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CONJECTURAL ALGEBRAIC FORMULAS FOR REPRESENTATIONS OF GL_n[†]

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0. Introduction.

0.0. Let F be a non-archimedean local field. Due to the recent work of Harris and Taylor [HT], we know that the local Langlands conjecture is true. In other words, for any local field F we know the existence of the one-to-one correspondence $\phi_n\colon \Pi_n \to GL_n(F)^\frown$ between the set Π_n of n-dimensional representations of the Galois group $\mathfrak{G} = \mathrm{Gal}(\overline{F}/F)$ and the set $GL_n(F)^\frown$ of irreducible nondegenerate representations of the group $GL_n(F)$. In particular, one can associate an irreducible representation π_χ of the group $GL_n(F)$ to a pair (E, χ), where E is a commutative semisimple algebra over F of degree n and χ is a multiplicative character of the group E^*. However, we do not know any explicit construction for the representation π_χ. In our paper we propose an explicit "algebraic" construction for the representation π_χ at least for $n = 4$.

One can inductively characterize the correspondence ϕ_n in the following way. Suppose that we know the correspondence ϕ_{n-2}. Then for any $\sigma \in \Pi_n$ we can characterize the representation $\phi_n(\sigma)$ as the unique representation of $GL_n(F)$ such that for any representation $\rho \in \Pi_{n-2}$ we have

$$\Gamma(\phi_n(\sigma), \phi_{n-2}(\rho)) = \Gamma(\sigma \otimes \rho^*),$$

where $\Gamma(\phi_n(\sigma), \phi_{n-2}(\rho))$ is the Gamma function of Jacquet, Piateskii–Shapiro, Shalika [JPS] and $\Gamma(\sigma \otimes \rho^*)$ is the Gamma function of Langlands. More precisely, let $GL_n(F)^\frown_u \subset GL_n(F)^\frown$ and $\Pi_{n.u} \subset \Pi_n$ be the subsets of unitary representations. We denote by Γ_n the function on the set $GL_{n-2}(F)^\frown_u \times \Pi_{n,u}$ defined by $\Gamma_n(\pi, \rho) = \Gamma(\pi, \phi_{n-2}(\rho))$. To any maximal torus T in $GL_n(F)$ and any character χ of T we may associate an n-dimensional representation $\sigma_\chi \in \Pi_n$ and therefore a representation $\pi_\chi = \phi_n(\sigma_\chi) \in GL_n(F)^\frown$. Let (ρ_{n-2}, Wh_{n-2}) be the Whittaker representation of $GL_{n-2}(F)$. The representation of $GL_{n-2} \times T$ in the space $Wh_{n-2} \otimes L^2(T)$ decomposes in the direct integral

$$Wh_{n-2} \otimes L^2(T) = \oplus \int_{GL_{n-2}(F)^\frown_u \times T^\frown} (V_\pi \otimes \chi), \tag{0.1}$$

where V_π is the space of the representation π. Let A^T_{n-2} be the unitary operator in the space $Wh_{n-2} \otimes L^2(T)$ commuting with $GL_{n-2}(F) \times T$ that in the above decompositions is the multiplication by $\Gamma_n(\pi, \pi_\chi)$. As follows from [JPS], one can write explicit formulas for all π_χ if one knows the operator A^T_{n-2}.

The goal of this paper is to propose an algebraic formula for this operator in the case $n = 4$. More precisely, for any n we construct "algebraic" data $\boldsymbol{\mu}_n = (\mathbf{Z}, \mathbf{Y}, \mathbf{p}, \omega, \mathbf{f})$ that define an operator $A^{T_0}_{n-2}$ corresponding to the maximal split torus

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T_0 in GL_n. In this case the representation π_χ is the induced principal series representation. Next, for $n = 4$, we define an action of the symmetric group S_4 on $\boldsymbol{\mu}_4$. Then for any maximal torus T in GL_4 we can define the corresponding twist $\boldsymbol{\mu}_{T,4}$ of $\boldsymbol{\mu}_4$ and, therefore, an operator $\widehat{A}_2^T$ on the space $Wh_2 \otimes L^2(T)$. We conjecture that $\widehat{A}_2^T = A_2^T$. Moreover, we conjecture that for any n there exists an action of the symmetric group S_n on $\boldsymbol{\mu}_n$ such that $\widehat{A}_{n-2}^T = A_{n-2}^T$ for any maximal torus T in $GL_n(F)$.

Note that for $n = 3$ this conjecture was proved in [K1].

The same formulas work for a real field F.

0.1. Algebraic measures and twisting. We start with the notion of an algebraic measure. For the rest of paper we choose a nontrivial additive character ψ_F of the field F, $\psi_F : F \to \mathbb{C}^*$. We will denote algebraic varieties over F by bold letters (say, $\mathbf{X}$) and the sets of F-point by the corresponding italic letters (say, $X = \mathbf{X}(F)$). Similarly, morphisms of algebraic varieties will be denoted by bold letters (say, $\mathbf{f} : \mathbf{X}_1 \to \mathbf{X}_2$) and the induced mappings of the sets of F-points by the corresponding italic letters (say, $f : X_1 \to X_2$). For a smooth algebraic variety $\mathbf{Y}$ by $\mathcal{S}(Y)$ we denote the space of locally constant function on Y with compact support.

Let $\mathbf{Y}$ be an algebraic variety over F and μ a complex valued measure on Y. An algebraic presentation of μ is data $(\mathbf{Z}, \mathbf{Y}, \mathbf{p}, \omega, \mathbf{f})$, where $\mathbf{Z}$ is a smooth algebraic variety, $\mathbf{p} : \mathbf{Z} \to \mathbf{Y}$ a morphism, $\omega \in \Omega^r(\mathbf{Z})$, $r = \dim \mathbf{Z}$, a volume form (i.e., a differential form of the top degree) on $\mathbf{Z}$, and $\mathbf{f}$ an algebraic function on $\mathbf{Z}$ such that the measure μ is equal to the distribution

$$p_*(|\omega| \cdot (\psi_F \circ f)),$$

where $|\omega|$ is the measure on Z corresponding to ω, see [W]. In other words, for a function $\varphi \in \mathcal{S}(Y)$ we have

$$\int_Y \varphi \, \mu = \int_Z \varphi(p(z))\psi_F(f(z))|\omega|(z).$$

One has to be careful since in cases we are interested in the integral in the right-hand side of the last formula does not converge absolutely. Therefore, we must specify the integration process. We choose the following scheme.

For $a \in F$ let $\mathbf{Z}_a \subset \mathbf{Z}$ be the level variety $\mathbf{Z}_a \overset{\mathrm{df}}{=} \{\mathbf{f} = a\}$. The volume form ω on $\mathbf{Z}$ determined the volume form ω_a on $\mathbf{Z}_a$ by the formula

$$\omega_a \overset{\mathrm{df}}{=} \operatorname{Res} \frac{\omega}{f - a}.$$

DEFINITION 1. *Algebraic measure* is data $\boldsymbol{\mu} = (\mathbf{Z}, \mathbf{Y}, \mathbf{p}, \omega, \mathbf{f})$ such that the following conditions are satisfied.

(i) For any function $\varphi \in \mathcal{S}(Y)$ and for almost any $a \in F$ the integral

$$I_a(\varphi) \overset{\mathrm{df}}{=} \int_{Z_a} \varphi(p(z))|\omega_a|(z)$$

converges absolutely.

(ii) $I_a(\varphi)$ is a locally L^1-function of a.

(iii) The limit

$$I(\varphi) \overset{\mathrm{df}}{=} \lim_{n \to \infty} \int_{|a| \leq p^n} \psi_F(a) I_a(\varphi) \, |da|$$

exists.

(iv) There exists a complex valued measure μ on Y such that $I(\varphi) = \int_Y \varphi\mu$ for $\varphi \in \mathcal{S}(Y)$.

In the case μ is called the *realization* of $\boldsymbol{\mu}$ and $\boldsymbol{\mu}$ is called an *algebraic presentation* of μ.

Remark. In general, a measure μ can have several nonisomorphic algebraic presentations.

Let Γ be a finite group of F-automorphisms of $\mathbf{Y}$. A lifting of the action of Γ to $\boldsymbol{\mu}$ is an F-action of Γ on $\mathbf{Z}$ that commutes with $\mathbf{p}$ and preserves ω and $\mathbf{f}$. If μ is the realization of a Γ-invariant data $\boldsymbol{\mu}$, then μ itself is Γ-invariant.

Given a lifting of the action of Γ to $\boldsymbol{\mu}$, we can construct twisted forms of μ as follows.

Let $\mathfrak{G} = \mathrm{Gal}(\overline{F}/F)$. Elements of $H^1(\mathfrak{G}, \mathrm{Aut}\,\mathbf{Z})$ correspond to homomorphisms $\alpha\colon \mathfrak{G} \to \Gamma$ modulo conjucation by elements of Γ. To any such α we associate the twisted form $\mathbf{Z}_\alpha$ of $\mathbf{Z}$. This is an algebraic variety over F, which is isomorphic to $\mathbf{Z}$ over $\overline{F}$. The set of F-points of $\mathbf{Z}_\alpha$ is given by

$$Z_\alpha = \{x \in \mathbf{Z}(\overline{F}) \mid \zeta z = \overline{\alpha}(\zeta) z \text{ for } \zeta \in \mathfrak{G}\}.$$

Similarly, to α we can associate the twisted form $\mathbf{Y}_\alpha$ of $\mathbf{Y}$. Since the action of Γ on $\mathbf{Z}$ and $\mathbf{Y}$ preserves ω and commutes with $\mathbf{p}$ and $\mathbf{f}$, we get the twisted data $\boldsymbol{\mu}_\alpha = (\mathbf{Z}_\alpha, \mathbf{Y}_\alpha, \mathbf{p}_\alpha, \omega_\alpha, \mathbf{f}_\alpha)$.

In the case where $\boldsymbol{\mu}_\alpha$ defines an algebraic measure (i.e., the integrals in Definition 1 converge and the limit exists), we define the measure μ_α on Y_α as the realization of $\boldsymbol{\mu}_\alpha$. We emphasize that μ_α depends not just on μ and the action of Γ on $\mathbf{Y}$, but also on the lifting of this action to $\boldsymbol{\mu}$.

A measure on Y is a linear functional of the space of continuous function. We need a generalization to the case where the function are replaces with sections of a line bundle on $\mathbf{Y}$. More precisely, we consider the following situation. Let $\widetilde{\mathbf{Y}}$ be a variety with the free action $m\colon \mathbf{U} \times \widetilde{\mathbf{Y}} \to \widetilde{\mathbf{Y}}$ of a unipotent group $\mathbf{U}$, such that $\mathbf{Y} = \widetilde{\mathbf{Y}}/\mathbf{U}$, and let $\Psi\colon \mathbf{U} \to \mathbf{G}_a$ be a character of $\mathbf{U}$. Then we can consider the space $\mathcal{S}_\Psi(Y)$ of locally constant functions φ on $\widetilde{Y}$ such that

$$\varphi(u\widetilde{y}) = \psi_F(-\Psi(u))\varphi(\widetilde{y}), \qquad u \in U, \quad \widetilde{y} \in \widetilde{Y},$$

and the function $||\varphi||$ on Y has a compact support.

A Ψ-*measure* is a linear functional on $\mathcal{S}_\Psi(Y)$ that extends to a continuous functional on a space of continuous Ψ-equivariant functions on $\widetilde{Y}$ that are "compactly supported" (in the above sense). Let $\widetilde{\mathbf{Z}}$ be a manifold with a free action of $\mathbf{U}$, $\widetilde{\mathbf{p}}\colon \widetilde{\mathbf{Z}} \to \widetilde{\mathbf{Y}}$ an $\mathbf{U}$-equivariant map, $\widetilde{\omega}$ a $\mathbf{U}$-invariant volume form on $\widetilde{\mathbf{Z}}$, $\widetilde{\mathbf{f}}\colon \widetilde{\mathbf{Z}} \to \mathbb{A}^1$ a function such that

$$\mathbf{f}(u\widetilde{z}) = \Psi(u) + \mathbf{f}(\widetilde{z}).$$

Let us choose an invariant volume form du on $\mathbf{U}$.

Definition 2. We say that the data $\widetilde{\boldsymbol{\mu}} = (\widetilde{\mathbf{Z}}, \widetilde{\mathbf{Y}}, \widetilde{\mathbf{p}}, \widetilde{\omega}, \widetilde{\mathbf{f}})$ is an algebraic Ψ-measure if the following conditions are satisfied.

(i′) For $\varphi \in \mathcal{S}_\Psi(Y)$ and $a \in F$ denote

$$\widetilde{Z}_{a,\varphi} = \{z \in \widetilde{Z} : \widetilde{f}(z)\varphi(\widetilde{p}^*(z)) = a\}.$$

Then $\widetilde{Z}_{a,\varphi}$ is invariant under U. Let $Z_{a,\varphi} = \widetilde{Z}_{a,\varphi}/U$ and $|\omega_{a,\varphi}|$ the measure on $Z_{a,\varphi}$ induced by $\widetilde{\omega}/du$. We assume that for any $\varphi \in \mathcal{S}_\Psi(Y)$ and for almost any $a \in F$ the integral

$$I_a(\varphi) \overset{\text{df}}{=} \int_{Z_{a,\varphi}} \widetilde{f}(z)\varphi(\widetilde{p}^*(z))|\omega_{a,\varphi}|$$

converges absolutely.

(ii′) $I_a(\varphi)$ is a locally L^1-function of a.

(iii′) The limit

$$I(\varphi) \overset{\text{df}}{=} \lim_{n\to\infty} \int_{|a|\le p^n} \psi_F(a) I_a(\varphi)\,|da|$$

exists.

(iv′) There exists an (U, Ψ)-equivariant complex valued measure μ on $\widetilde{Y}$ such that $I(\varphi) = \int_Y \varphi\mu$ for $\varphi \in \mathcal{S}_\Psi(Y)$.

As before, we call the data $\widetilde{\boldsymbol{\mu}}$ an *equivariant presentation* of μ and μ the *realization* of $\widetilde{\boldsymbol{\mu}}$.

Similarly to the above, we can define twisting of equivariant measures.

0.2. Γ-factors. Let $\mathbf{U}_-$ be the lower unipotent subgroup in $\mathbf{GL}_\ell$ and $\Psi\colon \mathbf{U}_- \to \mathbf{G}_a$ the homomorphism given by

$$\Psi(u) = u_{21} + \cdots + u_{\ell,\,\ell-1}.$$

The *Whittaker representation* (ρ_ℓ, Wh_ℓ) of GL_ℓ is defined by the formula

$$\rho_\ell = \operatorname{Ind}_{U_-}^{GL_\ell}(\psi).$$

An irreducible representation π of GL_n is called *generic* if it occurs in the decomposition of ρ_n into irreducible components. It is known that any generic unitary π occurs in ρ_ℓ exactly once.

