;a) & \text{if} \quad 0 > a > b \end{cases}$$

Let $\tau(\Lambda)$ be the corresponding discrete series representation of $U(2,1)$. The following can be read off from [Lib, §6].

LEMMA 4.1. *Under the local theta correspondence we have* $\pi(\lambda) \leftrightarrow \tau(\Lambda)$, *where* $\Lambda = \theta(\lambda)$ *is as given above.*

Henceforward we shall only consider the first case: $a > b > 0$. This will be sufficient for our purposes here. It is straight forward to verify

LEMMA 4.2. *We have* $\tau(a,0;b)|_{SU(2,1)} = \sigma_{p,q}$ *if and only if*

$$a = p + q + 8, \quad b = q + 4$$

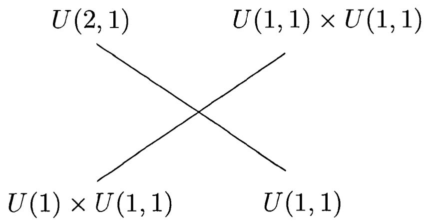
Fix a, b as above. A representation of $SU(1,1)$ occurs in the restriction of $\sigma_{p,q}$ if and only if it occurs in the restriction of $\tau(a,0;b)$. Let π_k be the holomorphic discrete series representation of $SU(1,1)$ with lowest weight $k \geq 2$. Suppose π_k occurs in the restriction of $\tau(a,0;b)$. Then some extension of π_k to $U(1) \times U(1,1)$ must also occur in the restriction of $\tau(a,0;b)$. We write such an extension as $\alpha \otimes \pi(\lambda_1, \lambda_2)$. Here α is an integer, identified with the character $t \mapsto t^\alpha$ of $U(1)$, and (λ_1, λ_2) is a Harish-Chandra parameter. We must have

$$\alpha + \lambda_1 + \lambda_2 = a + b \tag{4.1}$$

$$\lambda_1 - \lambda_2 = k - 1 \tag{4.2}$$

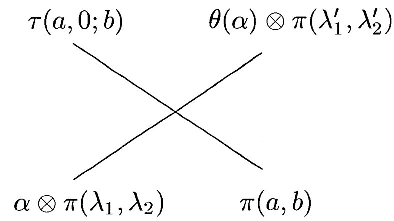
We shall make use of the seesaw diagram

(4.3)



There is a corresponding diagram for representations:

(4.4)



Here

$$\theta(\alpha) = \begin{cases} \pi(\alpha, 0), & \alpha \geq 0 \\ \pi(0, \alpha), & \alpha \leq 0 \end{cases}$$

When $\alpha = 0$ we understand $\theta(0)$ to be the limit of holomorphic discrete series representation with infinitesimal character $(0,0)$. In any case, the restriction of $\theta(\alpha)$ to $SU(1,1)$ is holomorphic of lowest weight $|\alpha|+1$. The parameters λ_1, λ_2 are in $\mathbb{Z}+1/2$ (since they parameterize discrete series of the linear group $U(1,1)$). For $\pi(\lambda_1, \lambda_2)$ to occur in the correspondence with $U(1,1)$, λ_1, λ_2 must be both positive or both negative (see [Pau]). In the first case, $(\lambda_1', \lambda_2') = (\lambda_1, \lambda_2)$; while in the second case $(\lambda_1', \lambda_2') = (\lambda_2, \lambda_1)$. The follwoing lemma can be easily deduced from results of Repka [Rep].

LEMMA 4.3. *The representation $\pi(a,b)$ occurs in $\theta(\alpha) \otimes \pi(\lambda_1', \lambda_2')$ if and only if*

$$\alpha \equiv p + k \pmod 2 \tag{4.5}$$

and either $\lambda_1, \lambda_2 < 0$, or $\lambda_1, \lambda_2 > 0$ and $|\alpha| + k \leq p + 4$.

