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ON PRINCIPAL G_2 BUNDLES*

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1. Introduction. Let G_2 be the exceptional group defined as the group of algebra automorphisms of the eight dimensional Cayley algebra C over an algebraically closed field k of characteristic p. Let C_0 denote the seven dimensional space of elements of trace 0 in C. Then G_2 acts irreducibly on C_0 if $p \neq 2, 3$. (See [3].) Let $E \to X$ be a principal G_2 bundle over a smooth projective variety X and $V \to X$ the rank seven vector bundle associated to E by the representation C_0 of G_2. In this article, we show that $E \to X$ is a semistable principal G_2 bundle if and only if $V \to X$ is a semistable vector bundle. In particular, this shows that the family of semistable G_2 bundles on X is a bounded family when char. $k > 17$. When G is a classical group and V the standard representation of G, a principal G bundle $E \to X$ is semistable iff the associated vector bundle $V \to X$ is semistable. That this is known to experts was pointed out by Usha Bhosle. (See [8], Proposition 4.2).

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2. Some representation theory and geometry. In this section, we denote by V the seven dimensional irreducible representation C_0 of the group G_2. There is a natural quadratic form on V induced by the quadratic form $n(a) = a\overline{a}, a \in C$ on the Cayley algebra C, where $\overline{a}$ is the usual involution on the Cayley algebra. Let $SO(V)$ denote the orthogonal group of this quadratic form on V. Let $\widetilde{P}_1, \widetilde{P}_2, \widetilde{P}_3$ denote the maximal parabolic subgroups of $SO(V)$ defined as the stablisers of rank one, rank two and rank three isotropic subspaces of V, respectively. Let P_1 and P_2 denote the two maximal parabolic subgroups of G_2, where P_1 is the intersection of $\widetilde{P}_1$ with G_2 for the natural inclusion of G_2 in $SO(V)$. Then we have

LEMMA *2.1.* $G_2/P_1 = SO(V)/\widetilde{P}_1 \subset P(V)$ where $P(V)$ is the projective space of lines in V.

Proof. By dimension count, dim $G_2/P_1 = 5$ and dim $SO(V)/\widetilde{P}_1 = 5$ and obviously $G_2/P_1 \subset SO(V)/\widetilde{P}_1$. □

Let $\Lambda^2 V$ denote the second exterior power of V. In characteristic zero, $\Lambda^2 V$ decomposes as a G_2 module as

$$\Lambda^2 V = V \oplus \Gamma_{01}$$

where Γ_{01} is the adjoint representation of G_2 (see [1], § 22.3). The same decomposition into irreducible G_2 module is also valid in characteristic p provided $2p > dim \Lambda^2 V = 21$, i.e., $p \geq 11$, by [5], Corollary 1.1.1 and Lemma 4.10.1.

We have $P(\Gamma_{01}) \subset P(\Lambda^2 V)$. Also, $G_2/P_2 \subset P(\Gamma_{01})$ and $SO(V)/\widetilde{P}_2 \subset P(\Lambda^2 V)$ (in all characteristics). We have

LEMMA *2.2.* Under the inclusion $P(\Gamma_{01}) \subset P(\Lambda^2 V), SO(V)/\widetilde{P}_2 \cap P(\Gamma_{01}) = G_2/P_2$ if $p \geq 11$ or char. 0.