Let T be a maximal torus in $GL_n(F)$ and χ a unitary character of T. The pair (T, χ) determines an n-dimensional representation ρ_χ of $\mathfrak{G}$, and by the Langlands correspondence (see [HT]), a unitary nondegenerate irreducible representation π_χ of $GL_n(F)$. Let σ be a generic unitary irreducible representation of GL_ℓ. Jacquet, Piateskii–Shapiro, and Shalika [JPS], associated to the pair (π_χ, σ) the number $\Gamma(\pi_\chi, \sigma)$, $|\Gamma(\pi_\chi, \sigma)| = 1$. Using the direct integral decomposition similar to (0.1), we combine the numbers $\Gamma(\pi_\chi, \sigma)$ for all unitary characters $\chi\colon T \to \mathbb{C}$ and all generic unitary irreducible representations σ of $GL_\ell(F)$ in a unitary operator A_ℓ^T in the space $Wh_\ell \otimes L^2(T)$ commuting with the action of GL_ℓ on the first factor multiplication by elements of T in the second factor.

Define the action m of the unipotent subgroup $\mathbf{U}_- \subset \mathbf{GL}_\ell$ on the space $\widetilde{\mathbf{Y}} = \mathbf{GL}_\ell \times \mathbf{T}$ by left multiplication on the first factor. Let $\mathbf{U}^{(2)} = \mathbf{U}_- \times \mathbf{U}_-$ and let $\Psi^{(2)}\colon \mathbf{U}^{(2)} \to \mathbf{G}_a$ be given by

$$\Psi^{(2)}(u, u') = \Psi(u) - \Psi(u').$$

The action m determines the action $m^{(2)}$ of $\mathbf{U}^{(2)}$ on $\widetilde{\mathbf{Y}} = \mathbf{GL}_\ell \times \mathbf{T} \times \mathbf{GL}_\ell \times \mathbf{T}$. Define the $\Psi^{(2)}$-measure $\mu(T, \ell)$ on $\widetilde{Y}$ by the formula

$$\int_{\widetilde{Y}} f_1(t_1, g_1)\overline{f_2(t_2, g_2)}\, \mu(T, \ell) = (A_\ell^T f_1, f_2)_{Wh_\ell \otimes L^2(T)}.$$

The first result of the paper (Proposition 5.1) is the construction of an algebraic presentation $\boldsymbol{\mu}_{T_0,n}$ of the measure $\mu(T, \ell)$ in the case where $n = 4$, $T = T_0$ is the split torus in GL_4, and $\ell = 2$.

0.3. Twisting of the algeraic presentation. For a unitary character $\chi\colon T_0 \to \mathbb{C}^*$ let π_χ be the corresponding irreducible unitary representation of the principal series. As is well known, for $w \in S_n$ the representation π_χ and π_{χ^w} are equivalent. Therefore, the constructed $\Psi^{(2)}$-measure $\mu(T_0, n-2)$ on Y is S_n-invariant.

Our second result (Proposition 5.2) is the construction of the lifting of the action of S_n on $\widehat{\mathbf{Y}}$ to the presentation $\boldsymbol{\mu}_{T_0,n}$ for $n = 4$.

Let T be a maximal torus in GL_4. Any such torus is obtained from the maximal split torus T_0 by an element of $H^1(\mathfrak{G}, \operatorname{Aut} T_0)$, i.e., by a homomorphism $\alpha\colon \mathfrak{G} \to S_n$. Using the lifting of S_4 we can define $\boldsymbol{\mu}_\alpha$ as the twisting of $\boldsymbol{\mu} = \boldsymbol{\mu}_{T_0,4}$.

CONJECTURE 1. *The data $\boldsymbol{\mu}_\alpha$ define a $\Psi^{(2)}$-measure μ_α on $\widetilde{Y} = GL_2 \times T \times GL_2 \times T$.*

The second conjecture is that this measure coincides with the measure defined by the Γ-function. More precisely, let A_α be the operator on the space $Wh_{n-2} \otimes L^2(T)$ corresponding to the measure μ_α.

CONJECTURE 2. *The operator A_α is unitary and in the direct integral decomposition* (0.1) *is given by the multiplication by $\Gamma(\pi_\chi, \sigma)$*

In other words, $A_\alpha = A^T_{n-2}$ for the operator A^T_{n-2} described in 0.0.

Finally, we conjecture that all of the above remains true for an arbitrary n.

CONJECTURE 3. *For an arbitrary $n \geq 4$ there exists a lifting of the action of S_n to the algebraic presentation $\boldsymbol{\mu} = \boldsymbol{\mu}_{T_0,n-2}$. For a maximal torus T in GL_n corresponding to a homomorphism $\alpha\colon \mathfrak{G} \to S_n$, the twisted data $\boldsymbol{\mu}_\alpha$ determine a $\Psi^{(2)}$-measure μ_α on $\widetilde{Y} = GL_{n-2} \times T \times GL_{n-2} \times T$. The corresponding operator in the space $Wh_{n-2} \otimes L^2(T)$ is unitary and in the decomposition* (0.1) *it is the multiplication by $\Gamma(\pi_\chi, \sigma)$.*

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1. Measures. In this section we present a general result about complex valued measures on vector bundles over smooth varieties over F. This result can be viewed as a formalization of the formula

$$\int_F \psi_F(xy)\, dy = \delta(x),$$

which is well known in the theory of distributions.

Let $\mathbf{M}$ be an m-dimensional algebraic variety over F. By $\Omega^m(M)$ we will always denote the space of volume forms on $\mathbf{M}$. Let $\mathcal{L}$ a one-dimensional vector bundle on $\mathbf{M}$, $\mathcal{L}^*$ the dual bundle, L, L^* the total spaces of $\mathcal{L}$ and $\mathcal{L}^*$, and $\pi\colon L \to M$, $\pi^*\colon L^* \to M$ the corresponding projections. Let also $\zeta\colon M \to L^*$ be the zero section of $\mathcal{L}^*$. For an open set $U \subset M$ we denote by L_U the total space of the restriction of $\mathcal{L}$ to U. Similarly, L^*_U is the total space of the restriction of $\mathcal{L}^*$ to U.

Let $\gamma \in \Gamma(U, \mathcal{L}^*)$ be a section of $\mathcal{L}^*$ and let $N_{\gamma,U} = \{x \in U \mid \gamma(x) = \zeta(x)\}$ be the subvariety of zeros of γ in U. Let $\mathcal{T}_M(N_{\gamma,U})$ and $\mathcal{L}^*(N_{\gamma,U})$ be the restrictions to $N_{\gamma,U}$ of the tangent bundle $\mathcal{T}_M$ and of the vector bundle $\mathcal{L}^*$. Denote by $T_M(N_{\gamma,U})$ and $L^*(N_{\gamma,U})$ the corresponding total spaces. Let x be a smooth point of $N_{\gamma,U}$, $y = \gamma(x)$ the corresponding point of L^*. The tangent space to L^* at y is canonically represented as the direct sum

$$T_{L^*,y} = T_{M,x} \oplus F, \tag{1.1}$$

where F is the one-dimensional tangent space to the fiber L_x^* of $\mathcal{L}^*$ over x. Therefore the composition of the differential of γ and the projection of $T_{L^*,y}$ to the second summand in (1.1) determines a morphism $\theta_x\colon T_{M,x} \to L_x^*$.

DEFINITION 1.1. (i) A section γ is said to be *generic* at a smooth point $x \in N_\gamma$ if θ_x is surjective.

(ii) A section γ of $\mathcal{L}^*$ is said to be *generic in* U if $N_{\gamma,U}$ is smooth and generic at all points of $N_{\gamma,U}$.

If γ is generic at x, then we can identify $\operatorname{Ker}\theta_x$ and the tangent space $T_{N_\gamma,x}$ to N_γ at x.

Let $U \subset M$ be an open set, $\gamma \in \Gamma(U,\mathcal{L}^*)$ a generic section. Our next goal is to construct a morphism $\eta(\gamma)\colon \Omega^{m+1}(L_U) \to \Omega^{m-1}(N_{\gamma,U})$.

Let γ be generic at a point $x \in N_{\gamma,U}$. Since $T_{L,x} = L_x \oplus T_{M,x}$, for the fiber of $\Omega^{m+1}(L_U)$ at x we have

$$\Omega^{m+1}(L_U)_x = L_x^* \otimes \Lambda^m(T_{M,x}^*).$$

The exact sequence

$$0 \to T_{N_\gamma,x} \to T_{M,x} \xrightarrow{\theta_x} L_x^* \to 0$$

shows that

$$\Lambda^m(T_{M,x}^*) = \Lambda^{m-1}(T_{N_\gamma,x}^*) \otimes L_x = \Omega^{m-1}(N_{\gamma,U})_x \otimes L_x,$$

so that we have

$$\Omega^{m+1}(U)_x = L_x^* \otimes \Omega^{m-1}(N_{\gamma,U})_x \otimes L_x.$$

Using the pairing $L_x^* \otimes L_x \to F$, we get the map

$$\Omega^{m+1}(U)_x \to \Omega^{m-1}(N_{\gamma,U})_x. \tag{1.2}$$

The collection of maps (1.2) for all $x \in N_{\gamma,U}$ yields the requred map $\eta(\gamma)$.

In coordinates the map $\eta(\gamma)$ is described as follows. Let $U \subset M$ be such that the restriction $\mathcal{L}|_U$ is the trivial line bundle. Choose a trivialization $L_U = U \times F$ and the dual trivialization $L_U^* = U \times F$. Denote by y the coordinate in the fibers of projection $L_U \to U$ and by y^* be the dual coordinate in the fibers of the projection $L_U^* \to U$. A section γ of $\mathcal{L}^*$ over U is given by a regular function $\theta(x)$ on U so that $\gamma(x) = (x,\theta(x)) \in U \times F$. For such a section, $N_{\gamma,U} = \{x \in U \mid \theta(x) = 0\}$ and γ is generic at a point $x \in N_{\gamma,U}$ if and only if $d\theta \neq 0$ at x.

A volume form $\omega \in \Omega^{m+1}(U)$ can be written as

$$\omega = \ell(p,y)\omega' \wedge dy, \tag{1.3}$$

where $\omega' \in \Omega^m(U)$, $\ell(p,y)$ is a function on $L_U = U \times F$. For such a form ω we have

$$\eta(\gamma)(\omega) = \operatorname*{Res}_{N_{\gamma,U}} \left\{ \frac{\ell(p,0)\omega'}{\theta} \right\}. \tag{1.4}$$

Clearly, the right-hand side of (1.4) does not depend of the representation of ω in the form (1.3).

DEFINITION 1.2. For an open $U \subset M$, a volume form $\omega \in \Omega^{m+1}(L_U)$ is said to be *fiberwise constant* if $t_b^*\omega = \omega$ for any $b \in \Gamma(U,\mathcal{L})$, where $t_b\colon L_U \to L_U$ is the fiberwise addition.

Denote by $\Omega_{fc}^{m+1}(L_U)$ the space of fiberwise constant volume forms on L_U. In coordinates $\omega \in \Omega_{fc}^{m+1}(L_U)$ if and only if in some (hence every) representation of ω in the form (1.3), ℓ does not depend on y. In this case we can take $\ell = \text{const}$.

Let γ be a section of $\mathcal{L}^*|_U$. Define the complex valued function ψ_γ on L_U by the formula

$$\psi_\gamma(z) = \psi_F(\langle\gamma(x), z\rangle),$$

where $z \in L_U$, $x = \pi(y) \in U$, and $\gamma(x) \in \mathcal{L}^*(x)$ is considered as a linear functional on $\mathcal{L}(x)$.

Let $\mathcal{L}$ be as before, $\omega \in \Omega^{m+1}(L)$, and f a complex valued locally constant function on L.

We say that f is locally integrable with respect to ω at a point $x \in M$ if for each sufficiently small compact neighborhood U of x the following condition holds.

• Choose a trivialization $L_U = U \times F$. Let $\mathcal{O}_1 \subset \mathcal{O}_2 \subset \cdots \subset F$ be a sequence of open compact subgroups such that $\bigcup \mathcal{O}_i = F$. Denote $V_i = U \times \mathcal{O}_i$. Then the limit

$$\lim_{i\to\infty} \int_{V_i} f|\omega| \tag{1.5}$$

exists.

It is clear that the limit (1.5) does not depend on the trivialization $\mathcal{L}$ over U in (i) and on the sequence $\{\mathcal{O}_i\}$ in (ii). We denote this limit by

$$\lim_{i\to\infty} \int_{V_i} f|\omega| = \int_{L_U} f|\omega|.$$

It is also clear that now we can define $\int_{L_U} f|\omega|$ for any open set $U \subset M$ and an arbitrary locally constant function f on L_U such that the projection of $p(\operatorname{supp} f) \subset M$ is compact and f is integrable with respect to ω at each point of $p(\operatorname{supp} f)$.

PROPOSITION 1.1. *Let $U \subset M$ be a compact open set. For any generic section γ of $\mathcal{L}^*|_U$, any locally compact complex valued function f on U, and any fiberwise constant volume form $\omega \in \Omega^{d+1}_{fc}(L_U)$, the function $\pi^*(f)\psi_\gamma$ is locally integrable with respect to ω at all points of U and the integral is given by*

$$\int_{L_U} \pi^*(f)\psi_\gamma|\omega| = \int_{N_{\gamma,U}} f|\eta(\gamma)(\omega)|,$$

where $N_{\gamma,U}$ is the set of zeros of γ in U and in the right hand side we take the restriction of f in $N_{\gamma,U}$.

Proof. First, we consider a case where M is an affine line with the coordinate x, $\mathcal{L}$ is a trivial one-dimensional bundle with the coordinate y along the fibers, and the section γ of $\mathcal{L}^*$ is given by a function $y^* = \theta(x)$, so that $\gamma(x) = (x, \theta(x))$. We have $N_\gamma = \{x \mid \theta(x) = 0\}$ and γ is generic at a point $x \in N_\gamma$ if and only if $\theta'(x) \neq 0$. If γ is generic, then N_γ consist of the finite number of isolated points. Assume that U is so small that $N_{\gamma,U}$ consists of a single point x_0. For a fiberwise constant form $a(x)\, dx\, dy \in \Omega^2(L_U)$ the value of zero-form $\eta(\gamma)(\omega)$ at $x_0 \in N$ is $a(x_0)/\theta'(x_0)$.