Now assume that the restriction of $\alpha \otimes \pi(\lambda_1, \lambda_2)$ to $T(E_{7,3})$ contains $(\mu)_{7,3}$ (see §2). This is just the condition

$$\lambda_1 + \lambda_2 - 2\alpha = \mu$$

which together with (4.1)-(4.2) determines λ_1, λ_2 and α completely, namely

$$\begin{cases} \alpha = \frac{1}{3}(p + 2q + 12 - \mu) \\ \lambda_1 = \frac{1}{6}(2p + 4q + 24 + \mu) + \frac{1}{2}(k-1) \\ \lambda_2 = \frac{1}{6}(2p + 4q + 24 + \mu) - \frac{1}{2}(k-1) \end{cases} \tag{4.6}$$

The condition (4.5) now translates to

$$\mu \equiv 3k - 2p + 2q \pmod 6$$

Accordingly we write

$$\mu = 3k - 2p + 2q + 6r \tag{4.7}$$

Then $r \in \mathbb{Z}$ and we have

$$\begin{cases} \alpha = -k + p + 4 - 2r \\ \lambda_1 = k + q + 4 + r - \frac{1}{2} \\ \lambda_2 = q + 4 + r + \frac{1}{2} \end{cases} \tag{4.8}$$

In terms of the parameters k, p, q, r we can formulate

PROPOSITION 4.4. *The restriction of $\sigma_{p,q}$ to $T(E_{7,3}) \times SU(1,1)$ contains*

$$(3k - 2p + 2q + 6r)_{7,3} \otimes \pi_k$$

if and only if either

$$0 \le r \le p + 4 - k$$

or

$$r \le -k - q - 4$$

5. Description of θ_k. We write the decomposition of the restriction of π_{min} to $E_{7,3} \times SU(1,1)$ as

$$\pi_{min}|_{E_{7,3} \times SU(1,1)} = \Big(\bigoplus_{|k| \ge 2} \theta_k \otimes \pi_k\Big) \bigoplus (\text{continuous spectrum}) \tag{5.1}$$

Here for $k \ge 2$, π_{-k} is the anti-holomorphic discrete series representation of $SU(1,1)$ of highest weight $-k$. Since π_{min} is self-contragredient, we see that θ_{-k} is the contragredient of θ_k. Therefore it suffices to describe θ_k for $k \ge 2$.

We have shown

LEMMA 5.1. *The $\mathfrak{k}$-type structure of θ_k is given by*

$$\theta_k|_{E_6 \times T(E_{7,3})} = \bigoplus \pi(p\omega_6 + q\omega_1) \otimes (\mu)_{7,3} \tag{5.2}$$

with

$$\mu = 3k - 2p + 2q + 6r \tag{5.3}$$

and either

$$0 \le r \le p + 4 - k \tag{5.4}$$

or

$$r \le -k - q - 4 \tag{5.5}$$

REMARK. With respect to the action of $\mathfrak{k}$ (and not just $\mathfrak{e}_{6,0}$) the highest weights of $\mathfrak{p}^{\pm}$ are $e_8 - e_6$ and $e_5 - e_7$ respectively. If $U(\lambda)$ denotes a representation of $\mathfrak{k}$ with highest weight λ then the typical $\mathfrak{k}$-type described in the above lemma can also be written as

$$U(p(e_5 - e_7) + q(e_8 - e_6)) \otimes (3k + 6r)_{7,3}$$

It follows from the lemma that θ_k is admissible. By Rallis and Schiffman [RaS] (see also [Lia]). we know that it is also quasi-simple. Its infinitesimal character is given as follows. Choose any positive root system. Let ρ be the half-sum of positive roots. Suppose α is the simple root such that one obtains a Dynkin diagram of type E_6 by taking out α from the Dynkin diagram of type E_7. Let ω be the fundamental weight dual to α. Then the infinitesimal character of θ_k is

$$(k - 10)\omega + \rho \tag{5.6}$$

If (5.4) (resp. (5.5)) is satisfied we shall say that the corresponding K-type is of type (5.4) (resp. (5.5)).

PROPOSITION 5.2. (a) *The representation θ_k contains the irreducible highest weight module σ_k with highest weight*

$$(\frac{k}{2} + 4)(e_6 - 2e_7 - e_8)$$

Note that this corresponds to the one-dimensional K-type $V(0) \otimes (-3k - 24)_{7,3}$ with $p = q = 0, r = -k - 4$. If $k \geq 10$ then σ_k is an anti-holomorphic discrete series representations.

(b) *The representation σ_k contains all K-types of type* (5.5), *but none of type* (5.4).

