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APERIODICITY IN QUANTUM AFFINE $\mathfrak{gl}_n$ †

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0. Introduction. In [BLM], the quantized enveloping algebra corresponding to $\mathfrak{gl}_n$ was studied using the geometry of pairs of n-step filtrations in a vector space. (For earlier related work see [J], [DJ].) Soon after [BLM] was written, I, and independently [GV], noticed that some aspects of [BLM] can be extended to the case of quantum affine $\mathfrak{gl}_n$ using the geometry of pairs of infinite periodic chains of lattices in a vector space over a power series field.

The purpose of this paper is to point out a major difference between the finite and affine case. Namely, in the finite case, the geometric approach gives a sequence of larger and larger algebras which are quotients of the quantum $\mathfrak{gl}_n$ with parameter q, which are realized as the spaces of invariant functions on a set of pairs of filtrations in a vector space over a finite field with q elements. The analogous geometrically defined algebras in the affine case are still receiving homomorphisms from the quantum affine $\mathfrak{gl}_n$ with parameter q, but this time the homomorphisms are not surjective, contrary to what is asserted in [GV, Sec.9]. This non-surjectivity statement is established in two ways, an elementary one (see 3.8) and a less elementary one, based on the theory of characteristic varieties (Sec. 6).

Most of this paper is concerned with the problem of describing the images of these homomorphisms. It turns out that these images are spanned by "intersection cohomology elements" indexed by certain matrices which are aperiodic in a suitable sense. (See Theorem 8.2.) These elements form a basis at infinity, or a crystal basis (up to signs) for these images. (See Theorem 8.4.)

1. The $\mathbf{Q}(v)$-algebra $\mathfrak{A}_{D,n,n}$.

1.1. Let $n \geq 1$ and $D \geq 0$ be integers. Let $\mathfrak{S}_{D,n}$ be the (finite) set of all $\mathbf{a} = (a_i)_{i\in\mathbf{Z}}$ with $a_i \in \mathbf{N}$ such that

(a) $a_i = a_{i-n}$ for all $i \in \mathbf{Z}$;

(b) for some (or any) $i_0 \in \mathbf{Z}$ we have $a_{i_0} + a_{i_0-1} + \cdots + a_{i_0-n+1} = D$.

1.2. Now let n' be another integer ≥ 1. Let $\mathfrak{S}_{D,n,n'}$ be the set of all matrices $A = (a_{i,j})_{(i,j)\in\mathbf{Z}\times\mathbf{Z}}$ with entries $a_{i,j} \in \mathbf{N}$ such that

(a) $a_{i,j} = a_{i-n,j-n'}$ for all $i, j \in \mathbf{Z}$;

(b1) for any $i \in \mathbf{Z}$, the set $\{j \in \mathbf{Z} | a_{i,j} \neq 0\}$ is finite;

(b2) for any $j \in \mathbf{Z}$, the set $\{i \in \mathbf{Z} | a_{i,j} \neq 0\}$ is finite;

(c1) for some (or any) $i_0 \in \mathbf{Z}$ we have $a_{i_0,*} + a_{i_0-1,*} + \cdots + a_{i_0-n+1,*} = D$;

(c2) for some (or any) $j_0 \in \mathbf{Z}$ we have $a_{*,j_0} + a_{*,j_0-1} + \cdots + a_{*,j_0-n'+1} = D$.

Here,

$$a_{i,*} = \sum_{j\in\mathbf{Z}} a_{ij}, \quad a_{*,j} = \sum_{i\in\mathbf{Z}} a_{ij}.$$

Note that, in the presence of (a), conditions (b1) and (b2) are equivalent and conditions (c1),(c2) are equivalent. For $A \in \mathfrak{S}_{D,n,n'}$ we set

$$r(A) = (a_{i,*})_{i\in\mathbf{Z}} \in \mathfrak{S}_{D,n}, \quad c(A) = (a_{*,j})_{j\in\mathbf{Z}} \in \mathfrak{S}_{D,n'},$$

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(d) $$d_A = \sum a_{i,j} a_{k,l},$$

where the sum is taken over a set of representatives Ξ for the orbits of the permutation $(i,j,k,l) \mapsto (i-n, j-n', k-n, l-n')$ of the set

$$\{(i,j,k,l) \in \mathbf{Z}^4 | i \geq k, \quad j < l\}.$$

From (a) it is clear that the sum in (d) is independent of the choice of Ξ. But we must check that only finitely terms in the sum are non-zero. It is enough to do this for a particular choice of Ξ, for example

$$\Xi = \{(i,j,k,l) \in \mathbf{Z}^4 | i \geq k, \quad j < l, \quad i \in [1,n]\}.$$

Let Ξ' be the set of elements of Ξ such that the corresponding term in the sum (d) is non-zero. If $(i,j,k,l) \in \Xi'$, then $i \in [1,n]$ and, since $a_{i,j} \neq 0$, we see from (b1) that j is forced to lie in a fixed finite subset of $\mathbf{Z}$. To show that k,l are also forced to lie in fixed finite sets we note that

$$\{(k,l) \in \mathbf{Z}^2 | a_{k,l} \neq 0, k \leq i_0, l > j_0\}$$

is a finite set for any fixed $(i_0, j_0) \in \mathbf{Z}^2$. (This follows easily from (a),(b1),(b2).) This proves the desired finiteness statement.

1.3. We fix a field $\mathbf{k}$, a prime number l invertible in $\mathbf{k}$ and an algebraic closure $\bar{\mathbf{Q}}_l$ of $\mathbf{Q}_l$. In the case where $\mathbf{k}$ is of characteristic $p > 0$, we assume chosen a square root $\sqrt{p}$ of p in $\bar{\mathbf{Q}}_l$. If, in addition, $\mathbf{k}$ is finite with $q = p^s$ elements for some integer $s \geq 1$, we set $\sqrt{q} = (\sqrt{p})^s$ and $q^{m/2} = (\sqrt{q})^m$ for any $m \in \mathbf{Z}$.

We shall write dim instead of $\dim_{\mathbf{k}}$.

Let V be a free $\mathbf{k}[\epsilon, \epsilon^{-1}]$-module of rank $D \geq 0$; here ϵ is an indeterminate. Let G be the group of automorphisms of the $\mathbf{k}[\epsilon, \epsilon^{-1}]$-module V. Let $\mathfrak{g}$ be the set of all endomorphisms of the $\mathbf{k}[\epsilon, \epsilon^{-1}]$-module V. A *lattice* in V is, by definition, a $\mathbf{k}[\epsilon]$-submodule L of V such that there exists a $\mathbf{k}[\epsilon]$-basis of L which is also a $\mathbf{k}[\epsilon, \epsilon^{-1}]$-basis of V.

Let $\mathcal{F}^n$ be the set of all collections $\mathbf{L} = (L_i)_{i \in \mathbf{Z}}$ where each L_i is a lattice in V, such that $L_{i-1} \subset L_i$ and $L_{i-n} = \epsilon L_i$ for all $i \in \mathbf{Z}$. Then G acts on $\mathcal{F}^n$ by $g : \mathbf{L} \mapsto g(\mathbf{L}) = \tilde{\mathbf{L}}$ where $\tilde{L}_i = g(L_i)$ for all $i \in \mathbf{Z}$. For $\mathbf{L} \in \mathcal{F}^n$ we set define $|\mathbf{L}| \in \mathfrak{S}_{D,n}$ by

$|\mathbf{L}| = \mathbf{a}, a_i = \dim L_i/L_{i-1} \quad \forall i.$

We sometimes write $|\mathbf{L}|_i$ instead of a_i. For $\mathbf{a} = (a_i) \in \mathfrak{S}_{D,n}$, we set

$\mathcal{F}_{\mathbf{a}} = \{\mathbf{L} \in \mathcal{F}^n || \mathbf{L}| = \mathbf{a}\}.$

Then $\mathcal{F}_{\mathbf{a}}$ for $\mathbf{a} \in \mathfrak{S}_{D,n}$ are exactly the G-orbits on $\mathcal{F}^n$. Now G acts on $\mathcal{F}^n \times \mathcal{F}^{n'}$ by $g : (\mathbf{L}, \mathbf{L}') \mapsto (g(\mathbf{L}), g(\mathbf{L}'))$. For $\mathbf{L} \in \mathcal{F}^n$, let $G_{\mathbf{L}} = \{g \in G | g(\mathbf{L}) = \mathbf{L}\}$.

LEMMA 1.4. *Given $(\mathbf{L}, \mathbf{L}') \in \mathcal{F}^n \times \mathcal{F}^{n'}$, we can find $\mathbf{k}$-subspaces $M_{i,j}$ of V indexed by $(i,j) \in \mathbf{Z} \times \mathbf{Z}$ such that*

$V = \oplus_{i,j} M_{i,j},$

$\epsilon M_{i,j} = M_{i-n, j-n'}$ *for all* i, j,

$L_i = \oplus_{i',j; i' \leq i} M_{i',j}$ *for all* i,

$L'_j = \oplus_{i,j'; j' \leq j} M_{i,j'}$ *for all* j.

We omit the proof.

LEMMA 1.5. *Let us associate to $(\mathbf{L}, \mathbf{L}') \in \mathcal{F}^n \times \mathcal{F}^{n'}$ the matrix $A = (a_{ij})_{i,j \in \mathbf{Z}}$ given by*

$$a_{ij} = \dim \frac{L_i \cap L'_j}{L_{i-1} \cap L'_j + L_i \cap L'_{j-1}} \in \mathbf{N}.$$

(a) We have $A \in \mathfrak{S}_{D,n,n'}$.

(b) $(\mathbf{L}, \mathbf{L}') \mapsto A$ defines a bijection between the set of G-orbits in $\mathcal{F}^n \times \mathcal{F}^{n'}$ and the set $\mathfrak{S}_{D,n,n'}$.

We prove (a). If $M_{i,j}$ are as in 1.4, we have $a_{ij} = \dim M_{i,j}$ for all i,j. In particular, $a_{ij} = a_{i-n,j-n'}$ for all i,j.

Clearly, $L_i/L_{i-1} \cong \oplus_{j\in\mathbf{Z}} M_{i,j}$ hence $\dim(L_i/L_{i-1}) = \sum_{j\in\mathbf{Z}} a_{ij}$. (In particular, $\sum_{j\in\mathbf{Z}} a_{ij} < \infty$ so that A satisfies condition 1.2(b1).) Hence, if i_0 is as in 1.2(c1), the sum in 1.2(c1) is equal to

$$\dim(L_{i_0}/L_{i_0-n}) = \dim(L_{i_0}/\epsilon L_{i_0}) = D.$$

This proves (a). We prove (b). Clearly, $(\mathbf{L}, \mathbf{L}') \mapsto A$ induces a well defined map ϕ from the set of G-orbits in $\mathcal{F}^n \times \mathcal{F}^{n'}$ to $\mathfrak{S}_{D,n,n'}$. We show that ϕ is injective. Assume that $(\tilde{\mathbf{L}}, \tilde{\mathbf{L}}')$ is another pair in $\mathcal{F}^n \times \mathcal{F}^{n'}$ with associated matrix equal to A. Let $M_{i,j}$ be associated to $(\mathbf{L}, \mathbf{L}')$ as in Lemma 1.4 and let $\tilde{M}_{i,j}$ be associated in a similar way to $(\tilde{\mathbf{L}}, \tilde{\mathbf{L}}')$. From our assumption we have $\dim M_{i,j} = \dim \tilde{M}_{i,j}$ for all i,j. We can define isomorphisms of $\mathbf{k}$-vector spaces $g_{i,j} : M_{i,j} \xrightarrow{\sim} \tilde{M}_{i,j}$ for all i,j such that $g_{i-n,j-n'}(\mu) = \epsilon g_{i,j}(\epsilon^{-1}\mu)$ for all $\mu \in M_{i-n,j-n'}$. Then $g = \oplus_{i,j} g_{i,j} : V \to V$ belongs to G. Also, $g(\mathbf{L}) = \tilde{\mathbf{L}}, g(\mathbf{L}') = \tilde{\mathbf{L}}'$. This shows that ϕ is injective. We show that ϕ is surjective. Given $A \in \mathfrak{S}_{D,n,n'}$, we can find a direct sum decomposition $V = \oplus_{i,j} M_{i,j}$ as a $\mathbf{k}$-vector space where $M_{i,j}$ are $\mathbf{k}$-subspaces such that $\dim M_{i,j} = a_{ij}$ and $\epsilon M_{i,j} = M_{i-n,j-n'}$ for all i,j. We then define $\mathbf{L}, \mathbf{L}'$ in $\mathcal{F}^n, \mathcal{F}^{n'}$ by

$L_i = \oplus_{i',j;i'\le i} M_{i',j}$ for $i \in \mathbf{Z}$,

$L'_j = \oplus_{i,j';j'\le j} M_{i,j'}$ for $j \in \mathbf{Z}$.

It is clear that the matrix associated to $(\mathbf{L}, \mathbf{L}')$ is just A. The lemma is proved.

1.6. For any $A \in \mathfrak{S}_{D,n,n'}$, we denote by $\mathcal{O}_A$ the G-orbit on $\mathcal{F}^n \times \mathcal{F}^{n'}$ corresponding to A as in 1.5. Note that, if $(\mathbf{L}, \mathbf{L}') \in \mathcal{O}_A$, then $\mathbf{L} \in \mathcal{F}_{r(A)}, \mathbf{L}' \in \mathcal{F}_{c(A)}$ and

$$\text{(a)} \qquad \dim \frac{L_k}{L_k \cap L'_l} = \sum_{r\le k; s>l} a_{rs}, \quad \dim \frac{L'_l}{L_k \cap L'_l} = \sum_{r>k; s\le l} a_{rs}.$$

Let $\mathbf{a} \in \mathfrak{S}_{D,n}, \mathbf{b} \in \mathfrak{S}_{D,n'}, \mathbf{L} \in \mathcal{F}_{\mathbf{a}}$. The action of $G_{\mathbf{L}}$ on $\mathcal{F}_{\mathbf{b}}$ (restriction of the G-action) has orbits

$$X_A^{\mathbf{L}} = \{\mathbf{L}' \in \mathcal{F}_{\mathbf{b}} | (\mathbf{L}, \mathbf{L}') \in \mathcal{O}_A\}$$

indexed by

$$\text{(b)} \qquad \{A \in \mathfrak{S}_{D,n,n'} | r(A) = \mathbf{a}, c(A) = \mathbf{b}\}.$$

1.7. Assume now that $\mathbf{k}$ is finite with q elements. Let n'' be another integer ≥ 1. Let $A \in \mathfrak{S}_{D,n,n'}, A' \in \mathfrak{S}_{D,n',n''}, A'' \in \mathfrak{S}_{D,n,n''}$ be such that $r(A) = r(A'') = \mathbf{a}, c(A) = r(A') = \mathbf{b}, c(A') = c(A'') = \mathbf{c}$. We denote by $\nu_{A,A',A'';q}$ the number of elements in the (finite) set

$$\{\mathbf{L}' \in \mathcal{F}_{\mathbf{b}} | (\mathbf{L}, \mathbf{L}') \in \mathcal{O}_A, (\mathbf{L}', \mathbf{L}'') \in \mathcal{O}_{A'}\}$$

where $(\mathbf{L}, \mathbf{L}'')$ is fixed in $\mathcal{O}_{A''}$. Clearly, $\nu_{A,A',A'';q}$ is independent of the choice of $(\mathbf{L}, \mathbf{L}'') \in \mathcal{O}_{A''}$.

If $A \in \mathfrak{S}_{D,n,n'}$ then the matrix ${}^tA$, whose (i,j)-entry is $a_{j,i}$, belongs to $\mathfrak{S}_{D,n',n}$. Moreover, if $(\mathbf{L}, \mathbf{L}') \in \mathcal{O}_A$, then $(\mathbf{L}', \mathbf{L}) \in \mathcal{O}_{{}^tA}$. It follows easily that

$$\text{(a)} \qquad \nu_{A,A',A'';q} = \nu_{{}^tA',{}^tA,{}^tA'';q}$$

for any A, A', A'' as above.

1.8. Let $\mathcal{A} = \mathbf{Z}[v, v^{-1}]$ where v is an indeterminate.

For A, A', A'' as above, we can find $\nu_{A,A',A''} \in \mathcal{A}$ such that

$$\nu_{A,A',A'';q} = \nu_{A,A',A''}|_{v=q^{1/2}}$$

for any prime power q. (This is a well known property of affine Hecke algebras of type A). More precisely, $\nu_{A,A',A''}$ is necessarily a polynomial in v^2. From 1.7(a), we deduce

(a) $$\nu_{A,A',A''} = \nu_{{}^tA',{}^tA,{}^tA''}.$$

1.9. Let $\mathfrak{A}_{D,n,n'}$ be the $\mathbf{Q}(v)$-vector space with basis $\{e_A | A \in \mathfrak{S}_{D,n,n'}\}$. This vector space has a second basis

$\{[A] | A \in \mathfrak{S}_{D,n,n'}\}, \quad [A] = v^{-d_A} e_A.$

We define a $\mathbf{Q}(v)$-bilinear pairing $\mathfrak{A}_{D,n,n'} \times \mathfrak{A}_{D,n',n''} \to \mathfrak{A}_{D,n,n''}$ by

$e_A \cdot e_{A'} = \sum_{A''} \nu_{A,A',A''} e_{A''}$ if $c(A) = r(A')$,

$e_A \cdot e_{A'} = 0$ if $c(A) \neq r(A')$.

The sum is taken over all $A'' \in \mathfrak{S}_{D,n,n''}$ such that $r(A) = r(A''), c(A') = c(A'')$. (The sum has only finitely many non-zero terms.)

For any $A \in \mathfrak{A}_{D,n,n'}, A' \in \mathfrak{A}_{D,n',n''}, A'' \in \mathfrak{A}_{D,n'',n'''}$ we have from definitions

$(e_A \cdot e_{A'}) \cdot e_{A''} = e_A \cdot (e_{A'} \cdot e_{A''})$.

For $\mathbf{a} = (a_i) \in \mathfrak{S}_{D,n}$, the set of all pairs $(\mathbf{L}, \mathbf{L}')$ with $\mathbf{L} = \mathbf{L}' \in \mathcal{F}_\mathbf{a}$ is equal to $\mathcal{O}_A$ where $A \in \mathfrak{S}_{D,n,n}$ is defined by $a_{i,j} = \delta_{ij} a_i$. We denote this A by $\mathbf{i_a}$. Clearly,

$d_{\mathbf{i_a}} = 0$ and $e_{\mathbf{i_a}} = [\mathbf{i_a}]$.

From the definition we have

$e_{\mathbf{i_a}} \cdot e_{A'} = e_{A'}$ for any $A' \in \mathfrak{S}_{n,n'}$ such that $r(A') = \mathbf{a}$,

$e_{A''} \cdot e_{\mathbf{i_a}} = e_{A''}$ for any $A'' \in \mathfrak{S}_{n',n}$ such that $c(A'') = \mathbf{a}$.

In particular, the above multiplication defines on $\mathfrak{A}_{D,n,n}$ a structure of associative algebra (over $\mathbf{Q}(v)$). This algebra has a unit element, namely $\sum_\mathbf{a} e_{\mathbf{i_a}} = \sum_\mathbf{a} [\mathbf{i_a}]$, where $\mathbf{a}$ runs over the finite set $\mathfrak{S}_{D,n}$. Similarly, the above multiplication defines a left $\mathfrak{A}_{D,n,n}$-module structure on $\mathfrak{A}_{D,n,n'}$ and a right $\mathfrak{A}_{D,n',n'}$-module structure on $\mathfrak{A}_{D,n,n'}$. These module structures commute with each other.

1.10. Let $\mathfrak{A}_{D,n,n';\mathcal{A}}$ be the $\mathcal{A}$-submodule of $\mathfrak{A}_{D,n,n'}$ spanned by $\{e_A | A \in \mathfrak{S}_{D,n,n'}\}$ or equivalently by $\{[A] | A \in \mathfrak{S}_{D,n,n'}\}$. Clearly, $\mathfrak{A}_{D,n,n;\mathcal{A}}$ is an $\mathcal{A}$-subalgebra of $\mathfrak{A}_{D,n,n}$ and $\mathfrak{A}_{D,n,n';\mathcal{A}}$ is a $\mathfrak{A}_{D,n,n;\mathcal{A}}$-submodule of $\mathfrak{A}_{D,n,n'}$.