Let $\{\mathcal{O}_i\}$ be an increasing sequence of open compact subgroups in F, $\cup\mathcal{O}_i = F$, and $V_i = U \times \mathcal{O}_i \subset U \times F = L_U$. Then

$$\begin{aligned}\int_{V_i} \pi^*(f)(z)\psi_\gamma(z)|\omega| &= \int_{U\times\mathcal{O}_i} \psi_F(y\theta(x))|a(x)|\,|dx|\,|dy| \\ &= \int_U f(x)|a(x)|\,|dx| \int_{\mathcal{O}_i} \psi_F(y\theta(x))|dy|.\end{aligned}$$

Denote $\mathcal{O}_i^\perp = \{y \in F \mid \psi_F(yu) = 1 \text{ for all } u \in \mathcal{O}_i\}$. Then

$$\int_{\mathcal{O}_i} \psi_F(y\theta(x))|dy| = \begin{cases} \operatorname{meas}\mathcal{O}_i & \text{if } \theta(x) \in \mathcal{O}_i^\perp, \\ 0 & \text{otherwise.} \end{cases}$$

Next, denote $\widehat{\mathcal{O}}_i^\perp = \{y \in F \mid \theta(y) \in \mathcal{O}_i^\perp\}$. Then

$$\int_{V_i} \pi^*(f)(z)\psi_\gamma(z)|\omega| = (\operatorname{meas}\mathcal{O}_i)\int_{U\cap\widehat{\mathcal{O}}_i^\perp} f(x)|a(x)|\,|dx|.$$

For a sufficiently large i, $\widehat{\mathcal{O}}_i^\perp$ is a small neighborhood of points $x_0 \in N_{\gamma,U}$ and $\cap_i\widehat{\mathcal{O}}_i^\perp = \{x_0\}$. In particlular, $\widehat{\mathcal{O}}_i^\perp \subset U$ and

$$\lim_{i\to\infty}(\operatorname{meas}\mathcal{O}_i)(\operatorname{meas}\widehat{\mathcal{O}}_i^\perp) = 1/|\theta(x_0)|.$$

Therefore

$$\begin{aligned}\lim_{i\to\infty}\int_{V_i} \pi^*(z)f(x)\psi_\gamma(z)|\omega| &= \lim_{i\to\infty}(\operatorname{meas}\mathcal{O}_i)\int_{\widehat{\mathcal{O}}_i^\perp} f(x)|a(x)|\,|dx| \\ &= \frac{f(x_0)|a(x_0)|}{|\theta'(x_0)|}\end{aligned}$$

and the right-hand side equals

$$\int_{N_{\gamma,U}} f\eta(\gamma)(\omega).$$

In the general case we can argue locally on M. Let $U \subset M$ be such that $\mathcal{L}$ and $\mathcal{L}^*$ are trivial over U. Choose local coordinates $x_1,\ldots,x_m$ in U such that γ is given by the equation of the form $y^* = \theta(x_1)$ and $N_{\gamma,U} = \{x_1,\ldots,x_m\} \mid x_1 = x_1^{(0)}\}$ for a single $x_0 \in F$. For a fiberwise constant form $\omega = a(x)dx_1\wedge\cdots\wedge dx_m\wedge dy^* = \omega_1\wedge dy*$ on L_U we have

$$\operatorname*{Res}_{N_{\gamma,U}}\frac{\omega_1}{\theta} = \sum_{x_1^{(0)}\in N_1}\frac{a(x_1^{(0)},x_2,\ldots,x_m)dx_2\wedge\cdots\wedge dx_m}{\theta'(x_1^{(0)})}.$$

To complete the proof one computes $\int f(z)\psi_\gamma(z)|\omega|$ using the arguments similar to those employed in the case $m=1$ above.

2. Γ-factors and measures.

2.1. Subgroups of the group GL_n. We recall some notation from the introduction and give also some new ones.

- $\mathbf{Q}$ is the subgroup of $\mathbf{GL}_n$ consisting of the matrices with the first row of the form $(*\,0\,0\ldots0)$.
- $\mathbf{U}_-$ is the lower unipotent subgroup of $\mathbf{GL}_n$, $\psi(u) = \psi_F(u_{21}+\cdots+u_{n\,n-1})$ a nondegenrate character (one-dimensional complex representation of U_- .
- σ: $\mathbf{GL}_n \to \mathbf{GL}_n$ is the involution given by the formula $\sigma(g) = (ag^\top a^{-1})^{-1}$, where $^\top$ denotes the reflection with respect to the second (nonprincipal) diagonal in $\mathbf{GL}_n$ and $a = \operatorname{diag}(1,-1,1,\ldots,(-1)^{n+1})$. We have $\sigma(\mathbf{U}_-) = U_-$ and $\psi\circ\sigma = \psi$.
- $\mathbf{R} = \mathbf{Q}\cap\mathbf{Q}^\sigma$.
- By Ind we will always understand the *unitary induction.*

2.2. Generic representations of GL_n. Let (π, V) be a unitary representation of GL_n. Let (π^a, V^a) be the smooth model of (π, V) (see [BZ]). Recall that V^a consists of all vectors $v \in V$ such that $\operatorname{Stab} v$ is an open compact subgroup of GL_n, and π^a is the restriction of π to V^a. It is known that V^a is dense in V.

DEFINITION 2.1. An irreducible unitary representation (π, V) of GL_n is said to be *generic* if V^a admits a (U_-, ψ)-equivariant linear functional φ, i.e.,

$$\varphi(u_- v) = \psi(u_-)\varphi(v), \qquad v \in V^a, \quad u_- \in U_-.$$

It is known [BZ] that for a generic irreducible unitary representation the functional φ is unique up to a scalar factor.

2.3. Standard representation of Q. Denote by $\mathbf{C} \subset \mathbf{Q} \subset \mathbf{GL}_n$ the center of GL_n. For a unitary character $\mathcal{E}\colon C \to \mathbb{C}^*$ introduce the *standard representation* of Q by the formula

$$(\rho_{Q,\mathcal{E}}, L_{Q,\mathcal{E}}) \stackrel{\text{df}}{=} \operatorname{Ind}_{CU_-}^{Q}(\mathcal{E} \cdot \psi).$$

The respresentation $\rho_{Q,\mathcal{E}}$ is irreducible. The following result proved in [BZ] is the basis of our construction of Γ-factors for GL_n.

THEOREM 2.1. *Let (π, V) be a generic unitary irreducible representation of GL_n, $\mathcal{E}$ the central character of π. The restriction of π to Q is equivalent to $\rho_{Q,\mathcal{E}}$.*

2.4. Two restrictions to Q. Let (π, V) be a generic unitary irreducible representation of GL_n with central character $\mathcal{E}$. By Theorem 2.1 there exists a (unique up to a factor) unitary operator $\alpha_1\colon V \to L_{Q,\mathcal{E}}$ establishing the equivalence $\pi\,|_Q \simeq \rho_{Q,\mathcal{E}}$.

Similarly, let $\sigma\colon GL_n \to GL_n$ be the involution defined in 2.1. Then $\pi^\sigma \stackrel{\text{df}}{=} \pi \circ \sigma$ is also a generic unitary irreducible representation of GL_n. Applying Theorem 2.1 again, we get the unitary operator $\alpha_2\colon V \to L_{Q,\mathcal{E}^\sigma}$ establishing the equivalence $\pi^\sigma\,|_Q \simeq \rho_{Q,\mathcal{E}^\sigma}$, where $\mathcal{E}^\sigma = \mathcal{E} \circ \sigma$ is the central character of π^σ.

Since π and π^σ act in the same space V, we have the operator $\beta_\pi = \alpha_2 \circ \alpha_1^{-1}\colon L_{Q,\mathcal{E}} \to L_{Q,\mathcal{E}^\sigma}$. It is a unitary operator satisfying the condition

$$\beta_\pi \circ \rho_{Q,\mathcal{E}}(r) = \rho_{Q,\mathcal{E}^\sigma}(r^\sigma) \circ \beta_\pi, \qquad r \in R = Q \cap Q^\sigma. \tag{2.1}$$

Note that the unitary operator β_π satisfying (2.1) is defined uniquely up to a multiplicative factor c_π with $|c_\pi| = 1$.

2.5. An auxiliary operator. We construct the operator $\kappa_\mathcal{E}\colon L_{Q,\mathcal{E}} \to L_{Q,\mathcal{E}^\sigma}$ satisfying the intertwining condition similar to (2.1). The operator $\kappa_\mathcal{E}$ will depend on $\mathcal{E}$ but not on π.

Denote by $w_Q \in Q$ the following permutation matrix:

$$w_Q = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 1 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Next, denote $U_Q = w_Q U_- w_Q^{-1}$ and let $\psi_Q\colon U_Q \to \mathbb{C}^*$ be given by the formula

$$\psi_Q(u) = \psi(w_Q^{-1} u w_Q), \qquad u \in U_Q.$$

Denote also $U_R = U_Q \cap R$, and $\psi_R = \psi_Q|_R\colon U_R \to \mathbb{C}^*$.

In this section we will often represent elements of $g \in GL_n$ by block 3×3-matrices according to the decomposition $n = 1 + (n-2) + 1$ of rows and columns. In such representation, $q \in Q$, $r \in R$, $u_Q \in U_Q$, and $u_R \in U_R$ have the form

$$q = \begin{pmatrix} * & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix}, \qquad r = \begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix},$$
$$u_Q = \begin{pmatrix} 1 & 0 & 0 \\ * & u' & * \\ * & 0 & 1 \end{pmatrix}, \qquad u_R = \begin{pmatrix} 1 & 0 & 0 \\ * & u' & 0 \\ * & 0 & 1 \end{pmatrix}.$$

Here $*$ denotes possibly nonzero positions and u' is a lower unipotent matrix of order $n-2$. The characters ψ_Q and ψ_R are given by the formulas

$$\psi_Q(u) = \psi_F(u_{n1} + u_{32} + \cdots + u_{n-1\,n-2} + u_{2n}),$$
$$\psi_R(u) = \psi_F(u_{n1} + u_{32} + \cdots + u_{n-1\,n-2}).$$

Introduce also the subgroup A as follows:

$$A = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & E_{n-2} & * \\ 0 & 0 & 1 \end{pmatrix} \right\},$$

where E_{n-2} is the identity matrix of order $n-2$. Notice that $A \subset U_Q$ and $\psi_A(a) = \psi_F(a_{2\,n})$ for $a \in A$.

Let us define the following representations.

(i) A representation of the group Q:

$$(\widehat{\rho}_{Q,\mathcal{E}}, \widehat{L}_{Q,\mathcal{E}}) \stackrel{\text{df}}{=} \operatorname{Ind}_{CU_Q}^{Q}(\mathcal{E}\cdot\psi_Q).$$

(ii) A representation of the group R:

$$(\rho_{R,\mathcal{E}}, L_{R,\mathcal{E}}) \stackrel{\text{df}}{=} \operatorname{Ind}_{CU_R}^{R}(\mathcal{E}\cdot\psi_R). \tag{2.2}$$

Next we introduce the following linear operators between the spaces of these representations:

$$C_1\colon L_{Q,\mathcal{E}} \to \widehat{L}_{Q,\mathcal{E}}, \quad (C_1 f)(q) = f(w_Q^{-1} q),$$
$$C_2\colon \widehat{L}_{Q,\mathcal{E}} \to L_{R,\mathcal{E}}, \quad C_2(f) = f\,|_R\,.$$

Finally, define the operator $C_3\colon L_{R,\mathcal{E}} \to L_{R,\mathcal{E}^\sigma}$ as follows. Let $B \subset U_Q$ be the $(n-2)$-dimensional commutative subgroup of matrices with the block representation of the form

$$b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & E_{n-2} & 0 \\ 0 & * & 1 \end{pmatrix},$$

where E_{n-2} is the identity matrix of order $n-2$. Let $f \in L_{R,\mathcal{E}}$ be a smooth (i.e., locally constant) function compactly supported on $CU_R \setminus R$. Denote

$$\widetilde{f}(r) = \int_B f(br^\sigma)|db|$$

one can verify that the integral converges absolutely, that $\widetilde{f} \in L_{R,\mathcal{E}}$, and that the mapping $f \mapsto \widetilde{f}$ defined on smooth compactly supported functions extends to a unitary operator $C_3 \colon L_{R,\mathcal{E}} \to L_{R,\mathcal{E}^\sigma}$.

Note that all operators C_1, C_2, C_3 depend on $\mathcal{E}$.

LEMMA 2.1. (i) *The operator* C_1 *establishes an equivalence of representations* $\rho_{Q,\mathcal{E}} \simeq \widehat{\rho}_{Q,\mathcal{E}}$.

(ii) *The operator* C_2 *establishes the equivalence of representations of* R*:* $\widehat{\rho}_{Q,\mathcal{E}}\,|_R \simeq (\rho_{R,\mathcal{E}}, L_{R,\mathcal{E}})$.

(iii) *The operator* C_3 *satisfies the condition*

$$\rho_{R,\mathcal{E}^\sigma}(r^\sigma) \circ C_3 = C_3 \circ \rho_{R,\mathcal{E}}(r), \quad r \in R.$$

Proof. (i) and (iii) are clear and (ii) follows from the fact that $U_Q R$ is dense in Q.

DEFINITION 2.1. Define the operator $\kappa_\mathcal{E} \colon L_{Q,\mathcal{E}} \to L_{Q,\mathcal{E}^\sigma}$ by the formula

$$\kappa_\mathcal{E} = C_1^{-1} \circ C_2^{-1} \circ C_3 \circ C_2 \circ C_1.$$

The operator $\kappa_\mathcal{E}$ is unitary and one easily verifies that $\kappa_{\mathcal{E}^\sigma} \circ \kappa_\mathcal{E} = \mathrm{id}$. By Lemma 2.1, $\kappa_\mathcal{E}$ satisfies the condition

$$\kappa_\mathcal{E} \circ \rho_{Q,\mathcal{E}}(r) = \rho_{Q,\mathcal{E}^\sigma}(r^\sigma) \circ \kappa_\mathcal{E} \qquad r \in R. \tag{2.3}$$

Explicit formula for $\kappa_\mathcal{E}$ is given as follows. Let $q = w_Q^{-1} a r$, $a \in A$, $r \in R$. Then

$$(\kappa_\mathcal{E} f)(q) = \int_B f(w_Q^{-1} b r^\sigma)|db|. \tag{2.4}$$

DEFINITION 2.2. Define the operator $\beta_\pi^\kappa \colon L_{Q,\mathcal{E}} \to L_{Q,\mathcal{E}}$ by the formula

$$\beta_\pi^\kappa = \kappa_\mathcal{E}^{-1} \circ \beta_\pi.$$

By (2.1) and (2.3), the operator β_π^κ commutes with $\rho_{Q,\mathcal{E}}(r)$ for $r \in R$.

2.6. The isomorphism of spaces of operators. Denote by $M \simeq F^* \times GL_{n-2} \subset R$ the subgroup of the matrices m of the form

$$m = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & \lambda \end{pmatrix},$$

where $\lambda \in F^*$, $A \in GL_{n-2}$. Denote $U_M = U_- \cap M$, $\psi_M = \psi_R|_M$, and

$$(\rho_{M,\mathcal{E}}, L_{M,\mathcal{E}}) \stackrel{\mathrm{df}}{=} \mathrm{Ind}_{CU_M}^M(\mathcal{E} \cdot \psi_M).$$

For two unitary representations (ρ_1, L_1), (ρ_2, L_2) of a group G by $\mathrm{Hom}_G(L_1, L_2)$ we denote the space of continuous linear operators $V_1 \to V_2$ commuting with the action of G.

Our goal in this subsection is to construct an isomorphism of linear spaces

$$\alpha_\mathcal{E} \colon \mathrm{Hom}_R(L_{Q,\mathcal{E}}, L_{Q,\mathcal{E}}) \simeq \mathrm{Hom}_M(L_{M,\mathcal{E}}, L_{M,\mathcal{E}}). \tag{2.5}$$

(i) By Lemma 2.1 (i), (ii), the restriction of $\rho_{Q,\mathcal{E}}$ to R is equivalent to the representation $(\rho_{R,\mathcal{E}}, L_{R,\mathcal{E}})$ with the equivalence established by the operator $C_2 \circ C_1 \colon L_{Q,\mathcal{E}} \to$

$L_{R,\mathcal{R}}$. Therefore,

$$\operatorname{Hom}_R(L_{Q,\mathcal{E}}, L_{Q,\mathcal{E}}) \simeq \operatorname{Hom}_R(L_{R,\mathcal{E}}, L_{R,\mathcal{E}}). \tag{2.6}$$

(ii) Denote by H the subgroup of the block matrices h of the form

$$h = \begin{pmatrix} 1 & 0 & 0 \\ a & E_{n-2} & 0 \\ c & b & \lambda \end{pmatrix}, \tag{2.7}$$

where E_{n-2} is the identity matrix, a and b are $(n-2)$-dimensional vectors, $c \in F$, and $\lambda \in F^*$. Let $U_H \subset H$ be the subgroup consisting of the matrices $h \in H$ with $b = 0$ and $\lambda = 1$ in (2.7), and $\psi_H \colon U_H \to \mathbb{C}^*$ the character given by

$$\psi_H(h) = \psi_F(c).$$

Let (ρ_H, L_H) be the induced representation

$$(\rho_H, L_H) \overset{\text{df}}{=} \operatorname{Ind}_{U_H}^H(\psi_H).$$

It is easy to prove that the representation ρ_H is irreducible. Furthermore, since the adjoint action of M on H preserves the subgroup H and the character U_H, we have the representation I of M in L_H given by the formula $(I(m)f)(h) = f(m^{-1}hm)$.

