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THE CUBIC SHIMURA CORRESPONDENCE*

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1. Introduction. In the theory of metaplectic groups a *Shimura correspondence* describes a relationship between the set of automorphic forms on a metaplectic cover of a group and those on a non-metaplectic group. The first example was discovered by G. Shimura in 1972 (see [22], [23]) using the converse theorem of Hecke theory. A little later Niwa [16] and Shintani [24], [25] discovered an intimate relationship between Shimura's correspondence and the theory of general theta functions. This discovery opened the way both to some remarkable applications – see, for example [30] – and to a wide range of generalizations and applications in connection with automorphic forms of higher rank – see [15].

The theory of theta functions appears to be rather special and, in the case of metaplectic covers of order exceeding 2, inapplicable. Y. Flicker [3] proved a Shimura correspondence for n-fold covers of GL_2, both locally and globally. This is done at the level of characters using the Trace Formula. It can be extended to some extent to other groups but we shall not go into these matters here. What is important for our purposes is to note that this method yields very little more precise information – for example about the Fourier coefficients of the metaplectic forms.

Recently there has been some very remarkable progress towards a more precise theory in the case of three-fold covers. This has its origins in investigations of Kazhdan and Savin ([11], [19]) of the representation theoretic basis of the properties of the theta representation. It has been developed in a global context by D. Ginzburg, S. Rallis and D. Soudry, [4] (see also [5]). It describes a Shimura correspondence for $\widetilde{SL}_2$ and for $\widetilde{SL}_3$ based on special properties of a group of type G_2.

The objective of this paper is to discuss a much more elementary method for constructing integral kernels for the Shimura correspondence. Although this paper was spurred on by the work of Ginzburg, Rallis and Soudry it appears to be independent of it in the sense that there is no apparent connection between the results found here and those of Ginzburg, Rallis and Soudry. Our method is one which is entirely carried out in the context of GL_2. In fact, we carry it out only over the field $\mathbb{Q}(\omega)(\omega^2 + \omega + 1 = 0)$. It is based on combining Hecke theory with Selberg's theory of point-pair invariants; it was inspired by a remark of Selberg's ([21], p. 188). Unfortunately it can be carried out only in the context of the highest level possible.

It is not clear if the method described here has a representation-theoretic basis; it seems unlikely. On the other hand the results which we obtain here are very explicit. Although we shall develop them only in the context of the cubic Shimura correspondence they could be applied in a much more general context. For example, if one applies them in the case of the 1-fold (i.e. trivial) covering one is led to theta functions associated with an indefinite ternary quadratic form of the type described in [29]. In the case of a 2-fold cover one is *not* led to standard theta-functions. In this case there are some technical difficulties of the same kind as we shall now describe in the cubic case.

What we construct is a correspondence between automorphic forms rather than

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between representations. For the method to be applicable one needs some kind of Multiplicity One Theorem, which describes the levels of the forms. In the classical case this is given by the Atkin-Lehner theory of new-forms but this theory has no analogue in the context of metaplectic groups. What we shall do here is to prove a very special result of this type. It is based on a construction in [17] and it appears to have more significance than is apparent there. Nevertheless no deeper reasons for the success of this method are available.

This part of the paper will be given in group-theoretic language, as is appropriate. It takes up §§2,3. In the remainder of the paper we adopt a more "classical" approach based on Selberg's method of point-pair invariants.

This method is, as indicated above, very explicit. It should lead, if not to theoretical results about the Fourier coefficients of cubic metaplectic forms, at least to a method by which these can be numerically evaluated. At the present time nothing beyond the elementary Hecke theory (cf. [12], [13]) is known, nor are there any numerical examples available. For this reason it would be very desirable to develop techniques for determining these coefficients.

I would like to thank L.Möhring for some very helpful conversations especially concerning the problems discussed at the end of this paper. He also pointed out the existence of the paper [26] to me.

2. The metaplectic cover of GL_2. In this section we shall recall some salient aspects of Kubota's theory of the metaplectic cover of GL_2 and develop aspects particular to the ground field $\mathbb{Q}_3(\sqrt{-3})$. Recall that F is a local field containing the n^{th} roots of 1 then we can define a 2-cocycle on $GL_2(F)$ as follows:

Let $x\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = c$ if $c \neq 0$, $= d$ if $c = 0$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(F)$. Let $(\ ,\ )_{n,F}$ denote the Hilbert symbol of order n in F, with values in the group of n^{th} roots of 1 in $F, \mu_n(F)$. Later, when no confusion should occur, we shall write $(\ ,\ )$ in place of $(\ ,\ )_{n,F}$. The Kubota 2-cocycle is given by

$$\sigma(g_1,\, g_2) = \left(\frac{x(g_1g_2)}{x(g_1)}, \frac{x(g_1g_2)}{x(g_2)\det(g_1)}\right)_{n,F}.$$

If F is non-archimedean and R_F denotes the ring of integers of F we set, for $\begin{pmatrix} a & b \\ a & d \end{pmatrix}$ $\in GL_2(R_F)$,

$$\kappa\left(\begin{pmatrix} a & b \\ a & d \end{pmatrix}\right) = \left(c, \frac{d}{\det\begin{pmatrix} a & b \\ a & d \end{pmatrix}}\right)_{n,F}, \quad \text{if } 0 < |c|_F < 1,$$
$$= 1, \qquad \text{otherwise.}$$

Kubota showed that if n is invertible in R_F one has

$$\sigma(g_1,\, g_2) = \kappa(g_1\, g_2)/\kappa(g_1)\kappa(g_2),$$

which gives an explicit splitting of σ over $GL_2(R_F)$. In fact Kubota also proves that the same is true on $\{g \in GL_2(R_F) : g \equiv I \pmod{n^2}\}$ even if n is not invertible in R_F. This is only a rough lower estimate for a group on which σ splits.

Let us now consider the case where $F = \mathbb{Q}_3(\sqrt{-3})$ and $n = 3$. For convenience set $\lambda = \sqrt{-3} = 2\omega + 1$. Let $F_0 = \mathbb{Q}_3$ and $K_0 = GL_2(R_{F_0})$. Let $K(2)$ be the principal congruence subgroup of level 2 in $GL_2(R_F)$, i.e.

$$K(2) = \{g \in GL_2(R_F) : g \equiv I \pmod{\lambda^2}\}.$$

We note that $K(2) \cap K_0$ is the principal congruence subgroup of level 1 in K_0 (as $\lambda^2 = -3$) and so

$$[K_0 : K(2) \cap K_0] = (3^2 - 1)(3^2 - 3) = 48.$$

Thus we have $[K_0 \cdot K(2) : K(2)] = 48$; as K_0 normalizes $K(2)$ we see that $K_0 \cdot K(2)$ is a group. On the other hand $[GL_2(R_F) : K(2)] = 3^4(3^2-1)(3^2-3) = 81 \cdot 48$. Hence $K_0 \cdot K(2)$ is of index 81 in $GL_2(R_F)$; we write $K_{81} := K_0 \cdot K(2)$.