For any prime power q, let $\mathfrak{A}_{D,n,n';q} = \bar{\mathbf{Q}}_l \otimes_\mathcal{A} \mathfrak{A}_{D,n,n';\mathcal{A}}$, where $\bar{\mathbf{Q}}_l$ is regarded as $\mathcal{A}$-algebra via $v \mapsto q^{1/2}$. Then $\mathfrak{A}_{D,n,n;q}$ is a $\bar{\mathbf{Q}}_l$-algebra in a natural way and $\mathfrak{A}_{D,n,n';q}$ is a left $\mathfrak{A}_{D,n,n;q}$-module in a natural way.

The element $1 \otimes e_A$ (resp. $1 \otimes [A]$) of $\mathfrak{A}_{D,n,n';q}$ is denoted again by e_A (resp. $[A]$). Note that $\{e_A | A \in \mathfrak{S}_{D,n,n'}\}$ and $\{[A] | A \in \mathfrak{S}_{D,n,n'}\}$ are bases of the $\bar{\mathbf{Q}}_l$-vector space $\mathfrak{A}_{D,n,n';q}$.

We may identify $\mathfrak{A}_{D,n,n';q}$ with the vector space of functions $f : \mathcal{F}^n \times \mathcal{F}^{n'} \to \bar{\mathbf{Q}}_l$ (in the case $\sharp\mathbf{k} = q$) that are constant on the orbits of G and are 0 outside a finite set of G-orbits, in such a way that e_A corresponds to the function which is identically 1 on $\mathcal{O}_A$ and is zero on $\mathcal{O}_{A'}$ for any $A' \neq A$. Then the multiplication on $\mathfrak{A}_{D,n,n;q}$ can be interpreted as the convolution of functions $f_1, f_2 \mapsto f_1 \cdot f_2$:

$$(f_1 \cdot f_2)(\mathbf{L}, \mathbf{L}'') = \sum_{\mathbf{L}' \in \mathcal{F}^n} f_1(\mathbf{L}, \mathbf{L}') f_2(\mathbf{L}', \mathbf{L}'')$$

for $\mathbf{L}, \mathbf{L}'' \in \mathcal{F}^n$.

LEMMA 1.11. *The* $\mathbf{Q}(v)$*-linear involution* $\Psi : \mathfrak{A}_{D,n,n} \to \mathfrak{A}_{D,n,n}$ *given by* $[A] \mapsto [{}^tA]$ *for all* $A \in \mathfrak{S}_{D,n,n}$ *is an algebra anti-automorphism.*

We have
$[A] \cdot [A'] = \sum_{A''} v^{-d_A - d_{A'} + d_{A''}} \nu_{A,A',A''} [A'']$ if $c(A) = r(A')$,
$[A] \cdot [A'] = 0$ if $c(A) \neq r(A')$.
In the sum over A'' we may restrict ourselves to A'' such that $r(A'') = r(A), c(A'') = c(A')$. Using 1.8(a), we see that it is enough to prove that

(a) $$-d_A - d_{A'} + d_{A''} = -d_{{}^tA'} - d_{{}^tA} + d_{{}^tA''}$$

whenever

(b) $$c(A) = r(A'), r(A'') = r(A), c(A'') = c(A').$$

By definition,

$$d_A = \sum a_{i,j} a_{k,l}, \quad d_{{}^tA} = \sum a_{j,i} a_{l,k}$$

where the sum is taken over

(c) $$\{(i,j,k,l) \in \mathbf{Z}^4 | i \geq k, \quad j < l, \quad i \in [1,n]\}$$

or equivalently over

(d) $$\{(i,j,k,l) \in \mathbf{Z}^4 | i \geq k, \quad j < l, \quad k \in [1,n]\}.$$

It will be convenient to take the sum over (c) for d_A and over (d) for $d_{{}^tA}$. In (d) we make the change of variable $(i,j,k,l) \mapsto (l,k,j,i)$ and we see that $d_{{}^tA} = \sum a_{i,j} a_{k,l}$, sum over

$$\{(i,j,k,l) \in \mathbf{Z}^4 | i > k, \quad j \leq l, \quad j \in [1,n]\}.$$

Therefore

$$d_A - d_{{}^tA} = \sum{}' a_{i,j} a_{k,l} - \sum{}'' a_{i,j} a_{k,l},$$

where Σ' is sum over $\{\{(i,j,k,l) \in \mathbf{Z}^4 | i = k \in [1,n], \quad j < l,\}$ and Σ'' is sum over $\{\{(i,j,k,l) \in \mathbf{Z}^4 | i > k, \quad j = l \in [1,n]\}$. The first sum equals

$$\frac{1}{2} \sum_{i,j,l; i \in [1,n]} a_{i,j} a_{i,l} - \sum_{i,j; i \in [1,n]} a_{i,j}^2;$$

the second sum equals

$$\frac{1}{2} \sum_{i,j,k; j \in [1,n]} a_{i,j} a_{k,j} - \sum_{i,j; j \in [1,n]} a_{i,j}^2.$$

Note that $\sum_{i,j; i \in [1,n]} a_{i,j}^2 = \sum_{i,j; j \in [1,n]} a_{i,j}^2$. Hence

$$d_A - d_{{}^tA} = \frac{1}{2} \sum_{i \in [1,n]} a_{i,*}^2 - \frac{1}{2} \sum_{j \in [1,n]} a_{*,j}^2.$$

We now use this identity for A, A', A'' and we find

$$d_A - d_{{}^tA} + d_{A'} - d_{{}^tA'} - d_{A''} + d_{{}^tA''} = \frac{1}{2}\sum_{i\in[1,n]} a_{i,*}^2 - \frac{1}{2}\sum_{j\in[1,n]} a_{*,j}^2$$

$$+\frac{1}{2}\sum_{i\in[1,n]} {a'_{i,*}}^2 - \frac{1}{2}\sum_{j\in[1,n]} {a'_{*,j}}^2 - \frac{1}{2}\sum_{i\in[1,n]} {a''_{i,*}}^2 + \frac{1}{2}\sum_{j\in[1,n]} {a''_{*,j}}^2;$$

this is zero since, by (b), $a_{*,i} = a'_{i,*}, a''_{i,*} = a_{i,*}, a''_{*,i} = a'_{*,i}$ for all i. The lemma is proved.

LEMMA 1.12. *Let $\mathbf{L}, \mathbf{L}' \in \mathcal{F}^n$. Let $g_k : L_k/(L_k \cap L'_k) \to L_{k-1}/(L_{k-1} \cap L'_{k-1})$ be* $\mathbf{k}$*-linear maps defined for $k \in \mathbf{Z}$ such that $g_{k-n} = \epsilon g_k \epsilon^{-1}$ for all $k \in \mathbf{Z}$ and such that $j_{k-1}g_k = g_{k+1}j_k$ for all k, where $j_k : L_k/(L_k \cap L'_k) \to L_{k+1}/(L_{k+1} \cap L'_{k+1})$ is the canonical map. Then there exists $e \in \mathfrak{g}$ such that $e(L_k) \subset L_{k-1}$, $e(L'_k) \subset L'_{k-1}$ for all k and such that e induces (g_k).*

Let $M_{i,j}$ be attached to $\mathbf{L}, \mathbf{L}'$ as in 1.4. Giving (g_k) is the same as giving linear maps $g_{k;r,s;r',s'} : M_{r,s} \to M_{r',s'}$ defined for $r \le k < s$ and $r' \le k-1 < s'$ such that

$g_{k-n;r-n,s-n;r'-n,s'-n} = \epsilon g_{k;r,s;r',s'}\epsilon^{-1}$,
$g_{k+1;r,s;r',s'} = g_{k;r,s;r',s'}$, if $k, k+1 \in [r, s-1]$ and $k-1, k \in [r', s'-1]$,
$g_{k;r,s;r',s'} = 0$ if $k, k+1 \in [r, s-1]$ and $k-1 = r' < s'$,
$g_{k;r,s;r',s'} = 0$ if $k-1, k \in [r', s'-1]$ and $r < s = k+1$.

In particular, if $g_{k;r,s;r',s'}$ is defined and either $r' \ge r$ or $s' \ge s$, then $g_{k;r,s;r',s'} = 0$.

We define linear maps $e_{r,s;r',s'} : M_{r,s} \to M_{r',s'}$ for any r, s, r', s' as follows. If there exists k such that $r \le k < s$ and $r' \le k-1 < s'$, we set $e_{r,s;r',s'} = g_{k;r,s;r',s'}$. (This is independent of the choice of k.) Otherwise, we set $e_{r,s;r',s'} = 0$. Then

$e_{r-n,s-n;r'-n,s'-n} = \epsilon e_{r,s;r',s'}\epsilon^{-1}$,
$e_{r,s;r',s'} = 0$ unless $r' < r$ and $s' < s$.

We define $e \in \mathfrak{g}$ so that its $(r, s; r', s')$ component with the decomposition $V = \oplus_{i,j} M_{i,j}$ is $e_{r,s;r',s'} = 0$. Clearly, e has the required properties. The lemma is proved.

We shall also need the following variant of Lemma 1.12.

LEMMA 1.13. *Let $\mathbf{L}, \mathbf{L}' \in \mathcal{F}^n$. Assume that $L'_k \subset L_k$ for all k. Let $u_k : L_k/L'_k \to L_k/L'_k$ be* $\mathbf{k}$*-linear isomorphisms defined for $k \in \mathbf{Z}$ such that $u_{k-n} = \epsilon u_k \epsilon^{-1}$ for all $k \in \mathbf{Z}$ and such that $j_k u_k = u_{k+1} j_k$ for all k, where $j_k : L_k/L'_k \to L_{k+1}/L'_{k+1}$ is the canonical map. Then there exists $e \in G_{\mathbf{L}} \cap G_{\mathbf{L}'}$ such that e induces (u_k).*

Let $M_{i,j}$ be attached to $\mathbf{L}, \mathbf{L}'$ as in 1.4. Giving (u_k) is the same as giving linear maps $u_{k;r,s;r',s'} : M_{r,s} \to M_{r',s'}$ defined for $r \le k < s$ and $r' \le k < s'$ such that

$u_{k-n;r-n,s-n;r'-n,s'-n} = \epsilon u_{k;r,s;r',s'}\epsilon^{-1}$,
$u_{k+1;r,s;r',s'} = u_{k;r,s;r',s'}$, if $k, k+1 \in [r, s-1]$ and $k, k+1 \in [r', s'-1]$,
$u_{k;r,s;r',s'} = 0$ if $k, k+1 \in [r, s-1]$ and $k+1 = r' < s'$,
$u_{k;r,s;r',s'} = 0$ if $k, k+1 \in [r', s'-1]$ and $r < s = k$.

In particular, if $u_{k;r,s;r',s'}$ is defined and either $r' > r$ or $s' > s$, then $u_{k;r,s;r',s'} = 0$.

We define linear maps $e_{r,s;r',s'} : M_{r,s} \to M_{r',s'}$ for any r, s, r', s' as follows.

If there exists k such that $r \le k < s$ and $r' \le k < s'$, we set $e_{r,s;r',s'} = u_{k;r,s;r',s'}$. (This is independent of the choice of k.) Otherwise, we set $e_{r,s;r',s'} = 0$. Then

$e_{r-n,s-n;r'-n,s'-n} = \epsilon e_{r,s;r',s'}\epsilon^{-1}$,
$e_{r,s;r',s'} = 0$ unless $r' \le r$ and $s' \le s$.

We define $e \in \mathfrak{g}$ so that its $(r, s; r', s')$ component with respect to the decomposition $V = \oplus_{i,j} M_{i,j}$ is $e_{r,s;r',s'} = 0$. Then $e \in \mathfrak{g}$ and it induces (u_k). Applying the same construction to (u_k^{-1}) instead of (u_k) we obtain $e' \in \mathfrak{g}$ which induces (u_k^{-1}). From the definitions we have $ee' = e'e = 1$. Thus, $e \in G$. More precisely, $e \in G_{\mathbf{L}} \cap G_{\mathbf{L}'}$. The lemma is proved.

2. The algebra $\mathbf{U}_D$.

2.1. Clearly, $\mathfrak{S}_{1,n,n'}$ consists of the matrices $E^{k,l} = (e_{i,j}^{k,l}) \in \mathfrak{S}_{1,n,n'}$ (with $k, l \in \mathbf{Z}$) where

$e_{i,j}^{k,l} = 1$, if $i = k + sn, j = l + sn'$ for some $s \in \mathbf{Z}$,

$e_{i,j}^{k,l} = 0$, otherwise.

Note that $E^{k,l} = E^{k+n,l+n'}$.

2.2. Let $\tilde{\mathfrak{S}}_n$ be the set of all $\mathbf{b} = (b_i)_{i\in\mathbf{Z}}$ with $b_i \in \mathbf{Z}$ such that $b_i = b_{i-n}$ for all $i \in \mathbf{Z}$. We regard $\tilde{\mathfrak{S}}_n$ as an (abelian) group with addition component by component. We have $\mathfrak{S}_{D,n} \subset \tilde{\mathfrak{S}}_n$. For $\mathbf{a}, \mathbf{b} \in \tilde{\mathfrak{S}}_n$ we set

$$\mathbf{a} \cdot \mathbf{b} = \sum a_i b_i \in \mathbf{Z},$$

where the sum is taken over a set of representatives for the residue classes modulo n in $\mathbf{Z}$. This is a non-singular pairing on $\tilde{\mathfrak{S}}_n$. For any $\mathbf{b} \in \tilde{\mathfrak{S}}_n$, we set

$$K_{\mathbf{b}} = \sum_{\mathbf{a}\in\mathfrak{S}_{D,n}} v^{\mathbf{a}\cdot\mathbf{b}}[\mathbf{i}_{\mathbf{a}}].$$

This may be regarded as an element of $\mathfrak{A}_{D,n,n}$, or $\mathfrak{A}_{D,n,n;\mathcal{A}}$, or $\mathfrak{A}_{D,n,n;q}$ (in the last case we substitute $v = q^{1/2}$.)

2.3. In the remainder of this section we assume that $n' = n \geq 2$. Let $i \in \mathbf{Z}$. Let $\mathbf{a}, \mathbf{a}' \in \mathfrak{S}_{D,n}$. We say that $\mathbf{a} \smile_i \mathbf{a}'$ if $\mathbf{i}_{\mathbf{a}'} = \mathbf{i}_{\mathbf{a}} + E^{i+1,i+1} - E^{i,i}$ or equivalently, if

$a'_{i+1} = a_{i+1} + 1, a'_i = a_i - 1$,

$a'_j = a_j$ for j such that $j \neq i \mod n$ and $j \neq i+1 \mod n$.

In this case,

$$\{(\mathbf{L}, \mathbf{L}') \in \mathcal{F}_{\mathbf{a}} \times \mathcal{F}_{\mathbf{a}'} | L_j = L'_j \text{ if } j \neq i \mod n, \quad L'_i \subset L_i\} = \mathcal{O}_A,$$

where

$$A = \mathbf{i}_{\mathbf{a}} - E^{i,i} + E^{i,i+1} \in \mathfrak{S}_{D,n,n}.$$

Note that $\dim(L_i/L'_i) = 1$ for $(\mathbf{L}, \mathbf{L}') \in \mathcal{O}_A$. In particular, if $\mathbf{L} \in \mathcal{F}_{\mathbf{a}}$, then $X_A^{\mathbf{L}}$ is in natural bijection with the set of hyperplanes in L_i/L_{i-1}. Moreover,

$$d_A = a_i - 1 = a'_i, \quad [A] = v^{-a'_i} e_A.$$

We set ${}_{\mathbf{a}}\mathbf{e}_{\mathbf{a}'} = A$. On the other hand,

$$\{(\mathbf{L}, \mathbf{L}') \in \mathcal{F}_{\mathbf{a}'} \times \mathcal{F}_{\mathbf{a}} | L_j = L'_j \text{ if } j \neq i \mod n, \quad L_i \subset L'_i\} = \mathcal{O}_{A'},$$

where

$$A' = {}_{\mathbf{a}'}\mathbf{f}_{\mathbf{a}} = \mathbf{i}_{\mathbf{a}'} - E^{i+1,i+1} + E^{i+1,i} \in \mathfrak{S}_{D,n,n}.$$

Note that $\dim(L'_i/L_i) = 1$ for $(\mathbf{L}, \mathbf{L}') \in \mathcal{O}_{A'}$. In particular, if $\mathbf{L} \in \mathcal{F}_{\mathbf{a}'}$, then $X_{A'}^{\mathbf{L}}$ is in natural bijection with the set of lines in L_{i+1}/L_i. Moreover,

$$d_{A'} = a'_{i+1} - 1 = a_{i+1}, \quad [A'] = v^{-a_{i+1}} e_{A'}.$$

We set ${}_{\mathbf{a}'}\mathbf{f}_{\mathbf{a}} = A'$. From the definitions we have ${}^t({}_{\mathbf{a}}\mathbf{e}_{\mathbf{a}'}) = {}_{\mathbf{a}'}\mathbf{f}_{\mathbf{a}}$, hence

(a) $$\Psi[{}_{\mathbf{a}}\mathbf{e}_{\mathbf{a}'}] = [{}_{\mathbf{a}'}\mathbf{f}_{\mathbf{a}}].$$

2.4. Let $i \in \mathbf{Z}$. We set

$$E_i = \sum [{}_{\mathbf{a}}\mathbf{e}_{\mathbf{a}'}], \quad F_i = \sum [{}_{\mathbf{a}'}\mathbf{f}_{\mathbf{a}}],$$

where the sums are taken over all $\mathbf{a}, \mathbf{a}'$ in $\mathfrak{S}_{D,n}$ such that $\mathbf{a} \smile_i \mathbf{a}'$. (Note that each one of $\mathbf{a}, \mathbf{a}'$ determines uniquely the other.) We may regard E_i, F_i as elements of $\mathfrak{A}_{D,n,n}$ or $\mathfrak{A}_{D,n,n;\mathcal{A}}$ or $\mathfrak{A}_{D,n,n;q}$. Note that E_i, F_i depend only on the residue class of $i \mod n$. If (for $\mathbf{k}$ finite with q elements) we regard E_i, F_i as functions on $\mathcal{F}^n \times \mathcal{F}^n$, as in 1.10, we have

$E_i(\mathbf{L}, \mathbf{L}') = q^{-|\mathbf{L}'|_i/2}$

if $L'_i \subset L_i, \dim(L_i/L'_i) = 1$ and $L'_j = L_j$ for $j \neq i \mod n$; $E_i(\mathbf{L}, \mathbf{L}') = 0$, otherwise;

$F_i(\mathbf{L}, \mathbf{L}') = q^{-|\mathbf{L}'|_{i+1}/2}$

if $L_i \subset L'_i, \dim(L'_i/L_i) = 1$ and $L'_j = L_j$ for $j \neq i \mod n$; $F_i(\mathbf{L}, \mathbf{L}') = 0$, otherwise.

From 2.3(a) we deduce

(a) $$\Psi(E_i) = F_i, \Psi(F_i) = E_i.$$

PROPOSITION 2.5. *For* $\mathbf{b}, \mathbf{b}' \in \tilde{\mathfrak{S}}_n$ *and* $i \in \mathbf{Z}$ *we have*

(a) $K_{\mathbf{b}} \cdot K_{\mathbf{b}'} = K_{\mathbf{b}+\mathbf{b}'}$,

(b) $K_{\mathbf{b}} \cdot E_i = v^{b_i - b_{i+1}} E_i \cdot K_{\mathbf{b}}, \quad K_{\mathbf{b}} \cdot F_i = v^{-b_i + b_{i+1}} F_i \cdot K_{\mathbf{b}}$.

The proof is immediate.

The following result can be deduced from [BLM, 5.6(e)]; but we will give a self-contained proof.

PROPOSITION 2.6. *For any* $i \in \mathbf{Z}$ *we define* $\mathbf{a} \in \tilde{\mathfrak{S}}_n$ *by* $a_i = 1, a_{i+1} = -1, a_j = 0$ *for* $j \neq i, i+1 \mod n$. *We have*

$$E_i \cdot F_i - F_i \cdot E_i = \frac{K_{\mathbf{a}} - K_{-\mathbf{a}}}{v - v^{-1}}.$$

For $i, k \in \mathbf{Z}$, $i \neq k \mod n$, *we have* $E_i \cdot F_k - F_k \cdot E_i = 0$.

It suffices to prove the analogous equalities in $\mathfrak{A}_{D,n,n;q}$. Therefore in the rest of the proof we assume that $\mathbf{k}$ is finite with q elements. We write

$$E_i \cdot F_k = \sum_A N'_A e_A, \quad F_k \cdot E_i = \sum_A N''_A e_A$$

where N'_A, N''_A are scalars. Let $N_A = N'_A - N''_A$. Let $(\mathbf{L}, \mathbf{L}') \in \mathcal{O}_A \subset \mathcal{F}_{\mathbf{a}'} \times \mathcal{F}_{\mathbf{a}''}$.