Proof. Consider the K-type with $p = q = 0$ and $r = -k - 4$. Let v_0 be any non-zero vector belonging to this one dimensional space. Since $\mathfrak{e}_{6,0}$ annihilates v_0, we see that the map

$$X \mapsto X \cdot v_0 \qquad (X \in \mathfrak{p}^+)$$

is an $\mathfrak{e}_{6,0}$-homomorphism from $\mathfrak{p}^+$ into θ_k. Thus if $\mathfrak{p}^+ v_0$ is non-zero it must be of type $V(\omega_1) \otimes (-3k - 22)$. But clearly this is not any one of the K-types described in Proposition 4.4. Thus $\mathfrak{p}^+$ must annihilate v_0. This proves (a).

Since σ_k is a highest weight representation, any of its K-types must have highest weight of the form

$$(\frac{k}{2} + 4)(e_6 - 2e_7 - e_8) + \sum n_j \beta_j$$

where the β_j's are roots in $\mathfrak{p}^-$, and the n_j's are non-negative integers. Suppose this is a K-type $V(p\omega_6 + q\omega_1) \otimes (\mu)_{7,3}$ of type (5.4). Then

$$\mu = 3k - 2p + 2q + 6r = -3k - 24 - 2\sum n_j$$

so

$$p = 3k + q + 3r + 12 + \sum n_j \geq 3k + q + 12 + \sum n_j$$

On the other hand by taking inner product with α_6 we see

$$p \le \text{coefficient of } e_5 - e_7 \le \sum n_j$$

Thus we have a contradiction which shows σ_k can not contain any K-type of type (5.4).

Now write

$$\theta_k = \sigma_k \oplus \sigma_k'$$

To finish the proof of (b) we need to show that σ_k' does not contain any K-type satifying (5.5). Suppose it does. We choose such a K-type satisfying

$$W = V(p\omega_6 + q\omega_1) \otimes (\mu)_{7,3}$$

with μ maximal, where $\mu = 3k - 2p + 2q + 6r$ with $r \le -k - q - 4$. Consider the map

$$\mathfrak{p}^+ \otimes W \longrightarrow \theta_k, \qquad X \otimes v \mapsto X \cdot v$$

This is a K-homomorphism. Suppose that the image of this map is non-zero (i.e. $\mathfrak{p}^+$ does not annihilate W). Then $\mathfrak{p}^+ \otimes W$ must contain an irreducible constituent of the form described by Lemma 5.1. But the higest weights in $\mathfrak{p}^+ \otimes W$ are all of the form

$$(\text{highest weight of } W) + (\text{a weight in } \mathfrak{p}^+)$$

From this we conclude that the only irreducible representation contained in $\mathfrak{p}^+ \otimes W$ and of the kind described in Lemma 5.1 is

$$V(p\omega_6 + (q+1)\omega_1) \otimes (\mu + 2)_{7,3}.$$

Since μ was assumed to be maximal, this K-type must be of the form (5.4). But

$$\mu + 2 = 3k - 2p + 2(q+1) + 6r.$$

So we must have $r \ge 0$ which is a contradiction. The contradiction shows that $\mathfrak{p}^+$ annihilates W. But then W generates an irreducible highest weight module contained in θ_k. The infinitesimal character of a highest weight module is easily obtained from the highest weight. Since W is different from $V(0) \otimes (-3k - 24)$, one can easily check that the infinitesimal character can not be that given by (5.6). So we conclude that in fact W does not exist. This proves the proposition.

6. Description of σ_k'.

LEMMA 6.1. *The representation σ_k' has a unique lowest K-type*

$$V(p\omega_6) \otimes (3k - 2p)_{7,3} \tag{6.1}$$

where $p = \max(k - 4, 0)$.

Proof. Recall that $\mathfrak{h} \simeq \mathfrak{h}^* \simeq \mathbb{C}^8$ with standard inner product

$$< x, y >= x_1 y_1 + \cdots + x_8 y_8$$

The roots and weights for various subalgebras of $\mathfrak{e}_{8,4}$ are all embedded in $\mathfrak{h}^*$, and the restriction of $<,>$ can be used as the norm that comes into the definition of the lowest K-type [Vog, Chapter 5]. With this understanding we have

$$\omega = \omega_6 = (0, 0, 0, 0, 1, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{3})$$

$$\omega^* = \omega_1 = (0,0,0,0,0,-\frac{2}{3},-\frac{2}{3},\frac{2}{3})$$

and $(\mu)_{7,3}$ corresponds to the weight

$$(0,0,0,0,0,-\frac{\mu}{6},\frac{\mu}{3},\frac{\mu}{6})$$

Furthermore the half-sum of positive roots of $\mathfrak{e}_{6,0}$ is

$$\rho_c = (0,1,2,3,4,-4,-4,4)$$

Therefore if λ denote the highest weight of $V(p\omega_6 + q\omega_1) \otimes (\mu)_{7,3}$ a simple calculation shows

$$< \lambda + 2\rho_c, \lambda + 2\rho_c >= (p+8)^2 + \frac{1}{3}(p+2q+24)^2 + \frac{1}{6}\mu^2 + 56$$