(iii) We have $R = H \ltimes M$ and $U_R = U_H \ltimes U_M$. Also, the restriction of ψ_R to $\widehat{H}$ and to U_M coincides with ψ_H and ψ_M respectively. For each $m \in M$ we have $mU_Hm^{-1} = U_H$ and $\psi_H(mhm^{-1}) = \psi_H(h)$ for all $h \in U_H$. Therefore, regarding $L_H \otimes L_{M,\mathcal{E}}$ as the space of $(U_H \times CU_M, \psi_H \cdot (\mathcal{E}\,\psi_M))$-equivariant functions on $H \times M$, we see that the mapping $f \mapsto (f_1)(h,m) = f(mh)$ establishes an isomorphism of linear spaces

$$L_{R,\mathcal{E}} \xrightarrow{\sim} L_H \otimes L_{M,\mathcal{E}}. \tag{2.8}$$

The group R acts on the spaces on both sides of the last formula: by $\rho_{R,\mathcal{E}}$ on the left space, and by the formula

$$r = hm \mapsto (\rho_H(h) \circ I(m)) \otimes \rho_{M,\mathcal{E}}(m)$$

on the right space, and the isomorphism (2.8) intertwines these actions of R. Taking into account that ρ_H is irreducible, we obtain

$$\operatorname{Hom}_H(L_{R,\mathcal{E}}, L_{R,\mathcal{E}}) \simeq \operatorname{Hom}(L_{M,\mathcal{E}}, L_{M,\mathcal{E}}). \tag{2.9}$$

(iv) The group M acts on both sides of (2.9) by the formula

$$a \mapsto \rho_R(m) \circ a \circ \rho_R(m^{-1}), \qquad a\colon L_{R,\mathcal{E}} \to L_{R,\mathcal{E}}$$

for the left-hand side and a similar formula for right-hand side, and the isomorphism (2.9) intertwines these actions. Taking M-invariant elements, we obtain the isomorphism

$$\operatorname{Hom}_R(L_{R,\mathcal{E}}, L_{R,\mathcal{E}}) \simeq \operatorname{Hom}_M(L_{M,\mathcal{E}}, L_{M,\mathcal{E}}). \tag{2.10}$$

Combining it with (2.6), we get the required isomorphism $\alpha_{\mathcal{E}}$ in (2.5).

2.7. The operator Γ_π. Applying the isomorphism $\alpha_{\mathcal{E}}$ in (2.5) to the operator $\beta^\kappa_\pi \colon L_{Q,\mathcal{E}} \to L_{Q,\mathcal{E}}$ we obtain an operator $\Gamma_\pi \colon L_{M,\mathcal{E}} \to L_{M,\mathcal{E}}$ commuting with $\rho_{M,\mathcal{E}}$. We call it the Γ-operator corresponding to the generic unitary representation π. Let

$$\rho_{M,\mathcal{E}} = \oplus \int_\Theta \theta \, d\mu(\theta)$$

be the direct integral decomposition of $\rho_{M,\mathcal{E}}$ into irreducible components, each occuring with multiplicity one. In this decomposition the operator Γ_π is the multiplication by almost everywhere defined function $\Gamma(\pi, \cdot)$ on Θ.

Since $M = C \times GL_{n-2}$, the formula $\theta \mapsto \mathcal{E}^{-1} \otimes \theta$ establishes a bijection between the set Θ of irreducible components of $\rho_{M,\mathcal{E}}$ and the set of generic unitary irreducible representations of GL_{n-2}.

Denote by $\gamma(\pi, \tau)$ the Gamma factor of [JPS] at the point $s = 1/2$ (see [JPS], (3.1)).

PROPOSITION 2.1. *For any generic unitary representation π of the group GL_n we have $\Gamma(\pi, \theta) = \theta(-1)^{n-1}\gamma(\pi, \mathcal{E}^{-1} \otimes \theta)$ for almost every $\theta \in \Theta$.*

Proof. See Appendix.

3. Γ-measure corresponding to the principal series.

3.1. Standard realization of the principal series. Let $AFl = U_+ \setminus GL_n$ be the affine flag manifold and $(M, \Pi) = \mathrm{Ind}_{U_+}^{GL_n}(1)$ the principal series representation of GL_n. On AFl, we consider the left action of the split torus $T_0 \subset GL_n$ given by the formula $x \mapsto tx$. This formula makes sense because T_0 normalizes U_+. The action of T_0 commutes with the action of GL_n on AFl, so we can regard Π as a representation of the direct product $GL_n \times T_0$ according to the formula

$$\Pi(g,t)f(x) = f(t^{-1}xg)\Delta_{B_+}(t), \qquad x \in AFl, \quad g \in GL_n, \quad t \in T_0,$$

where Δ_{B_+} is the modulus,

$$\Delta_{B_+}(t) = \left| \frac{d(tut^{-1})}{du} \right|^{1/2}, \qquad t \in T_0,$$

and du is the invariant volume form on U_+.

Let $\widetilde{B} \subset GL_n \times T_0$ be the following subgroup:

$$\widetilde{B} = \{(b,t) : b \in B_+, \, t \in T_0, \, t^{-1}b \in U_+\}.$$

Then

$$\Pi \simeq \mathrm{Ind}_{\widetilde{B}}^{GL_n \times T_0}(1)$$

(isomorphism of representations of $GL_n \times T_0$).

3.2. Irreducible principal series representations. Let $\chi \colon T_0 \to \mathbb{C}$ be a unitary character of T_0. Regarding χ as a character of B_+ via the isomorphism $T_0 \simeq B_+/U_+$, denote

$$(\pi_\chi, V_\chi) = \mathrm{Ind}_{B_+}^{GL_n}(\chi).$$

Recall that Ind denotes the unitary induction, so that the space of the representation π_χ consists of B_+-homogeneous functions on GL_n of degree $(\chi\Delta_{B_+})(b)$ for $b \in B_+$.

The following two propositions summarize the well-known results about principal series representations.

PROPOSITION 3.1. (i) *Each* π_χ *is a generic unitary representation of the group* GL_n.

(ii) *The Weyl group* S_n *(the symmetric group of order* n*) acts on* T_0*, hence also on the unitary dual* $(T_0)^\frown$ *to* T_0*. Representations* π_{χ_1} *and* π_{χ_2} *are equivalent if and only if* $\chi_1 = (\chi_2)^w$ *for some* $w \in S_n$.

Together with the representation π_χ we consider the representation $\widetilde{\pi}_\chi$ of the group $GL_n \times T_0$ defined by the formula $\widetilde{\pi}_\chi = \pi_\chi \otimes \chi^{-1}$.

PROPOSITION 3.2. *We have the direct integral decomposition of representations of* $GL_n \times T_0$*:*

$$\Pi = \oplus \int_{(T_0)^\frown} \widetilde{\pi}_\chi \, d\chi,$$

where $d\chi$ *is the Haar measure on* $(T_0)^\frown$.

3.3. Restriction to Q. Recall the definition of the representation $\rho_{Q,\mathcal{E}}$ of the group Q (see 2.1). By Proposition 3.1(i) and Theorem 2.1 we have

$$\pi_\chi|_Q \simeq \rho_{Q,\mathcal{E}},$$

where $\mathcal{E}(\lambda) = \chi(\operatorname{diag}(\lambda, \dots, \lambda))$ is the central character of π_χ. Therefore, we have

$$\widetilde{\pi}_\chi|_{Q\times T_0} \simeq \rho_{Q,\mathcal{E}}, \times \chi^{-1}. \tag{3.1}$$

Introduce the subgroup $C_1 \simeq F^*$ as follows: $C_1 = \{(\lambda E, \lambda E)\} \subset GL_n \times T_0$, where E is the identity element. Clearly, C_1 is in the kernel of the representation Π. Denote by $\widetilde{\psi}$ the one-dimesional reprepsentation $\widetilde{\psi}(\lambda u_-, \lambda E) = \psi(u_-)$ of the group $C_1 U_-$, and let

$$(\widetilde{\rho}_Q, \widetilde{L}_Q) \stackrel{\text{df}}{=} \operatorname{Ind}_{C_1 U_-}^{Q\times T_0}(\widetilde{\psi}).$$

For any $\mathcal{E} \in (F^*)^\frown$ let

$$(R_{T_0,\mathcal{E}}, L_{T_0,\mathcal{E}}) \stackrel{\text{df}}{=} \operatorname{Ind}_{\{\lambda E\}}^{T_0}(\mathcal{E})$$

be the $\mathcal{E}$-homogeneous part of the regular representation of T_0. Then

$$\widetilde{\rho}_Q = \oplus \int_{(F^*)^\frown} (\rho_{Q,\mathcal{E}} \otimes R_{T_0,\mathcal{E}^{-1}}) \, d\mathcal{E}. \tag{3.2}$$

On the other hand, by Proposition 3.2 and formula (3.1) we have

$$\Pi|_{Q\times T_0} = \oplus \int_{(F^*)^\frown} (\rho_{Q,\mathcal{E}} \otimes R_{T_0,\mathcal{E}^{-1}}) \, d\mathcal{E}.$$

Therefore, the representations $\Pi\,|_{Q\times T_0}$ and $\widetilde{\rho}_Q$ of the group $Q \times T_0$ are equivalent. Since these representations are reducible, there are many isomorphisms of these representations. For our purposes we must choose a particular isomorphism

$$\varphi \colon \Pi\,|_{Q\times T_0} \to \widetilde{\rho}_Q \tag{3.3}$$

constructed in [K] using the Jacquet functors. Rather than going into details of Jacquet functors, we present explicit formulas for φ. Before doing this, we need some preparation.

Consider the subgroups $\widetilde{B}$ and $\widetilde{Q} \stackrel{\text{df}}{=} Q \times T_0$ in $GL_n \times T_0$. Since $\widetilde{B}\widetilde{Q}$ is dense in $GL_n \times T_0$, the restriction $\Pi|_{\widetilde{Q}}$ is equivalent to the induced representation

$$(\Pi_1, M_1) \stackrel{\text{df}}{=} \operatorname{Ind}_{B_1}^{\widetilde{Q}}(1), \qquad B_1 \stackrel{\text{df}}{=} \widetilde{B} \cap \widetilde{Q},$$

in the space M_1 of left B_1-invariant functions on $\widetilde{Q}$. The operator $M \to M_1$ establishing this equivalence sends a function $f \in M$ on $GL_n \times T_0$ to its restriction to $\widetilde{Q} = Q \times T_0$. Hence, both representations $\Pi|_{\widetilde{Q}}$ and $\widetilde{\rho}_Q$ act in the spaces M_1 and $\widetilde{L}_Q$ of functions on the group $\widetilde{Q}$. In these realizations, explicit formulas for the operator φ are described as follows.

Let $L_S \subset \widetilde{L}_Q$ be the dense subspace consisting of smooth functions with compact support modulo C_1U_- (the Schwarz space). Let also $U_1 = U_+ \cap Q$ and $T_0^{\text{diag}} = \{(t,t) : t \in T_0\} \subset Q \times T_0$, so that $B_1 = (U_1 \times \{1\}) \cdot T_0^{\text{diag}} \subset Q \times T_0$.

Proposition 3.3 (see [K], Lemma 3.1.10). (i) *For $f \in L_S$ the integral*

$$(\varphi f)(q) = \int_{U_1 \times T_0 / C_1} f(utq, t)\, du\, dt$$

(which makes sense since f is invariant under C_1) converges absolutely.

(ii) The mapping $f \mapsto \varphi(f)$ extends to a unitary operator $\varphi\colon \widetilde{L}_Q \to M_1$ intertwining $\widetilde{\rho}_Q$ and $\Pi|_{Q\times T_0}$.

(iii) *For any $w \in W(T_0) = S_n$ we have*

$$\varphi \circ w = \mathcal{F}_w \circ \varphi,$$

where S_n acts on $\widetilde{L}$ by the formula

$$wf(q,t) = f(q, t^{w^{-1}}),$$

and $\mathcal{F}_w$ is the Fourier–Weyl operator in the space $M_1 \simeq M$ (for the definition of $\mathcal{F}_w$, see [GG] *or* [KL]*).*

Proof of (i). The proof follows from the directly verified fact that the composition

$$U_1 \hookrightarrow Q \to U_- \setminus Q$$

is a proper map.

3.4. Representing operators by measures. In this paper we will often replace operators between spaces of induced representations of a group G by the corresponding measures. The general construction is described as follows.

Let G be a topological group, H_1, H_2 two subgroups of G, and θ_1, θ_2 unitary characters of H_1, H_2 respectively. Let

$$(\rho_1, V_1) = \operatorname{Ind}_{H_1}^G(\theta_1), \quad (\rho_2, V_2) = \operatorname{Ind}_{H_2}^G(\theta_2)$$

be two irreducible representations and $E\colon V_1 \to V_2$ a linear operator. Define the left-$(H_1 \times H_2, \theta_1^{-1}\theta_2)$-equivariant complex valued measure μ_E on $G \times G$ by the formula

$$\int_{(H_1\backslash G)\times(H_2\backslash G)} f_1(x_1)\overline{f_2(x_2)}\mu_E(x_1,x_2) = (Ef_1, f_2)_{V_2}, \tag{3.4}$$

where $(\ ,\ )_{V_2}$ is the inner product in V_2. Since the linear combinations of the products $f_1(x_1)\overline{f_2(x_2)}$ are dense in the appropriate Hilbert space of left-$(H_1 \times H_2, \theta_1\theta_2^{-1})$-equivariant functions on $G \times G$, formula (1.3) determines μ_E uniquely.

If E is an intertwining operator, i.e., commutes with the action of G in V_1 and V_2, then the measure μ_E is right-invariant with respect to the diagonal action of G on $G \times G$.

In explicit computations it is often convenient to use another realization of the measure associated to an operator. As before, assume that $E\colon V_1 \to V_2$ is an operator between the spaces of two induced representations. Choose a section $s_i\colon X_i \to G$ of the projection $p_i\colon G \to H_i \backslash G$, $i = 1, 2$. Restricting functions in V_i to $X_i \subset G$, we can regard V_i as a space of functions on X_i (with appropriate inner product). Define the measure μ'_E on $X_1 \times X_2$ by the formula similar to (3.4):

$$\int_{X_1 \times X_2} f_1(x_1) f_2(x_2) \mu'_E(x_1, x_2) = (Ef_1, f_2)_{V_2}. \tag{3.5}$$

Again, formula (3.5) determines the measure μ'_E uniquely.