We have $N'_A = q^{c'/2} \sharp X'$ where $c' = -(a'_i - 1) - a''_{k+1}$ and X' is the set of all $\mathcal{L}' \in \mathcal{F}^n$ such that

$\mathcal{L}'_j = L_j$ for $j \neq i \mod n$, $\mathcal{L}'_j = L'_j$ for $j \neq k \mod n$,

$\mathcal{L}'_i \subset L_i$, $\mathcal{L}'_k \subset L'_k$, $\dim(L_i/\mathcal{L}'_i) = \dim(L'_k/\mathcal{L}'_k) = 1$.

We have $N''_A = q^{c''/2} \sharp X''$ where $c'' = -(a'_{k+1} - 1) - a''_i$ and X'' is the set of all $\mathcal{L}'' \in \mathcal{F}^n$ such that

$\mathcal{L}''_j = L_j$ for $j \neq k \mod n$, $\mathcal{L}''_j = L'_j$ for $j \neq i \mod n$,

$L_k \subset \mathcal{L}''_k$, $L'_i \subset \mathcal{L}''_i$, $\dim(\mathcal{L}''_k/L_k) = \dim(\mathcal{L}''_i/L'_i) = 1$.

Assume first that $i \neq k \mod n$. Note that X', X'' are empty (hence $N_A = 0$) unless

(a) $L_j = L'_j$ for $j \neq i, k \mod n$,

$L_{i-1} \subset L'_i \subset L_i \subset L'_{i+1}, \quad L'_{k-1} \subset L_k \subset L'_k \subset L_{k+1}$,

$\dim(L_i/L'_i) = 1, \dim(L'_k/L_k) = 1$.

On the other hand, if (a) is satisfied, then both X', X'' consists of exactly one point and $c' = c''$, hence again $N_A = 0$.

Assume next that $i = k$. If $L_j \neq L'_j$ for some j such that $j \neq i, k \mod n$, then clearly both X', X'' are empty hence $N_A = 0$. Assume now that $L_j = L'_j$ for all j such that $j \neq i, k \mod n$. We consider two cases: $L_i \neq L'_i$ and $L_i = L'_i$.

First case: $L_i \neq L'_i$. Then either both X', X'' are empty or both have exactly one element. Indeed, assume for example that X' is non-empty and let $\mathcal{L}' \in X'$. Since $\mathcal{L}'_i$ has codimension 1 in both L_i, L'_i (which are distinct), we must have $\mathcal{L}'_i = L_i \cap L'_i$; hence X' has exactly one element. But then $\mathcal{L}''_i = L_i + L'_i$ together with $\mathcal{L}''_j = \mathcal{L}'_j$ for $j \neq i \mod n$ defines an element of X''. Similarly, if X'' is non-empty then it has exactly one element and X' must be non-empty. Our claim follows. Note also that, if X', X'' are non-empty, then $c' = c''$. We see that in this case we have $N_A = 0$.

Second case: $L_i = L'_i$. Then $A = \mathbf{i}_{\mathbf{a}'}$. Now $\sharp X'$ is the number of codimension one subspaces in L_i/L_{i-1}, hence it equals $\frac{q^{a'_i}-1}{q-1}$ and $\sharp X''$ is the number of one dimensional subspaces in L_{i+1}/L_i hence it equals $\frac{q^{a'_{i+1}}-1}{q-1}$. Hence

$$\sharp X' - \sharp X'' = \frac{q^{a'_i} - 1}{q-1} - \frac{q^{a'_{i+1}} - 1}{q-1} = \frac{q^{a'_i} - q^{a'_{i+1}}}{q-1}.$$

In this case, $c' = c'' = -a'_i - a'_{i+1} + 1$. Hence $N_A = \frac{q^{(a'_i - a'_{i+1})/2} - q^{(-a'_i + a'_{i+1})/2}}{q^{1/2} - q^{-1/2}}$. The proposition follows.

2.7. Let $\mathbf{U}_D$ be the subalgebra of $\mathfrak{A}_{D,n,n}$ generated by the elements E_i, F_i for various i and by the elements $K_{\mathbf{b}}$ for various $\mathbf{b} \in \tilde{\mathfrak{S}}_n$.

Let $\mathbf{U}'_D$ be the subalgebra of $\mathfrak{A}_{D,n,n}$ generated by the elements $E_i, F_i \in \mathfrak{A}_{D,n,n}$ for various i and by the elements $\mathbf{i}_{\mathbf{a}}$ for various $\mathbf{a} \in \mathfrak{S}_{D,n}$.

LEMMA 2.8. $\mathbf{U}_D = \mathbf{U}'_D$.

Clearly, $K_{\mathbf{b}} \in \mathbf{U}'_D$ for all $\mathbf{b} \in \tilde{\mathfrak{S}}_n$. Hence $\mathbf{U}_D \subset \mathbf{U}'_D$. We now prove the reverse inclusion. From

$K_{\mathbf{b}} = \sum_{\mathbf{a} \in \mathfrak{S}_{D,n}} v^{\mathbf{a} \cdot \mathbf{b}} [\mathbf{i}_{\mathbf{a}}] \in \mathbf{U}_D$

for all $\mathbf{b} \in \tilde{\mathfrak{S}}_n$, we deduce, by a Vandermonde determinant argument, that $[\mathbf{i}_{\mathbf{a}}] \in \mathbf{U}_D$ for all $\mathbf{a} \in \mathfrak{S}_{D,n}$. It follows that $\mathbf{U}'_D \subset \mathbf{U}_D$. The lemma follows.

2.9. Let $\mathbf{U}_{D,\mathcal{A}}$ be the $\mathcal{A}$-subalgebra of $\mathbf{U}_D = \mathbf{U}'_D$ generated by the elements $\mathbf{i}_{\mathbf{a}}$ for various $\mathbf{a} \in \mathfrak{S}_{D,n}$ and by the elements $E_i^s/[s]!, F_i^s/[s]! \in \mathfrak{A}_{D,n,n}$ for various i and various $s \in \mathbf{N}$. Here

$$[s]! = \prod_{t=1}^{s} \frac{v^t - v^{-t}}{v - v^{-1}}.$$

3. An example.

3.1. In this section we assume that $n \geq 2$ and $n' \geq 1$. For $A \in \mathfrak{S}_{D,n,n'}$ and $i, p \in \mathbf{Z}$, we set

$a_{i,\geq p} = \sum_{j;j \geq p} a_{i,j}, \quad a_{i,>p} = \sum_{j;j>p} a_{i,j},$
$a_{i,\leq p} = \sum_{j;j \leq p} a_{i,j}, \quad a_{i,<p} = \sum_{j;j<p} a_{i,j}.$

3.2. Let $\mathfrak{A}^{\infty}_{D,n,n'}$ be the $\mathbf{Q}(v)$-vector space of all formal (possibly infinite) $\mathbf{Q}(v)$-linear combinations of elements e_A, $A \in \mathfrak{S}_{D,n,n'}$. For $i \in \mathbf{Z}$ we define $\mathbf{Q}(v)$-linear maps $\tau_i, \sigma_i : \mathfrak{A}_{D,n,n'} \to \mathfrak{A}_{D,n,n'}$ by

$$\tau_i(e_A) = \sum_{p \in \mathbf{Z}; a_{i+1,p} \geq 1} \frac{v^{2a_{i,\geq p}+2} - v^{2a_{i,>p}}}{v^2 - 1} e_{A+E^{i,p}-E^{i+1,p}},$$

$$\sigma_i(e_A) = \sum_{p \in \mathbf{Z}; a_{i,p} \geq 1} \frac{v^{2a_{i+1,\leq p}+2} - v^{2a_{i+1,<p}}}{v^2 - 1} e_{A-E^{i,p}+E^{i+1,p}}.$$

Here $E^{k,l}$ are defined as in 2.1, in terms of n, n'.

For $i \in \mathbf{Z}$ we define $\mathbf{Q}(v)$-linear maps $\tau'_i, \sigma'_i : \mathfrak{A}^{\infty}_{D,n,n'} \to \mathfrak{A}^{\infty}_{D,n,n'}$ by

$$\tau'_i(\sum_A x_A e_A) = \sum_A x_A \tau'_i(e_A), \quad \sigma'_i(\sum_A x_A e_A) = \sum_A x_A \sigma'_i(e_A),$$

where A runs over $\mathfrak{S}_{D,n,n'}$, $x_A \in \mathbf{Q}(v)$ and

$$\tau'_i(e_A) = \sum_{p \in \mathbf{Z}} \frac{v^{2a_{i+1,\leq p}} - v^{2a_{i+1,<p}}}{v^2 - 1} e_{A+E^{i,p}-E^{i+1,p}},$$

$$\sigma'_i(e_A) = \sum_{p \in \mathbf{Z}} \frac{v^{2a_{i,\geq p}} - v^{2a_{i,>p}}}{v^2 - 1} e_{A-E^{i,p}+E^{i+1,p}}.$$

Note that, if p is such that $(A - E^{i,p} + E^{i+1,p})$ has some entry < 0, then $a_{i,p} = 0$, hence $a_{i,\geq p} = a_{i,>p}$ and the coefficient of $e_{A-E^{i,p}+E^{i+1,p}}$ in $\sigma'_i(e_A)$ is zero. Thus, $\sigma'_i(e_A)$ is a well defined element of $\mathfrak{A}^{\infty}_{D,n,n'}$. Similarly, $\tau'_i(e_A)$ is a well defined element of $\mathfrak{A}^{\infty}_{D,n,n'}$.

We show that the infinite sum $\sum_A x_A \sigma'_i(e_A)$ is well defined. It is enough to show that, for any $A' \in \mathfrak{S}_{D,n,n'}$, there are only finitely many $A \in \mathfrak{S}_{D,n,n'}$ such that the coefficient of $e_{A'}$ in $\sigma'_i(e_A)$ is non-zero. It is also enough to show that, for any $A' \in \mathfrak{S}_{D,n,n'}$, there are only finitely many $p \in \mathbf{Z}$ such that $A' + E^{i,p} - E^{i+1,p} \in \mathfrak{S}_{D,n,n'}$. The last condition implies that $a'_{i+1,p} > 0$ and this is satisfied only by finitely many p.

Similarly, we see that the infinite sum $\sum_A x_A \tau'_i(e_A)$ is well defined. Hence the linear maps τ'_i, σ'_i are well defined.

3.3. Let $\langle , \rangle : \mathfrak{A}_{D,n,n'} \times \mathfrak{A}^{\infty}_{D,n,n'} \to \mathbf{Q}(v)$ be the $\mathbf{Q}(v)$-bilinear form given by

$$\langle \sum_A x_A e_A, \sum_{A'} y_{A'} e_{A'} \rangle = \sum_A x_A y_A.$$

(The first sum is finite, the second sum is possibly infinite, the third sum is finite.)

LEMMA 3.4. *For $\xi \in \mathfrak{A}_{D,n,n'}, \xi' \in \mathfrak{A}^{\infty}_{D,n,n'}$ we have*

$$\langle \tau_i(\xi), \xi' \rangle = \langle \xi, \sigma'_i(\xi') \rangle, \quad \langle \sigma_i(\xi), \xi' \rangle = \langle \xi, \tau'_i(\xi') \rangle.$$

We may assume that $\xi = e_A, \xi' = e_{A'}$ where $A, A' \in \mathfrak{S}_{D,n,n'}$. We have

$$\langle \tau_i(A), A' \rangle = \sum \frac{v^{2a_{i,\geq p}+2} - v^{2a_{i,>p}}}{v^2 - 1}$$

where the sum is taken over all $p \in \mathbf{Z}$ such that $A + E^{ip} - E^{i+1,p} = A'$. (For such p we automatically have $a_{i+1,p} > 0$.) For each p in the sum we have $a_{i,\geq p} + 1 = a'_{i,\geq p}$ and $a_{i,>p} = a'_{i,>p}$. Hence

$$\langle \tau_i(A), A' \rangle = \sum \frac{v^{2a'_{i,\geq p}} - v^{2a'_{i,>p}}}{v^2 - 1}$$

where the sum is taken over all $p \in \mathbf{Z}$ such that $A' - E^{ip} + E^{i+1,p} = A$ or equivalently $\langle \tau_i(A), A' \rangle = \langle A, \sigma'_i(A') \rangle$. This proves the first equality in the lemma. The proof of the second equality is entirely similar. The lemma is proved.

The following result, which is an affine analogue of [BLM, 3.2] relates the operators τ_i, σ_i with the left $\mathfrak{A}_{D,n,n}$-module structure of $\mathfrak{A}_{D,n,n'}$ (see 1.9).

PROPOSITION 3.5. *Let $i \in \mathbf{Z}$.*

(a) Let $A \in \mathfrak{S}_{D,n,n'}$ and let $\mathbf{a}' = r(A)$. If there exists $\mathbf{a} \in \mathfrak{S}_{D,n}$ such that $\mathbf{a} \smile_i \mathbf{a}'$ (that is, if $a'_{i+1} > 0$), then $e_{({}_{\mathbf{a}}\mathbf{e}_{\mathbf{a}'})} \cdot e_A = \tau_i(e_A)$. If no such $\mathbf{a}$ exists, (that is, if $a'_{i+1} = 0$), then $\tau_i(e_A) = 0$.

(b) Let $A' \in \mathfrak{S}_{D,n,n'}$ and let $\mathbf{a} = r(A')$. If there exists $\mathbf{a}' \in \mathfrak{S}_{D,n}$ such that $\mathbf{a} \smile_i \mathbf{a}'$, (that is, if $a_i > 0$), then $e_{({}_{\mathbf{a}'}\mathbf{f}_{\mathbf{a}})} \cdot e_{A'} = \sigma_i(e_A)$. If no such $\mathbf{a}'$ exists, (that is, if $a_i = 0$), then $\tau_i(e_A) = 0$.

We prove (a). The second assertion of (a) is immediate. Hence we may assume that there exists $\mathbf{a} \in \mathfrak{S}_{D,n}$ such that $\mathbf{a} \smile_i \mathbf{a}'$. We have

$$e_{({}_{\mathbf{a}}\mathbf{e}_{\mathbf{a}'})} \cdot e_A = \sum_B N_B e_B$$

where B runs over $\mathfrak{S}_{D,n,n'}$ and $N_B \in \mathcal{A}$. We assume that $\mathbf{k}$ is finite with q elements and we compute $N_B^0 = N_B|_{v=q^{1/2}}$. Let $(\mathbf{L}, \mathbf{L}') \in \mathcal{O}_B$. Let Z be the set of all lattices U in V such that $L_{i-1} \subset U \subset L_i$ and $\dim(L_i/U) = 1$. For $U \in Z$, let $\mathbf{L}^U \in \mathcal{F}^n$ be defined by $L_k^U = L_k$ for $k \neq i \mod n$ and $L_k = \epsilon^{-t}U$ for $k = i + nt$. Then $U \mapsto \mathbf{L}^U$ is a bijection between Z and the set of all $\tilde{\mathbf{L}} \in \mathcal{F}^n$ such that $(\mathbf{L}, \tilde{\mathbf{L}}) \in \mathcal{O}_{({}_{\mathbf{a}}\mathbf{e}_{\mathbf{a}'})}$.

For each $p \in \mathbf{Z}$, let Z_p be the subset of Z defined by the conditions

$L_i \cap L'_j = U \cap L'_j$ for $j < p$, $L_i \cap L'_j \neq U \cap L'_j$ for $j \geq p$

or equivalently by the conditions

$L_i + L'_j \neq U + L'_j$ for $j < p$, $L_i + L'_j = U + L'_j$ for $j \geq p$.

The subsets Z_p form a partition of Z. Let $U \in Z_p$. If $(\mathbf{L}^U, \mathbf{L}') \in \mathcal{O}_A$, then $B = A + E^{i,p} - E^{i+1,p}$. Conversely, if $B = A + E^{i,p} - E^{i+1,p}$ and $U \in Z$ then $U \in Z_p$ and $(\mathbf{L}^U, \mathbf{L}') \in \mathcal{O}_A$. We see that

$N_B^0 = \sharp Z_p$, if $B = A + E^{i,p} - E^{i+1,p}$ for some p,

$N_B^0 = 0$, otherwise.

We have

$$\begin{aligned}\sharp Z_p &= \sharp\{U | L_{i-1} + (L_i \cap L'_{p-1}) \subset U\} - \sharp\{U | L_{i-1} + (L_i \cap L'_p) \subset U\} \\ &= (q-1)^{-1}(q^{\dim(L_i/(L_{i-1}+(L_i\cap L'_{p-1})))} - q^{\dim(L_i/(L_{i-1}+(L_i \cap L'_p)))}) \\ &= \frac{q^{b_{i,\geq p}} - q^{b_{i,>p}}}{q-1} = \frac{q^{a_{i,\geq p}+1} - q^{a_{i,>p}}}{q-1}.\end{aligned}$$

This proves (a). We prove (b). The second assertion of (b) is immediate. Hence we may assume that there exists $\mathbf{a}' \in \mathfrak{S}_{D,n}$ such that $\mathbf{a} \smile_i \mathbf{a}'$. We have

$$e_{({}_{\mathbf{a}'}\mathbf{f}_{\mathbf{a}})} \cdot e_{A'} = \sum_B N'_B e_B$$

where B runs over $\mathfrak{S}_{D,n,n'}$ and $N'_B \in \mathcal{A}$. We assume that $\mathbf{k}$ is finite with q elements and we compute $N'^0_B = N'_B|_{v=q^{1/2}}$. Let $(\mathbf{L}, \mathbf{L}') \in \mathcal{O}_B$. Let Z' be the set of all lattices U in V such that $L_i \subset U \subset L_{i+1}$ and $\dim(U/L_i) = 1$. For $U \in Z'$, let $\mathbf{L}^U \in \mathcal{F}^n$ be defined by $L_k^U = L_k$ for $k \neq i \mod n$ and $L_k = \epsilon^{-t}U$ for $k = i + nt$. Then $U \mapsto \mathbf{L}^U$ is a bijection between Z' and the set of all $\tilde{\mathbf{L}} \in \mathcal{F}^n$ such that $(\mathbf{L}, \tilde{\mathbf{L}}) \in \mathcal{O}_{({}_{\mathbf{a}'}\mathbf{f}'_{\mathbf{a}})}$.

For each $p \in \mathbf{Z}$, let Z'_p be the subset of Z' defined by the conditions

$L_i \cap L'_j = U \cap L'_j$ for $j < p$, $L_i \cap L'_j \neq U \cap L'_j$ for $j \geq p$

or equivalently by the conditions

$L_i + L'_j \neq U + L'_j$ for $j < p$, $L_i + L'_j = U + L'_j$ for $j \geq p$.
The subsets Z'_p form a partition of Z'. Let $U \in Z'_p$. If $(\mathbf{L}^U, \mathbf{L}') \in \mathcal{O}_{A'}$, then $B = A' - E^{i,p} + E^{i+1,p}$. Conversely, if $B = A' - E^{i,p} + E^{i+1,p}$ and $U \in Z'$ then $U \in Z'_p$ and $(\mathbf{L}^U, \mathbf{L}') \in \mathcal{O}_{A'}$. We see that
$N'^{\,0}_B = \sharp Z'_p$, if $B = A' - E^{i,p} + E^{i+1,p}$ for some p,
$N'^{\,0}_B = 0$, otherwise.
We have

$$\begin{aligned}\sharp Z'_p &= \sharp\{U|U \subset L_i + (L_{i+1} \cap L'_p)\} - \sharp\{U|U \subset L_i + (L_{i+1} \cap L'_{p-1})\} \\ &= (q-1)^{-1}(q^{\dim(L_i/(L_i+(L_{i+1}\cap L'_p)))} - q^{\dim(L_i/(L_i+(L_{i+1}\cap L'_{p-1})))}) \\ &= \frac{q^{b_{i+1,\leq p}} - q^{b_{i,<p}}}{q-1} = \frac{q^{a'_{i,\leq p}+1} - q^{a'_{i,<p}}}{q-1}.\end{aligned}$$

This proves (b). The proposition is proved.

3.6. We now assume that $n = n' = D = 2$. Let $i \in \{0,1\}$ and let $p \in \mathbf{Z}$. For each $s \in \mathbf{Z}$, we consider the element $E^{i,p-s} + E^{i+1,p+s+1} \in \mathfrak{S}_{2,2,2}$, and we form the (infinite) linear combination

$$\rho_{i,p} = \sum_{s\in\mathbf{Z}} (-1)^s e_{E^{i,p-s}+E^{i+1,p+s+1}} \in \mathfrak{A}^{\infty}_{2,2,2}.$$

Note that $\rho_{1,p+1} = \rho_{0,p}$ for any p.