This is a quadratic form and it is elementary to verify that, subject to the conditions (5.3)-(5.4), the minimum is reached precisely when

$$p = \max(k-4, 0), \quad q = 0, \quad r = 0$$

This is what the lemma says.

Note that when $k \geq 4$ the lowest K-type specified in the above lemma is

$$V((k-4)\omega_6) \otimes (k+8)_{7,3}$$

with highest weight

(6.2) $$(0,0,0,0,k-4,-\frac{k}{2},4,\frac{k}{2})$$

Let

(6.3) $$x_0 = (0,0,0,0,2,-1,0,1) \in \mathfrak{h} \cap \mathfrak{e}_{7,3}$$

Let $\mathfrak{q} = \mathfrak{q}(x_0)$ be the parabolic subalgebra of $\mathfrak{e}_{7,3}$ defined as the sum of eigenspaces for $ad(x_0)$ with non-negative eigenvalues. Set

(6.4) $$\gamma = (0,0,0,0,1,-\frac{1}{2},0,\frac{1}{2}) \in \mathfrak{h}^*$$

For $\lambda = (k-10)\gamma$ define the representations $\mathcal{R}^j_{\mathfrak{q}}(\lambda) = \mathcal{R}^j_{\mathfrak{q}}((k-10)\gamma)$ as in [Vog]. It turns out that the parameter $(k-10)\gamma$ is always in the fair range. So $\mathcal{R}^j_{\mathfrak{q}}((k-10)\gamma) = 0$ unless $j = S = \dim(\mathfrak{u} \cap \mathfrak{k})$, where $\mathfrak{u}$ is the nilpotent radical of $\mathfrak{q}$. We will see that $S = 16$ here. If $k \geq 10$ then $\mathcal{R}^S_{\mathfrak{q}}((k-10)\gamma) = A_{\mathfrak{q}}((k-10)\gamma)$ is a unitary representation with non-zero cohomology [VoZ].

LEMMA 6.2. *For $k \geq 4$ the representations σ'_k and $\mathcal{R}^S_{\mathfrak{q}}((k-10)\gamma)$ have exactly the same infinitesimal character and K-type structure.*

Proof. Let $\mathfrak{l}$ be the centralizer of x_0. Let $\mathfrak{u}$ be the sum of eigenspaces for $ad(x_0)$ with positive eigenvalues. We have the Levi decomposition $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$. It is easy to see that $[l,l]$ is of type E_6, and in fact it comes from a real form of Cartan type EIII. This is the real form that gives rise to the exceptional hermitian domain of type E_6. Thus we may write $[l,l] = \mathfrak{e}_{6(-14)}$.

We construct a set of simple roots for $\mathfrak{e}_{7,3}$ such that all roots in $\mathfrak{u}$ are positive for the system determined by the simple roots. Take the first 5 simple roots $\alpha_1, \cdots, \alpha_5$ of $\mathfrak{e}_{8,4}$ listed in §2. Then we add $\alpha_6' = e_7 - e_4$. These form a system of simple roots for $\mathfrak{e}_{6(-14)}$. Next we add $\alpha_7' = e_5 - e_7$. These together form a system of simple roots for $\mathfrak{e}_{7,3}$. With respect to this system, we see that γ is nothing but the 7-th fundamental weight. Let ρ be the half sum of positive roots with respect to the positive system we just defined. The representation $R_{\mathfrak{q}}^S((k-10)\gamma)$ has infinitesimal character $(k-10)\gamma+\rho$, which agrees with the infinitesimal charcater of θ_k (and hence of σ_k') given by (5.6).

It is easy to check that

$$\delta_k \equiv (k-10)\gamma + 2\rho(\mathfrak{u}\cap\mathfrak{p}) = (0,0,0,0,k-4,-\frac{k}{2},4,\frac{k}{2})$$

which is the highest weight of the lowest K-type specified in Lemma 6.1.