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Proof. By a dimension count, we see that dim $SO(7)/\widetilde{P}_2 = 7$. Let $Y = P(\Gamma_{01}) \cap SO(7)/\widetilde{P}_2$. Since the embedding $SO(7)/\widetilde{P}_2 \subset P(\Lambda^2 V)$ is nondegenerate, $SO(7)/\widetilde{P}_2$ is not contained in $P(\Gamma_{01})$ and hence Y is a proper closed subvariety of $SO(7)/\widetilde{P}_2$. Hence dim $Y \leq 6$. Suppose dim $Y = 6$. Then Y is a divisor in $SO(7)/\widetilde{P}_2$ and if H is a hyperplane in $P(\Lambda^2 V)$ with $P(\Gamma_{01}) \subset H$, then $Y = SO(7)/\widetilde{P}_2 \cap H$. But a hyperplane section of $SO(7)/\widetilde{P}_2$ in $P(\Lambda^2 V)$ is a divisor linearly equivalent to the positive line bundle on $SO(7)/\widetilde{P}_2$ defined by the fundamental weight λ_2 corresponding to $\widetilde{P}_2$. Thus Y defines a section $\sigma \in H^0(SO(7)/\widetilde{P}_2, \lambda_2) = \Lambda^2 V$. On the other hand, G_2 acts on Y, so σ is a section left invariant under G_2 action on $H^0(SO(7)/\widetilde{P}_2, \lambda_2) = \Lambda^2 V$, hence we obtain a one dimensional subspace (viz. generated by σ) of $\Lambda^2 V$ invariant under G_2. However, according to the decomposition $\Lambda^2 V = V \oplus \Gamma_{01}$ into irreducible G_2 modules (for $p > 11$, or char. 0) there is no trivial G_2 submodule of $\Lambda^2 V$, so dim Y cannot be 6, and so dim $Y \leq 5$.

Also, Y being a complete variety on which G_2 acts algebraically, Y contains a closed G_2 orbit. The only G_2 homogeneous spaces are $G_2/P_1, G_2/P_2$, both of dimension 5, and G_2/B (where $B = P_1 \cap P_2$ is the Borel subgroup) of dimension 6, so dim $Y \geq 5$. From these considerations, we obtain dim $Y = 5$, and since G_2/P_2 is the unique closed G_2 homogeneous subvariety of $P(\Gamma_{01})$ of dimension 5 (if char $p > 11$ by [5]), we obtain that $Y = G_2/P_2$. □

We now consider $\Lambda^3 V$. We have the decomposition in char 0

$$\Lambda^3 V = V \oplus \Gamma_{20} \oplus 1$$

into irreducible G_2 modules, where Γ_{20} is the module with highest weight $2\gamma_1$, with γ_1 the fundamental weight corresponding to the parabolic P_1, and 1 denotes the trivial one dimensional module. The same decomposition also holds in characteristic p, if $2p >$ dim $\Lambda^3 V = 35$, i.e. $p > 17$, by [5], Corollary 1.1.1 and Lemma 4.10.1.

We have $SO(V)/\widetilde{P}_3 \subset P(\Lambda^3 V)$ and we have

LEMMA *2.3.* $SO(V)/\widetilde{P}_3 \cap P(\Gamma_{20}) = G_2/P_1$ if char $= p > 17$ or char. 0.

Proof. By a dimension count, we see that dim $SO(V)/\widetilde{P}_3 = 6$. Also the embedding $SO(V)/\widetilde{P}_3 \subset P(\Lambda^3 V)$ is nondegenerate, and so $SO(V)/\widetilde{P}_3$ is not contained in $P(\Gamma_{20})$. Therefore, dim $SO(V)/\widetilde{P}_3 \cap P(\Gamma_{20}) \leq 5$, and by an argument similar to the one in Lemma 2.2, we conclude that dim $(SO(V)/\widetilde{P}_3) \cap P(\Gamma_{20}) = 5$. Since G_2/P_1 is the unique closed G_2-homogenous subvariety of $P(\Gamma_{20})$ of dimension 5 (if $p > 17$ or char. 0 again by [5]), we obtain the conclusion of the lemma. □

3. The vector bundle. Let $E \to X$ be a principal bundle with structure group G_2, where X is either a complete nonsingular curve, or a nonsingular projective variety with a given polarisation, over an algebraically closed field of characteristic p. Let C_0 be the seven dimensional space of Cayley numbers of trace zero over the given field. Then if $p \neq 2, G_2$ acts irreducibly on C_0, and let $V \to X$ be the associated vector bundle of rank seven. We have

THEOREM *3.1.* If $E \to X$ is a semistable G_2 bundle, then $V \to X$ is a semistable vector bundle if the characteristic $p > 17$ (or char. =0).