Next we note that the Kubota function κ is defined on $K(2)$. We extend this definition to K_{81} by setting

$$\tilde{\kappa}(gg') = \sigma(g, g')\kappa(g') \quad \text{if } g \in K_0,\ g' \in K(2).$$

We have:

LEMMA 2.1. *The function $\tilde{\kappa}$ is well-defined. Moreover, for $\gamma_1,\ \gamma_2 \in K_{81}$ we have*

$$\sigma(\gamma_1,\ \gamma_2) = \tilde{\kappa}(\gamma_1\ \gamma_2)/\tilde{\kappa}(\gamma_1)\tilde{\kappa}(\gamma_2).$$

Proof. If $gg' = g_1\ g_1'$ then $g_1 = g\delta,\ g_1' = \delta^{-1}g'$ where $\delta \in K_0 \cap K(2)$. We see, using the cocycle relation, that we have to verify

$$\sigma(g\delta^{-1}, \delta) = 1$$

and

$$\kappa(g') = \sigma(\delta^{-1}, g')^{-1}\kappa(\delta^{-1}g')$$

in order that $\tilde{\kappa}$ be well-defined. The first of these follows as the Hilbert symbol restricted to $\mathbb{Q}_3^\times \times \mathbb{Q}_3^\times$ is trivial. The second follows, as one can verify, using the relations

$$1 + \lambda^4 R_F \subset F^{\times 3}$$

and

$$(1 + \lambda^2 R_F,\ 1 + \lambda^2 R_F)_{3,F} = 1,$$

that κ splits σ on $K(2)$. Since $\kappa(\delta) = 1$ as before the second relation follows.

In order to prove the second statement we have to verify that, if $\gamma_1 = g_1\ g_1', \gamma_2 = g_2\ g_2'$ with $g_1,\ g_2 \in K_0,\ g_1',\ g_2' \in K(2)$

$$\sigma(g_1,\ g_1')\kappa(g_1') \cdot \sigma(g_2,\ g_2')\kappa(g_2') \cdot \sigma(g_1 g_1',\ g_2 g_2') = \sigma(g_1 g_2,\ g_2^{-1} g_1' g_2 g_2')\kappa(g_2^{-1} g_1' \cdot g_2 \cdot g_2').$$

If we use the fact that κ splits σ on $K(2)$ and that $\sigma(g_1,\ g_2) = 1$ this simplifies to

$$\kappa(g_2^{-1} g_1' g_2) = \sigma(g_1',\ g_2) \cdot \sigma(g_2, g_2^{-1} g_1' g_2)^{-1} \cdot \kappa(g_1').$$

This shows the behaviour of κ under conjugation. To prove it one first notes the general fact that if $g \in K_0$ then

$$\kappa^g(g') = \kappa(g^{-1}g'g)\sigma(g, g^{-1}g'g)/\sigma(g', g) \qquad (g' \in K(2))$$

also splits σ on $K(2)$. It follows that κ^g/κ is a character of order 3 on $K(2)$. Next we observe that the same construction applied to κ^g yields

$$(\kappa^g)^h(g') = \kappa^{hg}(g').$$

If we write $\chi_g(g') = \kappa^g(g')/\kappa(g')$ then

$$\chi_{hg}(g') = \chi_g(h^{-1}g'h)\chi_h(g'),$$

for $g,\ h \in K_0,\ g' \in K(2)$. To verify that all the χ_g are trivial it therefore suffices to prove that this is so for g of the form $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ $(x \in R_{F_0})$, $\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$ $(\alpha \in R_{F_0}^{\times})$ or $E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The first case is straightforward to verify, the second is not particularly difficult. The third case is treated by noting that, if $b,\ c \neq 0$

$$\chi_E\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = (b, a/(ad-bc))_{3,F} \cdot (b,d)_{3,F}(d-(ad-bd))_{3,F}.$$

Since $\Delta \equiv ad \pmod{\lambda^2}$ it follows that this expression is 1. The general case (b, d possibly 0) follows at once.

This completes the proof of the lemma. It should be noted that this argument (like the statement) is merely a variant on [17], §2. □

We can now begin to formulate the main theorem of this section. Denote by $\tilde{G} = \widetilde{GL}_2(F)$ the 3-fold metaplectic cover of $G = GL_2(F)$ in the case $F = \mathbb{Q}_3(\sqrt{-3})$. This is defined by σ; we write elements of $\tilde{G}$ as pairs $(g,\zeta),\ g \in G,\ \zeta \in \mu_3(F)$ and the multiplication is given by

$$(g_1,\ \zeta_1)\cdot(g_2,\zeta_2) = (g_1g_2,\ \zeta_1\zeta_2\sigma(g_1,g_2)).$$

The map $K_{81} \to \tilde{G};\ k \mapsto (k,\ \tilde{\kappa}(\kappa))$ is, by Lemma 2.1, a homomorphism. Let K_{81}^* be the image of this map; K_{81}^* is isomorphic to K_{81}.

The centre of $\tilde{G}$ is $\tilde{Z} = \left\{\left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \zeta\right) : \alpha \in F^{\times 3},\ \zeta \in \mu_3(F)\right\}$ (see [10] 0.1). We fix an injective homomorphism $\epsilon : \mu_3(F) \to \mathbb{C}^{\times}$. Let χ be an extension of ϵ to $\tilde{Z}$, trivial on $K_{81}^* \cap \tilde{Z}$. The choice of χ is not important for our purposes and we could take

$$\chi\left(\left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \zeta\right)\right) = \epsilon(\zeta).$$

We shall now define a Hecke algebra $H = H_\chi(\tilde{G}; K_{81}^*)$. This consists of locally constant functions φ on $\tilde{G}$ which satisfy

1. $\varphi(gz) = \chi(z)\varphi(g)$ $\quad (g \in \tilde{G},\ z \in \tilde{Z})$;
2. $\varphi(k_1gk_2) = \varphi(g)$ $\quad (k_1,\ k_2 \in K_{81}^*,\ g \in \tilde{G})$;
3. φ is compactly supported modulo $\tilde{Z}$.

We fix a Haar measure on $\tilde{G}/\tilde{Z}$; this group is unimodular and the Haar measure is therefore left and right invariant. It is determined by the measure of $K_{81}^*\tilde{Z}/\tilde{Z}$. We define H to have the algebra structure determined by convolution on $\tilde{G}/\tilde{Z}$, i.e.

$$(\varphi_1 \star \varphi_2)(g) = \int_{\tilde{G}/\tilde{Z}} \varphi_1(gh^{-1})\varphi_2(h)dh.$$

Then we have:

THEOREM 2.2. *The algebra H is commutative. If $\varphi \in H$, $g \in \tilde{G}$ and $\varphi(g) \neq 0$ then there exist k_1, $k_2 \in K_{81}^*$ so that $k_1 g k_2$ is of the form $\left(\begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}, \zeta\right)$ with α_1, $\alpha_2 \in F^{\times 3}$, $\zeta \in \mu_3(F)$. Conversely, if α_1, α_2, ζ are as above then we can define φ_1 by*

$$\varphi_1\left(zk_1 \left(\begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}, \zeta\right) k_2\right) = \chi(z)$$

and

$$\varphi_1(g) = 0 \text{ if } g \in \tilde{G} - K_{81}^* \left(\begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}, \zeta\right) K_{81}^*;$$

then we have $\varphi_1 \in H$.

Proof. The final statement is relatively straightforward and so we shall leave it to the reader. We shall first prove the second statement, and from this we deduce the commutativity of H.

Let $K = GL_2(R_F)$. Then the set of elements

$$\tau = \begin{pmatrix} 1 + \lambda A & \lambda B \\ \lambda C & 1 + \lambda D \end{pmatrix} \qquad (A,\ B,\ C,\ D = 0, \pm 1)$$

form a set of representatives for K/K_{81} or $K_{81}\backslash K$. One sees this as there are $3^4 = 81$ such elements and no element of the form $\tau_1^{-1}\tau_2, \tau_1 \neq \tau_2$ lies in K_{81}. Suppose that $g \in \tilde{G}$. Then by the Cartan decomposition there exist τ_1, τ_2 as above, h_1, $h_2 \in F^\times$, $\zeta \in \mu_3(F)$, k_1, $k_2 \in K_{81}^*$ so that

$$g = k_1\ \tau_1 \left(\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}, \zeta\right) \tau_2 k_2.$$

We can, and shall, assume that $|h_1| \leq |h_2|$. By modifying h_1, h_2, ζ if necessary we can then assume that τ_1 has the form $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ and τ_2 the form $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ where b, $c = 0$, $\pm\lambda$. We now seek k_1', $k_2' \in K_{81}$

$$k_1' \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} k_2'$$

and examine if the corresponding equation holds in K_{81}^*. If it does not then $\varphi(g) = 0$ and we have a contradiction.