Let τ be an irreducible representation of K which acts on the space Z. The weight δ_k defines a one dimensional representation of $L\cap K$. Let

$$S(\mathfrak{u}\cap\mathfrak{p}) = \sum_{m=0}^{\infty} S^m(\mathfrak{u}\cap\mathfrak{p})$$

be the symmetric algebra of $\mathfrak{u}\cap\mathfrak{p}$. The proof of the generalized Blattner formula in [Vog] shows that the multiplicity of τ in $R_{\mathfrak{q}}^S((k-4)\omega)$ is equal to the dimension of the space

$$\mathrm{Hom}_{L\cap K}(H^0(\mathfrak{u}\cap\mathfrak{k}, Z), S(\mathfrak{u}\cap\mathfrak{p})\otimes\delta_k)$$

(Combine (6.3.15) and (6.3.20) of [Vog]). Since $H^0(\mathfrak{u}\cap\mathfrak{k}, Z)$ is just the space of $\mathfrak{u}\cap\mathfrak{k}$ invariants in Z, the highest weight vectors of the $L\cap K$ module $H^0(\mathfrak{u}\cap\mathfrak{k}, Z)$ are precisely the highest weight vectors of the K module Z. Thus it suffice to decompose $S^m(\mathfrak{u}\cap\mathfrak{p})$, $m = 0,1,2,\cdots$ as $L\cap K$ modules.

It is easy to check that the semi-simple part of $\mathfrak{l}\cap\mathfrak{k}$ is $\mathfrak{so}(10)$ generated by the first 5 simple root $\alpha_1,\cdots,\alpha_5$ list in §2. Let $\mathfrak{t}$ be the center of $\mathfrak{l}\cap\mathfrak{k}$. It acts on $\mathfrak{u}\cap\mathfrak{p}^+$ by $(e_8-e_6)|_{\mathfrak{t}}$ and on the one dimensional space $\mathfrak{u}\cap\mathfrak{p}^-$ by $(e_5-e_7)|_{\mathfrak{t}}$. Since e_5-e_7 is orthogonal to all roots in $\mathfrak{so}(10)$ it defines a one-dimensional representation of all of $\mathfrak{l}\cap\mathfrak{k}$. With respect to the action of $\mathfrak{so}(10)$, $\mathfrak{u}\cap\mathfrak{p}^+$ is isomorphic to the standard module $\mathbb{C}^{10}$. Thus as a module for $\mathfrak{l}\cap\mathfrak{k} = \mathfrak{so}(10)+\mathfrak{t}$ we have

$$\mathfrak{u}\cap\mathfrak{p} = (\mathbb{C}^{10}\otimes(e_8-e_6)|_{\mathfrak{t}})\bigoplus(e_5-e_7)|_{\mathfrak{t}}$$

Therefore

$$S^m(\mathfrak{u}\cap\mathfrak{p}) = \sum_{l=0}^{m} S^l(\mathbb{C}^{10})\otimes[l(e_8-e_6)+(m-l)(e_5-e_7)]|_{\mathfrak{t}}$$

The restriction of e_8-e_6 to $\mathfrak{so}(10)\cap\mathfrak{h}$ defines the first fundamental weight for $\mathfrak{so}(10)$ which we denote by ω_1'. Now

$$S^l(\mathbb{C}^{10}) = \mu_l\oplus\mu_{l-2}\cdots$$

where μ_j is the irreducible representation of $\mathfrak{so}(10)$ with highest weight $j\omega_1'$. We obtain

$$(6.5)\qquad S(\mathfrak{u}\cap\mathfrak{p})\otimes\delta_k = \sum(S^j(\mathbb{C}^{10})\otimes[l(e_8-e_6)+(m-l)(e_5-e_7)]|_{\mathfrak{t}})\otimes\delta_k$$

where the sum is over all integers m, l, j with

$$0 \leq j \leq l \leq m, \qquad j \equiv l \pmod 2$$

Set

$$r = \frac{l-j}{2}, \quad q = j, \quad p = r + (m-l) + k - 4$$

Then $0 \leq r \leq p + 4 - k$ and one checks easily that the highest weight of the typical summand in (6.5) is precisely the highest weight of the representation $V(p\omega_6 + q\omega_1) \otimes (\mu)_{7,3}$ with $\mu = 3k - 2p + 2q + 6r$, that appears in (5.2) and satisfies (5.4). This proves the lemma.