LEMMA 3.7. *For any $j \in \mathbf{Z}$ we have $\tau'_j \rho_{i,p} = 0, \sigma'_j \rho_{i,p} = 0$.*

From the definitions, we have
$\tau'_i \rho_{i,p} = \sum_s (-1)^s e_{E^{i,p-s}+E^{i,p+s+1}} = 0$,
$\tau'_{i+1} \rho_{i,p} = \sum_s (-1)^s e_{E^{i+1,p+s+1}+E^{i+1,p-s+2}} = 0$,
$\sigma'_i \rho_{i,p} = \sum_s (-1)^s e_{E^{i+1,p-s}+E^{i+1,p+s+1}} = 0$,
$\sigma'_{i+1} \rho_{i,p} = \sum_s (-1)^s e_{E^{i,p-s}+E^{i,p+s-1}} = 0$.
The lemma is proved.

3.8. Let $\mathfrak{A}'_{2,2,2}$ be the subspace consisting of all vectors $\xi \in \mathfrak{A}_{2,2,2}$ that satisfy
$\langle \xi, \rho_{i,p} \rangle = 0$ for all $i \in \{0,1\}$ and all $p \in \mathbf{Z}$.
This is a proper subspace of $\mathfrak{A}_{2,2,2}$ (an intersection of countably many hyperplanes). From Lemmas 3.4 and 3.7 we see that

(a) $$\tau_i(\mathfrak{A}'_{2,2,2}) \subset \mathfrak{A}'_{2,2,2}, \quad \sigma_i(\mathfrak{A}'_{2,2,2}) \subset \mathfrak{A}'_{2,2,2}.$$

Consider the subspace $\mathbf{U}'_2$ of $\mathfrak{A}_{2,2,2}$ defined in 2.7. Using Lemma 3.5, we see that $\mathbf{U}'_2$ is spanned as a $\mathbf{Q}(v)$-vector space by elements of the form

(b) $$T_1 T_2 \dots T_N(a)$$

where a is one of the elements $e_{E^{0,0}+E^{1,1}}, e_{2E^{0,0}}, e_{2E^{1,1}}$ and T_s is either τ_i or σ_i for some i.

We show that any element of the form (b) belongs to $\mathfrak{A}'_{2,2,2}$. We argue by induction on N. For $N = 0$ our assertion is obvious. Assume now that $N \geq 1$. Then our element is of the form $\tau_i \xi$ or $\sigma_i \xi$ where ξ is known by induction to belong to $\mathfrak{A}'_{2,2,2}$. We then use (a) and our statement is proved.

We see therefore that $\mathbf{U}'_2 \subset \mathfrak{A}'_{2,2,2}$. In other words (see Lemma 2.8), we have $\mathbf{U}_2 \subset \mathfrak{A}'_{2,2,2}$ and *the algebra $\mathfrak{A}_{2,2,2}$ is not generated by the elements $E_i, F_i, K_{\mathbf{b}} \in \mathfrak{A}_{2,2,2}$.*

Exactly the same argument shows that the algebra $\mathfrak{A}_{2,2,2;q}$ is not generated by the elements $E_i, F_i, K_{\mathbf{b}} \in \mathfrak{A}_{2,2,2;q}$.

Thus, the surjectivity statement in [GV, Thm. 9.2] is false.

4. The elements $\{A\}$.

4.1. In this section we assume that $\mathbf{k}$ is algebraically closed. Let $\mathbf{a} \in \mathfrak{S}_{D,n}, \mathbf{b} \in \mathfrak{S}_{D,n'}, \mathbf{L} \in \mathcal{F}_{\mathbf{a}}$. For $i_0, j_0 \in \mathbf{Z}$, the subsets

$$\{\mathbf{L}' \in \mathcal{F}_{\mathbf{b}} | \epsilon^p L_{i_0} \subset L'_{j_0} \subset \epsilon^{-p} L_{i_0}\}$$

of $\mathcal{F}_{\mathbf{b}}$ (with $p = 1, 2, 3, \dots$) are naturally projective algebraic varieties, each one included in the next; they are all $G_{\mathbf{L}}$-stable and their union is $\mathcal{F}_{\mathbf{b}}$. For each A in the set

(a) $$\{A \in \mathfrak{S}_{D,n,n'} | r(A) = \mathbf{a}, c(A) = \mathbf{b}\},$$

we can define $\bar{X}_A^{\mathbf{L}}$ as the closure of $X_A^{\mathbf{L}}$ in one of these projective varieties for large enough p. This is a well defined $G_{\mathbf{L}}$-stable projective variety, independent of the choices of i_0, j_0, p. Let

(b) $$d'_A = \dim(X_A^{\mathbf{L}}) = \dim(\bar{X}_A^{\mathbf{L}}).$$

For A, A_1 in the set (a), we write $A_1 \le A$ if $X_{A_1}^{\mathbf{L}} \subset \bar{X}_A^{\mathbf{L}}$; we then define P to be the simple perverse sheaf on $\bar{X}_A^{\mathbf{L}}$ whose restriction to $X_A^{\mathbf{L}}$ is $\bar{\mathbf{Q}}_l[d'_A]$ and we define $\mathcal{H}^{s'}$ be the s'-th cohomology sheaf of P. Let $\mathcal{H}_y^{s'}$ be the stalk of $\mathcal{H}^{s'}$ at a point $y \in X_{A_1}^{\mathbf{L}}$. Let

(c) $$\Pi_{A_1,A} = \sum_{s \in \mathbf{Z}} \dim \mathcal{H}_y^{s - d'_{A_1}}(P) v^s \in \mathcal{A},$$

(d) $$\{A\} = \sum_{A_1; A_1 \le A} \Pi_{A_1,A}[A_1] \in \mathfrak{A}_{D,n,n'}.$$

Note that

(e) $$\Pi_{A,A} = 1 \text{ and } \Pi_{A_1,A} \in v^{-1}\mathbf{Z}[v^{-1}] \text{ if } A_1 < A.$$

(We write $A_1 < A$ if $A_1 \le A$ and $A_1 \ne A$.) Hence the elements $\{A\}$ with $A \in \mathfrak{S}_{D,n,n'}$ form a $\mathbf{Q}(v)$-basis of $\mathfrak{A}_{D,n,n'}$ and an $\mathcal{A}$-basis of $\mathfrak{A}_{D,n,n';\mathcal{A}}$.

4.2. Let $A_1 \in \mathfrak{S}_{D,n,n'}, A_2 \in \mathfrak{S}_{D,n',n''}$ be such that $r(A_1) = \mathbf{a}, c(A_1) = r(A_2) = \mathbf{b}, c(A_2) = \mathbf{c}$. Let $\mathbf{L} \in \mathcal{F}_{\mathbf{a}}$. Then

$$Z = \{(\mathbf{L}', \mathbf{L}'') \in \mathcal{F}_{\mathbf{b}} \times \mathcal{F}_{\mathbf{c}} | \mathbf{L}' \in \bar{X}_{A_1}^{\mathbf{L}}, \mathbf{L}'' \in \bar{X}_{A_2}^{\mathbf{L}'}\}$$

is naturally an irreducible projective variety (it is a closed subset of the projective variety

$$\{(\mathbf{L}', \mathbf{L}'') \in \mathcal{F}_{\mathbf{b}} \times \mathcal{F}_{\mathbf{c}} | \epsilon^p L_{i_0} \subset L'_{j_0} \subset \epsilon^{-p} L_{i_0}, \epsilon^p L_{i_0} \subset L''_{k_0} \subset \epsilon^{-p} L_{i_0}\}$$

for large enough p, where $i_0, j_0, k_0 \in \mathbf{Z}$ are fixed). Moreover,

$$Z_0 = \{(\mathbf{L}', \mathbf{L}'') \in \mathcal{F}_{\mathbf{b}} \times \mathcal{F}_{\mathbf{c}} | \mathbf{L}' \in X_{A_1}^{\mathbf{L}}, \mathbf{L}'' \in X_{A_2}^{\mathbf{L}'}\}$$

is an open smooth dense subvariety of Z. Note that $G_{\mathbf{L}}$ acts on Z and Z_0 by conjugation on both factors.

Let Z' be the image of Z under the second projection $\mathcal{F}_{\mathbf{b}} \times \mathcal{F}_{\mathbf{c}} \to \mathcal{F}_{\mathbf{c}}$. Then Z' is a projective subvariety of $\mathcal{F}_{\mathbf{c}}$ stable under the action of $G_{\mathbf{L}}$. Let $\pi : Z \to Z'$ be the natural morphism (restriction of the second projection).

For each A in the finite set

(a) $$\{A \in \mathfrak{S}_{D,n,n''} | r(A) = \mathbf{a}, c(A) = \mathbf{c}, \quad X_A^{\mathbf{L}} \subset Z'\}$$

we denote by P_A the simple perverse sheaf on Z' with support equal to $\bar{X}_A^{\mathbf{L}}$ and whose restriction to $X_A^{\mathbf{L}}$ is $\bar{\mathbf{Q}}_l[d'_A]$. Any simple $G_{\mathbf{L}}$-equivariant perverse sheaf on Z' is isomorphic to a unique P_A as above. (Note that $G_{\mathbf{L}}$ acts on Z' through a quotient which is an algebraic group hence we can talk about $G_{\mathbf{L}}$-equivariant perverse sheaves on Z'.)

Let $\mathcal{I}$ be the simple perverse sheaf on Z whose restriction to Z_0 is $\bar{\mathbf{Q}}_l[d]$ where $d = \dim Z = \dim Z_0$. By the decomposition theorem [BBD], the direct image complex $\pi_*\mathcal{I}$ on Z' is isomorphic to a direct sum of simple perverse sheaves on Z' (with shifts), which are necessarily $G_{\mathbf{L}}$-equivariant. Thus, we have

$$\pi_*\mathcal{I} = \oplus_{A;\delta} P_A[\delta]^{\oplus N_{A,\delta}}$$

where A runs through the set (a), δ runs over the integers and $N_{A,\delta} \in \mathbf{N}$. We set

$$\gamma_{A_1,A_2,A} = \sum_\delta N_{A,\delta} v^\delta \in \mathcal{A}.$$

It is clear that $\gamma_{A_1,A_2,A}$ is independent of the choice of $\mathbf{L} \in \mathcal{F}_{\mathbf{a}}$. In the case where $A \in \mathfrak{S}_{D,n,n''}$ satisfies $r(A) = \mathbf{a}, c(A) = \mathbf{c}$ and $X_A^{\mathbf{L}} \not\subset Z'$, we set $\gamma_{A_1,A_2,A} = 0$.

Let $^- : \mathcal{A} \to \mathcal{A}$ be the ring involution defined by $\overline{v^m} = v^{-m}$ for all $m \in \mathbf{Z}$. This extends to a field involution of $\mathbf{Q}(v)$ denoted again by $^-$. From the relative hard Lefschetz theorem of Deligne [BBD, 5.4.10], it follows that

(b) $\gamma_{A_1,A_2,A} \in \mathcal{A}$ is fixed by $^-: \mathcal{A} \to \mathcal{A}$.

The following result is an afine analogue of [BLM, 2.2].

LEMMA 4.3. *For any $A \in \mathfrak{S}_{D,n,n'}$ we have $d_A = d'_A$. Here d_A is as in 1.2(d) and d'_A is as in 4.1(b).*

We fix $\mathbf{L} \in \mathcal{F}_{\mathbf{a}}$ where $\mathbf{a} = r(A)$. Note that $G_{\mathbf{L}}$ acts transitively on $X_A^{\mathbf{L}}$ and if $\mathbf{L}' \in X_A^{\mathbf{L}}$, then the stabilizer of $\mathbf{L}'$ in $G_{\mathbf{L}}$ is $G_{\mathbf{L}} \cap G_{\mathbf{L}'}$. For $N \geq 1$, let H_N be the set of all $g \in G_{\mathbf{L}}$ such that $g = 1$ on $L_0/\epsilon^N L_0$; this is a normal subgroup of $G_{\mathbf{L}}$ such that $G_{\mathbf{L}}/H_N$ is a connected algebraic group. We can choose N large enough so that H_N acts trivially on $X_A^{\mathbf{L}}$; in particular, we have $H_N \subset G_{\mathbf{L}} \cap G_{\mathbf{L}'}$. We see that $X_A^{\mathbf{L}}$ may be identified with the space of cosets of the algebraic group $G_{\mathbf{L}}/H_N$ by the closed subgroup $(G_{\mathbf{L}} \cap G_{\mathbf{L}'})/H_N$. Hence

$$\dim X_A^{\mathbf{L}} = \dim(G_{\mathbf{L}}/H_N) - \dim((G_{\mathbf{L}} \cap G_{\mathbf{L}'})/H_N).$$

Let X be the set of all $T \in \mathfrak{g}$ such that $T(L_i) \subset L_i$ for all i. Let X' be the set of all $T \in \mathfrak{g}$ such that $T(L_i) \subset L_i$ and $T(L'_i) \subset L'_i$ for all i. Let X_N be the set of all maps T in X such that $T(L_0) \subset \epsilon^N L_0$. Then X is a Lie algebra with respect to the commutator of endomorphisms, X' is a Lie subalgebra of X and X_N is an ideal of X contained in X'. Moreover, X/X_N is naturally the Lie algebra of $G_{\mathbf{L}}/H_n$ and X'/X_N is naturally the Lie algebra of $(G_{\mathbf{L}} \cap G_{\mathbf{L}'})/H_N$. It follows that

$$\dim(X_A^{\mathbf{L}}) = \dim(X/X_N) - \dim(X'/X_N) = \dim(X/X').$$

Let $M_{i,j}$ be associated to $\mathbf{L}, \mathbf{L}'$ as in 1.4. Then X consists of all collections $(T_{i,j,k,l})$ where $T_{i,j,k,l} : M_{i,j} \to M_{k,l}$ are $\mathbf{k}$-linear maps defined for $(i,j,k,l) \in \mathbf{Z}^4$ with $i \geq k$, such that $T_{i-n,j-n',k-n,j-n'} = \epsilon T_{i,j,k,l}\epsilon^{-1}$. On the other hand, X' consists of all collections $(T_{i,j,k,l})$ where $T_{i,j,k,l} : M_{i,j} \to M_{k,l}$ are $\mathbf{k}$-linear maps defined for $(i,j,k,l) \in \mathbf{Z}^4$ with $i \geq k, j \leq l$, such that $T_{i-n,j-n',k-n,j-n'} = \epsilon T_{i,j,k,l}\epsilon^{-1}$. Hence X/X' is the space consisting of all collections $(T_{i,j,k,l})$ where $T_{i,j,k,l} : M_{i,j} \to M_{k,l}$ are $\mathbf{k}$-linear maps defined for $(i,j,k,l) \in \mathbf{Z}^4$ with $i \geq k, j > l$, such that $T_{i-n,j-n',k-n,j-n'} = \epsilon T_{i,j,k,l}\epsilon^{-1}$. Hence $\dim(X/X') = d_A$. The lemma is proved.

4.4. The varieties $\bar{X}^{\mathbf{L}}_A$ are very closely connected with the affine Schubert varieties for the group $GL_D(\mathbf{k}[\epsilon,\epsilon^{-1}])$. More precisely, for any A there exists a fibre bundle over $\bar{X}^{\mathbf{L}}_A$ with smooth fibres (isomorphic to a suitable flag manifold in a product of groups $GL_m(\mathbf{k})$) whose total space is an affine Schubert variety for $GL_D(\mathbf{k}[\epsilon,\epsilon^{-1}])$. It follows that the results in [KL2] on the stalks and eigenvalues of Frobenius for the intersection cohomology of affine Schubert varieties (in our case, of type A) imply analogous results for $\bar{X}^{\mathbf{L}}_A$. Moreover, the elements $\gamma_{A_1,A_2,A}$ can be interpreted in terms of multiplication of suitable elements in an affine Hecke algebra. Using these facts as well as Lemma 4.3, we see that the following properties of $\gamma_{A_1,A_2,A}$ hold.

(a) The relation $A_1 \leq A$ is independent of the field $\mathbf{k}$; $\Pi_{A_1,A}$ are independent of the field $\mathbf{k}$ and of l.

(b) $\gamma_{A_1,A_2,A}$ are independent of the field $\mathbf{k}$ and of l.

(c) $$\sum_{\substack{C_1,C_2\\ C_1\leq A_1\\ C_2\leq A_2}} \Pi_{C_1,A_1}\Pi_{C_2,A_2}v^{-d_{C_1}-d_{C_2}}\nu_{C_1,C_2,A} = v^{-d_A}\sum_{B;A\leq B}\Pi_{A,B}\gamma_{A_1,A_2,B}$$

for any A_1, A_2, A as in 4.2. (The last sum has only finitely many non-zero terms.)

In establishing the properties above we also use the following fact. In the case where $\mathbf{k}$ is an algebraic closure of the finite field with p elements, the coefficient of v^i in $\gamma_{A_1,A_2,A}$ is naturally the dimension of a $\bar{\mathbf{Q}}_l$-vector space with a natural action of the Frobenius map.

All eigenvalues of this linear map are equal to $p^{i/2}$.

The proof is essentially the same as that of [L4, Thm. 5.4].

4.5. For $A_1 \in \mathfrak{S}_{D,n,n'}, A_2 \in \mathfrak{S}_{D,n',n''}$ we have
$\{A_1\}\cdot\{A_2\} = \sum_A \gamma_{A_1,A_2,A}\{A\}$, if $c(A_1) = r(A_2)$,
$\{A_1\}\cdot\{A_2\} = 0$, if $c(A_1) \neq r(A_2)$.
The sum is taken over all $A \in \mathfrak{S}_{D,n,n''}$ such that $r(A_1) = r(A), c(A_2) = c(A)$. (The sum has only finitely many non-zero terms.) This follows from 4.4(c).

4.6. Let $\mathfrak{S}^-_{D,n,n}$ be the set of all $B \in \mathfrak{S}_{D,n,n}$ such that $b_{i,j} = 0$ for all $i > j$. Let $\mathfrak{S}^+_{D,n,n}$ be the set of all $B \in \mathfrak{S}_{D,n,n}$ such that $b_{i,j} = 0$ for all $i < j$.

LEMMA 4.7. *Let $B \in \mathfrak{S}_{D,n,n}$ and let $(\mathbf{L},\mathbf{L}') \in \mathcal{O}_B$.*

(a) We have $B \in \mathfrak{S}^-_{D,n,n}$ if and only if $L'_i \subset L_i$ for all i.

(b) We have $B \in \mathfrak{S}^+_{D,n,n}$ if and only if $L_i \subset L'_i$ for all i.

This follows immediately from definitions.

4.8. Let $A \in \mathfrak{S}_{D,n,n}$. We define $A^- \in \mathfrak{S}^-_{D,n,n}, A^+ \in \mathfrak{S}^+_{D,n,n}$ by
$a^-_{i,j} = a_{i,j}$ if $i < j$; $a^-_{i,j} = 0$ if $i > j$; $a^-_{i,i} = \sum_{j\in\mathbf{Z};i\geq j} a_{i,j}$;
$a^+_{i,j} = a_{i,j}$ if $i > j$; $a^+_{i,j} = 0$ if $i < j$; $a^+_{i,i} = \sum_{k\in\mathbf{Z};k\leq i} a_{k,i}$.
Note that $r(A) = r(A^-), c(A) = c(A^+), c(A^-) = r(A^+)$.