There is an obvious one-to-one correspondence between measures on $X_1 \times X_2$ and left-$(H_1 \times H_2, \theta_1^{-1}\theta_2)$-equivariant measures on $G \times G$ which sends μ_E to μ'_E.

In what follows we will not distinguish between the measures μ_E and μ'_E. It will be clear from the context (or stated explicitly) which of these two measures is used.

3.5. Properties of measures corresponding to operators.

DEFINITION 3.1. Let X be a space with a positive measure ν. A complex valued measure μ on $X \times X$ is said to be a *ν-operator measure* (or simply an operator measure) if

$$\langle \mu f_1 \times \overline{f}_2 \rangle \leq C \|f_1\|_{L_2} \cdot \|f_2\|_{L_2}$$

for all $f_1, f_2 \in L_2(X, \nu)$.

LEMMA 3.1. *Formula* (3.4) *establishes a one-to-one correspondence between the bounded operators* $E\colon L_2(X,\nu) \to L_2(X,\nu)$ *and the operator measures* μ_E *on* $X \times X$.

Proof. Clear.

We will need the following properties of operator measures.

I. Let $X' \subset X$ be a subset such that $\nu(X \setminus X') = 0$ and ν' the restriction of ν to X'. For an operator measure μ on $X \times X$ denote by μ' the restriction of μ to $X' \times X'$. Then $\mu \longleftrightarrow \mu'$ is an one-to-one correspondence between operator measures on $X \times X$ and $X' \times X'$.

II. Let $(X, \nu) = (X_1 \times X_2, \nu_1 \times \nu_2)$. Let μ_i be a ν_i-operator measure on $X_i \times X_i$, $i = 1, 2$. Then $\mu = \mu_1 \times \mu_2$ is a ν-operator measure on X.

Similarly we can define operator measures on sections of product line bundles on $X \times X$.

Now we turn to the construction of the Γ-measure corresponding to the principal series representation Π. This construction is similar to the construction of the Γ-factors described in §2, but is performed "for all $\chi \in (T_0)\hat{}$ simulteneously." At the end of the section we show how this construction is formulated in terms of measures.

3.6. The operator β_Π. Extend this involution σ from 2.1 to $GL_n \times T_0$ by the formula

$$\sigma(g, t) = (g^\sigma, t^\sigma).$$

Since the subgroup $\widetilde{B} \subset GL_n \times T_0$ is invariant under σ, we can define the operator $\Pi(\sigma)$ in the space M by the formula

$$\Pi(\sigma) f(x) = f(x^\sigma).$$

Now we set

(3.6) $$\beta_\Pi = \varphi^{-1} \circ \Pi(\sigma) \circ \varphi \colon \widetilde{L}_Q \to \widetilde{L}_Q.$$

Clearly, β_Π is a unitary operator in the space $\widetilde{L}_Q$ satisfying the condition

(3.7) $$\beta_\Pi \circ \widetilde{\rho}_Q(r,t) = \widetilde{\rho}_Q(r^\sigma, t^\sigma) \circ \beta_\Pi, \qquad r \in R = Q \cap Q^\sigma, \quad t \in T_0.$$

Formula (3.7) implies, in particular, that the operator β_Π intertwines irreducible principal series representations π_χ and π_{χ^σ} in Π.

PROPOSITION 3.4. *In decomposition* (3.2) *we have*

$$\beta_\Pi = \oplus \int_{(T_0)^\frown} (\beta_{\pi_\chi} \otimes \sigma),$$

where the operator β_{π_χ} *is defined in* 2.5 *and* $\sigma \colon L_{T_0,\mathcal{E}} \to L_{T_0,\mathcal{E}^{-1}}$ *is induced by the action of* σ *on* T_0.

Proof. Clear.

3.7. The operator κ. Recall the direct integral decomposition (3.2) of the representation $(\widetilde{\rho}_Q, \widetilde{L}_Q)$ of the group $\widetilde{Q}$,

(3.8) $$\widetilde{L}_Q = \oplus \int (L_{Q,\mathcal{E}} \otimes L_{T_0,\mathcal{E}^{-1}})\, d\mathcal{E}.$$

DEFINITION 3.2. In the decomposition (3.8), let us define the operator $\kappa \colon \widetilde{L}_Q \to \widetilde{L}_Q$ by the formula

(3.9) $$\kappa = \oplus \int (\kappa_{\mathcal{E}} \otimes \sigma_{T_0})\, d\mathcal{E},$$

where $\sigma_{T_0} : L_{T_0,\mathcal{E}^{-1}} \to L_{T_0,\mathcal{E}}$ is given by $\sigma F(t) = F(t^\sigma)$.

PROPOSITION 3.5. (i) *Formula* (3.9) *yields a unitary operator* $\kappa \colon \widetilde{L}_Q \to \widetilde{L}_Q$ *such that* $\kappa^2 = \mathrm{id}$.

(ii) *The operator* κ *satisfies the condition*

(3.10) $$\kappa \circ \widetilde{\rho}_Q(r,t) = \widetilde{\rho}_Q(r^\sigma, t^\sigma) \circ \kappa \qquad r \in R = Q \cap Q^\sigma, \quad t \in T_0.$$

Proof. Immediately follows from the corresponding properties of the operators $\kappa_{\mathcal{E}}$.

Similarly to formula (2.4), we can give an explicit formula for the operator κ. Let $q = w_Q^{-1} a r$ with $a \in A$, $r \in R$, and let $t \in T_0$. Then

(3.11) $$(\kappa f)(q,t) = \psi_Q(a) \int_B f(w_Q^{-1} b r^\sigma, t^\sigma) |db|.$$

3.8. The isomorphism α. In §2 (see 2.7), we have constructed the isomorphisms

$$\alpha_{\mathcal{E}} \colon \mathrm{Hom}_R(L_{Q,\mathcal{E}}, L_{Q,\mathcal{E}}) \simeq \mathrm{Hom}_M(L_{M,\mathcal{E}}, L_{M,\mathcal{E}}), \quad \mathcal{E} \in (F^*)^\frown.$$

Here we repeat this construction in the framework of the "tensored with T_0" space $\widetilde{L}_Q$ instead of $L_{Q,\mathcal{E}}$. We will use the notation of §2. In addition, we denote $\widetilde{R} = R \times T_0$,

$\widetilde{M} = M \times T_0$. Then $C_1 = \{(\lambda I, \lambda I)\} \subset \widetilde{M} \subset \widetilde{R}$ and we can define

$$(\widetilde{\rho}_M, \widetilde{L}_M) \stackrel{\text{df}}{=} \operatorname{Ind}_{C_1 U_M}^{\widetilde{M}}(\psi_M),$$
$$(\widetilde{\rho}_R, \widetilde{L}_R) \stackrel{\text{df}}{=} \operatorname{Ind}_{C_1 U_R}^{\widetilde{R}}(\psi_R).$$

Our goal is to construct the isomorphism

$$\alpha\colon \operatorname{Hom}_{\widetilde{R}}(\widetilde{L}_Q, \widetilde{L}_Q) \simeq \operatorname{Hom}_{\widetilde{M}}(\widetilde{L}_M, \widetilde{L}_M).$$

In addition to the direct integral decomposition (3.11), we have a similar decomposition

$$\widetilde{M}_Q = \oplus \int (M_{Q,\mathcal{E}} \otimes L_{T_0,\mathcal{E}^{-1}})\, d\mathcal{E}. \tag{3.12}$$

for $\widetilde{M}_Q$. Clearly,

$$\operatorname{Hom}_{\widetilde{R}}(\widetilde{L}_Q, \widetilde{L}_Q) = \oplus \int (\operatorname{Hom}_R(L_{Q,\mathcal{E}}, L_{Q,\mathcal{E}}) \otimes \operatorname{Hom}_{T_0}(L_{T_0,\mathcal{E}}, L_{T_0,\mathcal{E}}))\, d\mathcal{E},$$

and similarly for $\widetilde{M}_Q$:

$$\operatorname{Hom}_{\widetilde{M}}(\widetilde{L}_M, \widetilde{L}_M) = \oplus \int (\operatorname{Hom}_R(L_{M,\mathcal{E}}, L_{M,\mathcal{E}}) \otimes \operatorname{Hom}_{T_0}(L_{T_0,\mathcal{E}}, L_{T_0,\mathcal{E}}))\, d\mathcal{E}.$$

In terms of these decompositions, we define the operator α as the direct integral

$$\alpha = \oplus \int (\alpha_{\mathcal{E}} \otimes \operatorname{Id})\, d\mathcal{E}.$$

Repeating the construction of 2.6, we see that the isomorphism α is the composition of two isomorphisms,

$$\alpha_{Q\to R}\colon \operatorname{Hom}_{\widetilde{R}}(\widetilde{L}_Q, \widetilde{L}_Q) \simeq \operatorname{Hom}_{\widetilde{R}}(\widetilde{L}_R, \widetilde{L}_R)$$

and

$$\alpha_{R\to M}\colon \operatorname{Hom}_{\widetilde{R}}(\widetilde{L}_R, \widetilde{L}_R) \simeq \operatorname{Hom}_{\widetilde{M}}(\widetilde{L}_M, \widetilde{L}_M).$$

3.9. Transformation of measures. Let μ_β be the $(C_1U_- \times C_1U_-, \psi^{-1}\cdot\psi)$-equivariant measure on $\widetilde{Q} \times \widetilde{Q}$ corresponding to the operator β_Π and μ_Γ be the $(U_M \times U_M, \psi_M^{-1}\cdot\psi_M)$-equivariant measure on $\widetilde{M} \times \widetilde{M}$ corresponding to the operator $\alpha(\kappa\circ\beta_\Pi)$. In this subsection we express μ_Γ in terms of μ_β. The measure μ_Γ is obtained from μ_β in the following three steps:

$$\mu_\beta \Longrightarrow \mu_\beta^\kappa \stackrel{\alpha_{Q\to R}}{\Longrightarrow} \mu_R \stackrel{\alpha_{R\to M}}{\Longrightarrow} \mu_\Gamma, \tag{3.13}$$

where μ_β^κ corresponds to the operator $\kappa\circ\beta_\Pi$, μ_R corresponds to the operator $\alpha_{Q\to R}(\kappa\circ\beta_\Pi)$, and μ_Γ corresponds to the operator $\alpha_{R\to M}(\alpha_{Q\to R}(\kappa\circ\beta_\Pi)) = \alpha(\kappa\circ\beta_\Pi)$. In the next three lemmas we present the formulas for each of the steps in (3.13).

Recall the definition of the subgroup A in 2.5.

LEMMA 3.2. *The measure μ_β^κ is the unique $(C_1U_- \times C_1U_-, \psi^{-1}\cdot\psi)$-equivariant operator measure on $\widetilde{Q} \times \widetilde{Q}$ whose restriction to the open set*

$$Q \times T_0 \times w_Q^{-1} AR \times T_0 \subset Q \times T_0 \times Q \times T_0 = \widetilde{Q} \times \widetilde{Q} \tag{3.14}$$

is given by the formula

$$\mu^{\kappa}_{\beta}(q,t;w_Q^{-1}ar,t') = \psi_Q^{-1}(a)\mu_{\beta}(q,t;w_Q^{-1}r^{\sigma},t^{\sigma}). \tag{3.15}$$

Proof. Formula (3.15) for the restriction of μ^{κ}_{β} to the open set (3.14) follows from (3.11). The uniqueness of the extension follows from Property I in 3.5.

Next we construct μ_R from μ^{Γ}_{β}. Consider the auxiliary measure

$$\mu'(q_1,t_1;q_2,t_2) = \mu^{\kappa}_{\beta}(w_Q^{-1}q_1,t_1;w_Q^{-1}q_2,t_2)$$

on $\widetilde{Q}\times\widetilde{Q}$. This measure is $(C_1U_Q\times C_1U_Q,\psi_Q^{-1}\cdot\psi_Q)$-equivariant. Therefore, its restriction to the open dense set $AR\times T_0\times AR\times T_0\subset\widetilde{Q}\times\widetilde{Q}$ has the form

$$\mu'(q_1,t_1;q_2,t_2) = \psi_Q^{-1}(a_1)\psi_Q(a_2)da_1'\,da_2'\,\mu''(r_1,t_1;r_2,t_2),$$

for $q_i = a_ir_i$, $a_i\in A$, $r_i\in R$, $i=1,2$, where μ'' is a $(C_1U_R\times C_1U_R,\psi_R^{-1}\cdot\psi_R)$-equivariant operator measure on $\widetilde{R}\times\widetilde{R}$.

LEMMA 3.3. *The measure μ_R corresponding to $\alpha_{Q\to R}(\mu^{\kappa}_{\beta})$ coincides with the measure $\mu''(r_1,t_1;r_2,t_2)$.*

Proof. Clear.

Now we construct μ_{Γ} from μ_R. We have the semidirect product decompositions $\widetilde{R}=H\ltimes\widetilde{M}$ and $C_1U_R=U_H\ltimes C_1U_M$ (see 2.6), hence, in particlar, $\widetilde{R}=H\times\widetilde{M}$ as spaces.

Denote by the δ_{H/U_H} the $(U_H\times U_H,\psi_H^{-1}\cdot\psi_H)$-equivariant measure on $H\times H$ corresponding to the identity operator in the space L_H of (U_H,ψ_H)-equivariant functions on H. Explicitely, δ_{H/U_H} is given as follows. Let

$$V_H=\left\{v=\begin{pmatrix}1&0&0\\0&E_{n-2}&0\\0&b&a\end{pmatrix}\right\}\subset H.$$

Then $H=U_HV_H$ and $U_H\cap V_H=\{1\}$. Let δ_{V_H} be the diagonal measure on $V_H\times V_H$, i.e.,

$$\int_{V_H\times V_H} f(v_1,v_2)\delta_{V_H} = \int_{V_H} f(v,v)\,|dv|.$$

Then

$$\delta_{H/U_H}(u_1v_1,u_2v_2)\overset{\mathrm{df}}{=}\psi_H(u_1^{-1}u_2)\delta_{V_H}(v_1,v_2).$$

LEMMA 3.4. *The measure μ_R is of the form $\mu_R=\delta_{U_H}\times\mu'''$ for a unique $(U_M\times U_M,\psi_M^{-1}\cdot\psi_M)$-equivariant operator measure μ''' on $\widetilde{M}\times\widetilde{M}$, and the measure μ_{Γ} corresponding to the operator $\alpha(\kappa\circ\beta_{\Pi})$ coincides with μ'''.*

Proof. The first statement holds for every operator measure μ corresponding to an operator $\Phi\in\mathrm{Hom}_H(\widetilde{L}_R,\widetilde{L}_R)$. The second statement follows from the explicit constriction of the isomorphism $\alpha_{R\to M}$ (see 2.6).