With respect to the positive root system for $\mathfrak{e}_{7,3}$ introduced during the proof of the above lemma, we have

$$\rho = (0, 1, 2, 3, 5, -\frac{17}{2}, 4, \frac{17}{2})$$

The restriction of ρ to the center of $\mathfrak{l}$ is equal to 9γ. It is then easy to see that $(k-10)\gamma$ is always in the fair range. For $k \geq 9$ it is in the weakly good range, so $\mathcal{R}^S_{\mathfrak{q}}((k-10)\gamma)$ is irreducible. Finally if $k \geq 10$ then $\mathcal{R}^S_{\mathfrak{q}}((k-10)\gamma) = A_{\mathfrak{q}}((k-10)\gamma)$ is a unitary representation with non-zero cohomology. In this case the infinitesimal character and full K-type structure is more than enough to determine the isomorphism class of the representation in question [VoZ, Proposition 6.1.]. So Lemma 6.2 implies

THEOREM 6.3. *For $k \geq 10$ we have $\sigma'_k \simeq A_{\mathfrak{q}}((k-10)\gamma)$.*

Note that

$$\dim(\mathfrak{u} \cap \mathfrak{p}^+) = 10, \quad \dim(\mathfrak{u} \cap \mathfrak{p}^-) = 1 \tag{6.6}$$

So the representation $A_{\mathfrak{q}}((k-10)\gamma)$ has non-zero cohomology in bi-degree $(10, 1)$. In this regard we observe that up to conjugation by K, there are exactly two θ-stable parabolics satisfying (6.6). If x'_0 is the parameter (cf. (6.3)) that defines the parabolic $\mathfrak{q}'$ different from $\mathfrak{q}$ and satisfies (6.6) then x_0 and x'_0 are conjugate to each other via an outer automophism of $E_{6,0}$. Also $\mathfrak{q}, \mathfrak{q}'$ and their complex conjugates together constitute all parabolics satisfying $\mathfrak{u} \cap \mathfrak{p} = 11$. Thus there are exactly 4 families of unitary representations with non-zero cohomology at the minimal degree $r_G = 11$ (cf. [VoZ, p. 38]), and that they have cohomology only at bi-degrees $(10, 1)$ and $(1, 10)$.

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REFERENCES

[Gan] WEE TECK GAN, *An automorphic theta module for quarternionic exceptional groups*, Preprint.

[GrW] B. GROSS AND N. WALLACH, *A distinguished family of unitary representations for the exceptional groups of real rank = 4*, in Lie Theory and Geometry (in honor of B. Kostant), Birkhäuser, 1994, pp. 289–304.

[HeS] H. HECHT AND W. SCHMID, *A proof of Blattner's conjecture,* Inventiones Math., 31 (1975), pp. 129–154.

[Hel] S. HELGASON, *Differential Geometry, Lie Groups, and Symmetric Spaces,* Academic Press, 1978.

[HPS] J. Huang, P. Pandzic, and G. Savin, *New dual pair correspondences,* Duke Math. Journal, 82 (1996), pp. 447–471.

[Lia] J-S. Li, *T The correspondence of infinitesimal characters for reductive dual pairs in simple Lie groups*, Duke Mathematical Journal, 97:2 (1999), pp. 347–377.

[Lib] J-S. Li, *Theta liftings for unitary representations with non-zero cohomology,* Duke Math. Journal, 61 (1990), pp. 913–937.

[Pau] A. Paul, *Howe correspondence for real unitary groups,* thesis, University of Maryland, 1996.

[Ras] S. Rallis and G. Schiffmann, *The orbit and theta correspondences for some dual pairs,* J. Math. Kyoto Univ., 35 (1995), pp. 423–493.

[Rep] J. Repka, *Tensor Products of Holomorphic Discrete Series,* Can. J. Math., XXXI:4 (1979), pp. 836–844.

[Rub] H. Rubenthaler, *Les Paires Duales Dans Les Algèbres De Lie Réductives,* Astérisque, 219, 1994.

[Vog] D. Vogan, *Representations of Real Reductive Lie Groups,* Birkhauser, Boston, 1981.

[VoZ] D. Vogan and G. Zuckerman, *Unitary representations with non-zero cohomology,* Compositio Math., 53 (1984), pp. 51–90.