Proof. Suppose $V \to X$ is not semistable (For the notion of semistability of principal bundles and vector bundles, see [4], [2] and [6]). Let $B \subset V$ be the β-subbundle of V (see [4]). B is a semistable bundle of positive degree. Since $G_2 \subset SO(C_0) = SO(7)$ for the natural quadratic form on C_0 (see [3], Proposition 2, page

11), the vector bundle V carries a nondegenerate quadratic form and hence we obtain an isomorphism $V \simeq V^*$. The composite map $B \to V \to V^* \to B^*$ is zero since $\mu(B) > \mu(B^*)$, where $\mu(B)$ is the slope $\frac{\deg(B)}{\mathrm{rank}\,(B)}$ and hence $B \subset Ker(V \to B^*) = B^{\perp}$, where $B^{\perp}$ denotes the perpendicular of B with respect to the quadratic form on V. This shows that B is isotropic, and hence rank $B = 1, 2$ or 3.

Case 1. Rank $B = 1$.

In this case, the line bundle $B \subset V$ defines a section $X \to P(V)$ where $P(V)$ denotes the projective bundle of lines in V and since B is isotropic this section actually factors as $X \to E(SO(7)/\widetilde{P}_1) \subset P(V)$ where $E(SO(7)/\widetilde{P}_1)$ is the bundle associated to E for the action of G_2 on $SO(7)/\widetilde{P}_1$. But $SO(7)/\widetilde{P}_1 = G_2/P_1$ (by Lemma 2.1), so the line bundle $B \subset V$ actually defines a section $\sigma : X \to E(G_2/P_1) \subset P(V)$. Let T denote the tangent bundle along the fibres of $E(G_2/P_1) \to X$. Then $detT = \mathcal{O}_{P(V)}(1)^{\otimes m}$ for a positive m (which can be explicitly computed but not necessary for us). It follows that

$$deg\sigma^*T = mdeg\sigma^*\mathcal{O}_{P(V)}(1) = mdegB^* < 0.$$

This section $\sigma : X \to E(G_2/P_1)$ defines a reduction of E to the parabolic P_1 contradicting the semistability of E. Hence rank B cannot be 1.

Case 2: Rank $B = 2$. In this case, det $B \subset \Lambda^2 V$ and this defines a section $\sigma : X \to P(\Lambda^2 V)$. Consider the decomposition

$$\Lambda^2 V = V \oplus E(\Gamma_{01})$$

where $E(\Gamma_{01})$ is associated to the adjoint representation of G_2. Then $\mu(detB) = 2\mu(B) > \mu(B) = \mu_{\max}(V)$, so the map $detB \to V$ is zero. Hence $detB \subset E(\Gamma_{01})$. Also, B is isotropic, so the section $\sigma : X \to P(\Lambda^2 V)$ defined by B actually lies in $E(SO(7)/\widetilde{P}_2) \cap P(E(\Gamma_{01}))$. Now applying Lemma 2.2, we obtain

$$E(SO(7)/\widetilde{P}_2) \cap P(E(\Gamma_{01})) = E(G_2/P_2)$$

and the section σ factors

$$\sigma : X \to E(G_2/P_2) \subset P(\Lambda^2 V).$$

Again letting T denote the tangent bundle along the fibres of $E(G_2/P_2) \to X$, we obtain $detT = \mathcal{O}_{P(\Lambda^2 V)}(1)^{\otimes m}$ for a positive constant m. Hence

$$\begin{aligned} deg\sigma^*T &= mdeg\sigma^*\mathcal{O}_{P(\Lambda^2 V)}(1) \\ &= mdegB^* < 0 \end{aligned}$$

Once more, $\sigma : X \to E(G_2/P_2)$ defines a reduction of structure group of E to P_2 contradicting the semistability of E.

Case 3. Rank $B = 3$.