LEMMA 4.9. *We have $d_A = d_{A^-} + d_{A^+}$.*

Using the definitions we have

$$d_{A^-}+d_{A^+}=\sum_{\substack{i,j,k,l\\ i\ge k;j<l}} a^-_{ij}a^-_{kl}+\sum_{\substack{ijkl\\ i\ge k;j<l}} a^+_{ij}a^+_{kl}$$

$$=\sum_{\substack{i,j,k,l\\ i<j;k<l\\ i\ge k;j<l}} a_{ij}a_{kl}+\sum_{\substack{i,j,k,l\\ i\ge j;k<l\\ i\ge k;i<l}} a_{ij}a_{kl}+\sum_{\substack{i,j,k,l\\ i<j;k\ge l\\ i\ge k;j<k}} a_{ij}a_{kl}+\sum_{\substack{i,j,k,l\\ i\ge j;k\ge l\\ i\ge k;i<k}} a_{ij}a_{kl}$$

$$+\sum_{\substack{i,j,k,l\\ i>j;k>l\\ i\ge k;j<l}} a_{ij}a_{kl}+\sum_{\substack{i,j,k,l\\ i\le j;k>l\\ i\ge k;i<l}} a_{ij}a_{kl}+\sum_{\substack{i,j,k,l\\ i>j;k\le l\\ i\ge k;j<k}} a_{ij}a_{kl}+\sum_{\substack{i,j,k,l\\ i\le j;k\le l\\ i\ge k;i<k}} a_{ij}a_{kl}$$

where the four indices in the sum are taken modulo simultaneous translation by a multiple of n. Among the last eight sums, the third, fourth, sixth and eighth are empty. Hence

$$d_{A^-}+d_{A^+}=\sum_{\substack{i,j,k,l\\ k\le i<j<l}} a_{ij}a_{kl}+\sum_{\substack{i,j,k,l\\ j<l<k\le i}} a_{ij}a_{kl}+\sum_{\substack{i,j,k,l\\ j\le i<l\\ k\le i}} a_{ij}a_{kl}+\sum_{\substack{i,j,k,l\\ j<l\le i\\ k\le l}} a_{ij}a_{kl}$$

$$=\sum_{\substack{i,j,k,l\\ i\ge k;j<l}} a_{ij}a_{kl}=d_A.$$

The lemma is proved.

4.10. Let $(\mathbf{L},\mathbf{L}'')\in\mathcal{O}_A$. We define $\tilde{\mathbf{L}}\in\mathcal{F}$ by $\tilde{L}_i=L_i\cap L''_i$ for all i. From the definitions we have that

(a) $(\mathbf{L},\tilde{\mathbf{L}})\in\mathcal{O}_{A^-}$, $(\tilde{\mathbf{L}},\mathbf{L}'')\in\mathcal{O}_{A^+}$.

PROPOSITION 4.11. *$\{A^-\}\cdot\{A^+\}=\{A\}$ plus an $\mathcal{A}$-linear combination of elements $\{A_1\}$ with $A_1<A$.*

Let $\mathbf{a}=r(A)$. Let $\mathbf{L}\in\mathcal{F}_{\mathbf{a}}$. Let $Z,Z',Z_0,\pi:Z\to Z',\mathcal{I}$ be as in 4.2 with $A_1=A^-,A_2=A^+$.

Clearly, Z is irreducible and by Lemma 4.3, it has dimension $d_{A^-}+d_{A^+}$, that is, $\dim Z=d_A$ (see Lemma 4.9). From 4.10(a) we see that $X^{\mathbf{L}}_A\subset Z'$. Hence $\bar{X}^{\mathbf{L}}_A\subset Z'$. Now $\bar{X}^{\mathbf{L}}_A$ is irreducible, projective, and by Lemma 4.3 it has dimension d_A. Since Z' is projective, irreducible of dimension $\le d_A$, it follows that $Z'=\bar{X}^{\mathbf{L}}_A$. Let Y be the inverse image of $X^{\mathbf{L}}_A$ under $\pi:Z\to\bar{X}^{\mathbf{L}}_A$. Note that Y is an open dense subset of Z. Hence it is an irreducible variety of dimension d_A. Now $G_{\mathbf{L}}$ acts naturally on Y and $X^{\mathbf{L}}_A$ compatibly with $\pi':Y\to X^{\mathbf{L}}_A$ (restriction of π). Since $G_{\mathbf{L}}$ acts transitively on $X^{\mathbf{L}}_A$, and $\dim Y=\dim X^{\mathbf{L}}_A$, it follows that $G_{\mathbf{L}}$ acts transitively on Y. Hence, the stabilizer $G_{\mathbf{L}}\cap G_{\mathbf{L}'}$ of any point $\mathbf{L}'\in X^{\mathbf{L}}_A$ acts transitively on the fibre of π' at $\mathbf{L}'$ (a finite set). This finite set must then be a single point, since the action of $G_{\mathbf{L}}\cap G_{\mathbf{L}'}$ is through a quotient which is a connected algebraic group. Thus, π' is bijective. Since both Z_0 and Y are open dense in Z and $G_{\mathbf{L}}$ acts on Z_0 and acts transitively on Y, it follows that $Y\subset Z_0$. The arguments above show that the restriction of $\pi_*\mathcal{I}$ from $\bar{X}^{\mathbf{L}}_A$ to $X^{\mathbf{L}}_A$ is just $\bar{\mathbf{Q}}_l[d_A]$. It follows that the simple perverse sheaf on $\bar{X}^{\mathbf{L}}_A$ which equals $\bar{\mathbf{Q}}_l[d_A]$ on $X^{\mathbf{L}}_A$ appears with multiplicity one and without a shift in $\pi_*\mathcal{I}$. The proposition follows.

PROPOSITION 4.12. *The anti-automorphism $\Psi:\mathfrak{A}_{D,n,n}\to\mathfrak{A}_{D,n,n}$ (see 1.11) carries $\{A\}$ to $\{{}^tA\}$ for any $A\in\mathfrak{S}_{D,n,n}$.*

We must prove that

(a) $$\Pi_{{}^tA_1,{}^tA} = \Pi_{A_1,A} \text{ for any } A_1, A \text{ in } \mathfrak{S}_{D,n,n}.$$

(When $A_1 \not\leq A$ we set $\Pi_{A_1,A} = 0$.) Since $\Pi_{A_1,A}$ are (up to normalization) special cases of the polynomials $P_{y,w}$ of [KL1] (where y, w are elements of a Coxeter group or, rather, an extension of a Coxeter group), it is enough to apply the property

$$P_{y,w} = P_{y^{-1},w^{-1}}$$

of those polynomials, which follows easily from the definition in terms of Coxeter groups.

4.13. Let $^-\colon \mathfrak{A}_{D,n,n} \to \mathfrak{A}_{D,n,n}$ be the (involutive) group homomorphism defined by

$$\overline{f\{A\}} = \bar{f}\{A\} \text{ for all } A \in \mathfrak{S}_{D,n,n}, f \in \mathbf{Q}(v)$$

where $f \mapsto \bar{f}$ is the field involution of $\mathbf{Q}(v)$ such that $\bar{v} = v^{-1}$. This is a ring homomorphism, by 4.2(b). It keeps fixed each of the elements $[A]$ where A is either $\mathbf{i_a}$ as in 1.9 or ${}_{\mathbf{a}}\mathbf{e}_{\mathbf{a}'}, {}_{\mathbf{a}'}\mathbf{f}_{\mathbf{a}}$ as in 2.3. (These A are minimal for $\leq$ hence they satisfy $[A] = \{A\}$.) Hence it keeps fixed E_i, F_i and, more generally, $E_i^s/[s]!, F_i^s/[s]!$ for various i and various $s \in \mathbf{N}$. Hence it restricts to involutions $^-\colon \mathbf{U}_D \to \mathbf{U}_D$ and $^-\colon \mathbf{U}_{D,\mathcal{A}} \to \mathbf{U}_{D,\mathcal{A}}$.

5. Cyclic quivers.

5.1. In this section we assume that $\mathbf{k} = \mathbf{C}$.

Let $\mathbf{L}, \mathcal{L} \in \mathcal{F}^n$. Let B be such that $(\mathbf{L}, \mathcal{L}) \in \mathcal{O}_B$. Assume that $\mathcal{L}_k \subset L_k$ for all k. We define a representation of a cyclic quiver as follows. The vertices of the cyclic quiver are the elements of $\mathbf{Z}/n\mathbf{Z}$. To the vertex corresponding to the residue class of $k \mod n$ we associate the vector space $L_k/\mathcal{L}_k$. (We identify canonically $L_k/\mathcal{L}_k, L_{k'}/\mathcal{L}_{k'}$ for k, k' in the same residue class via the isomorphism $\epsilon^{(k-k')/n} : L_k/\mathcal{L}_k \xrightarrow{\sim} L_{k'}/\mathcal{L}_{k'}$.) To have a representation of the cyclic quiver we need also a linear map $L_k/\mathcal{L}_k \to L_{k+1}/\mathcal{L}_{k+1}$ for each k. These are just the maps induced by the inclusion $L_k \subset L_{k+1}$. These linear maps are compatible with the identification above.

For any $p \geq 1$ we consider the kernel $K_{k,p}$ of the p-fold composition

$$L_k/\mathcal{L}_k \to L_{k+1}/\mathcal{L}_{k+1} \to \cdots \to L_{k+p}/\mathcal{L}_{k+p}$$

We have $K_{k,p} = (L_k \cap \mathcal{L}_{k+p})/\mathcal{L}_k$ and its dimension is

$$\dim L_k/(L_k \cap \mathcal{L}_k) - \dim L_k/(L_k \cap \mathcal{L}_{k+p}) = \sum_{\substack{r\leq k\\ s>k}} b_{rs} - \sum_{\substack{r\leq k\\ s>k+p}} b_{rs} = \sum_{\substack{r\leq k\\ k<s\leq k+p}} b_{rs}.$$

(See 1.6(a).) Note that for large enough p, the composition above is zero (that is, $L_k \subset \mathcal{L}_{k+p}$); in other words, our representation of the cyclic quiver is *nilpotent.*

5.2. The indecomposable nilpotent representations of our cyclic quiver are classified up to isomorphism by pairs (t, m) where t is an integer defined up to translation by a multiple of n and $m \in \{1, 2, \ldots\}$. The representation corresponding to t, m is denoted by $V_{t,m}$. It has a basis $e_t, e_{t+1}, \ldots, e_{t+m-1}$ with e_j of degree $j \mod n$ and we have $e_t \to e_{t+1} \to \cdots \to e_{t+m-1} \to 0$ in the representation. Let $\mu_{t,m}$ be the number of summands of the representation in 5.1 that are isomorphic to $V_{t,m}$.

PROPOSITION 5.3. *For any $k \in \mathbf{Z}$ and any $p \geq 1$ we have $b_{k,k+p} = \mu_{k,p}$.*

For $V_{t,m}$, the kernel of the p-fold composition starting at degree k, (analogous to the one in 5.1) has dimension

$$\sharp(u \in [t, t+m-1] | u = k \mod n, t + (m-1) - u < p).$$

Hence

$$\dim K_{k,p} = \sum_{\substack{t \in [0,n-1] \\ m \geq 1}} \sharp(u \in [t, t+m-1] | u = k \mod n, t + (m-1) - u < p) \mu_{t,m}$$

so that

$$\sum_{\substack{r \leq k \\ k < s \leq k+p}} b_{rs} = \sum_{t \in [0,n-1]; m \geq 1; u \in [t,t+m-1]; u=k \mod n, t+(m-1)-u<p} \mu_{t,m}.$$

This holds for any $p \geq 1$. But it also holds for $p = 0$: both sides are 0. We write this for p and $p - 1$ (where $p \geq 1$) and substract one equality from the other:

(a) $$\sum_{\substack{r \leq k \\ s = k+p}} b_{rs} = \sum_{\substack{t \in [0,n-1]; m \geq 1 \\ u \in [t,t+m-1]; u=k \mod n \\ t+(m-1)-u=p-1}} \mu_{t,m} = \sum_{\substack{t \in [0,n-1] \\ u \geq t; u=k \mod n}} \mu_{t,p+u-t}.$$

Replacing here k by $k-1$ and p by $p+1$ we obtain

$$\sum_{\substack{r \leq k-1 \\ s = k+p}} b_{rs} = \sum_{\substack{t \in [0,n-1] \\ u \geq t; u=k-1 \mod n}} \mu_{t,p+u-t+1} = \sum_{\substack{t \in [0,n-1] \\ u' > t; u'=k \mod n}} \mu_{t,p+u'-t}.$$

Substracting from (a), we get

$$b_{k,k+p} = \sum_{\substack{t \in [0,n-1] \\ u=t; u=k \mod n}} \mu_{t,p+u-t} = \mu_{k,p}.$$

The proposition is proved.

5.4. Let (W_k) be a collection of finite dimensional $\mathbf{k}$-vector spaces indexed by $k \in \mathbf{Z}/n\mathbf{Z}$. Let E_W^{nil} be the set of all of all nilpotent representations

(a) $$(W_k \xrightarrow{y_k} W_{k+1})_{k \in \mathbf{Z}/n\mathbf{Z}}$$

of our cyclic quiver.

In the remainder of this subsection and in 5.5 we assume that $n \geq 2$. We say that the representation (a) is *aperiodic* (cf. [R], [L1]) if for any $m \geq 1$ there exists $t \in \mathbf{Z}/n\mathbf{Z}$ such that $V_{t,m}$ does not appear as a direct summand of the representation.

The following condition is equivalent to the aperiodicity of (a). (See [L1] for a proof.)

(b) *If $g_k : W_k \to W_{k-1}$ are linear maps defined for $k \in \mathbf{Z}/n\mathbf{Z}$ such that $y_{k-1} g_k = g_{k+1} y_k$ for all k, then for any k there exists $p \geq 1$ such that the composition $W_k \xrightarrow{g_k} W_{k-1} \xrightarrow{g_{k-1}} W_{k-2} \to \dots \xrightarrow{g_{k-p+1}} W_{k-p}$ is zero.*

An element $A \in \mathfrak{S}_{D,n,n}$ is said to be *aperiodic* if for any $p \in \mathbf{Z} - \{0\}$ there exists $k \in \mathbf{Z}$ such that $a_{k,k+p} = 0$. Let $\mathfrak{S}_{D,n,n}^{\text{ap}}$ be the set of aperiodic elements in $\mathfrak{S}_{D,n,n}$. Clearly,

(c) the condition that A is aperiodic is equivalent to the condition that A^- and A^+ are aperiodic.
The definition of aperiodicity of A is justified by the following result.

COROLLARY 5.5. *The representation of the cyclic quiver described in 5.1 is aperiodic if and only if for any $p \geq 1$ there exists $k \in \mathbf{Z}$ such that $b_{k,k+p} = 0$, that is, if B is aperiodic.*

5.6. Let $\nu = (\nu_k)$ be a collection of natural numbers indexed by $k \in \mathbf{Z}$ such that $\nu_{k-n} = \nu_k$ for all k. Let Σ_ν be the set of all tableaux $(\mu_{t,p})_{t,p \in \mathbf{Z}, p>0}$ with entries in $\mathbf{N}$ such that

$$\mu_{t,p} = \mu_{t-n,p} \text{ for all } t,p, \qquad \sum_{t,p; t \leq k < t+p} \mu_{t,p} = \nu_k \text{ for all } k \in \mathbf{Z}.$$

Let (W_k) be as in 5.4 with $\dim(W_k) = \nu_k$ for all $k \in \mathbf{Z}/n\mathbf{Z}$. The group

$$G_W = \prod_{k \in \mathbf{Z}/n\mathbf{Z}} GL(W_k)$$

acts naturally on E_W^{nil} with finitely many orbits. It is well known that the set of orbits is in natural bijection with the set Σ_ν. The orbit of a point in E_W^{nil} corresponds to $(\mu_{t,p})$ where $\mu_{t,p}$ is the number of indecomposable summands of the representation of the cyclic quiver that are isomorphic to $V_{t,p}$.

COROLLARY 5.7. *Let $\mathbf{a} \in \mathfrak{S}_{D,n}, \mathbf{L} \in \mathcal{F}_{\mathbf{a}}$. Let ν be as in 5.6. Let*

$$\mathcal{X} = \{\mathcal{L} \in \mathcal{F}_{\mathbf{a}} | \mathcal{L}_k \subset L_k, \dim(L_k/\mathcal{L}_k) = \nu_k \quad \forall k\}.$$

Note that, if $\mathcal{L} \in \mathcal{X}$ then the $\mathbf{k}$-vector space $L_k/\mathcal{L}_k$ is isomorphic to W_k for any $k \in \mathbf{Z}/n\mathbf{Z}$ hence we may transport (via such an isomorphism) the representation of the cyclic quiver on $(L_k/\mathcal{L}_k)$ to a representation on (W_k). This defines a map

$$\iota : \{B \in \mathfrak{S}_{D,n,n} | X_B^{\mathbf{L}} \subset \mathcal{X}\} \to \Sigma_\nu.$$

(ι carries B where $\mathcal{L} \in X_B^{\mathbf{L}}$ to the parameter of the corresponding G_W-orbit in E_W^{nil}.) ι is injective. Its image is the set of all $(\mu_{t,p}) \in \Sigma_\nu$ such that
$\mu_{t,1} + \mu_{t,2} + \mu_{t,3} + \cdots \leq a_t$ for all t.

Assume that $(\mu_{t,p}) = \iota(B)$. Using 5.3, we have

$$a_t = b_{t,*} = b_{t,t} + b_{t,t+1} + b_{t,t+2} + \cdots = b_{t,t} + \mu_{t,1} + \mu_{t,2} + \mu_{t,3} + \ldots.$$

Thus, $a_t - (\mu_{t,1} + \mu_{t,2} + \mu_{t,3} + \ldots) = b_{t,t} \geq 0$.

Conversely, assume that $(\mu_{t,p}) \in \Sigma_\nu$ is such that $\mu_{t,1} + \mu_{t,2} + \mu_{t,3} + \cdots \leq a_t$ for all t. We define $B \in \mathfrak{S}_{D,n,n}$ by
$b_{i,j} = \mu_{i,j-i}$ for $i < j$,
$b_{t,t} = a_t - (\mu_{t,1} + \mu_{t,2} + \mu_{t,3} + \ldots)$
$b_{i,j} = 0$ for $i > j$.
We have $r(B) = \mathbf{a}$. Hence there exists $\mathcal{L} \in \mathcal{F}^n$ such that $(\mathbf{L}, \mathcal{L}) \in \mathcal{O}_B$. By 4.7 we have $\mathcal{L}_k \subset L_k$ for all k. It is clear that $X_B^{\mathbf{L}} \subset \mathcal{X}$ and that $\iota(B) = (\mu_{t,p}) \in \Sigma_\nu$. The corollary is proved.

In the remainder of this section we assume that $\mathbf{k}$ is algebraically closed.

LEMMA 5.8. *Let $\mathcal{U}_{\mathbf{a}}$ be the subset of E_W^{nil} consisting of those representations such that the corresponding $(\mu_{t,p}) \in \Sigma_\nu$ satisfies $\mu_{t,1} + \mu_{t,2} + \mu_{t,3} + \cdots \leq a_t$ for all t. Then $\mathcal{U}_{\mathbf{a}}$ is open in E_W^{nil}.*

A representation (y_k) as in 5.4(a) belongs to $\mathcal{U}_{\mathbf{a}}$ if and only if

$\dim \operatorname{coker}(y_{t-1} : W_{t-1} \to W_t) \le a_t$
for all t. This is clearly an open condition. The lemma is proved.

LEMMA 5.9. *Let $B \in \mathfrak{S}^-_{D,n,n'}, \mathbf{a} = r(B)$. Assume that $(\mathbf{L}, \mathbf{L}') \in \mathcal{O}_B$ (hence $L'_k \subset L_k$ for all k), and $\dim(L_k/L'_k) = \nu_k$ for all k, where $\nu = (\nu_k)$ is as in 5.6. Let G' be the group of all $g \in G_{\mathbf{L}}$ such that $g \in G_{\mathbf{L}'}$ and g induces the identity map on L_k/L'_k for all k. Then $\dim(G_{\mathbf{L}}/G') = \sum_{i\in[1,n]} a_i\nu_i$.*

Arguing as in the proof of Lemma 4.3 we see that $\dim(G_{\mathbf{L}}/G') = \dim(X/X'')$ where X is as in that proof and X'' is the set of all $T \in \mathfrak{g}$ such that $T(L_i) \subset L'_i$ for all i. Let $M_{i,j}$ be associated to $\mathbf{L}, \mathbf{L}'$ as in 1.4. We have $M_{i,j} = 0$ unless $i \le j$. Then X consists of all collections $(T_{i,j,k,l})$ where $T_{i,j,k,l} : M_{i,j} \to M_{k,l}$ are $\mathbf{k}$-linear maps defined for $(i,j,k,l) \in \mathbf{Z}^4$ with $l \ge k \le i \le j$ such that $T_{i-n,j-n',k-n,j-n'} = \epsilon T_{i,j,k,l}\epsilon^{-1}$. On the other hand, X'' consists of all collections $(T_{i,j,k,l})$ where $T_{i,j,k,l} : M_{i,j} \to M_{k,l}$ are $\mathbf{k}$-linear maps defined for $(i,j,k,l) \in \mathbf{Z}^4$ with $k \le l \le i \le j$ such that $T_{i-n,j-n',k-n,j-n'} = \epsilon T_{i,j,k,l}\epsilon^{-1}$. Hence X/X'' is the space consisting of all collections $(T_{i,j,k,l})$ where $T_{i,j,k,l} : M_{i,j} \to M_{k,l}$ are $\mathbf{k}$-linear maps defined for $(i,j,k,l) \in \mathbf{Z}^4$ with $k \le i \le j$, $i < l$, such that $T_{i-n,j-n',k-n,j-n'} = \epsilon T_{i,j,k,l}\epsilon^{-1}$. Hence

$$\dim(X/X'') = \sum_{\substack{i,j,k,l\\ i\in[1,n]\\ k\le i\le j\\ i<l}} b_{i,j}b_{k,l} = \sum_{\substack{i,k,l\\ i\in[1,n]\\ k\le i<l}} b_{i,*}b_{k,l}.$$

For each $i \in [1,n]$ we have

$$\sum_{\substack{k,l\\ k\le i<l}} b_{k,l} = \sum_{u;u>i} (b_{*,u} - b_{u,*}) = \sum_{u;u>i} (\dim(L'_u/L'_{u-1}) - \dim(L_u/L_{u-1}))$$
$$= \sum_{u;u>i} (-\dim(L_u/L'_u) + \dim(L_{u-1}/L'_{u-1})) = \dim(L_i/L'_i) = \nu_i.$$

Hence $\dim(X/X'') = \sum_{i\in[1,n]} a_i\nu_i$. The lemma is proved.