Finally, we combine Lemmas 3.2–3.4 and give an expression for μ_{Γ}. Consider he map

$$S\overset{\mathrm{df}}{=}(A\times H\times M)\times T_0\times(A\times H\times M)\times T_0\to\widetilde{Q}\times\widetilde{Q}$$

given by the formula

$$(a_1, u_1, m_1; t_1; a_2, u_2, m_2; t_2) \mapsto (w_Q^{-1} a_1 h_1 m_1, t_1; w_Q^{-1} a_2 h_2^\sigma m_2^\sigma, t_2^\sigma) \in Q \times T_0 \times Q \times T_0 = \widetilde{Q} \times \widetilde{Q}.$$

This map is an embedding of S into an open dense subset $\widetilde{S} \subset \widetilde{Q} \times \widetilde{Q}$. Lemmas 3.2–3.4 immediately show that the restriction $\mu' = \mu_\beta|_{\widetilde{S}}$ of the operator measure μ_β to $\widetilde{S} \simeq S$ is given by the formula

$$\mu'(a_1, u_1, m_1; t_1; a_2, u_2, m_2; t_2) = \delta_{H/U_H}(h_1, h_2)\psi_Q^{-1}(a_1)\psi_Q(a_2)\widehat{\mu}(m_1, t_1; m_2, t_2)$$

for some $(U_M \times U_M, \psi_M^{-1} \cdot \psi_M)$-equivariant operator measure $\widehat{\mu}$ on $\widetilde{M} \times \widetilde{M}$, and that the measure μ_Γ corresponding to the operator $\alpha(\kappa \circ \beta_\Pi)$, equals $\widehat{\mu}$.

4. The measure μ_β.

4.1. The measure μ_β. In this section we compute explicitely the measure μ_β corrsponding to the operator $\beta_\Pi\colon \widetilde{L}_Q \to \widetilde{L}_Q$ defined by formula (3.6). Hence, μ_β is a $(C_1U_- \times C_1U_-, \psi \cdot \psi^{-1})$-equivariant complex valued measure on $\widetilde{Q} \times \widetilde{Q}$ given by the formula

$$\langle \mu_\beta, f_1 \times \overline{f_2} \rangle = (\beta_\Pi f_1, f_2)_{\widetilde{L}_Q} \tag{4.1}$$

for $f_1, f_2 \in \widetilde{L}_Q$.

Since φ and $\Pi(\sigma)$ in the definition (3.6) of β_Π are unitary operators, and $\Pi(\sigma)$ is an involution, we have

$$(\beta_\Pi f_1, f_2)_{\widetilde{L}_Q} = (\varphi f_1, \Pi(\sigma)\varphi f_2)_{M_1}, \qquad f_1, f_2 \in \widetilde{L}_Q \tag{4.2}$$

Let $B_- \subset Q \subset GL_n$ be the subgroup of lower triangular matrices. Denote by the same letter the corresponding subgroup $B_- \times \{1\}$ in $\widetilde{Q} \subset GL_n \times T_0$. We have $B_- \cap B_1 = \{1\}$ and B_-B_1 is dense in $\widetilde{Q}$. Therefore, the inner product in M_1 can be written in the form

$$(F_1, F_2)_{M_1} = \int_{B_-} F_1(b_-)\overline{F_2(b_-)}\,|db_-|, \tag{4.3}$$

where db_- is a fixed left invariant volume form on B_- and the integral converges absolutely.

Denote $Z = U_1 \times T_0/C \times U_1 \times T_0/C \times B_-$. Recall that by $L_S \subset \widetilde{L}_Q$ we denoted the space of smooth (C_1U_-)-equivariant fiunctions on $\widetilde{Q}$ that are compact modulo C_1U_-. Taking into account the formulas for the operator φ (Proposition 3.3(i, ii)) we formally obtain from (4.1) and (4.2) that

$$\langle \mu_\beta, f_1 \times \overline{f_2} \rangle = \int_Z f_1(u_1t_1b_-, t_1)\overline{f_2(u_2t_2b_-^\sigma, t_2)}\,|du_1\, dt_1\, du_2\, dt_2\, db_-|, \tag{4.4}$$

where $f_1, f_2 \in L_S$ are considered as functions on $\widetilde{Q} = Q \times T_0$ invariant under C_1.

Since the integral (4.4) does not converge absolutely, we must specify the order of integration. Fix $f_1, f_2 \in L_S$ and denote the intgrand in (4.4) by $F = F_{f_1,f_2}(u_1, t_1; u_2, t_2; b_-)$. Then Proposition 3.3(i)–(ii) show that for a fixed $b_- \in B_-$ there exists a compact set $K(b_-) = K_{f_1,f_2}(b_-) \subset U_1 \times T_0/C \times U_1 \times T_0/C$ (depending on the supports of f_1 and f_2) such that $F(u_1, t_1; u_2, t_2; b_-)$ vanishes for $(u_1, t_1; u_2, t_2) \notin K(b_-)$.

PROPOSITION 4.1. *For $f_1, f_2 \in L_S$ we have*

$$(4.5) \qquad \langle \mu_\beta, f_1 \times \overline{f_2} \rangle = \int_{B_-} db_- \int_{K(b_-)} F_{f_1,f_2}(u_1, t_1; u_2, t_2; b_-) |\, du_1\, dt_1\, du_2\, dt_2\, db_- |$$

and the integral over B_- converges absolutely.

Proof. Formula (4.5) follows from Proposition 3.3 (i)–(ii). The absolute convergence of the integral over B_- is the absolute convergence of the integral in (4.3).

4.2. Reduction of the measure μ_β. In this subsection we use the construction described in §1 (Proposition 1.1) to show that the measure μ_β is supported on a of codimension one submanifold $Y_1 \subset Y$ and compute the restriction of μ_β to Y_1.

In the group B_-, introduce the following subsets:

$$B' = \left\{ r(a) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & E & 0 \\ a & 0 & 1 \end{pmatrix}, \ a \in F \right\},$$
$$B'' = \{ b_- \in B_- \mid (b_-)_{n\,1} = 0 \}.$$

Then $B_- = B'B''$ with the unique decomposition and for $b_- = r(a)b''$ we have $db_- = da\, db''$ for a unique volume form db'' on B''. With this decomposition of B_- we will regard $Z = U_1 \times T_0/C \times U_1 \times T_0/C \times B_-$ as the total space of the trivial line bundle over $Z_0 = U_1 \times T_0/C \times U_1 \times T_0/C \times B''$ with a fixed trivialization. We identify each fiber with $B' \simeq F$, so that $Z = Z_0 \times B'$.

Fix $f_1, f_2 \in L_S$ and recall that by $F(z) = F_{f_1,f_2}(z)$ we denoted the integrand in (4.4) and in (4.5), so that the right-hand side of (4.5) is

$$(4.6) \qquad \int_{B_-} db_- \int_{K_{f_1,f_2}(b_-)} F_{f_1,f_2}(u_1, t_1; u_2, t_2; b_-) |\omega(z)|,$$

where $\omega = du_1\, dt_1\, du_2\, dt_2\, db_-$ is a volume form on Z. Using the intertwining property (3.7) of the operator β_Π with $r = r_a \in B'$, we easily see that for $z = (z_0, r_a) \in Z_0 \times B'$ we have

$$F(z) = \psi_F(\gamma(z_0)a) F_0(z_0),$$

where F_0 is the restriction of F to $Z_0 = Z_0 \times \{0\} \subset Z$ and the function γ (a section of the trivial bundle) is given by the formula

$$\gamma(z_0) = (u_1)_{2\,n} (t_1)_n (t_1)_1^{-1} + (-1)^{n+1} (u_2)_{2\,n} (t_2)_n (t_2)_1^{-1}$$

for $z_0 = (u_1, t_1; u_2, t_2; b'') \in Z_0$.

Denote by $N = N_\gamma \subset Z_0$ the set of zeros of γ.

LEMMA 4.1. *N is a smooth subvariety of Z_0, the section γ is generic at all points of N, and the form ω in (4.6) is fiberwise constant.*

Proof. Direct verification.

Denote by $\omega_1 = \eta(\gamma)\omega$ the induced form on N (see (1.4)), and by F_1 the restriction of F to N. We want to prove that

$$(4.7) \qquad \int_Z F(z)|\omega| = \int_N F_1(z)|\omega_1|.$$

However, we cannot apply Proposition 1.1 directly since Z_0 and N are not compact and the integral on the left-hand side of (4.7) is a iterated integral as explained in 4.1.

Nevertheless, formula (4.7) holds id the integral on the right-hand side in interpreted as an iterated integral similarly to the integral on the left, but "intersected with N".

Namely, for $b'' \in B''$ denote

$$M(b'') = \{(u_1, t_1; u_2, t_2) \text{ such that } (u_1, t_1; u_2, t_2; b'') \in N\}$$

and let $\omega_{b''} = \omega_1/db''$ be the corresponding volume form on $M(b'')$. Denote the integral (4.6) by I.

PROPOSITION 4.2. *We have*

$$I = \int_{B''} |db''| \int_{M(b'')} F_1(u_1, t_1; u_2, t_2; b'')|\omega_{b''}| \tag{4.8}$$

and both the inner and the outer integral on the right-hand side converge absolutely.

Proof. Recall that for a fixed $b_- \in B_-$ the inner integral in (4.6) is taken over the compact set $K(b_-) \subset U_1 \times T_0/C \times U_1 \times T_0/C$. It is easy to see that if $b_- = r_a b''$, then the set $K(b_-)$ can be taken not depending on a. Therefore, we denote it by $K(b'')$.

Now we take an arbitrary sequence of open compact sets $\mathcal{B}_n \subset B''$ such that

$$\mathcal{B}_1 \subset \cdots \subset \mathcal{B}_n \subset \cdots, \quad \bigcup \mathcal{B}_n = B''. \tag{4.9}$$

Let also $\mathcal{O}_m \subset B' = F$ be given by

$$\mathcal{O}_m = \{r_a | \, |a| \leq p^m\}.$$

Denote by $\Phi(b_-)$ the inner integral in (4.6) and let

$$I_{mn} = \int_{\mathcal{O}_m \times \mathcal{B}_n} \Phi(b_-) \, |db_-|.$$

Since the integral over B_- in (4.6) converges absolutely, we have

$$I = \lim_{m,n \to \infty} I_{mn} = \lim_{n \to \infty} (\lim_{m \to \infty} I_{mn}).$$

Denote

$$K_n = \bigcup_{b'' \in \mathcal{B}_n} K(b'') \subset U_1 \times T_0/C \times U_1 \times T_0/C.$$

This is a compact set and

$$\begin{aligned} I_{mn} &= \int_{\mathcal{O}_m \times \mathcal{B}_n} da \, db'' \int_{K_n} F(z_0, r_a) \, |du_1 \, dt_1 \, du_2 \, dt_2| \\ &= \int_{\mathcal{O}_m} da \int_{K_n \times \mathcal{B}_n} F(z_0, r_a)|\omega_0|. \end{aligned}$$

Since $K_n \times \mathcal{B}_n$ is a compact set, we can apply Proposition 1.1 and obtain

$$\lim_{m \to \infty} I_{mn} = \int_{(K_n \times \mathcal{B}_n) \cap N} F_1(z)|\omega_1|.$$

Since $\{\mathcal{B}_n\}$ is an arbitrary sequence of open sets satisfying (4.9), Proposition 4.2 is proved.

COROLLARY. *The restriction $\mu_\beta^{(1)}$ of the measure μ_β to Y_1 is given by formula* (4.8).

4.3. Another presentation of the measure $\mu_\beta^{(1)}$. To construct an algebraic presentation we need to transform formula (4.8) for the measure $\mu_\beta^{(1)}$.

Let us choose an arbitrary semialgebraic section $X \hookrightarrow \widetilde{Q}$ of the projection $\widetilde{Q} \to C_1 U_- \setminus \widetilde{Q}$, so that each $\widetilde{q} \in \widetilde{Q}$ we have a unique decomposition

$$\widetilde{q} = u_-(\widetilde{q})x(\widetilde{q}), \qquad u_-(\widetilde{q}) \in U_-, \quad x(\widetilde{q}) \in X.$$

According to the remark at the end of 3.4, we can define the measure μ_β as a measure on the set $Y = X \times X$.

Denote by $Y_1 \subset Y$ the subvariety of those $y = (q_1, t_2; q_2, t_2)$ for which

$$\widetilde{\gamma}(y) \overset{\text{df}}{=} (q_1)_{11}^{-1}(q_1)_{2n} + (-1)^{n+1}(q_2)_{11}^{-1}(q_2)_{2n} = 0. \tag{4.10}$$

Then the measure μ_β is supported on Y_1 (see 4.2) and $\mu_\beta^{(1)}$ is the restriction of μ_β to Y_1, i.e.,

$$\int_Y F\mu_\beta = \int_{Y_1} F\,|_{Y_1}\, \mu_\beta^{(1)}$$

for smooth compactly supported functions F on Y.

First of all, formula (4.4) immediately shows that the measure μ_β is supported on the subset $Y_0 \subset Y$ consisting of the quadruples $(q_1, t_1; q_2, t_2)$ such that

$$\det q_1 \det q_2 = \det t_1 \det t_2. \tag{4.11}$$

For two elements $g_1, g_2 \in GL_n$ we write $g_1 \overset{U_-}{\sim} g_2$ if g_1 and g_2 belong to the same double coset modulo U_-, i.e.,

$$g_1 = u_{-1} g_2 u_{-2}, \qquad u_{-1}, u_{-2} \in U_-.$$

For $y = (q_1, t_2; q_2, t_2) \in Y$ denote

$$b(y) = q_1 q_2^\top \in GL_n, \qquad t(y) = t_1 t_2^\top \in T_0. \tag{4.12}$$

Note that $b(y)_{1n} = 0$ for $y \in Y$ and, by (4.11), for $y \in Y_0$ we have

$$\det b(y) = \det t(y).$$

Now we define Z' as a subset of the direct product $Y_0 \times U_+$ consisting of the pairs (y, u) such that

$$b(y) \overset{U_-}{\sim} t(y)u, \tag{4.13}$$

i.e.,

$$u_{-1} b(y) u_{-2}^\top = t(y)u, \qquad u_{-1}, u_{-2} \in U_-. \tag{4.14}$$

For $(u_1, t_1; u_2, t_2; b_-) \in Z = U_1 \times T_0/C \times U_- \times T_0/C \times B_-$ we define $\theta(z) \in Y \times U_+$ as follows:

$$\theta(z) = (q_1, t_1; q_2, t_2; u) \in X \times X \times U_+,$$

where

$$\begin{aligned}(q_1, s_1) &= x(t_1 u_1 b_-, t_1),\\ (q_2, s_2) &= x(t_2 u_2 b_-^\sigma, t_2),\\ u &= t(y)^{-1} u_1 t(y) u_2^\top.\end{aligned} \tag{4.15}$$

Clearly, $u \in U_+$.

PROPOSITION 4.3. *We have $\theta(z) \in Z'$.*

Proof. Define $(c_1 u_{-1}, c_1), (c_2 u_{-2}, c_2) \in C_1 U_-$ by the formulas

$$(c_1 u_{-1}, c_1) = u_-(t_1 u_1 b_-, t_1), \qquad (c_2 u_{-2}, c_2) = u_-(t_2 u_2 b_-^\sigma, t_2),$$

so that in $\widetilde{Q} = Q \times T_0$ we have

$$(t_1 u_1 b_-, t_1) = (c_1 u_{-1}, c_1)(q_1, s_1), \tag{4.16}$$

$$(t_2 u_2 b_-^\sigma, t_2) = (c_2 u_{-2}, c_2)(x_2, s_2). \tag{4.17}$$

Applying the antiinvolution $\top$ to (4.17), multiplying by (4.16), and using formulas (4.11) and (4.15), we get

$$\begin{aligned} t_1 t_2^\top &= ct(y) \\ t_1 t_2^\top u &= cu_{-1} b(y) u_{-2}^\top, \end{aligned} \tag{4.18}$$

where $c = c_1 c_2$. Therefore, so that $b(y) \overset{U_-}{\sim} t(y)u$. Since $u \in U_+$, we have $q(z) \in Z'$. Proposition 4.3 is proved.