In this case,

$$\Lambda^3 B = detB \subset \Lambda^3 V = V \oplus E(\Gamma_{20}) \oplus \mathcal{O}$$

and the map det $B \to V$ is zero because $\mu(detB) = 3\mu(B) > \mu(B) = \mu_{\max}(V)$. So is the map $detB \to \mathcal{O}$. Hence $detB \subset E(\Gamma_{20})$. Since B is isotropic, we deduce as before, applying Lemma 2.3, that $detB \subset \Lambda^3 V$ defines a section $\sigma : X \to P(\Lambda^3 V)$ which factors as

$$\sigma : X \to E(G_2/P_1) \subset P(E(\Gamma_{20})) \subset P(\Lambda^3 V).$$

If we again denote by T the tangent bundle algong the fibres of $E(G_2/P_1) \to X$, then $detT = \mathcal{O}_{P(E(\Gamma_{20}))}(1)^{\otimes m}$ for a positive constant m and

$$\begin{aligned} deg\sigma^* T &= mdeg\sigma^* \mathcal{O}_{P(E(\Gamma_{20}))}(1) = mdeg\sigma^* \mathcal{O}_{P(\Lambda^3 V)}(1) \\ &= mdegB^* < 0 \end{aligned}$$

and again $\sigma : X \to E(G_2/P_1)$ defines a reduction of structure group of E to P_1 contradicting semistability of E. □

REMARK *3.2.* Suppose V is a semistable vector bundle in the notation of the above theorem. Let $E_1 \subset E$ be a reduction of structure group of E to the maximal parabolic subgroup P_1 of G_2. Then this reduction defines an isotropic line sub-bundle $L \subset V$, and conversely, an isotropic line subbundle $L \subset V$ defines a reduction of structure group of E to P_1 (by Lemma 2.1). If T denots the tangent bundle along the fibres of $E(G_2/P_1) \to X$ and $\sigma : X \to E(G_2/P_1)$ defines the reduction to P_1, then $\det T = \mathcal{O}_{P(V)}(1)^{\otimes m}$ for a positive integer m, and

$$\deg \sigma^* T = m \deg \sigma^* \mathcal{O}_{P(V)}(1) = -m \deg L.$$

Since V is semistable, $\deg L \leq 0$, and hence $\deg \sigma^* T \geq 0$, verifying the semistability criterion for σ.

Now let $E_2 \subset E$ be a reduction of structure group to P_2, defined by a section $\sigma : X \to E(G_2/P_2)$. Since $E(G_2/P_2) \subset E(SO(7)/\tilde{P}_2) \subset P(\wedge^2 V)$, the section σ defines a rank two istropic subbundle $S \subset V$. Letting T denote the tangent bundle along the fibres of $E(G_2/P_2) \to X$, we obtain $\sigma^* \det T = m\ \sigma^* \mathcal{O}_{P(\wedge^2 V)}(1)$ for a positive integer m, and we get

$$\begin{aligned} \deg \sigma^* T &= m \deg \sigma^* \mathcal{O}_{P(\wedge^2 V)}(1) \\ &= -m \deg \det S \end{aligned}$$

Since V is semistable, $\deg S \leq 0$, and so $\deg \sigma^* T \geq 0$.

Thus V is semistable implies that E is a semistable G_2 bundle.

COROLLARY *(3.3).* The family of semistable G_2 bundles on X is bounded in characteristic $p > 17$ if $\dim X \leq 2$.

Proof. If $E \to X \times S$ is a family of semistable G_2 bundles, an C_o is the seven dimensional representation of G_2 considered above, then the associated vector bundle $E(C_o) \to X \times S$ is a family of semistable vector bundles on X parametrised by S. The corollary follows from the Main Theorem in [9]. □

REMARK *3.4.* The proof that a bundle associated to a semistable bundle is also semistable in char. 0 given in [7] uses Kempf's theorem on the rationality of the instability flag.

REMARK *3.5.* In Theorem (3.1) above, I believe that the condition char $k = p > 17$ can be improved.

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