We begin with $k_1' = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{pmatrix} \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}$ and $k_2' = \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix}\begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ where θ_1, $\theta_2 \equiv 1(\lambda^2)$. In this case we find that the equation can be lifted if $(\theta,\ h_2)_{3,F} \cdot (\theta_2,\ h_1)_{3,F}^{-1} = 1$ and we conclude that h_1, $h_2 \in (1 + \lambda^2 R_F)\lambda^{3\mathbb{Z}}$. By a simple modification we may assume that h_1, $h_2 \in \lambda^{3\mathbb{Z}}$. In this calculation we use the fact that $(\theta_1,\ \theta_2)_{3,F} = 1$ if θ_1, $\theta_2 \equiv 1 \pmod{\lambda^2}$. Next we take

$$k_1' = \left(\begin{pmatrix} 1 + \lambda^2 b & -b^2\lambda^2 \\ \lambda^2 & 1 - b\lambda^2 \end{pmatrix},\quad (\lambda^2,\ 1 - b\lambda^2)_{3,F}\right).$$

One finds that

$$k_2' = \left(\begin{pmatrix} 1 & 0 \\ \frac{h_1}{h_2}\lambda^2 & 1 \end{pmatrix}, 1 \right)$$

and that the condition that the equation above can be lifted is that $(\lambda^2,\ 1-b\lambda^2)_{3,F} = 1$. As $(\lambda,\ 1\pm\lambda^3)_{3,F} \neq 1$ we conclude that a lift is only possible if $b=0$. An analogous argument shows that $c=0$.

Now it follows that if we let φ_m be the element of H supported on $K_{81}^* \left(\begin{pmatrix} \lambda^{3m} & 0 \\ 0 & 1 \end{pmatrix}, 1 \right) K_{81}^*$ and with the value 1 at $\left(\begin{pmatrix} \lambda^{3m} & 0 \\ 0 & 1 \end{pmatrix}, 1 \right)$ for $m \geq 0$ then every element of H is a linear combination of finitely many φ_m. We shall show that $\varphi_m * \varphi_1$ is of the form $c\varphi_{m+1} + \sum_{j\leq m} c_j\varphi_j$ with $c \neq 0$. As φ_0 is a non-zero multiple of the identity in H it will then follow that H is a polynomial algebra generated by φ_1.

Let $\delta_1, \dots, \delta_r$ be a set of representatives for

$$K_{81}^*/K_{81}^* \cap \left(\begin{pmatrix} \lambda^3 & 0 \\ 0 & 1 \end{pmatrix}, 1 \right) K_{81}^* \left(\begin{pmatrix} \lambda^{-3} & 0 \\ 0 & 1 \end{pmatrix}, 1 \right);$$

then

$$K_{81}^* \left(\begin{pmatrix} \lambda^3 & 0 \\ 0 & 1 \end{pmatrix}, 1 \right) K_{81}^* = \bigcup_{j=1}^{r} \delta_j \left(\begin{pmatrix} \lambda^3 & 0 \\ 0 & 1 \end{pmatrix}, 1 \right) K_{81}^*.$$

Suppose that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_{81}$ is such that

$$\begin{pmatrix} \lambda^3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda^{-3} & 0 \\ 0 & 1 \end{pmatrix} \in K_{81}.$$

Then we see that $c \equiv 0 \pmod{\lambda^3}$. It follows that $a \equiv \pm 1 \pmod{\lambda^2}$, $d \equiv \pm 1 \pmod{\lambda^2}$ and $b \equiv 0,\ \pm 1 \pmod{\lambda^2}$. We conclude that

$$\begin{pmatrix} \lambda^3 & 0 \\ 0 & 1 \end{pmatrix} K_{81} \begin{pmatrix} \lambda^{-3} & 0 \\ 0 & 1 \end{pmatrix} \cap K_{81} =$$
$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,\ d \equiv \pm 1(\lambda^2), c \equiv 0,\ \pm 1(\lambda^2),\ b = \lambda^3 b_1 \text{ with } b_1 \equiv 0,\ \pm 1(\lambda^3) \right\}.$$

Let $K(j) = \{g \in GL_2(R_F) : g \equiv I \pmod{\lambda^j}\}$. Then we see that $\begin{pmatrix} \lambda^3 & 0 \\ 0 & 1 \end{pmatrix} K_{81}$ $\begin{pmatrix} \lambda^{-3} & 0 \\ 0 & 1 \end{pmatrix} \cap K_{81} \supset K(5)$. Since

$$[K_{81} : K(2)] = 48,\ [K(2) : K(5)] = 3^{12}$$

and, from what we have just seen,

$$[\begin{pmatrix} \lambda^3 & 0 \\ 0 & 1 \end{pmatrix} K_{81} \begin{pmatrix} \lambda^{-3} & 0 \\ 0 & 1 \end{pmatrix} \cap K_{81} : K(5)] = 4 \cdot 3^{11};$$

we conclude that

$$[K_{81} : \begin{pmatrix} \lambda^3 & 0 \\ 0 & 1 \end{pmatrix} K_{81} \begin{pmatrix} \lambda^{-3} & 0 \\ 0 & 1 \end{pmatrix} \cap K_{81}] = 36.$$

We shall now write

$$K_H = \begin{pmatrix} \lambda^3 & 0 \\ 0 & 1 \end{pmatrix} K_{81} \begin{pmatrix} \lambda^{-3} & 0 \\ 0 & 1 \end{pmatrix} \cap K_{81}.$$

We shall now determine a special set of $\delta_j (1 \le j \le 36)$ of representations K_{81}/K_H. First of all suppose that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_{81}$, $d \not\equiv 0 \pmod{\lambda^2}$. We can multiply by an element of K_H to bring this to the form $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$. There are 27 such elements and we can take, as a set of representatives, $\begin{pmatrix} 1 & b_0 + b_2\lambda^2 + b_4\lambda^4 \\ 0 & 1 \end{pmatrix}$ with b_0, b_2, $b_4 = 0, \pm 1$. If $d \equiv 0 \pmod{\lambda^2}$ then b, $c \equiv \pm 1 \pmod{\lambda^2}$. We can therefore arrange first of all that $a = 0$, and then that b, $c = 1$. We obtain as a set of representatives

$$\begin{pmatrix} 0 & 1 \\ 1 & d_2\lambda^2 + d_4\lambda^4 \end{pmatrix} \quad d_2,\ d_4 = 0,\ \pm 1.$$

Since σ and $\tilde{\kappa}$ are invariant under conjugation by $\begin{pmatrix} \lambda^3 & 0 \\ 0 & 1 \end{pmatrix}$ we therefore have for our set of $36 = 27 + 9$ representatives

$$\left(\begin{pmatrix} 1 & b_0 + b_2\lambda^2 + b_4\lambda^4 \\ 0 & 1 \end{pmatrix}, 1 \right) \qquad b_0,\ b_2,\ b_4 = 0,\ \pm 1$$

and

$$\left(\begin{pmatrix} 0 & 1 \\ 1 & d_2\lambda^2 + d_4\lambda^4 \end{pmatrix}, 1 \right) \qquad d_2,\ d_4 = 0,\ \pm 1.$$