Denote by $p' \colon Z' \to Y_0$ the projection and let $Z_1' \subset Z'$ be the preimage of Y_1, so that Z_1' consists of those $z = (q_1, t_1; q_2, t_2; u)$ for which

$$\widetilde{\gamma}(p(z')) = 0.$$

Now we define open dense subset $Y_2 \subset Y_1$, $Z_2 \subset Z_1$, and $Z_2' \subset Z_1'$ by the conditions:

(4.19)
$$\begin{aligned} Y_2 &= \{y \in Y_1 \ \text{ such that } (b(y))_{1\, n-1} = (b(y))_{2n} \neq 0\} \\ Z_2 &= \{(u_1, t_1; u_2, t_2; b_-) \in Z_1 \ \text{ such that } (u_1)_{2n} \neq 0,\ (u_1)_{2n} \neq 0\}; \\ Z_2' &= \{(y, u) \in Z_1' \ \text{ such that } y \in Y_2.\} \end{aligned}$$

Clearly, θ maps Z_2 to Z_2'.

PROPOSITION 4.4. *$\theta \colon Z_2 \to Z_2'$ is a bijection.*

Proof. We construct the inverse map $\theta' \colon Z_2' \to Z_2$. In this construction we assume that $B_- \hookrightarrow GL_n \times T_0$ is a genuine section of the projection $GL_n \times T_0 \to B_1 \backslash GL_n \times T_0$ (for the notation, see 3.1). To make all the arguments precise we must replace B_- by a semialgebraic section $\widehat{B}_- \subset GL_n \times T_0$ such that $\widehat{B}_-$ is invariant under the left multiplication by elements of B'.

Let $z = (q_1, t_1; q_2, t_2; u) \in Z_2'$, i.e., conditions (4.11), (4.13) and (4.19) hold. For each pair (u_{-1}, u_{-2}) satisfying (4.14) we define $u_1 \in U_1$ and $b_- \in B_-$ from the (unique) decomposition

$$u_1 t_1 b_- = u_{-1} q_1$$

(with t_1, u_{-1}, q_1 known). From condition (4.19) one easily gets that the family of pairs (u_{-1}, u_{-2}) satisfying (4.14) is one-dimensional and there exists exactly one pair for which the corresponding b_- is in B''. Below by (u_{-1}, u_{-2}) we will mean this particular pair.

Now we set

$$u_2 = (u')^\top (t(y)^{-1} u_1 t(y))^\sigma. \tag{4.20}$$

From the definition (4.15) of u we get

$$u_2 t_2 b_-^\sigma = u_{-2} x_2.$$

In particular, $u_2 \in Q$ and $u_2 \in U' \subset U_+$, so that $u_2 \in Q \cap U_+ = U_1$. Therefore,

$$\theta'(z') \overset{\mathrm{df}}{=} z = (u_1, t_1; u_2, t_2, b_-) \in Z_1$$

and $p(z) = y$. Hence, $z \in Z_2$. It is also clear that $q \circ \theta' = \mathrm{id}_{Z_2'}$.

For $y \in Y_2'$ we denote by $U_y \subset U_+$ the fiber of the projection $p' \colon Z_2' \to Y_2$. This is a dense open subset in a closed codimension n subvariety $\overline{U_y}$ in U_+ given by the n equations

$$\Phi_1^{(y)}(u) = \cdots = \Phi_n^{(y)}(u) = 0, \qquad u \in U_+. \tag{4.21}$$

The first two of these equations are

$$u_{1n} = 0, \qquad u_{1\,n-1} = u_{2n}.$$

Introduce the volume form $\omega_{Y_1} = \mathrm{Res}\,\frac{\omega_Y}{\widetilde{\gamma}}$ ($\widetilde{\gamma}$ is defined by (4.10)) and let ω_{Y_2} be the restriction of ω_{Y_1} to Y_2. Introduce also a volume form ω_{U_y} on each fiber U_y by the formula

$$\omega_{U_y} = \mathrm{Res}\,\frac{\bigwedge_{i<j} du_{ij}}{\Phi_1^{(y)} \ldots \Phi_n^{(y)}}.$$

The forms ω_{Y_2} and ω_{U_y}, $y \in Y_2$, determine a volume form ω' on Z_2'.

Finally, introduce an algebraic function f' on Z_2' by the formula

$$f'(y, u) = \Psi(u_{-1} u_{-2}),$$

where u_{-1}, u_{-2} are taken from (4.14). Note that although u_{-1} and u_{-2} in (4.14) are not unique, the condition $\widetilde{\gamma}(y) = 0$ implies that f' is well defined on Z_2'.

CONJECTURE. *Denote by $\mu_\beta^{(2)}$ the restriction of $\mu_\beta^{(1)}$ to the open dense subset $Y_2 \subset Y_1$. Then the data*

$$\boldsymbol{\mu} = (\mathbf{Z}_2', \mathbf{Y}_2, \mathbf{p}', \omega', \mathbf{p}')$$

is an algebraic presentation of $\mu_\beta^{(2)}$.

Remarks. 1. If the conjecture is true, than it is easy to prove that the measure realized by the data $\boldsymbol{\mu}$ coincides with $\mu_\beta^{(2)}$.

2. In all examples intersection of the level sets of $\mathbf{f}'$ in $\mathbf{Z}_2'$ with the fibers of $\mathbf{p}'$ are open subsets of Calabi–Yau manifolds and the form ω' extends to a regular form on the completion.

For $n = 3$ the above conjecture is obviously true because each fiber U_y consists of just one point. In the next section we prove this conjecture for $n = 4$.

5. Explicit formulas for GL_4.

5.1. The measure $\mu_\beta^{(2)}$. Now we consider the case $n = 4$. We write a matrix $u \in U$ in the form

$$u = \begin{pmatrix} 1 & a_1 & a_2 & a_3 \\ 0 & 1 & a_4 & a_5 \\ 0 & 0 & 1 & a_6 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

so that $\omega_U(u) = \bigwedge_{i=1}^6 da_i$.

Fix $y = (q_1, t_1; q_2, t_2) \in Y_2$. Let the Bruhat decomposition of $b(u) \in GL_4$ be $b(y) = u'_{-1}\Lambda u'_{-2}$ with $u'_{-1}, u'_{-2} \in U_-$,

$$\Lambda = \begin{pmatrix} 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \\ 0 & \nu_1 & 0 & 0 \\ \nu_2 & 0 & 0 & 0 \end{pmatrix}.$$

Let $t(y) \in T_0$ be the diagonal matrix $t(y) = \mathrm{diag}(\tau_1, \tau_2, \tau_3, \tau_4)$.

The condition (4.13) is equivalent to the following equalities:

$$a_3 = 0, \quad \tau_1 a_2 = \tau_2 a_5 = \lambda, \quad \tau_1\tau_2\tau_3(a_1a_4a_6 - a_1a_6 - a_2a_5) = \lambda^2\nu_1,$$

so that the fiber $U_y \subset U_+$ over y is the affine plane, and we can take $s_1 = \tau_1 a_1$ and $s_2 = \tau_3 a_6$ as coordinates on this plane.

The function $f'(z')$ on Z'_2 is given by the formula

$$f'(z') = \Psi(u_{-1}u_{-2}{}^{\top})(R(s_1, s_2; \Lambda, t)), \tag{5.1}$$

where $R(s_1, s_2; \Lambda, t)$ is a rational function in s_1, s_2 depending on parameters Λ, t. Explicitly,

$$R(s_1, s_2, \Lambda, t) = \lambda^{-1}\big[s_1 + s_2 + \lambda(\tau_1 + \tau_2)s_1^{-1} + \lambda(\tau_3 + \tau_4)s_2^{-1} + \nu_1^{-1}\tau_3\tau_4 s_1 s_2^{-1} + \nu_1^{-1}\tau_1\tau_2 s_2 s_1^{-1} + \lambda^2\nu_1 s_1^{-1}s_2^{-1}\big].$$

Finally, the form ω_{U_y} on the fiber $U_y \subset U_+$ is given by

$$\omega_{U_y} = \frac{da_1\, da_4\, da_6}{d(\tau_1\tau_2\tau_3(a_1a_4a_6 - a_1a_6 - a_2a_5))} = \frac{ds_1\, ds_2}{s_1 s_2}.$$

PROPOSITION 5.1. *The data $\boldsymbol{\mu} = (\mathbf{Z}'_2, \mathbf{Y}_2, \mathbf{p}', \omega', \mathbf{f}')$ give an algebraic presentation of the measure $\mu_\beta^{(2)}$.*

Proof. We must prove that conditions (i)–(iv) of Definition 0.1 are satisfied.

Instead of proving (i)–(iii) we prove a stronger result that similar properties hold at each fiber U_y of $p'\colon Z'_2 \to Y_2$. Take a point $y \in Y_2$.

The factor $\Psi(u_{-1}u_{-2}{}^{\top})$ in formula (5.1) is constant on the fiber U_y, so we can ignore it.

For almost each $\xi \in F$ the level set $R + \xi$ of the function R inside the affine plane $\mathbb{A}^2$ with coordinates s_1, s_2 is (the affine part of) a smooth elliptic curve E_ξ given by the equation $F_\xi(s_1, s_2; \Lambda, t, \xi) = 0$, where

$$F_\xi(s_1, s_2; \Lambda, t, \xi) \stackrel{\mathrm{df}}{=} s_1 s_2(R(s_1, s_2; \Lambda, t) + \xi) = s_1^2 s_2 + s_1 s_2^2 + \nu_1^{-1}\tau_3\tau_4 s_1^2 + \xi s_1 s_2 + \nu_1^{-1}\tau_1\tau_2 s_2^2 + \lambda(\tau_3 + \tau_4)s_1 + \lambda(\tau_1 + \tau_2)s_2 + \lambda^2\nu_1.$$

The form ω_ξ on the level set E_ξ is given by the formula

$$\omega_\xi = \operatorname{Res} \frac{ds_1 ds_2}{F_\xi(s_1, s_2; \Lambda, t, \xi)}.$$

It is easy to see that ω_ξ is a (unique up to a scalar factor) regular differential on E_ξ. Therefore, for almost all $\xi \in F$ the integral

$$I(\xi) \overset{\mathrm{df}}{=} \int_{E_\xi} |\omega_\xi|$$

converges.

Next we must prove that $I(\xi)$ is a locally L^1-function of ξ. The singularities of $I(\xi)$ occur at the point where E_ξ becomes a singular curve. It is easy to see from explicit expression for ω_ξ that for each such point $\overline{\xi}$ the form $\omega_{\overline{\xi}}$ is nonsingular at each generic point of the (possibly reducible) curve $E_{\overline{\xi}}$. By a general theorem (see [S]), this implies that

$$I(\xi) = O(|\log(\xi - \overline{\xi})|^k)$$

for some k as $\xi \to \overline{\xi}$ and $I(\xi)$ is an L^1-function near $\overline{\xi}$.

Finally, as $|\xi| \to \infty$, the curve E_ξ degenerate into the curve E_∞ given by the equation $s_1 s_2 = 0$, and the form ω_∞ on E_∞ is regular at both generic points of E_∞. The same general result easily implies that as $|\xi| \to \infty$, the function $I(\xi))$ depends only on $|\xi|$. This implies that the sequence of integrals

$$\int_{|\xi| \leq p^n} I(\xi)\, |d\xi|.$$

stabilizes.

So, we proved that "fiberwise versions" of properties (i)–(iii) of Definition 0.1 hold. In particular, this implies that if $f_1, f_2 \in L_S$, then the integral

$$\int_{Z'_2} \psi_F(f'(z'))(p')^*(f_1 \times \overline{f_2})(z')|\omega'(z')| \tag{5.2}$$

converges in the sense of Definition 0.1. Using the convergence of the integrals in formula (4.8) one can see that the integral (5.2) equals $\langle \mu_\beta^{(1)}, f_1 \times \overline{f_2} \rangle$. Proposition 5.1 is proved.

5.2. S_4-invariance. In this section we construct an action of the group S_4 on the data $\boldsymbol{\mu} = (\mathbf{Z}'_2, \mathbf{Y}_2, \mathbf{p}', \omega', \mathbf{f}')$.

Recall that the action of S_4 on Y (and the induced action on Y_2) is given by $w(x_1, t_2; x_2, t_2) = (x_1, t_1^w; x_2, t_2^{w^\sigma})$. We must construct the action of S_4 on Z'_2 compatible with the action of S_4 on Y_2 and preserving the function f' and the form ω'.

Let $\Sigma_i(\tau_1, \tau_2, \tau_3, \tau_4)$, $i = 1, 2, 3, 4$, be the i-th elementary symmetric function. For the proof, it suffice to rewrite the formulas the curve E_ξ and for the form ω_{E_ξ} in terms of S_4-invariant combinations Σ_i of τ_i.

We pass from the coordinates s_1, s_2 on the plane to homogeneous coordinates S_1, S_2, S_3 such that $s_1 = S_1/S_3$, $s_2 = (S_2 - S_1)/S_3$. In these coordinates we have

$$S_3^3 F_\xi = P_1(S_2, S_3) S_1^2 + P_2(S_2, S_3) S_1 + P_3(S_2, S_3),$$

where

$$P_1(S_2,S_3) = -S_2 + (\nu_1^{-1}\tau_1\tau_2 + \nu_1^{-1}\tau_3\tau_4 - \xi)S_3,$$
$$P_2(S_2,S_3) = S_2^2 + (-\xi + 2\nu_1^{-1}\tau_1\tau_2)S_2S_3 + S_3^2\lambda(-\tau_1 - \tau_2 + \tau_3 + \tau_4),$$
$$P_3(S_2,S_3) = \nu_1 S_2(\nu_1^{-1}\tau_1 S_2 + \lambda S_3)(\nu_1^{-1}\tau_2 S_2 + \lambda S_3).$$

Direct computations show that the discriminant $\Delta = P_2^2 - 4P_1P_3$ is a symmetric polynomial in $\tau_1, \tau_2, \tau_3, \tau_4$. Expressed in terms of elementary symmetric functions Σ_i, it is

$$\Delta = S_2^4 + 2\xi S_2^3 S_3 + (\xi^2 + 2\Sigma_1 + 4\rho\Sigma_4)S_2^2S_3^2 + (2\xi\Sigma_1 + 4\rho^{-1} - 4\rho\Sigma_3)S_2S_3^3 + (2\xi\rho^2 + \Sigma_1^2 - 4\Sigma_2)S_3^4,$$

where $\rho = \lambda^2\nu_1$.

Taking instead of S_1 the variable S given by

$$S = 2P_1(S_2,S_3)S_1 - P_2(S_2,S_3), \tag{5.3}$$

we obtain that the equation of E_ξ is

$$S^2 - \Delta(S_2,S_3) = 0.$$

Hence, E_ξ is invariant under the action of S_4.

Using the change of variables (5.3) one gets, in an obvious way, the action of S_4 on $\mathbf{Z}_2'$.