5.10. Let $\mathbf{a} \in \mathfrak{S}_{D,n}, \mathbf{L} \in \mathcal{F}_{\mathbf{a}}$. Let ν and (W_k) be as in 5.6. Let $\tilde{\mathcal{X}}$ be the set of all pairs $(\mathcal{L}, \phi)$ where $\mathcal{L} \in \mathcal{X}$ (see 5.7) and $\phi = (\phi_k)_{k\in\mathbf{Z}/n\mathbf{Z}}$ is a collection of vector space isomorphisms $\phi_k : L_k/\mathcal{L}_k \xrightarrow{\sim} W_k$. (As in 5.1, we may regard $L_k/\mathcal{L}_k$ as depending only on the residue class of k modulo n.) We can regard $\tilde{\mathcal{X}}$ as an algebraic variety in a natural way. We have a diagram of algebraic varieties

$$\mathcal{X} \xleftarrow{\alpha} \tilde{\mathcal{X}} \xrightarrow{\beta} \mathcal{U}_{\mathbf{a}} \xrightarrow{\gamma} E^{\mathrm{nil}}_W;$$

here $\alpha(\mathcal{L}, \phi) = \mathcal{L}$, $\mathcal{U}_{\mathbf{a}}$ is as in 5.8, γ is the inclusion $\mathcal{U}_{\mathbf{a}} \subset E^{\mathrm{nil}}_W$ and $\beta(\mathcal{L}, \phi) = (y_k)$ where y_k is the composition

$$W_k \xrightarrow{\phi_k^{-1}} L_k/\mathcal{L}_k \to L_{k+1}/\mathcal{L}_{k+1} \xrightarrow{\phi_{k+1}} W_{k+1}$$

(the middle map is induced by the inclusion $L_k \subset L_{k+1}$). Now $G_{\mathbf{L}} \times G_W$ acts on $\tilde{\mathcal{X}}$ by $(g, g_1) : (\mathcal{L}, \phi) \mapsto (\mathcal{L}', \phi')$ where $\mathcal{L}' = g(\mathcal{L})$ and ϕ'_k is the composition

$$L_k/\mathcal{L}'_k \xrightarrow{g^{-1}} L_k/\mathcal{L}_k \xrightarrow{\phi_k} W_k \xrightarrow{g_1} W_k.$$

This action is compatible under α, γ, β with the action of $G_{\mathbf{L}} \times G_W$ on $\mathcal{X}$ (trivial on G_W and already known on $G_{\mathbf{L}}$) and with the action of $G_{\mathbf{L}} \times G_W$ on $\mathcal{U}_{\mathbf{a}}, E^{\mathrm{nil}}_W$ (trivial on $G_{\mathbf{L}}$ and already known on G_W). Note that α is in fact a principal bundle with group G_W. On the other hand, from 5.7 we see that β is surjective and that

(a) α, β establish bijections between the set $G_{\mathbf{L}}$-orbits on $\mathcal{X}$, the set of $G_{\mathbf{L}} \times G_W$-orbits on $\tilde{\mathcal{X}}$ and the set of G_W-orbits on $\mathcal{U}_{\mathbf{a}}$.

Although β is not in general a locally trivial fibration, we have the following.

LEMMA 5.11. *The morphism $\beta : \tilde{\mathcal{X}} \to \mathcal{U}_{\mathbf{a}}$ is smooth with connected fibres.*

We will verify the following statement:

(a) *the fibres of β are exactly the orbits of the $G_{\mathbf{L}}$-action on $\tilde{\mathcal{X}}$ and they are all connected of the same dimension.*

The fact that the $G_{\mathbf{L}}$-action on $\tilde{\mathcal{X}}$ is fibre preseving has been already noted (and is obvious). Now let $(\mathcal{L}, \phi), (\mathcal{L}', \phi')$ be two points in the same fibre of β. We want to show that there exists $g \in G_{\mathbf{L}}$ which carries $(\mathcal{L}, \phi)$ to $(\mathcal{L}', \phi')$. From 5.10(a) we see that some element of $G_{\mathbf{L}}$ carries $\mathcal{L}$ to $\mathcal{L}'$. Hence we can assume that $\mathcal{L} = \mathcal{L}'$. Since $\beta(\mathcal{L}, \phi) = \beta(\mathcal{L}, \phi')$, the compositions

$$W_k \xrightarrow{\phi_k^{-1}} L_k/\mathcal{L}_k \to L_{k+1}/\mathcal{L}_{k+1} \xrightarrow{\phi_{k+1}} W_{k+1}$$

$$W_k \xrightarrow{\phi_k'^{-1}} L_k/\mathcal{L}_k \to L_{k+1}/\mathcal{L}_{k+1} \xrightarrow{\phi'_{k+1}} W_{k+1}$$

coincide. Hence if $u_k = \phi_k'^{-1}\phi_k$, then (u_k) satisfies the assumptions of Lemma 1.13. By that lemma, we can find $e \in G_{\mathbf{L}} \cap G_{\mathbf{L}'}$ such that e induces (u_k). We then have $e^{-1}(\mathcal{L}, \phi) = (\mathcal{L}, \phi')$.

Thus, any fibre of β is a homogeneous space for $G_{\mathbf{L}}$. If $(\mathcal{L}, \phi) \in \tilde{\mathcal{X}}$, the stabilizer of $(\mathcal{L}, \phi)$ in $G_{\mathbf{L}}$ is the group of all g in $G_{\mathbf{L}} \cap G_{\mathcal{L}}$ such that g induces the identity map on $L_k/\mathcal{L}_k$ for all k. Using Lemma 5.9, we see that the dimension of the $G_{\mathbf{L}}$-orbit of $(\mathcal{L}, \phi)$ (hence the dimension of the fibre containing $(\mathcal{L}, \phi)$) is equal to $\sum_{i \in [1,n]} a_i \nu_i$, which is independent of $(\mathcal{L}, \phi)$. Finally note that $G_{\mathbf{L}}$ acts on $\tilde{\mathcal{X}}$ through a quotient which is a connected algebraic group, hence all its orbits are connected. This proves (a). The lemma is proved.

5.12. In the remainer of this section we assume that $n \geq 2$. In the setup of 5.10, we consider a sequence of integers

(a) $i_\bullet = (i_1, i_2, \dots, i_N)$

such that

(b) $\sharp\{s \in [1, N] | i_s = k \mod n\} = \nu_k$ for all $k \in \mathbf{Z}$.

To this sequence we associate a morphism $\rho : \dot{E}_W^{\text{nil}} \to E_W^{\text{nil}}$ in the following way (a special case of [L1, 1.5]).

$\dot{E}_W^{\text{nil}}$ consists of all collections $W_k^s, y_k : W_k \to W_{k+1}$, $(k \in \mathbf{Z}/n\mathbf{Z}, s \in [0, N])$ where W_k^s is a vector subspace of W_k and we have

$W_k^0 = W_k$, $W_k^N = 0$,

$W_k^s = W_k^{s-1}$ for $k \neq i_s \mod n, s \in [1, N]$,

$W_k^s \subset W_k^{s-1}, \dim(W_k^{s-1}/W_k^s) = 1$ for $k = i_s \mod n, s \in [1, N]$;

y_k is a linear map such that $y_s(W_k^s) \subset W_{k+1}^s$ for $k \in \mathbf{Z}/n\mathbf{Z}, s \in [0, N]$.

We define ρ by $(W_k^s, y_k) \mapsto (y_k)$. This is a proper morphism and $\dot{E}_W^{\text{nil}}$ is a smooth irreducible variety. The direct image of $\bar{\mathbf{Q}}_l$ under ρ is denoted by $K'_{i_\bullet}$.

Similarly, to $i_\bullet$ we associate a morphism $\kappa : \dot{\mathcal{X}} \to \mathcal{X}$ as follows.

$\dot{\mathcal{X}}$ consists of all collections $(\mathbf{L}^s)_{s \in [0,N]}$ where $\mathbf{L}^s \in \mathcal{F}^n$ and we have

$\mathbf{L}^0 = \mathbf{L}$,

$\mathbf{L}_k^s = \mathbf{L}_k^{s-1}$ for $k \neq i_s \mod n, s \in [1, N]$,

$\mathbf{L}_k^s \subset \mathbf{L}_k^{s-1}, \dim(\mathbf{L}_k^{s-1}/\mathbf{L}_k^s) = 1$ for $k = i_s \mod n, s \in [1, N]$.

We define κ by $(\mathbf{L}^s) \mapsto \mathbf{L}^N$. Then $\dot{\mathcal{X}}$ is either empty or a smooth irreducible variety and κ is a proper morphism. Let $K_{i_\bullet}$ be the direct image of $\bar{\mathbf{Q}}_l$ under κ. (If $\dot{\mathcal{X}}$ is empty, then $K_{i_\bullet} = 0$.)

5.13. Next, to $i_\bullet$ we associate a morphism $\lambda : \dot{\tilde{\mathcal{X}}} \to \tilde{\mathcal{X}}$ as follows.

$\dot{\tilde{\mathcal{X}}}$ consists of all collections $(\mathbf{L}^s)_{s\in[0,N]}$, $\phi = (\phi_k)_{k\in\mathbf{Z}/n\mathbf{Z}}$ where $(\mathbf{L}^s) \in \dot{\mathcal{X}}$ and $\phi_k : L_k/\mathcal{L}_k \xrightarrow{\sim} W_k$ are vector space isomorphisms.

We define λ by $((\mathbf{L}^s), \phi) \mapsto (\mathbf{L}^N, \phi)$. Then $\dot{\tilde{\mathcal{X}}}$ is either empty or a smooth irreducible variety and λ is a proper morphism. Let $K''_{i_\bullet}$ be the direct image of $\bar{\mathbf{Q}}_l$ under λ. (If $\dot{\tilde{\mathcal{X}}}$ is empty, then $K''_{i_\bullet} = 0$.) We define a commutative diagram

$$\begin{array}{ccccc}
\dot{\mathcal{X}} & \xleftarrow{\dot{\alpha}} & \dot{\tilde{\mathcal{X}}} & \xrightarrow{(\gamma\beta)^{\cdot}} & \dot{E}_W^{\mathrm{nil}} \\
{\scriptstyle\kappa}\downarrow & & {\scriptstyle\lambda}\downarrow & & {\scriptstyle\rho}\downarrow \\
\mathcal{X} & \xleftarrow{\alpha} & \tilde{\mathcal{X}} & \xrightarrow{\gamma\beta} & E_W^{\mathrm{nil}}
\end{array}$$

as follows. We set $\dot{\alpha}(\mathbf{L}^s, \phi) = (\mathbf{L}^s)$ and $(\beta\gamma)^{\cdot}(\mathbf{L}^s, \phi) = (W_k^s, y_k)$ where W_k^s is the image of $\mathbf{L}_k^s/\mathbf{L}_k^N$ under $\phi_k : L_k/\mathbf{L}_k^N \xrightarrow{\sim} W_k$ and $(y_k) = \beta(\mathbf{L}^N, \phi)$.

One checks easily that both squares in the diagram are cartesian. It follows that

(a) $\alpha^* K_{i_\bullet} = K''_{i_\bullet} = (\gamma\beta)^* K'_{i_\bullet}$

5.14. For each

(a) $B \in \mathfrak{S}^-_{D,n,n}$ such that $X_B^{\mathbf{L}} \in \mathcal{X}$

we denote by o_B the corresponding G_W-orbit in E_W^{nil}. Note that o_B is contained in the open subset $\mathcal{U}_{\mathbf{a}}$ of E_W^{nil}.

Let P_B be the irreducible $G_{\mathbf{L}}$-equivariant perverse sheaf on $\mathcal{X}$ whose support is the closure of $X_B^{\mathbf{L}}$. Let P''_B be the irreducible $G_{\mathbf{L}} \times G_W$-equivariant perverse sheaf on $\tilde{\mathcal{X}}$ whose support is the closure of $\alpha^{-1}(X_B^{\mathbf{L}})$. Let P'_B be the irreducible G_W-equivariant perverse sheaf on E_W^{nil} whose support is the closure of o_B.

By the decomposition theorem [BBD], we have

$K'_{i_\bullet} = \oplus_{B;\delta} P'_B[\delta]^{\oplus M'_{B,i_\bullet,\delta}} \oplus \tilde{K}$

$K_{i_\bullet} = \oplus_{B;\delta} P_B[\delta]^{\oplus M_{B,i_\bullet,\delta}}$

$K''_{i_\bullet} = \oplus_{B;\delta} P''_B[\delta]^{\oplus M''_{B,i_\bullet,\delta}}$

where B runs through the set (a), δ runs over the integers and

$M'_{B,i_\bullet,\delta}, M_{B,i_\bullet,\delta}, M''_{B,i_\bullet,\delta} \in \mathbf{N}$;

$\tilde{K}$ is a direct sum of perverse sheaves (with shifts) on E_W^{nil} with support in the closed subset $E_W^{\mathrm{nil}} - \mathcal{U}_{\mathbf{a}}$ of E_W^{nil}. Restricting to the open subset $\mathcal{U}_{\mathbf{a}} \subset E_W^{\mathrm{nil}}$ we obtain

$$K'_{i_\bullet}|_{\mathcal{U}_{\mathbf{a}}} = \oplus_{B;\delta} (P'_B)|_{\mathcal{U}_{\mathbf{a}}}[\delta]^{\oplus M'_{B,i_\bullet,\delta}}.$$

Since $\beta : \tilde{\mathcal{X}} \to \mathcal{U}_{\mathbf{a}}$ is smooth with connected fibres, we have $\beta^*((P'_B)|_{\mathcal{U}_{\mathbf{a}}}) = P''_B[d_1]$, where d_1 is independent of B. (See [BBD, 4.2.5].) Since α is a principal fibration, we have $\alpha^*(P_B) = P''_B[d_2]$ where d_2 is independent of B. Using these equalities together with 5.13(a), we deduce that

$$\oplus_{B;\delta} P''_B[\delta + d_1]^{\oplus M'_{B,i_\bullet,\delta}} = \oplus_{B;\delta} P''_B[\delta]^{\oplus M''_{B,i_\bullet,\delta}} = \oplus_{B;\delta} P''_B[\delta + d_2]^{\oplus M_{B,i_\bullet,\delta}}.$$

Hence

(b) $M_{B,i_\bullet,\delta-d_2} = M'_{B,i_\bullet,\delta-d_1}$ for all B, δ.

5.15. We set
$\chi(i_\bullet) = [s_1]![s_2]!\dots[s_r]! \in \mathcal{A}$
where $s_1, s_2, \dots$ are defined by

$$i_1 = i_2 = \dots = i_{s_1} \neq i_{s_1+1} = \dots = i_{s_1+s_2} \neq i_{s_1+s_2+1} \dots .$$

5.16. Assume now that B in 5.14(a) is aperiodic. Then, by 5.5, o_B consists of aperiodic points. By [L2, 5.9] and [L1, 7.3], there exist sequences $i^u_\bullet$ as in 5.12(a),(b), with $u = 1, 2, \dots, h$ and elements $f_{u,\delta} \in \mathcal{A}$ such that for B' in 5.14(a),

$$\sum_{u\in[1,h],\delta\in\mathbf{Z}} f_{u,\delta}\chi(i^u_\bullet)^{-1} M_{P_{B'},i^u_\bullet,\delta} v^\delta$$

equals 1 if $B' = B$ and 0 if $B' \neq B$. (All but finitely many $f_{u,\delta}$ are 0.) Moreover, $\chi(i^u_\bullet)^{-1} M_{P_{B'},i^u_\bullet,\delta} \in \mathcal{A}$ for all u. Using now 5.14(b), we deduce that

$$\sum_{u\in[1,h],\delta\in\mathbf{Z}} f'_{u,\delta}\chi(i^u_\bullet)^{-1} M'_{P_{B'},i^u_\bullet,\delta} v^\delta$$

equals 1 if $B' = B$ and 0 if $B' \neq B$; here $f'_{u,\delta}$ equals $f_{u,\delta}$ times a power of v depending on u but not on δ.

This can be interpreted as follows: the element $\{B\} \in \mathfrak{A}_{D,n,n}$ is an $\mathcal{A}$-linear combination of products

$$\chi(i_\bullet)^{-1} E_{i_N} E_{i_{N-1}} \dots E_{i_1}[\mathbf{i_a}]$$

for various $i_\bullet$ as in 5.12(a),(b). Thus, we have the following result.

PROPOSITION 5.17. *Assume that $B \in \mathfrak{S}^-_{D,n,n}$ is aperiodic. Then $\{B\} \in \mathbf{U}_{D,\mathcal{A}}$.*

Using 4.12 and the fact that $\Psi : \mathfrak{A}_{D,n,n} \to \mathfrak{A}_{D,n,n}$ maps $\mathbf{U}_D$ to itself and $\mathbf{U}_{D,\mathcal{A}}$ to itself, we deduce from 5.17 the following result.

PROPOSITION 5.18. *Assume that $B \in \mathfrak{S}^+_{D,n,n}$ is aperiodic. Then $\{B\} \in \mathbf{U}_{D,\mathcal{A}}$.*

6. Singular supports.

6.1. In this section we assume that $n \geq 2$ and that $\mathbf{k} = \mathbf{C}$.

For any $\mathbf{L} \in \mathcal{F}^n$, let $E_\mathbf{L}$ be the set of all $f \in \mathfrak{g}$ such that $f(L_k) \subset L_{k-1}$ for all $k \in \mathbf{Z}$. This is naturally a $\mathbf{k}[\epsilon]$-module.

For $\mathbf{b} \in \mathfrak{S}_{D,n}$, let $T^*\mathcal{F}_\mathbf{b}$ be te set of all pairs $(\mathbf{L}, e)$ where $\mathbf{L} \in \mathcal{F}_\mathbf{b}$ and $e \in E_\mathbf{L}$.

We will think of $T^*\mathcal{F}_\mathbf{b}$ as the "cotangent bundle" of $\mathcal{F}_\mathbf{b}$. This can be justified by the fact that although $\mathcal{F}_\mathbf{b}$ is an infinite dimensional object, it is in a sense (explained in [KT]), a limit of smooth algebraic varieties. (The discussion in [KT] applies to any affine flag manifold.)

We fix $\mathbf{a} \in \mathfrak{S}_{D,n}, \mathbf{L} \in \mathcal{F}_\mathbf{a}$. For any $A \in \mathfrak{S}_{D,n,n}$ such that $r(A) = \mathbf{a}, c(A) = \mathbf{b}$, we set

$$\mathcal{C}_A = \{(\mathbf{L}', e) \in T^*\mathcal{F}_\mathbf{b} | \mathbf{L}' \in X^\mathbf{L}_A, e \in E_\mathbf{L} \cap E_{\mathbf{L}'}\}.$$

We will think of $\mathcal{C}_A$ as the "conormal bundle" to the $G_\mathbf{L}$-orbit $X^\mathbf{L}_A$ in $\mathcal{F}_\mathbf{b}$. This is again justified by the discussion in [KT].