Now let $k > m$; we shall compute $\varphi_1 * \varphi_m \left(\begin{pmatrix} \lambda^{3k} & 0 \\ 0 & 1 \end{pmatrix}, 1 \right)$. This is

$$\begin{aligned} &\int_{\tilde{G}/\tilde{Z}} \varphi_1(h) \varphi_m \left(h^{-1} \left(\begin{pmatrix} \lambda^{3k} & 0 \\ 0 & 1 \end{pmatrix}, 1 \right) \right) dh \\ &= \operatorname{meas}(K_{81}^* \tilde{Z}/\tilde{Z}) \sum_{j=1}^{36} \varphi_m \left(\left(\begin{pmatrix} \lambda^{-3} & 0 \\ 0 & 1 \end{pmatrix}, 1 \right) \delta_j^{-1} \left(\begin{pmatrix} \lambda^{3k} & 0 \\ 0 & 1 \end{pmatrix}, 1 \right) \right). \end{aligned}$$

Under the assumption that $k|m$ one sees that the only term giving a non-zero contribution is where $\delta_j = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 1 \right)$ and this only when $k = m + 1$. We have therefore proved our assertion and, with it, the theorem. □

We obtain from the proof the following corollary:

COROLLARY 2.3. *Let φ_1 be the element of H which is supported on $K_{81}^* \left(\begin{pmatrix} \lambda^3 & 0 \\ 0 & 1 \end{pmatrix}, 1 \right) K_{81}^*$ and such that $\varphi_1 \left(\begin{pmatrix} \lambda^3 & 0 \\ 0 & 1 \end{pmatrix}, 1 \right) = 1$. Then H is the algebra of polynomials in φ_1.*

We can extend the argument above to give the following result:

PROPOSITION 2.4. *Let $m \ge 1$; then*

$$K_{81}^* \left(\begin{pmatrix} \lambda^{3m} & 0 \\ 0 & 1 \end{pmatrix}, 1 \right) K_{81}^*$$

is the union of

$$\left(\begin{pmatrix} \lambda^{3m} & b_0 + b_2\lambda^2 + b_3\lambda^3 + \cdots + b_{3m-1}\lambda^{3m-1} + b_{3m+1}\lambda^{3m+1} \\ 0 & 1 \end{pmatrix}, 1\right) K_{81}^*$$

and

$$\left(\begin{pmatrix} 1 & 0 \\ d_2\lambda^2 + d_3\lambda^3 + \cdots + d_{3m-1}\lambda^{3m-1} + d_{3m+1}\lambda^{3m+1} & \lambda^{3m} \end{pmatrix}, 1\right) K_{81}^*,$$

where $b_0, b_2, \ldots, b_{3m-1}, b_{3m+1}, d_2, d_3, \ldots, d_{3m-1}, d_{3m+1}$ *run through the set* $0, \pm 1$.

The proof is more or less as above and so we shall not give the details here.

3. The Shimura correspondence. In this section we shall develop the correspondence of the results of §2 for the Shimura correspondence, first locally and then globally.

We begin by studying some aspects of the representation theory of $\widetilde{GL_2}(F)$. Let

$$H = \left\{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : a,\ d \in F^\times\right\}$$

and

$$H_3 = \left\{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : a,\ d \in F^{\times 3}\right\};$$

let $\tilde{H},\ \tilde{H}_3$ be the corresponding covers. One has $[H : H_3] = 81^2 = 3^6$. Let

$$H_* = \left\{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : a,\ d \in \pm\lambda^{3\mathbb{Z}}(1 + \lambda^2 R_F)\right\}.$$

Then $[H : H_*] = 81$. Then $\tilde{H}_3$ is the centre of $\tilde{H}$ and $\tilde{H}_*$ is a maximal abelian subgroup of $\tilde{H}$. Let ω be a quasicharacter of $\tilde{H}_3$ extending χ and let ω_* be an extension to $\tilde{H}_*$. We shall assume that ω is trivial on $\tilde{H}_3 \cap K_{81}^*$ and one can chose ω_* to be trivial on $\tilde{H}_* \cap K_{81}^*$. Let $N = \left\{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in F\right\}$ and $B = HN,\ B_* = H_*N$ and $B_3 = H_3N$. Again we let $B,\ \tilde{B}_*,\ \tilde{B}_3$ be the correspoding covers. As in [10],I.1 we induce ω_* from $\tilde{B}_*$ (trivial on the canonical lift of N) to $\tilde{G}$: This yields a representation $V(\omega)$ of $\tilde{G}$ which does not depend on the choice of ω_*. Let $\mu\left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right) = |a/d|^{1/2}$ constant functions $f : \tilde{G} \to \mathbb{C}$:

$$f(hng) = (\omega_*\mu)(h)f(g),$$

where $h \in \tilde{H}_*,\ n \in N^*$ (the canonical lift on N) and $g \in \tilde{G}$ endowed with the action of $\tilde{G}$ on the right.

PROPOSITION 3.1. *The subspace of K_{81}^*-invariant vectors in $V(\omega)$ is 1-dimensional. In the model of $V(\omega)$ given above any such vector is supported on $\tilde{B}_*K_{81}^*$.*

Proof. The proof of this is analogous to, but a little more involved than, the analogous case in the "unramified" case - cf. [10], I.1. We shall leave the details to the reader. □

Let us now write the action of $\tilde{G}$ on $V(\omega)$ as left multiplication. Let φ_m be as in §2. Then a calculation using Proposition 2.4 shows the following:

PROPOSITION 3.2. *Let $v_0 \in V(\omega)$ be K_{81}^*-invariant. Then*

$$\int_{\tilde{G}/\tilde{Z}} \varphi_m(g) g\, v_0\, dg = \operatorname{meas}(K_{81}^* \tilde{Z}/\tilde{Z}) \tilde{t}_m^0(\omega) v_0,$$

where

$$\tilde{t}_m(\omega) = 3^{\frac{3m}{2}} \left(\omega\left(\begin{pmatrix} \lambda^{3m} & 0 \\ 0 & 1 \end{pmatrix}^* \right) + \omega\left(\begin{pmatrix} 1 & 0 \\ 0 & \lambda^{3m} \end{pmatrix}^* \right) + \frac{2}{3} \sum_{1 \le j \le m/3} \omega\left(\begin{pmatrix} \lambda^{3(m-j)} & 0 \\ 0 & \lambda^{3j} \end{pmatrix}^* \right) \right)$$

with $g^* = (g,\ 1)$.

The analogous result in the "unramified" case was indicated – with a different normalization – by Kubota in [12], p. 59.

We may now conclude from the standard theory of spherical functions (see, for example, [6] or [18]), these calculations and Propositions 2.2 and 2.3 the following theorem:

THEOREM 3.3. *Let V be an irreducible representation of $\tilde{G}$ with central quasicharacter χ and a non-trivial K_{81}^*-invariant vector v. Then V is a quotient of some $V(\omega)$ as above with $\omega | \tilde{H}_3 \cap K_{81}^* = 1$.*

A corresponding result holds for G. Now let

$$\omega_\alpha(x) = \omega\left(\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}^* \right)$$

for $x \in F^{\times 3}$. If $\omega | \tilde{H}_3 \cap K_{81}^* = 1$ then $\omega_\alpha(x)$ is of the form $|x|_F^s$ with $s \in \mathbb{C}$. The representation $V(\omega)$ is reducible if and only if $\omega_\alpha(x) = |x|_F^{\pm 1/3}$; see [10] th. I. 2.9; only if $\omega_\alpha(x) = |x|^{1/3}$ do we have a quotient with a K_{81}^*-invariant vector. Under this condition there are two possible ω's since

$$\omega\left(\begin{pmatrix} x^2 & 0 \\ 0 & 1 \end{pmatrix}^* \right) = \chi\left(\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}^* \right) \omega_\alpha(x)$$

for $x \in F^{\times 3}$.