PROPOSITION 5.2. *The described action of S_4 on $\mathbf{Z}_2'$ determines the action of S_4 on the data $\boldsymbol{\mu} = (\mathbf{Z}_2', \mathbf{Y}_2, \mathbf{p}', \omega', \mathbf{f}')$.*

Proof. By construction, the action of S_4 commutes with $\mathbf{p}'$ and $\mathbf{f}'$. The invariance of the form ω_ξ on E_ξ under the action of S_4 is easy, and it implies the invariance of ω'.

Using Proposition 5.2, one can easily write does the twisted data $\boldsymbol{\mu}_\alpha$ that conjecturally deteremine the measure μ_α defined by the Γ-function for GL_4 (see Conjectures 1 and 2 in the introduction).

Appendix. Comparison with the JPS gamma factor.

by J. Cogdell

In this appendix the proof of Proposition 2.1 is given.

PROPOSITION A.1. *Let $\tau = \mathcal{E}^{-1} \otimes \theta$ be a generic irreducible unitary representation of GL_{n-2}. Then*

$$\Gamma(\pi,\theta) = \omega_{\widetilde{\tau}}(-1)^{n-1}\gamma(\pi \times \widetilde{\tau}, 1/2, \psi),$$

where $\omega_{\widetilde{\tau}}$ is the central character of $\widetilde{\tau}$, $\widetilde{\tau}$ is the contragredient of τ (which, since τ is unitary, is $\overline{\tau}$), and γ is the gamma factor of [JPS].

In this appendix we use a slightly different set up from the one in the main body of the paper (see [JPS]). In particular, we will use the following notation.

- w_n is the $n \times n$ permutation matrix $w_n = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$ with 1's along the skew diagonal.

• For the outer involution of GL_n in this appendix we take the involution given by $g^\sigma = w_n {}^t g^{-1} w_n$.

• Q is the stabilizer of the point $(0 : \cdots : 0 : 1) \in \mathbb{P}^n$ and so consists of matrices whose last row is $(0, \ldots, 0, *)$.

• $R = Q \cap Q^\sigma$ is then the standard parabolic subgroup associated to the partition $n = 1 + (n-2) + 1$ containing the upper triangular unipotent subgroup U.

• $\psi = \psi_n$ is the standard non-degenerate character of U.

Let $w_Q = \begin{pmatrix} 0 & 1 & 0 \\ I_{n-2} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then

$$U_Q = w_Q U w_Q^{-1} = \left\{ \begin{pmatrix} 1 & 0 & c \\ a' & u' & a \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

with $\psi_Q(u) = \psi_0(u_{1,n} + u_{2,3} + \cdots + u_{n-2,n-1} + u_{n-1,1})$

$$U_R = U_Q \cap R = \left\{ \begin{pmatrix} 1 & 0 & c \\ 0 & u' & a \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

with $\psi_R = \psi_Q|_R$. Here, the matrices are in block form associated to the partition $(1, n-2, 1)$ of n.

The induced representations are all as in the main text, with the replacement of U_- by the upper unipotent subgeoup U.

We want to analyze a particular element of $\mathrm{Hom}_R(\rho_{R,\mathcal{E}}, \rho_{R,\mathcal{E}})$ beginning with a irreducible unitary generic representation (π, V_π) of GL_n with central character $\mathcal{E}$.

Step 1: To get our first realization of $\rho_{R,\mathcal{E}}$ we do the following. We first pass from V_π to its Whittaker model $\mathcal{W}(\pi, \psi)$ and then restrict these functions to Q:

$$v \in V_\pi \mapsto W_v(q)$$

where W_v is the Whittaker function associated to v and $q \in Q$. The space

$$\{W_v(q) \mid W_v \in \mathcal{W}(\pi, \psi), q \in Q\}$$

gives a realization of $\rho_{Q,\mathcal{E}}$.

To get a realization of $\rho_{R,\mathcal{E}}$ we apply your maps C_1 and then C_2:

$$W_v(q) \mapsto W_v(w_Q^{-1} q) \mapsto W_v(w_Q^{-1} r)$$

with now $r \in R$. So our first realization of $\rho_{R,\mathcal{E}}$ is on the space

$$\{W_v(w_Q^{-1} r) \mid W_v \in \mathcal{W}(\pi, \psi), r \in R\}$$

with R acting by right translation.

Step 2: The element of $\mathrm{Hom}_R(\rho_{R,\mathcal{E}}, \rho_{R,\mathcal{E}})$ that we want to analyze is, using the notation in Section 2,

$$C_3 \circ C_2 \circ C_1 \circ \beta_\pi \circ C_1^{-1} \circ C_2^{-1}.$$

If we begin with $W_v(w_Q^{-1} r)$, then applying $C_1^{-1} \circ C_2^{-1}$ brings us back to $W_v(q)$. The map β_π in these models is the map

$$\beta_\pi : W_v(q) \mapsto \widetilde{W}_v(q w_n) = \widetilde{W}_{\pi(w_n) v}(q),$$

where, as in [JPS], we have set $\widetilde{W}(g) = W(w_n{}^t g^{-1})$. Applying C_1 and then C_2 to this gives

$$\widetilde{W}_{\pi(w_n)v}(q) \mapsto \widetilde{W}_{\pi(w_n)v}(w_Q^{-1}q) \mapsto \widetilde{W}_{\pi(w_n)v}(w_Q^{-1}r).$$

Now, applying the map C_3 gives

$$\widetilde{W}_{\pi(w_n)v}(w_Q^{-1}r) \mapsto \int_B \widetilde{W}_{\pi(w_n)v}(w_Q^{-1}\underline{b}r^\sigma)\, db,$$

where we now have

$$B = \left\{ \underline{b} = \begin{pmatrix} 1 & b & 0 \\ 0 & I_{n-2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| b \in F^{n-2} \right\}.$$

So, our element of $\mathrm{Hom}(\rho_{R,\mathcal{E}}, \rho_{R,\mathcal{E}})$ is

$$W_v(w_Q^{-1}r) \mapsto \int_B \widetilde{W}_{\pi(w_n)v}(w_Q^{-1}\underline{b}r^\sigma)\, db.$$

Step 3: We now pass from $\mathrm{Hom}(\rho_{R,\mathcal{E}}, \rho_{R,\mathcal{E}})$ to $\mathrm{Hom}(\rho_{M,\mathcal{E}}, \rho_{M,\mathcal{E}})$ using $\rho_{R,\mathcal{E}} \simeq \rho_H \otimes \rho_{M,\mathcal{E}}$. This isomorphism is effected by restricting the functions in $\rho_{R,\mathcal{E}}$ to M. To pass from $\rho_{R,\mathcal{E}}$ to $\rho_{M,\mathcal{E}}$ we must then twist these restrictions by the action of M on ρ_H. Let us write $M = C_n \times GL_{n-2}$ and correspondingly $m = cm'$ with $m' \in GL_{n-2}$ embedded as the center block in GL_n. Then the unitary action of m on ρ_H, if we realize this as functions on an appropriate space X, is $\varphi(x) \mapsto |\det(m')|^{1/2}\varphi(xm')$. This gives, in essence, $\rho_{M,\mathcal{E}} = |\det(m')|^{-1/2}\rho_{R,\mathcal{E}}|_M$.

So, our element of $\mathrm{Hom}(\rho_{M,\mathcal{E}}, \rho_{M,\mathcal{E}})$, in our models, takes the form

$$\mathcal{E}(c)|\det(m')|^{-1/2}W_v(w_Q^{-1}m') \mapsto \mathcal{E}(c)|\det(m')|^{-1/2} \int_B \widetilde{W}_{\pi(w_n)v}(w_Q^{-1}\underline{b}(m')^\sigma)\, db.$$

Step 4: This morphism should act as a scalar $\Gamma(\pi,\theta)$ on each irreducible component θ of $\rho_{M,\mathcal{E}}$. Each such component is of the form $\theta = \mathcal{E} \otimes \tau$ with τ an irreducible unitary generic representation of GL_{n-2}. To compute this scalar we want to project into the $\mathcal{E} \otimes \tau$ component by pairing $\rho_{M,\mathcal{E}}$ with the contragredient $(\mathcal{E} \otimes \tau)^\sim = \mathcal{E}^{-1} \otimes \widetilde{\tau} = \bar{\mathcal{E}} \otimes \bar{\tau}$. In this pairing, the central characters cancel. So we can effect the pairing by taking $\widetilde{\tau}$ in its ψ^{-1}-Whittaker model and integrating over $U_{n-2} \setminus GL_{n-2}$. (We do not worry about convergence of the integrals.)

Let $W_{\widetilde{\tau}}(g) \in \mathcal{W}(\widetilde{\tau}, \psi^{-1})$. Before applying the morphism we have

$$\begin{aligned} I &= \langle |\det(m')|^{-1/2}W_v(w_Q^{-1}m'), W_{\widetilde{\tau}}(m')\rangle \\ &= \int_{U_{n-2}\backslash GL_{n-2}} W_v\left(w_Q^{-1}\begin{pmatrix} 1 & & \\ & m' & \\ & & 1 \end{pmatrix}\right) W_{\widetilde{\tau}}(m')|\det(m')|^{-1/2}\, dm' \\ &= \int_{U_{n-2}\backslash GL_{n-2}} W_{\pi(w_Q^{-1})v}\begin{pmatrix} m' & \\ & I_2 \end{pmatrix} W_{\widetilde{\tau}}(m')|\det(m')|^{1/2-1}\, dm' \\ &= \Psi(W_{\pi(w_Q^{-1})v}, W_{\widetilde{\tau}}, 1/2), \end{aligned}$$

where $\Psi(W_{\pi(w_Q^{-1})v}, W_{\widetilde{\tau}}, 1/2)$ is as in [JPS].

After applying the morphism, we should get

$$\widetilde{I} = \langle |\det(m')|^{-1/2} \int_B \widetilde{W}_{\pi(w_n)v}(w_Q^{-1}\underline{b}(m')^\sigma)\; db, W_{\widetilde{\tau}}(m')\rangle$$

which, if the morphism is to act by the scalar $\Gamma(\pi,\theta)$ on this piece, should give

$$\begin{aligned}\widetilde{I} &= \Gamma(\pi,\theta)\langle |\det(m')|^{-1/2} W_v(w_Q^{-1}m'), W_{\widetilde{\tau}}(m')\rangle \\ &= \Gamma(\pi,\theta)\Psi(W_{\pi(w_Q^{-1})v}, W_{\widetilde{\tau}}, 1/2).\end{aligned}$$

Step 5: The final step is to identify $\widetilde{I}$ with the right-hand side of the $GL_n \times GL_{n-2}$ functional equation. If we write the integral $\widetilde{I}$ out it is

$$\begin{aligned}\widetilde{I} &= \int_{U_{n-2}\backslash GL_{n-2}} \int_B \widetilde{W}_{\pi(w_n)v}(w_Q^{-1}\underline{b}(m')^\sigma) db\; W_{\widetilde{\tau}}(m')|\det(m')|^{-1/2} dm' \\ &= \int\int \widetilde{W}_v \left(w_Q^{-1}\underline{b} \begin{pmatrix} 1 & & \\ & m' & \\ & & 1 \end{pmatrix}^\sigma w_n \right) db W_{\widetilde{\tau}}(m')|\det(m')|^{-1/2} dm'.\end{aligned}$$

We next have a few elementary calculations:

$$\widetilde{W}_v(g) = \widetilde{W}_{\pi(w_Q^{-1})v}(g w_Q)$$

$$\begin{pmatrix} 1 & & \\ & m' & \\ & & 1 \end{pmatrix}^\sigma = \begin{pmatrix} 1 & & \\ & (m')^\sigma & \\ & & 1 \end{pmatrix}$$

$$w_Q^{-1} \begin{pmatrix} 1 & b & \\ & I_{n-2} & \\ & & 1 \end{pmatrix} w_Q = \begin{pmatrix} I_{n-2} & & \\ b & 1 & \\ & & 1 \end{pmatrix} = \widetilde{b}$$

$$w_Q^{-1} \begin{pmatrix} 1 & & \\ & m' & \\ & & 1 \end{pmatrix} w_Q = \begin{pmatrix} m' & \\ & I_2 \end{pmatrix}$$

$$\begin{pmatrix} w_{n-2} & \\ & I_2 \end{pmatrix} w_Q^{-1} w_n w_Q = \begin{pmatrix} I_{n-2} & \\ & w_2 \end{pmatrix} = w_{n,n-2}$$

If we now use these calculations in our expression for $\widetilde{I}$, and set $v' = \pi(w_Q^{-1})v$, we obtain

$$\widetilde{I} = \int\int \widetilde{W}_{v'} \left(\widetilde{b} \begin{pmatrix} w_{n-2}{}^t(m')^{-1} & \\ & I_2 \end{pmatrix} w_{n,n-2} \right) db W_{\widetilde{\tau}}(m')|\det(m')|^{-1/2} dm'.$$

Now we change of variables $m' \mapsto w_{n-2}{}^t(m')^{-1}$ and note that $W_{\widetilde{\tau}}(w_{n-2}{}^t(m')^{-1}) = \widetilde{W_{\widetilde{\tau}}}(m')$. Then our expression can be written

$$\widetilde{I} = \int\int (\rho(w_{n,n-2})\widetilde{W}_{v'}) \begin{pmatrix} m' & & \\ bm' & 1 & \\ & & 1 \end{pmatrix} db \widetilde{W}_{\widetilde{\tau}}(m')|\det(m')|^{1/2} dm'.$$

Making the change of variables $b \mapsto b(m')^{-1}$ we finally obtain

$$\widetilde{I} = \int\int (\rho(w_{n,n-2})\widetilde{W}_{v'}) \begin{pmatrix} m' & & \\ b & 1 & \\ & & 1 \end{pmatrix} db \widetilde{W}_{\widetilde{\tau}}(m') |\det(m')|^{-1/2} dm'$$

$$= \int\int (\rho(w_{n,n-2})\widetilde{W}_{v'}) \begin{pmatrix} m' & & \\ b & 1 & \\ & & 1 \end{pmatrix} db \widetilde{W}_{\widetilde{\tau}}(m') |\det(m')|^{(1-1/2)-1} dm'$$

$$= \Psi(\rho(w_{n,n-2})\widetilde{W}_{v'}, \widetilde{W}_{\widetilde{\tau}}, 1-1/2; 1).$$

Thus we arrive at

$$\Gamma(\pi, \theta)\Psi(W_{v'}, W_{\widetilde{\tau}}, 1/2) = \Psi(\rho(w_{n,n-2})\widetilde{W}_{v'}, \widetilde{W}_{\widetilde{\tau}}, 1-1/2; 1).$$

By the local functional equation of [JPS] we have

$$\omega_{\widetilde{\tau}}(-1)^{n-1}\gamma(\pi \times \widetilde{\tau}, 1/2, \psi)\Psi(W_{v'}, W_{\widetilde{\tau}}, 1/2) = \Psi(\rho(w_{n,n-2})\widetilde{W}_{v'}, \widetilde{W}_{\widetilde{\tau}}, 1-1/2; 1).$$

Hence we have

$$\Gamma(\pi, \theta) = \omega_{\widetilde{\tau}}(-1)^{n-1}\gamma(\pi \times \widetilde{\tau}, 1/2, \psi)$$

as claimed.

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