6.2. We consider a sequence $i_1, i_2, \dots, i_{p+r}$ of integers and a sequence

(a) $$\mathbf{a}^0, \mathbf{a}^1, \dots, \mathbf{a}^p, \mathbf{a}^{p+1}, \dots, \mathbf{a}^{p+r}$$

in $\mathfrak{S}_{D,n}$ such that
$\mathbf{a}^0 \smile_{i_1} \mathbf{a}^1, \mathbf{a}^1 \smile_{i_2} \mathbf{a}^2, \dots, \mathbf{a}^{p-1} \smile_{i_p} \mathbf{a}^p,$

$\mathbf{a}^{p+1} \smile_{i_{p+1}} \mathbf{a}^p, \mathbf{a}^{p+2} \smile_{i_{p+2}} \mathbf{a}^{p+1}, \dots, \mathbf{a}^{p+r} \smile_{i_{p+r}} \mathbf{a}^{p+r-1}$.
We assume that $\mathbf{a}^0 = \mathbf{a}$. We set $\mathbf{a}^{p+r} = \mathbf{b}$.

Consider the following condition for $A \in \mathfrak{S}_{D,n,n}$ such that $r(A) = \mathbf{a}, c(A) = \mathbf{b}$:

(b) $\{A\}$ *appears with cofficient* $\neq 0$ *in the product*

$$\{{}_{\mathbf{a}^0}\mathbf{e}_{\mathbf{a}^1}\} \cdot \{{}_{\mathbf{a}^1}\mathbf{e}_{\mathbf{a}^2}\} \cdot \dots \cdot \{{}_{\mathbf{a}^{p-1}}\mathbf{e}_{\mathbf{a}^p}\} \cdot \{{}_{\mathbf{a}^p}\mathbf{f}_{\mathbf{a}^{p+1}}\} \cdot \dots \cdot \{{}_{\mathbf{a}^{p+r-1}}\mathbf{f}_{\mathbf{a}^{p+r}}\}$$

expressed in $\mathfrak{A}_{D,n,n}$ *as a linear combination of elements* $\{A'\}$.

Condition (b) can be expressed geometrically as follows.

Recall that we have fixed $\mathbf{L} \in \mathcal{F}_{\mathbf{a}^0}$. We consider the set Y of all sequences

$$(\mathbf{L}^0, \mathbf{L}^1, \dots, \mathbf{L}^{p+r}) \in \mathcal{F}_{\mathbf{a}^0} \times \mathcal{F}_{\mathbf{a}^1} \times \dots \times \mathcal{F}_{\mathbf{a}^{p+r}}$$

such that $\mathbf{L}^0 = \mathbf{L}$,

(c) $L_k^t = L_k^{t-1}$ for $k \neq i_t \mod n, t \in [1, p+r]$,

(d) $L_k^t \subset L_k^{t-1}, \dim(L_k^{t-1}/L_k^t) = 1$ for $k = i_t \mod n, t \in [1,p]$,

(e) $L_k^{t-1} \subset L_k^t, \dim(L_k^t/L_k^{t-1}) = 1$ for $k = i_t \mod n, t \in [p+1, p+r]$.

Then Y is a smooth projective variety (an iterated projective space bundle over a point.) Let $\pi : Y \to \mathcal{F}_{\mathbf{b}}$ be defined by

$$\pi(\mathbf{L}^0, \mathbf{L}^1, \dots, \mathbf{L}^{p+r}) = \mathbf{L}^{p+r}.$$

Consider the direct image $\pi_*(\mathbf{C}[d])$ where $d = \dim Y$. Then A satisfies (b) if and only if some shift of the $G_{\mathbf{L}}$-equivariant simple perverse sheaf supported by $\bar{X}_A^{\mathbf{L}}$ appears as a direct summand in $\pi_*(\mathbf{C}[d])$. (This follows from 4.5.) In that case, the "conormal bundle" $\mathcal{C}_A$ is contained in the "singular support" of $\pi_*(\mathbf{C}[d])$ (a subset of $T^*\mathcal{F}_{\mathbf{b}}$).

6.3. The cotangent space of Y at $(\mathbf{L}^0, \mathbf{L}^1, \dots, \mathbf{L}^{p+r})$ is naturally the cokernel of the map

$$(E_{\mathbf{L}^0} \cap E_{\mathbf{L}^1}) \oplus (E_{\mathbf{L}^1} \cap E_{\mathbf{L}^2}) \oplus \dots \oplus (E_{\mathbf{L}^{p+r-1}} \cap E_{\mathbf{L}^{p+r}}) \xrightarrow{\theta} E_{\mathbf{L}^1} \oplus \dots \oplus E_{\mathbf{L}^{p+r}}$$

given by

$$\begin{aligned}&(e_{01}, e_{02}, \dots, e_{p+r-1,p+r})\\ &\mapsto (e_{01} - e_{12}, e_{12} - e_{23}, \dots, e_{p+r-2,p+r-1} - e_{p+r-1,p+r}, e_{p+r-1,p+r}).\end{aligned}$$

Now π induces a linear map from the cotangent space to $\mathcal{F}_{\mathbf{b}}$ at $\mathbf{L}^{p+r}$ to the cotangent space to Y at $(\mathbf{L}^0, \mathbf{L}^1, \dots, \mathbf{L}^{p+r})$ given by

(a) $$E_{\mathbf{L}^{p+r}} \to \operatorname{coker}(\theta), \quad e \mapsto (0, 0, \dots, 0, e).$$

The kernel of this linear map consists of those $e \in E_{\mathbf{L}^{p+r}}$ such that there exists $(e_{01}, e_{02}, \dots, e_{p+r-1,p+r})$ as above with

$$e_{01} = e_{12} = e_{23} = \dots = e_{p+r-1,p+r} = e_{p+r-1,p+r} = e.$$

Thus, the kernel of the linear map (a) is the image of the obvious imbedding

$$E_{\mathbf{L}^0} \cap E_{\mathbf{L}^1} \cap E_{\mathbf{L}^2} \cap \dots \cap E_{\mathbf{L}^{p+r}} \to E_{\mathbf{L}^{p+r}}.$$

We shall apply the estimate [KS] for the singular support of a direct image to $\pi_*(\mathbf{C}[d])$ (even though $\mathcal{F}_{\mathbf{b}}$ is infinite dimensional, see the remarks in 6.1). We see that the singular support of $\pi_*(\mathbf{C}[d])$ is contained in the set of all pairs $(\mathbf{L}', e)$ where $\mathbf{L}' \in \mathcal{F}_{\mathbf{b}}$, and

(b) $e \in E_{\mathbf{L}^0} \cap E_{\mathbf{L}^1} \cap E_{\mathbf{L}^2} \dots \cap E_{\mathbf{L}^{p+r}}$ for some $(\mathbf{L}^0, \mathbf{L}^1, \dots, \mathbf{L}^{p+r}) \in Y$ with $\mathbf{L}^{p+r} = \mathbf{L}'$.

LEMMA 6.4. *Let $\mathbf{L}' \in \mathcal{F}_\mathbf{b}$ and assume that $e \in E_\mathbf{L} \cap E_{\mathbf{L}'}$ satisfies condition 6.3(b). Then for any $k \in \mathbf{Z}$ there exists $s \geq 1$ such that the compositions*

$L_k/(L_k \cap L'_k) \xrightarrow{e} L_{k-1}/(L_{k-1} \cap L'_{k-1}) \xrightarrow{e} \dots \xrightarrow{e} L_{k-s}/(L_{k-s} \cap L'_{k-s})$,

$L'_k/(L_k \cap L'_k) \xrightarrow{e} L'_{k-1}/(L_{k-1} \cap L'_{k-1}) \xrightarrow{e} \dots \xrightarrow{e} L'_{k-s}/(L_{k-s} \cap L'_{k-s})$,

are zero.

Let $(\mathbf{L}^0, \mathbf{L}^1, \dots, \mathbf{L}^{p+r})$ be as in 6.3(b). Let $k \in \mathbf{Z}$. From 6.2(c),(d),(e) we see that:

(a) for $t \in [1,p]$, at least one of $L_k^{t-1}/L_k^t, L_{k-1}^{t-1}/L_{k-1}^t$ is zero, hence $e : L_k^{t-1}/L_k^t \to L_{k-1}^{t-1}/L_{k-1}^t$ is zero; hence $e(L_k^{t-1}) \subset L_{k-1}^t$;

(b) for $t \in [p+1, p+r]$, at least one of $L_k^t/L_k^{t-1}, L_{k-1}^t/L_{k-1}^{t-1}$ is zero, hence $e : L_k^t/L_k^{t-1} \to L_{k-1}^t/L_{k-1}^{t-1}$ is zero; hence $e(L_k^t) \subset L_{k-1}^{t-1}$.

From (a),(b) we deduce

(c) $$e^p(L_k^0) \subset L_{k-p}^p, \quad e^r(L_k^{p+r}) \subset L_{k-r}^p$$

respectively. From 6.2(c),(d) we deduce $L_{k'}^p \subset L_{k'}^0$ and $L_{k'}^p \subset L_{k'}^{p+r}$. Taking $k' = k-p$ or $k' = k-r$, we deduce $L_{k-p}^p \subset L_{k-p}^0 \cap L_{k-p}^{p+r}$ and $L_{k-r}^p \subset L_{k-r}^0 \cap L_{k-r}^{p+r}$. Hence (c) implies

$$e^p(L_k^0) \subset L_{k-p}^0 \cap L_{k-p}^{p+r}, \quad e^r(L_k^{p+r}) \subset L_{k-r}^0 \cap L_{k-r}^{p+r}.$$

The lemma follows.

PROPOSITION 6.5. *Assume that $A \in \mathfrak{S}_{D,n,n}$ satisfies 6.2(b). Then A is aperiodic.*

We choose $\mathbf{L}' \in \mathcal{F}_\mathbf{b}$. Let $\mathcal{L} \in \mathcal{F}^n$ be defined by $\mathcal{L}_i = L_i \cap L'_i$ for all $i \in \mathbf{Z}$.

Let $g_k : L_k/\mathcal{L}_k \to L_{k-1}/\mathcal{L}_{k-1}$ be $\mathbf{k}$-linear maps defined for $k \in \mathbf{Z}$ such that $g_{k-n} = \epsilon g_k \epsilon^{-1}$ for all $k \in \mathbf{Z}$ and such that $j_{k-1} g_k = g_{k+1} j_k$ for all k, where $j_k : L_k/\mathcal{L}_k \to L_{k+1}/\mathcal{L}_{k+1}$ is the canonical map.

By 1.12, we can find $e \in E_\mathbf{L} \cap E_{\mathbf{L}'}$ such that e induces (g_k). We have $(\mathbf{L}', e) \in \mathcal{C}_A$. Hence, from 6.3, 6.4, we see that the conclusion of Lemma 6.4 holds for $(\mathbf{L}', e)$. In particular, the maps (g_k) form a nilpotent representation of the opposite of the cyclic quiver in 5.1. By the criterion 5.4(b), it follows that the representation of the cyclic quiver given by the maps (j_k) is aperiodic. We can apply 5.5 to $\mathbf{L}, \mathcal{L}, A^-$ instead of $\mathbf{L}, \mathcal{L}, B$ and we deduce that for any $p \geq 1$ there exists $k \in \mathbf{Z}$ such that $a^-_{k,k+p} = 0$ (hence $a_{k,k+p} = 0$).

Now let $g'_k : L'_k/\mathcal{L}_k \to L'_{k-1}/\mathcal{L}_{k-1}$ be $\mathbf{k}$-linear maps defined for $k \in \mathbf{Z}$ such that $g'_{k-n} = \epsilon g'_k \epsilon^{-1}$ for all $k \in \mathbf{Z}$ and such that $j'_{k-1} g'_k = g'_{k+1} j'_k$ for all k, where $j'_k : L'_k/\mathcal{L}_k \to L'_{k+1}/\mathcal{L}_{k+1}$ is the canonical map.

By 1.12 (with $\mathbf{L}, \mathbf{L}'$ interchanged) we can find $e \in E_\mathbf{L} \cap E_{\mathbf{L}'}$ such that e induces (g'_k). We have $(\mathbf{L}', e) \in \mathcal{C}_A$. Hence, from 6.3, 6.4, we see that the conclusion of Lemma 6.4 holds for $(\mathbf{L}', e)$. In particular, the maps (g'_k) form a nilpotent representation of the opposite of the cyclic quiver in 5.1. By the criterion 5.4(b), it follows that the representation of the cyclic quiver given by the maps (j'_k) is aperiodic. We can apply 5.5 to $\mathbf{L}', \mathcal{L}, {}^tA^+$ instead of $\mathbf{L}, \mathcal{L}, B$ and we deduce that for any $p \geq 1$ there exists $k \in \mathbf{Z}$ such that $a^+_{k+p,k} = 0$ (hence $a_{k+p,k} = 0$). Equivalently, for any $p < 0$ there exists $k \in \mathbf{Z}$ such that $a_{k,k+p} = 0$. The proposition is proved.

COROLLARY 6.6. *$\mathbf{U}_D$ is contained in the subspace of $\mathfrak{A}_{D,n,n}$ spanned by the elements $\{A\}$ with $A \in \mathfrak{S}_{D,n,n}^{\mathrm{ap}}$ (see 5.4).*

From 2.6, it is easy to see that the $\mathbf{Q}(v)$-vector space $\mathbf{U}_D = \mathbf{U}'_D$ is spanned by the various products as in 6.2(b). The corollary follows.

7. Inner product on $\mathfrak{A}_{D,n,n'}$.

7.1. For any integer $c \geq 0$ we set
$c^\sharp = \prod_{t=1}^{c}(1 - v^{-2t}) \in \mathcal{A}$.
Assume now that $\mathbf{k}$ is finite with q elements. We set $c_q^\sharp = c^\sharp|_{v=\sqrt{q}} \in \bar{\mathbf{Q}}_l$. We define a bilinear form

$$(,) : \mathfrak{A}_{D,n,n';q} \times \mathfrak{A}_{D,n,n';q} \to \bar{\mathbf{Q}}_l$$

by

$$(f, \tilde{f}) = \sum_{\mathbf{L},\mathbf{L}'} \prod_{i\in[1,n]} \frac{1}{(|\mathbf{L}|_i)_q^\sharp} f(\mathbf{L},\mathbf{L}')\tilde{f}(\mathbf{L},\mathbf{L}')$$

for $f, \tilde{f} \in \mathfrak{A}_{D,n,n';q}$ regarded as functions $\mathcal{F}^n \times \mathcal{F}^{n'} \to \bar{\mathbf{Q}}_l$ (as in 1.10); in the sum, $\mathbf{L}$ runs over a set of representatives for the G-orbits in $\mathcal{F}^n$, $\mathbf{L}'$ runs over $\mathcal{F}^{n'}$. For $A, A' \in \mathfrak{S}_{D,n,n'}$, we have

$$(e_A, e_{A'}) = \delta_{A,A'} \prod_{i\in[1,n]} \frac{1}{(|\mathbf{L}|_i)_q^\sharp} \sharp(X_A^{\mathbf{L}}),$$

where $\mathbf{L} \in \mathcal{F}_{r(A)}$, hence

(a) $$([A], [A']) = \delta_{A,A'} \prod_{i\in[1,n]} \frac{1}{(|\mathbf{L}|_i)_q^\sharp} q^{-d_A} \sharp(X_A^{\mathbf{L}}).$$

In particular, the form $(,)$ is symmetric.

LEMMA 7.2. *Let $i \in \mathbf{Z}$ and let $f, \tilde{f} \in \mathfrak{A}_{D,n,n';q}$. Let $\mathbf{a}$ be as in 2.6 and let $\mathbf{b} \in \tilde{\mathfrak{S}}_n$. We have*

$$(E_i(f), \tilde{f}) = (f, \sqrt{q}K_{\mathbf{a}}F_i(\tilde{f})), \quad (F_i(f), \tilde{f}) = (f, \sqrt{q}K_{-\mathbf{a}}E_i(\tilde{f})),$$
$$(K_{\mathbf{b}}(f), \tilde{f}) = (f, K_{\mathbf{b}}(\tilde{f})).$$

To prove the third equality we may assume that $f = e_A, \tilde{f} = e_{A'}$. Then the desired equality is immediate from 7.1(a). The second equality follows immediately from the first and third equality. It remains to prove the first equality.

Let $\hat{\mathcal{F}}^n$ be the set of all collections $\hat{\mathbf{L}} = (L_j)$ where L_j are lattices in V defined for any $j \in \mathbf{Z}$ such that $j \neq i \mod n$ and such that $L_{j'} \subset L_j$ for all $j' \leq j$ with $j \neq i \mod n, j' \neq i \mod n$ and $L_{j-n} = \epsilon L_j$ for all $j \neq i \mod n$. We fix a set of representatives Ξ for the orbits of the obvious action of G on $\hat{\mathcal{F}}^n$. Let $\zeta : \mathcal{F}^n \to \hat{\mathcal{F}}^n$ be the map defined by attaching to $\mathbf{L} = (L_j)$ the collection obtained by forgetting all L_j with $j = i \mod n$. For $\mathbf{L} \in \mathcal{F}_{\mathbf{c}}$, the number of $\tilde{\mathbf{L}} \in \mathcal{F}_{\mathbf{c}}$ such that $\zeta(\tilde{\mathbf{L}}) = \zeta(\mathbf{L})$ is

$$\frac{N}{\prod_{t=1}^{|\mathbf{L}|_i}(q^t - 1)\prod_{t=1}^{|\mathbf{L}|_{i+1}}(q^t - 1)} = (\sqrt{q})^{-|\mathbf{L}|_i(|\mathbf{L}|_i+1)-|\mathbf{L}|_{i+1}(|\mathbf{L}|_{i+1}+1)} \frac{N}{(|\mathbf{L}|_i)_q^\sharp(|\mathbf{L}|_{i+1})_q^\sharp},$$

where $N = \prod_{t=1}^{|\mathbf{L}|_i+|\mathbf{L}|_{i+1}}(q^t - 1)$. Hence, for $f, \tilde{f} \in \mathfrak{A}_{D,n,n';q}$, we have

$$(f, \tilde{f})$$
$$= \sum_{\substack{\mathbf{L},\mathbf{L}'\in\mathcal{F}^n \\ \zeta(\mathbf{L})\in\Xi}} \prod_{j\in[1,n]} (\sqrt{q})^{|\mathbf{L}|_i(|\mathbf{L}|_i+1)+|\mathbf{L}|_{i+1}(|\mathbf{L}|_{i+1}+1)} \frac{(|\mathbf{L}|_i)_q^\sharp(|\mathbf{L}|_{i+1})_q^\sharp}{N(|\mathbf{L}|_j)_q^\sharp} f(\mathbf{L},\mathbf{L}')\tilde{f}(\mathbf{L},\mathbf{L}').$$

We have

$$(E_i f)(\mathbf{L},\mathbf{L}') = \sum (\sqrt{q})^{-|\mathbf{L}''|_i} f(\mathbf{L}'',\mathbf{L}')$$

sum over all $\mathbf{L}''$ such that $\zeta(\mathbf{L}'') = \zeta(\mathbf{L})$ and $L_{i-1} \subset L''_i \subset L_i$, $\dim(L_i/L''_i) = 1$,

$$(\sqrt{q}K_{\mathbf{a}}F_i\tilde{f})(\mathbf{L},\mathbf{L}') = \sum (\sqrt{q})^{-1-2|\mathbf{L}''|_{i+1}+|\mathbf{L}''|_i} \tilde{f}(\mathbf{L}'',\mathbf{L}')$$

sum over all $\mathbf{L}''$ such that $\zeta(\mathbf{L}'') = \zeta(\mathbf{L})$ and $L_i \subset L''_i \subset L_{i+1}$, $\dim(L''_i/L_i) = 1$. It follows that

$$\begin{aligned}&(E_i(f),\tilde{f})\\ &= \sum \prod_{\substack{j\in[1,n]\\ j\neq i,i+1\\ \text{mod } n}} \frac{(\sqrt{q})^{|\mathbf{L}|_i(|\mathbf{L}|_i+1)+|\mathbf{L}|_{i+1}(|\mathbf{L}|_{i+1}+1)-(|\mathbf{L}|_i-1)} f(\mathbf{L}'',\mathbf{L}')}{N(|\mathbf{L}|_j)^\sharp_q}\tilde{f}(\mathbf{L},\mathbf{L}'),\end{aligned}$$

sum over all $\mathbf{L},\mathbf{L}',\mathbf{L}'' \in \mathcal{F}^n$ such that $\zeta(\mathbf{L}) = \zeta(\mathbf{L}'') \in \Xi$ and $L_{i-1} \subset L''_i \subset L_i$, $\dim(L_i/L''_i) = 1$. Similarly,

$$\begin{aligned}&(f,\sqrt{q}K_{\mathbf{a}}F_i(\tilde{f}))\\ &= \sum \prod_{\substack{j\in[1,n]\\ j\neq i,i+1\\ \text{mod } n}} \frac{(\sqrt{q})^{|\mathbf{L}|_i(|\mathbf{L}|_i+1)+|\mathbf{L}|_{i+1}(|\mathbf{L}|_{i+1}+1)-1-2|\mathbf{L}''|_{i+1}+|\mathbf{L}''|_i}}{N(|\mathbf{L}|_j)^\sharp_q} f(\mathbf{L},\mathbf{L}')\tilde{f}(\mathbf{L}'',\mathbf{L}')\end{aligned}$$

sum over all $\mathbf{L},\mathbf{L}',\mathbf{L}'' \in \mathcal{F}^n$ such that $\zeta(\mathbf{L}) = \zeta(\mathbf{L}'') \in \Xi$ and $L_i \subset L''_i \subset L_{i+1}$, $\dim(L''_i/L_i) = 1$. Interchanging here L, L'' we obtain

$$\begin{aligned}&(f,\sqrt{q}K_{\mathbf{a}}F_i(\tilde{f}))\\ &= \sum \prod_{\substack{j\in[1,n]\\ j\neq i,i+1\\ \text{mod } n}} \frac{(\sqrt{q})^{(|\mathbf{L}|_i-1)|\mathbf{L}|_i+(|\mathbf{L}|_{i+1}+1)(|\mathbf{L}|_{i+1}+2)-1-2|\mathbf{L}|_{i+1}+|\mathbf{L}|_i}}{N(|\mathbf{L}|_j)^\sharp_q} f(\mathbf{L}'',\mathbf{L}')\tilde{f}(\mathbf{L},\mathbf{L}')\end{aligned}$$

sum over all $\mathbf{L},\mathbf{L}',\mathbf{L}'' \in \mathcal{F}^n$ such that $\zeta(\mathbf{L}) = \zeta(\mathbf{L}'') \in \Xi$ and $L_{i-1} \subset L''_i \subset L_i$, $\dim(L_i/L''_i) = 1$. It remains to observe that

$$\begin{aligned}&|\mathbf{L}|_i(|\mathbf{L}|_i+1)+|\mathbf{L}|_{i+1}(|\mathbf{L}|_{i+1}+1)-(|\mathbf{L}|_i-1)\\ &= (|\mathbf{L}|_i-1)|\mathbf{L}|_i+(|\mathbf{L}|_{i+1}+1)(|\mathbf{L}|_{i+1}+2)-1-2|\mathbf{L}|_{i+1}+|\mathbf{L}|_i.\end{aligned}$$

The lemma is proved.