If we define ω_0 and χ_0 by

$$\omega_0\left(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \right) = \omega\left(\begin{pmatrix} x^3 & 0 \\ 0 & y^3 \end{pmatrix}^* \right), \qquad (x,\ y \in F^\times),$$

and

$$\chi_0\left(\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \right) = \chi\left(\begin{pmatrix} x^3 & 0 \\ 0 & x^3 \end{pmatrix}^* \right), \qquad (x \in F^\times),$$

then the Shimura correspondence ([3]) shows that the principal series representation of G associated with ω_0 corresponds to $V(\omega)$. If $\omega_\alpha(x) = |x|_F^{1/3}$ then the one dimensional representation $g \mapsto \omega_0\left(\begin{pmatrix} \det(g) & 0 \\ 0 & 1 \end{pmatrix} \right) |\det(g)|_F^{-1/2}$ corresponds to the unique irreducible quotient $V_0(\omega)$ of $V(\omega)$; see [3].

We shall now translate these results to the language of classical automorphic forms. Let $\mathbb{H}^3$ denote the upper half-space $\mathbb{C} \times \mathbb{R}_+^\times$. Let Δ be the Laplace operator

$v^2(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial v^2})-v\frac{\partial}{\partial v}$ where $x+iy$ and v are the coordinates. Let $\Gamma = SL_2(\mathbb{Z}[\omega])$ where we identify ω with $e^{2\pi i/3}$; Γ acts on $\mathbb{H}^3$. Let $\Gamma(3) = \{g \in \Gamma : g \equiv I \pmod 3\}$ and $\Gamma_2 = SL_2(\mathbb{Z}) \cdot \Gamma(3)$. We can define, for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(3)$ the homomorphism χ by

$$\chi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \begin{aligned} &= \left(\tfrac{c}{a}\right)_3, \quad (c \neq 0), \\ &= 1, \qquad (c = 0). \end{aligned}$$

We can extend χ to Γ_2 by $\chi | SL_2(\mathbb{Z}) = 1$ (see [17], §2).

The group $GL_2(\mathbb{C})$ also operates on $\mathbb{H}^3$ by

$$g((z,\ v)) = \left(\frac{(az+b)(\overline{cz}+\overline{d}) + a\bar{c}v^2}{|cz+d|^2+|c|^2v^2},\ \frac{|\det(g)|v}{|cz+d|^2+|c|^2v^2}\right),$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; in fact this yields an action of $PGL_2(\mathbb{C})$ on $\mathbb{H}^3$. Note that this cannot be done in the case of the action of $SL_2(\mathbb{R})$ on $\mathbb{H}^2$, the upper half-plane. This means that we can examine the action of $GL_2(\mathbb{Z}[\omega])$ on Γ-automorphic functions. We note that $GL_2(\mathbb{Z}[\omega])/SL_2(\mathbb{Z})[\omega]) \cong \mu_6(\mathbb{Q}(w))$ by the determinant map.

Our objective is to compare Γ-automorphic forms with trivial character with Γ_2-automorphic forms with character χ. To do this we have to recall the definition of an automorphic form. First of all these are square-integrable functions f (on $\Gamma \setminus \mathbb{H}^3$ or $\Gamma_2 \setminus \mathbb{H}^3$) with the appropriate transformation property

$$\begin{aligned} f(\gamma w) &= f(w) && (\gamma \in \Gamma) \\ \text{(resp. } f(\gamma w) &= \chi(\gamma) f(w) && (\gamma \in \Gamma_2)). \end{aligned}$$

Moreover f should be an eigenfunction of Δ

$$\Delta f = -s(2-s)f;$$

we shall refer, loosely, to s as the parameter of f, and write it as s_f. It is only defined up to the equivalence defined by $s \mapsto 2-s$ on $\mathbb{C}$. Next we require, in the case of Γ, that the action of $GL_2(\mathbb{Z}[\omega])$ should map f into a multiple of itself. According to whether $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ acts as ± 1 we shall refer to f as even or odd. We note that as $\begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega^2 \end{pmatrix}\begin{pmatrix} \omega^{-1} & 0 \\ 0 & \omega \end{pmatrix}$ the action of $\begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix}$ is trivial.

In the case of Γ_1 it is only possible to consider the action of elements of determinant ± 1 as conjugation by $\begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix}$ does not preserve χ. We demand that $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ acts on f by inducing multiplication by ± 1. Again we speak of f being even or odd.

Next we shall consider the action of Hecke operators on f. Following tradition we shall define

$$T_\mu = \{g \in M_2(\mathbb{Z}[\omega]) : \det(g) = \mu\};$$

we do *not* assume that g is primitive, i.e. we do not exclude g being of the form νg_1 where $g_1 \in M_2(\mathbb{Z}[\omega])$ and $\nu \in \mathbb{Z}[\omega]$ is a non-unit. Then we define

$$f|T_\mu(w) = \sum_{g \in \Gamma \setminus T_\mu} f(g(w)).$$

It is clear that this is again Γ-invariant. We require that f satisfy

$$f|T_\mu = t_\mu(f)\cdot f$$

for some $t_\mu(f)\in\mathbb{C}$ and all $\mu\in\mathbb{Z}[\omega]-\{0\}$. The $t_\mu(f)$ are the *Hecke eigenvalues* of f. Let us write

$$T^0_\mu = \{g\in M_2(\mathbb{Z}[\omega]) : \det(g)=\mu,\ g \text{ primitive}\}.$$

Then we can define $f|T^0_\mu$ as above. It follows that f is also an eigenfunction of this operator. Let $t^0_\mu(f)$ denote the eigenvalue. Then

$$6t_\mu(f) = \sum_{\delta:\delta^2|\mu} t^0_{\mu|\delta^2}(f);$$

the factor 6 comes from the units. We recall also that

$$T_{\mu_1}T_{\mu_2} = \frac{1}{6}\cdot\sum_{\substack{\delta|\mu_1\\ \delta|\mu_2}} N(\delta)T_{\mu_1\mu_2/\delta^2}.$$

We recall also that if f is automorphic as above then it has a Fourier expansion of the form

$$f(z,v) = \hat{f}(0)v^{2-s} + \sum_{\mu\in\frac{1}{\lambda}\mathbb{Z}[w]} \hat{f}(\mu)vK_{s-1}(4\pi|\mu|v)e(\mu z),$$

where $K_*(*)$ is the standard Bessel function, $e(z)=e^{2\pi i(z+\overline{z})}$ and s is the parameter of f. We chose s to satisfy $Re(s)>1$ if $Re(s)\neq 1$; if $Re(s)=1$ then $\hat{f}(0)=0$. In fact it is known that in this case the only f with $\hat{f}(0)\neq 0$ are the constant functions, as follows from [2],p.243 and the Fourier expansion of the Eisenstein series in this case. The main theorem of Hecke theory asserts that

$$\hat{f}(\lambda^{-1})t_\mu(f) = N(\mu)^{\frac{1}{2}}\hat{f}(\mu\lambda^{-1}).$$

We also have that $t_\mu(f)$ is of the form $\alpha+\alpha'$ with $\alpha\alpha'=N(\mu)$. One expects that $|\alpha|=|\alpha'|=N(\mu)^{\frac{1}{2}}$ (generalized Ramanujan conjecture).