7.3. From the definition we have

(a) $$\sharp(X^{\mathbf{L}}_A) = \nu_{A,{}^tA,\mathbf{i}_{r(A)};q}.$$

Hence it is natural to define a $\mathbf{Q}(v)$-bilinear form

$$(,) : \mathfrak{A}_{D,n,n'} \times \mathfrak{A}_{D,n,n'} \to \mathbf{Q}(v)$$

by

(b) $$([A],[A']) = \delta_{A,A'} \prod_{i\in[1,n]} \frac{1}{(|\mathbf{L}|_i)^\sharp} v^{-2d_A}\nu_{A,{}^tA,\mathbf{i}_{r(A)}}$$

for any $A, A' \in \mathfrak{S}_{D,n,n'}$. Here $\mathbf{L} \in \mathcal{F}_{r(A)}$. This specializes, for any prime power q, to the bilinear form in 7.1, under $v = \sqrt{q}$.

LEMMA 7.4. *(a) For $A \in \mathfrak{S}_{D,n,n'}$ we have $([A],[A]) \in \mathbf{Q}(v) \cap (1 + v^{-1}\mathbf{Z}[[v^{-1}]])$.*
(b) For $A \neq A'$ in $\mathfrak{S}_{D,n,n'}$ we have $([A],[A']) = 0$.

(b) holds by definition. We prove (a). Note that $v^{-2d_A}\nu_{A,{}^tA,\mathbf{i}_{r(A)}} \in \mathcal{A}$. Since, for $\mathbf{k}$ an algebraic closure of the field with q elements, $X_A^{\mathbf{L}}$ is irreducible of dimension d_A (see Lemma 4.3), we see from the Lang-Weil estimates [LW] that the number of points of $X_A^{\mathbf{L}}$ over the field with q^s elements divided by q^{sd_A} tends to 1 as $s \to \infty$. Using 7.3(a) it follows that

$$v^{-2d_A}\nu_{A,{}^tA,\mathbf{i}_{r(A)}} \in 1 + v^{-1}\mathbf{Z}[v^{-1}].$$

It remains to use the fact that $\frac{1}{(|\mathbf{L}|_i)^!} \in \mathbf{Q}(v) \cap (1 + v^{-1}\mathbf{Z}[[v^{-1}]])$ (where $\mathbf{L} \in \mathcal{F}_{r(A)}$). The lemma is proved.

LEMMA 7.5. *(a) For $A \in \mathfrak{S}_{D,n,n'}$ we have $(\{A\},\{A\}) \in \mathbf{Q}(v) \cap (1+v^{-1}\mathbf{Z}[[v^{-1}]])$.*
(b) For $A \neq A'$ in $\mathfrak{S}_{D,n,n'}$ we have $(\{A\},\{A'\}) = 0$.
This follows from 7.4 using 4.1(d),(e).

From Lemma 7.2 we deduce the following "generic" version.

LEMMA 7.6. *Let $i \in \mathbf{Z}$ and let $f, \tilde{f} \in \mathfrak{A}_{D,n,n'}$. Let $\mathbf{a}$ be as in 2.6 and let $\mathbf{b} \in \tilde{\mathfrak{S}}_n$. We have*

$$(E_i(f), \tilde{f}) = (f, vK_{\mathbf{a}}F_i(\tilde{f})), \quad (F_i(f), \tilde{f}) = (f, vK_{-\mathbf{a}}E_i(\tilde{f})),$$
$$(K_{\mathbf{b}}(f), \tilde{f}) = (f, K_{\mathbf{b}}(\tilde{f})).$$

7.7. In the remainder of this section we assume that $n \geq 2$.

Consider the root datum (cf. [L3, 2.2]) consisting of the free abelian groups $Y = X = \tilde{\mathfrak{S}}_n$, with the non-singular pairing $Y \times X \to \mathbf{Z}$ described in 2.2 and with the (equal) imbeddings $I = \mathbf{Z}/n\mathbf{Z} \to Y$, $I = \mathbf{Z}/n\mathbf{Z} \to X$ given by $i \mapsto \mathbf{a}$ ($\mathbf{a}$ as in Proposition 2.6). (A root datum of affine type $\tilde{A}_{n-1}$ which is degenerate in the sense that the image of $I \to Y = X$ is linearly dependent.)

Let $'\mathbf{U}$ be the $\mathbf{Q}(v)$-algebra associated to this root datum in [L3, 3.1.1]. Thus, $\mathbf{U}'$ is the associative algebra with generators

$E_i (i \in I), F_i (i \in I), K_{\mathbf{b}} (\mathbf{b} \in \tilde{\mathfrak{S}}_n)$

subject to the relations given in 2.5, 2.6. Let $\mathbf{U}$ be the quotient of $'\mathbf{U}$ by the two-sided ideal generated by the quantum Serre relations in the E_i and in the F_i. (A Drinfeld-Jimbo quantized enveloping algebra.)

Now 2.5, 2.6 imply that we have a unique algebra homomorphism $'\mathbf{U} \to \mathbf{U}_D$ which takes $E_i, F_i, K_{\mathbf{b}}$ in $'\mathbf{U}$ to the elements with the same name in $\mathbf{U}_D$. The left $\mathfrak{A}_{D,n,n}$-module $\mathfrak{A}_{D,n,n'}$ (see 1.9) will be regarded by restriction of scalars as a (left) $\mathbf{U}_D$-module or as a $'\mathbf{U}$-module.

(a) This $'\mathbf{U}$-module is *integrable*

since E_i^D, F_i^D are zero on $\mathfrak{A}_{D,n,n'}$; the existence of a weight decomposition is clear. (The notion of integrability of $'\mathbf{U}$-modules is defined in the same way as the analogous notion for $\mathbf{U}$-modules, see [L3, 3.5.1].)

From (a) it follows that the $'\mathbf{U}$-module $\mathfrak{A}_{D,n,n'}$ factors through a $\mathbf{U}$-module. (The fact that the quantum Serre relations hold on $\mathfrak{A}_{D,n,n'}$ can be also verified by a direct computation, similar to the one in [BLM, 5.6]). In particular, we see that $\mathbf{U}_D$ is naturally a quotient algebra of $\mathbf{U}$.

7.8. Let $\mathbf{A} = \mathbf{Q}(v) \cap \mathbf{Q}[[v^{-1}]]$.

Let M be an integrable $\mathbf{U}$-module. Let $\tilde{E}_i : M \to M, \tilde{F}_i : M \to M$ be the Kashiwara operators. (See [L3, 16.1.4].)

A *signed basis at* ∞ or a *signed crystal basis* of M is a pair consisting of a free $\mathbf{A}$-submodule $M_{\mathbf{A}}$ of M such that $M = \mathbf{Q}(v) \otimes_{\mathbf{A}} M_{\mathbf{A}}$ and a signed basis $\mathfrak{B}$ of the $\mathbf{Q}$-vector space $M_{\mathbf{A}}/v^{-1}M_{\mathbf{A}}$ such that properties (a)-(d) below hold.

(a) $M_{\mathbf{A}}$ is stable under $\tilde{E}_i, \tilde{F}_i$ for all i; thus, $\tilde{E}_i, \tilde{F}_i$ act on $M_{\mathbf{A}}/v^{-1}M_{\mathbf{A}}$.

(b) $\tilde{E}_i(\mathfrak{B}) \subset \mathfrak{B} \cup \{0\}$ and $\tilde{F}_i(\mathfrak{B}) \subset \mathfrak{B} \cup \{0\}$ for all i.

(c) $M_{\mathbf{A}}$ is the sum of its intersections with the various weight spaces of M and each element of $\mathfrak{B}$ is contained in the image of one of these intersections in $M_{\mathbf{A}}/v^{-1}M_{\mathbf{A}}$.

(d) Given $\beta, \beta' \in \mathfrak{B}$ and $i \in I$, we have $\tilde{E}_i(\beta) = \beta'$ if and only if $\tilde{F}_i(\beta') = \beta$.

(This definition reduces to Kashiwara's definition of a crystal basis if $\mathfrak{B}$ is assumed to be a basis instead of a signed basis. Recall that a signed basis of a vector space consists of $\pm$ the elements of a basis.)

7.9. Let $\mathfrak{A}_{D,n,n';\mathbf{A}}$ be the $\mathbf{A}$-submodule of $\mathfrak{A}_{D,n,n'}$ spanned by the elements $[A], A \in \mathfrak{S}_{D,n,n'}$, or equivalently by the elements $\{A\}, A \in \mathfrak{S}_{D,n,n'}$. Let $\mathfrak{B}$ be the basis of the $\mathbf{Q}$-vector space $\mathfrak{A}_{D,n,n';\mathbf{A}}/v^{-1}\mathfrak{A}_{D,n,n';\mathbf{A}}$ formed by the images of the elements $[A]$ (or equivalently $\{A\}$) with $A \in \mathfrak{S}_{D,n,n'}$.

THEOREM 7.10. $(\mathfrak{A}_{D,n,n';\mathbf{A}}, \pm\mathfrak{B})$ *is a signed basis at infinity (or signed crystal basis) of the integrable* $\mathbf{U}$*-module* $\mathfrak{A}_{D,n,n'}$.

We apply the results in [L3, 16.2] to the basis $\{[A] | A \in \mathfrak{S}_{D,n,n'}\}$ of $\mathfrak{A}_{D,n,n'}$ and to the form $(,)$ on $\mathfrak{A}_{D,n,n'}$. These results are applicable in view of Lemmas 7.4, 7.6.

8. A basis of $\mathbf{U}_D$. In this section we assume that $n \geq 2$.

PROPOSITION 8.1. *Assume that* $A \in \mathfrak{S}^{\mathrm{ap}}_{D,n,n}$ *(see 5.4). Then* $\{A\} \in \mathbf{U}_{D,\mathcal{A}}$.

Our hypothesis implies that A^- and A^+ are aperiodic. If A is a minimal element for the partial order $\leq$, then from 4.11 we have $\{A^-\} \cdot \{A^+\} = \{A\}$. By 5.17, 5.18 we have $\{A^-\} \in \mathbf{U}_{D,\mathcal{A}}, \{A^+\} \in \mathbf{U}_{D,\mathcal{A}}$ hence $\{A\} \in \mathbf{U}_{D,\mathcal{A}}$.

Thus we may assume that A is not minimal for $\leq$ and that the proposition holds for any $A_1 \in \mathfrak{S}^{\mathrm{ap}}_{D,n,n}$ such that $A_1 \leq A, A_1 \neq A$. By 4.11 we have

$$\{A^-\} \cdot \{A^+\} = \{A\} + \sum_{A_1; A_1 < A} c_{A_1} \{A_1\}$$

where $c_{A_1} \in \mathcal{A}$. Since, by 5.17, 5.18, we have $\{A^-\} \cdot \{A^+\} \in \mathbf{U}_{D,\mathcal{A}}$, we see from 6.6 that $c_{A_1} \neq 0$ implies A_1 aperiodic. By the induction hypothesis, for all such A_1 we have $\{A_1\} \in \mathbf{U}_{D,\mathcal{A}}$. It follows that $\{A\} \in \mathbf{U}_{D,\mathcal{A}}$. The proposition is proved.

Combining 6.6 and 8.1 we obtain the following result.

THEOREM 8.2. *The elements* $\{A\}$, *with* $A \in \mathfrak{S}^{\mathrm{ap}}_{D,n,n}$, *form a* $\mathbf{Q}(v)$*-basis of* $\mathbf{U}_D$ *and an* $\mathcal{A}$*-basis of* $\mathbf{U}_{D,\mathcal{A}}$.

8.3. Let $\mathbf{U}_{D,\mathbf{A}}$ be the $\mathbf{A}$-submodule of $\mathbf{U}$ spanned by the elements $\{A\}, A \in \mathfrak{S}^{\mathrm{ap}}_{D,n,n}$. Let $\tilde{\mathfrak{B}}$ be the basis of the $\mathbf{Q}$-vector space $\mathbf{U}_{D,\mathbf{A}}/v^{-1}\mathbf{U}_{D,\mathbf{A}}$ formed by the images of the elements $\{A\}, A \in \mathfrak{S}^{\mathrm{ap}}_{D,n,n}$.

THEOREM 8.4. $(\mathbf{U}_{D,\mathbf{A}}, \pm\tilde{\mathfrak{B}})$ *is a signed basis at infinity (or a signed crystal basis) of the integrable* $\mathbf{U}$*-module* $\mathbf{U}_D$.

We apply the results in [L3, 16.2] to the basis $\{\{A\} | A \in \mathfrak{S}^{\mathrm{ap}}_{D,n,n}\}$ of $\mathbf{U}_D$ and to the restriction to $\mathbf{U}_D$ of the form $(,)$ on $\mathfrak{A}_{D,n,n}$. These results are applicable in view of Lemmas 7.5, 7.6.

9. Stabilization.

9.1. In this section we assume that $n \geq 2$.

One can show that there is a unique homomorphism $\phi_{D+n,D} : \mathbf{U}_{D+n} \to \mathbf{U}_D$ of algebras such that

$\phi_{D+n,D}(E_i) = E_i$, for all i,

$\phi_{D+n,D}(F_i) = F_i$, for all i,

$\phi_{D+n,D}(K_{\mathbf{b}}) = v^{\mathbf{b}\cdot\mathbf{b}_0/2} K_{\mathbf{b}}$, for all $\mathbf{b} \in \tilde{\mathfrak{S}}_n$,

where $\mathbf{b}^0 = (\dots, 1, 1, 1, \dots) \in \tilde{\mathfrak{S}}_n$. The existence of this homomorphism will be proved elsewhere. The uniqueness is obvious.

It is clear that $\phi_{D+n,D}(\bar{\xi}) = \overline{\phi_{D+n,D}(\xi)}$ for all $\xi \in \mathbf{U}_{D+n}$. (Here $^{-}$ is as in 4.13.)

CONJECTURE 9.2. *Let $A \in \mathfrak{S}^{\text{ap}}_{D+n,n,n}$. Let A' be the matrix defined by $a'_{i,j} = a_{i,j} - \delta_{i,j}$ for all $i, j \in \mathbf{Z}$.*

(a) If $a_{i,i} \geq 0$ for all i, then $\phi_{D+n,D}(\{A\}) = \{A'\}$.

(b) If $a_{i,i} = 0$ for some i, then $\phi_{D+n,D}(\{A\}) = 0$.

(Note that in case (a), we have $A' \in \mathfrak{S}^{\text{ap}}_{D,n,n}$.) From the definition it is clear that the conjecture holds at least if A is either $\mathbf{i_a}$ as in 1.9 or ${}_{\mathbf{a}}\mathbf{e}_{\mathbf{a}'}, {}_{\mathbf{a}'}\mathbf{f}_{\mathbf{a}}$ as in 2.3.

9.3. Let $\mathcal{S}^{\text{ap}}_n$ be the set of all matrices $A = (a_{i,j})_{(i,j)\in\mathbf{Z}\times\mathbf{Z}}$ with entries $a_{i,j} \in \mathbf{Z}$ such that

(a) $a_{i,j} \geq 0$ for all $i \neq j$;

(b) $a_{i,j} = a_{i-n,j-n}$ for all $i, j \in \mathbf{Z}$;

(c1) for any $i \in \mathbf{Z}$, the set $\{j \in \mathbf{Z} | a_{i,j} \neq 0\}$ is finite;

(c2) for any $j \in \mathbf{Z}$, the set $\{i \in \mathbf{Z} | a_{i,j} \neq 0\}$ is finite;

(d) for any $p \neq 0$ there exists $k \in \mathbf{Z}$ such that $a_{k,k+p} = 0$.

Note that, in the presence of condition (b), conditions (c1),(c2) are equivalent.

For $A \in \mathcal{S}^{\text{ap}}_n$ and $p \in \mathbf{Z}$ we define ${}_pA \in \mathcal{S}^{\text{ap}}_n$ so that its (i,j) entry is $a_{i,j}$, if $i \neq j$ and its (i,i) entry is $a_{i,i} + p$ for any i.

If $A, A' \in \mathcal{S}^{\text{ap}}_n$, we write $A \sim A'$ if $A' = {}_pA$ for some $p \in \mathbf{Z}$. This is an equivalence relation on $\mathcal{S}^{\text{ap}}_n$. Let $\mathcal{S}^{\text{ap}}_n/\sim$ be the set of equivalence classes.

Let $\mathbf{K}$ be the $\mathbf{Q}(v)$-vector space with basis elements indexed by $\mathcal{S}^{\text{ap}}_n/\sim$. For any $D \in \mathbf{N}$ we define a $\mathbf{Q}(v)$-linear map $\phi_D : \mathbf{K} \to \mathbf{U}_D$ by the following requirement. If a basis element of $\mathbf{K}$ is indexed by an element which can be represented by an element $A \in \mathfrak{S}_{D,n,n}$ (necessarily unique), then that basis element is mapped to $\{A\}$; otherwise, it is mapped to 0.

This linear map is well defined and surjective by Theorem 8.2. We have

$$\phi_{D+n,D}\phi_{D+n} = \phi_D.$$

We conjecture that there is a unique structure of associative algebra (without 1) on $\mathbf{K}$ so that ϕ_D is an algebra homomorphism for any $D \in \mathbf{N}$. We also conjecture that the algebra $\mathbf{K}$ with its basis indexed by $\mathcal{S}^{\text{ap}}_n/\sim$ is naturally the modified quantized enveloping algebra of affine $\mathfrak{sl}_n$ type with its canonical basis (see [L3, Ch. 23, Ch. 25]).

9.4. The results of this paper have analogues for the case of quantum (non-affine) $\mathfrak{gl}_n$. These analogues can be proved in the same (or easier) way as the results in the affine case; they can be also deduced from the results in the affine case.

One should replace V by a vector space of dimension D over $\mathbf{k}$ and $\mathcal{F}$ by the space of n-step filtrations of that vector space (as in [BLM]). The analogues of the results in Sec. 4 continue to hold, but one should use ordinary Schubert varieties instead of affine ones; those of Sec. 5 continue to hold if we use linear quivers instead of cyclic quivers. The aperiodicity condition plays no role in this case. The analogue of Conjecture 9.2 is again expected to be true and in fact it can be proved for $n = 2$, thus providing a support for the general conjecture.

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