In the case of metaplectic forms the situation is rather more complicated. We let

$$\tilde{T}^0_{\mu^3} = \{g\in M_2(\mathbb{Z}[\omega]) : g=\gamma_1\begin{pmatrix}\mu^3 & 0\\ 0 & 1\end{pmatrix}\gamma_2,\ \gamma_1,\ \gamma_2\in\Gamma_2\}.$$

We can also extend χ to $\tilde{T}^0_{\mu^3}$ by

$$\chi(\gamma_1\begin{pmatrix}\mu^3 & 0\\ 0 & 1\end{pmatrix}\gamma_2) = \chi(\gamma_1)\chi(\gamma_2);$$

this is the assertion of [12] Theorem 3, p. 31 and Theorem 2.2 above. Now, if $\tilde{f}$ is automorphic under Γ_2 with character χ we can define $\tilde{f}|\tilde{T}^0_{\mu^3}$ by

$$\tilde{f}|\tilde{T}^0_{\mu^3} = \sum_{g\in\Gamma_1\backslash\tilde{T}^0_{\mu^3}} \overline{\chi}(g)\tilde{f}(g(w)).$$

We shall also demand that $\tilde{f}$ be an eigenfunction of all such $\tilde{T}^0_{\mu^3}$, i.e. for a suitable $\tilde{t}^0_{\mu^3}(f)$ we have

$$\tilde{f}|\tilde{T}^0_{\mu^3} = \tilde{t}^0_{\mu^3}(f) \cdot \tilde{f}.$$

Although using [13], pp. 37–39 and Proposition 2.5 above one can develop a relationship between the Fourier coefficients of $\tilde{f}$ and $\tilde{t}^0_{\mu^3}(f)$ it is more complicated – especially in this language – as Γ_1 has 3 cusps, represented by 1, ω, ω^2; see [17], p. 127. The $\tilde{t}^0_{\mu^3}(f)$ do not determine the Fourier coefficients in as simple a manner as we had above. They only give certain linear relations between them. For our purposes we do not need these and so we shall not go into this point further here.

We can now formulate the cubic Shimura correspondence [3] in this language, where we make use of the results given earlier in this section. Let us also agree to identify two automorphic forms which are multiples of one another.

THEOREM 3.4. *There is a bijective correspondence between the automorphic forms f under Γ with trivial character, and those $\tilde{f}$ under Γ_2 with character χ. If f is even (resp. odd) then so is $\tilde{f}$. If $s(f)$ is the parameter of f, $s(\tilde{f})$ that of $\tilde{f}$ then $(s(f)-1) = \pm 3(s(\tilde{f})-1)$. One has*

$$N(\mu^3)^{-\frac{1}{2}}\tilde{t}^0_{\mu^3}(\tilde{f}) = N(\mu)^{-\frac{1}{2}}t^0_{\mu}(f).$$

The only non-cuspidal automorphic form for Γ is 1; this corresponds to θ ([17], Theorem 8.1). For all other f we have that $Re(s(f)) = 1$ and both f and $\tilde{f}$ are cuspidal.

4. The kernel. In this section we shall construct a kernel representing the cubic: Shimura correspondence as given in Theorem 3.4. The basis of this is the theory of spherical functions, or, in Selberg's language, point-pair invariants. Let, for $w = (z,v)$, $w' = (z',v') \in \mathbb{H}^3$

$$L(w,w') = \frac{|z-z'|^2 + (v-v')^2}{vv'};$$

one then has for $g \in \mathrm{GL}_2(\mathbb{C})$

$$L(g(w), g(w')) = L(w,w').$$

In fact $\mathbb{H}^3$ is a Riemannian space, $PGL_2(\mathbb{C})$ acts as isometries and L is closely related to the distance function. Any other function k satisfying $k(g(w),\ g(w')) = k(w,w')$ for all $g \in PG_2(\mathbb{C})$, $w, w' \in \mathbb{H}^3$ is of the form $k(w,w') = k_0(L(w,w'))$. If $k_0(\xi) = \mathcal{O}(\xi^{-A})$ for a suitable $A > 0$ as $\xi \to \infty$, and ϕ is an eigenfunction of Δ with eigenvalue $-s(2-s)$ then

$$\int_{\mathbb{H}^3} k(w,w')\phi(w')d\sigma(w') = h_{k_0}(s) \cdot \phi(w),$$

where $d\sigma((z,v)) = \frac{dm(z)\cdot dv}{v^3}$ and m is the 2-dimensional Lebesgue measure on $\mathbb{H}^3$. See [2] for a treatment of this. The function $h_{k_0}(s) = \pi \int\limits_0^\infty \int\limits_{\frac{(v-1)^2}{v}}^\infty k_0(\xi)d\xi v^{s-1}\frac{dv}{v}$. Suppose that $m \in \mathbb{N}$; we define a polynomial $p_m(x)$ by

$$p_m(v^{\frac{1}{m}} + v^{-\frac{1}{m}} - 2) = v + v^{-1} - 2.$$

If $m = 3$, the case that will us interest most,

$$p_m(x) = x^3 + 6x^2 + 9x.$$

In general we see that p_m is an increasing function and $p'_m(0) = m^2$. It follows that p_m has an inverse function q_m defined on $[0, \infty[$ and real analytic at 0. We shall now prove the following

PROPOSITION 4.1. *Let* $k_0(\xi) = \exp(-\alpha \cdot q_m(\xi))q'_m(\xi)$. *Then*

$$h_{k_0}(s) = \frac{2\pi m \cdot \exp(2\alpha)}{\alpha} K_{m(s-1)}(2\alpha).$$

Proof. In

$$h_{k_0}(s) = \pi \int_0^\infty \int_{v+v^{-1}-2}^\infty \exp(-\alpha q_m(\xi))q'_m(\xi)d\xi \cdot v^{s-1}\frac{dv}{v}$$

we substitute $\xi = p_m(x)$. This gives us

$$\begin{aligned} h_{k_0}(s) &= \pi \int_0^\infty \int_{v^{\frac{1}{m}}+v^{-\frac{1}{m}}-2}^\infty \exp(-\alpha x)dx \cdot v^{s-1}\frac{dv}{v} \\ &= \frac{\pi}{\alpha}\int_0^\infty \exp(-\alpha(v^{\frac{1}{m}} + v^{-\frac{1}{m}} - 2))v^{s-1}\frac{dv}{v} \\ &= \frac{\pi \cdot m}{\alpha}\exp(2\alpha)\int_0^\infty \exp(-\alpha(u + u^{-1}))u^{m(s-1)}\frac{du}{u} \\ &= \frac{2\pi m}{\alpha} \cdot \exp(2\alpha)K_{m(s-1)}(2\alpha) \end{aligned}$$

by [31], p. 183. This proves the assertion. □

We now define

$$k(w, w'; w_0) = \frac{v_0^2}{6\pi}\exp(-4\pi v_0 - 2\pi q_3(L(w, w'))v_0)q'_3(L(w, w'))e(z_0),$$

where $w_0 = (z_0, v_0)$. We write also $\mu w_0 = \begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix}(w_0)$ so that $\mu(z_0, v_0) = (\mu z_0, |\mu| v_0)$. Then the Selberg transform of $k(\cdot, \cdot; w_0)$ is

$$s \mapsto v_0 K_{3(s-1)}(4\pi v_0)e(z_0).$$

We shall now show how to construct out of a metaplectic form $\tilde{f}$ the form f which corresponds to it by Theorem 3.4. We have that

$$\begin{aligned} t_\mu(f) &= \frac{1}{6}\sum_{\delta^2|\mu} t^0_{\mu/\delta^2}(f) \\ &= \frac{1}{6}N(\mu)^{-1} \cdot \sum_{\delta^2|\mu} N(\delta)^2 \cdot \tilde{t}^0_{(\mu/\delta^2)^3}(\tilde{f}). \end{aligned}$$

Next

$$\int\limits_{\Gamma_2\backslash\mathbb{H}^3} \sum_{\gamma\in\Gamma_2} \chi(\gamma)k(w,\gamma(w');\frac{\mu}{\lambda}w_0)\tilde{f}(w')d\sigma(w')$$

is equal to

$$3^{-\frac{1}{2}}\tilde{f}(w)N(\mu)^{\frac{1}{2}}v_0K_{3(s(\tilde{f})-1)}\left(4\pi|\frac{\mu}{\lambda}|v_0\right)e\left(\frac{\mu}{\lambda}z_0\right),$$

where $s(\tilde{f})$ is the parameter of $\tilde{f}$. If we write $s(f)$ for the parameter of f, we have $3(s(\tilde{f})-1) = \pm(s(f)-1)$. We apply $\tilde{T}^0_{(\mu')^3}$ to both sides of the equation above and obtain

$$\int\limits_{\Gamma_2\backslash\mathbb{H}^3} \sum_{\gamma\in\tilde{T}^0_{(\mu')^3}} \chi(\gamma)k\left(w,\gamma(w');\frac{\mu}{\lambda}w_0\right)\tilde{f}(w')d\sigma(w')$$
$$= \tilde{t}^0_{(\mu')^3}(\tilde{f})\cdot\tilde{f}(w)\cdot 3^{-\frac{1}{2}}\cdot N(\mu)^{\frac{1}{2}}v_0K_{s(f)-1}\left(4\pi|\frac{\mu}{\lambda}|v_0\right)e\left(\frac{\mu z_0}{\lambda}\right),$$

from which we obtain

$$\frac{1}{6}\sum_{\delta^2|\mu}N(\mu)^{-2}N(\delta)^2\int\limits_{\Gamma_2\backslash\mathbb{H}^3}\sum_{\gamma\in\tilde{T}^0_{(\mu\backslash\delta^2)^3}}\chi(\gamma)k\left(w,\gamma(w');\frac{\mu}{\lambda}w_0\right)\tilde{f}(w')d\sigma(w)$$
$$= 3^{-\frac{1}{2}}N(\mu)^{-\frac{1}{2}}t_\mu(f)\tilde{f}(w)\cdot v_0K_{s(f)-1}\left(4\pi|\frac{\mu}{\lambda}|v_0\right)e\left(\frac{\mu}{\lambda}z_0\right).$$

If we now define

$$K(w,w';w_0) = \sum_{\delta,\mu}N(\mu)^{-2}N(\delta)^{-2}\sum_{\gamma\in\tilde{T}^0_{\mu^3}}\chi(\gamma)k\left(w,\gamma(w');\frac{\mu\delta^2}{\lambda}w_0\right),$$

then this would suggest that

$$\int\limits_{\Gamma_2\backslash\mathbb{H}^3} K(w,w';w_0)\tilde{f}(w')d\sigma(w') = 2\cdot 3^{\frac{1}{2}}\cdot f(w_0)\tilde{f}(w)$$

if f denotes that automorphic form with $\hat{f}(\frac{1}{\lambda}) = 1$ (the Hecke normalization). If $\tilde{f}$ is not a cusp form then, as we have seen, it is the θ of [17], Theorem 8.1 and in this case we would obtain a rather different formula, to which we shall return later.

We now have to justify the exchange of summation and integration above. To this end we shall now prove the following estimate:

PROPOSITION 4.2. *The series defining $K(w,w';w_0)$ converges absolutely and for $\gamma_1,\gamma_2\in SL_2(\mathbb{Z}[\omega]), A>0$ we have*

$$|K(\gamma_1(w_1),\gamma_2(w_2);w_0)|$$
$$\leq c_A\left(1+\frac{1}{v(w_1)v(w_2)}\frac{1}{v_0^3}\right)\left(1+\frac{v(w_1)}{v(w_2)}\frac{1}{v_0^3}\right)\left(1+\frac{v(w_2)}{v(w_1)}\frac{1}{v_0^3}\right)\left(1+\frac{v(w_1)v(w_2)}{v_0^3}\right)$$

when $v(w_0)\geq A$.

Proof. We begin with the identity

$$L(\left(w_1,\ \begin{pmatrix} a & b \\ c & d \end{pmatrix}(w_2)\right)+2=\frac{Q_{w_1\ w_2}(a,b,c,d)}{v_1\ v_2|ad-bc|}\qquad \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in GL_2(\mathbb{C})\right),$$

where Q_{w_1,w_2} is the Hermitian form with the matrix

$$\begin{bmatrix} |z_2|^2+v_2^2 & z_2 & -\overline{z}_1(|z_2|^2+v_2^2) & -z_2\overline{z}_1 \\ \overline{z}_2 & 1 & -\overline{z}_1\overline{z}_2 & -\overline{z}_2 \\ -z_1(|z_2|)^2+v_2^2) & -z_1z_2 & (|z_1|^2+v_1^2)(|z_2|^2+v_2^2) & z_2(|z_1|^2+v_1^2) \\ -z_1\overline{z}_2 & -z_1 & \overline{z}_2(|z_1|^2+v_1^2) & |z_1|^2+v_1^2 \end{bmatrix},$$

where $w_1=(z_1,v_1),\ w_2=(z_2,v_2)$. This has determinant $v_1^4\cdot v_2^4$ and in fact is

$$T\begin{pmatrix} v_2^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & v_1^2v_2^2 & 0 \\ 0 & 0 & 0 & v_1^2 \end{pmatrix}\overline{T}^t$$

with

$$T=\begin{pmatrix} 1 & z_2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -z_1 & -z_1z_2 & 1 & z_2 \\ 0 & -z_1 & 0 & 1 \end{pmatrix}.$$

The identity shows that

$$Q_{w_1w_2}(a,b,c,d)\geq 2v_1v_2|ad-bc|;$$

in particular, if $a,b,c,d\in\mathbb{Z}[\omega],\ ad-bc\neq 0$ then

$$Q_{w_1w_2}(a,b,c,d)\geq 2v_1v_2.$$

If now $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in M_2(\mathbb{Z}[\omega])$ with $\det(g)=\mu^3$ then the definition of k shows that $k\left(w_1,\ g(w_2);\ \frac{\mu\delta^2}{\lambda}w_0\right)$ is equal to

$$\frac{N(\mu)N(\delta)^2}{18\pi}\exp\left(-4\pi\frac{|\mu\delta^2|}{\sqrt{3}}v_0-2\pi q_3\left(\frac{Q(a,b,c,d)}{v_1v_2|\mu|^3}-2\right)\frac{|\mu\delta^2|}{\sqrt{3}}v_0\right)$$
$$\times q_3'\left(\frac{Q(a,b,c,d)}{v_1v_2|\mu|^3}-2\right)e\left(z\left(\frac{\mu\delta^2}{\lambda}w_0\right)\right),$$

where $Q=Q_{w_1w_2}$. We use this to estimate $K(\gamma_1(w_1),\ \gamma_2(w_2);w_0)$. If we substitute we see that the sum over δ is a standard theta-function. Since $\frac{|\mu|v_0}{\sqrt{3}}\geq\frac{A}{\sqrt{3}}$ the theta-function is bounded by the terms with $|\delta|=1$ and so we can ignore the summation over δ. What remains is dominated by

$$\sum_{\substack{a,b,c,d \\ ad-bc=\mu^3 \\ \mu\neq 0}} N(\mu)^{-1}\cdot\exp\left(-\frac{4\pi|\mu|v_0}{\sqrt{3}}-2\pi q_3\left(\frac{Q(a,b,c,d)}{v_1v_2|\mu|^3}-2\right)\frac{|\mu|v_0}{\sqrt{3}}\right)$$
$$\times q_3'\left(\frac{Q(a,b,c,d)}{v_1v_2|\mu|^3}-2\right).$$

Since we now sum over all possible a, b, c, d satisfying the restrictions above we can drop the γ_1, γ_2 from the discussion. Next we note that $N(\mu) \geq 1$ and $q_3'(\xi) \leq 1/9$. Next, if $x \geq 1$ then $p_m(x) \leq 16x^3$ and so $q_3(\xi) \geq (\frac{1}{16}\xi)^{\frac{1}{3}}$. If $x < 1$ then $p_m(x) \leq 16x$ and so $q_3(\xi) \leq \frac{1}{16}\xi$. This means that

$$\begin{aligned} q_3(\xi) \quad &\geq (\xi/16)^{\frac{1}{3}}, \quad \text{if } \xi \geq 16, \\ &\geq \xi/16, \qquad \text{if } \xi \leq 16. \end{aligned}$$

If now (a, b, c, d) is such that

$$\frac{Q(a,b,c,d)}{v_1 v_2 |\mu|^3} - 2 \leq 16,$$

then we drop the second expression in the exponential. Since

$$|\mu|^3 \geq \frac{Q(a,b,c,d)}{18\ v_1 v_2},$$

we see that the summand is bounded by

$$\frac{1}{9}\exp\left(-\frac{4\pi}{\sqrt{3}\cdot(18)^{\frac{1}{3}}}\frac{Q(a,b,c,d)^{\frac{1}{3}}}{(v_1 v_2)^{\frac{1}{3}}}\right)$$

in these cases. If $\xi \geq 18$ then we note that

$$\begin{aligned} q_3(\xi - 2) \quad &\geq ((\xi-2)/16)^{\frac{1}{3}} \\ &= \xi^{\frac{1}{3}}((1-\tfrac{2}{\xi})/16)^{\frac{1}{3}} \\ &\geq \xi^{\frac{1}{3}}/(18)^{\frac{1}{3}}. \end{aligned}$$

We therefore obtain the estimate if $\frac{Q(a,b,c,d)}{v_1 v_2 |\mu|^3} - 2 > 16$:

$$\frac{1}{9}\exp\left(-\frac{2\pi}{(18)^{\frac{2}{3}}\sqrt{3}}\frac{Q(a,b,c,d)^{\frac{1}{3}}}{v_1^{\frac{1}{3}} v_2^{\frac{1}{3}}} v_0\right).$$

We therefore see that our series is dominated by

$$c_A^{(1)} \sum_{\substack{a,b,c,d \\ ad-bc=\mu^3 \\ \mu\neq 0}} \exp\left(-c_1 \cdot \frac{Q(a,b,c,d)^{\frac{1}{3}}}{v_1^{\frac{1}{3}} v_2^{\frac{1}{3}}} v_0\right),$$

where $c_A^{(1)} > 0$ and $c_1 = \frac{2\pi}{(18)^{\frac{1}{3}}\sqrt{3}}$ is absolute.

Next let $N(R)$ be the number of (a, b, c, d) such that $Q(a, b, c, d) \leq R\ v_1 v_2$. Since

$$Q(a,b,c,d)/v_1 v_2 = |c|^2 v_1 v_2 + |a - cz_1|^2 \frac{v_2}{v_1} + |d + cz_2|^2 \cdot \frac{v_1}{v_2} + |b + az_2 - dz_1 - cz_1 z_2|^2 \cdot \frac{1}{v_1 v_2}$$

and as, for any fixed $\zeta \in \mathbb{C}$, the number of $\alpha \in \mathbb{Z}[\omega]$ with $|\alpha - \zeta|^2 \leq R$ is bounded by $k(1 + R)$ with $k = \frac{8\pi}{3\sqrt{3}}$ we see that $N(R) \leq k^4 \cdot p(R)$, where

$$p(R) = \left(1 + \frac{R}{v_1 v_2}\right)\left(1 + R \cdot \frac{v_1}{v_2}\right)\left(1 + R \cdot \frac{v_2}{v_1}\right)(1 + R \cdot v_1 v_2).$$

Thus our series is bounded by

$$c_A^{(1)} \int_0^\infty \exp(-c_1 R^{\frac{1}{3}} v_0) d\, N(R)$$

after we have allowed the summation to be over all $(a,b,c,d) \in \mathbb{Z}[\omega]^4$. Double integration by parts shows that this is bounded by

$$k^4 c_A^{(1)} \int_0^\infty \exp(-c_1 R^{\frac{1}{3}} v_0) p'(R) dR.$$

Since this converges we conclude that one original series was absolutely convergent. Moreover, if we replace R by R/v_0^3 we obtain the estimate asserted in the statement. This completes the proof of the proposition. □

As we shall need it later we record one corollary:

COROLLARY 4.3. *For w_0, w_1 fixed, $\gamma \in SL_2(\mathbb{Z}[\omega])$*

$$K(w_1,\ \gamma(w_2); w_0) = \mathcal{O}(1)$$

as $v(w_2) \to \infty$.

This corollary means that we determine how the integral operator with kernel K operates on both Eisenstein series and generalized theta functions. We first have to recall some of the fundamental notions.

First of all Γ has only the cusp at ∞ and we can construct a standard Eisenstein series $E(w,s)$ associated with this map. It has the Fourier expansion

$$\begin{aligned} E((z,v),s) &= v^s + \frac{1}{\sqrt{3}} \frac{Z_{\mathbb{Q}(\omega)}(s-1)}{Z_{\mathbb{Q}(\omega)}(s)} v^{2-s} \\ &\quad + \frac{2}{\sqrt{3}} \cdot Z_{\mathbb{Q}(\omega)}(s)^{-1} \sum_{\substack{\mu \in \lambda^{-1}\mathbb{Z}[\omega] \\ \mu \neq 0}} N(\mu)^{\frac{s-1}{2}} \sigma_{1-s}(\mu\lambda) v K_{s-1}(4\pi|\mu|v) e(\mu z), \end{aligned}$$

where $Z_{\mathbb{Q}(\omega)}(s) = (2\pi)^{-s}\Gamma(s)\zeta_{\mathbb{Q}(\omega)}(s)$ and $\sigma_t(\mu) = \sum_{d|\mu} N(d)^t$. The function $Z_{\mathbb{Q}(\omega)}$ satisfies the functional equation

$$Z_{\mathbb{Q}(\omega)}(s) = 3^{\frac{1}{2}-s} Z_{\mathbb{Q}(\omega)}(1-s)$$

and it follows that

$$\begin{aligned} \frac{1}{2} 3^{\frac{s}{2}} Z_{\mathbb{Q}(\omega)}(s) E((z,v),s) &= \frac{1}{2}(3^{\frac{s}{2}} Z_{\mathbb{Q}(\omega)}(s) v^s + 3^{\frac{2-s}{2}} Z_{\mathbb{Q}(\omega)}(2-s) v^{2-s}) \\ &+ \sum_{\substack{\mu \in \lambda^{-1}\mathbb{Z}[\omega] \\ \mu \neq 0}} N(\mu\lambda)^{\frac{s-1}{2}} \sigma_{1-s}(\mu\lambda) v \cdot K_{s-1}(4\pi|\mu|v) e(\mu z) \end{aligned}$$

is invariant under $s \mapsto 2-s$.

In the case of Γ_2 with character χ there is one essential cusp which we can take to be ∞. This means that there is one Eisenstein series associated with Γ_2 and χ; we shall denote it here by $\tilde{E}(w,s)$. Likewise there is an Eisenstein series associated with Γ_2 and χ^{-1} and this we shall denote by $\tilde{\tilde{E}}(w,s)$. The function $\tilde{E}(w,s)$ is denoted by $E_{II}(w,s)$ in [17], p. 132. Essentially [17], (3.21) gives the constant term of $\tilde{E}(w,s)$. It is slightly easier to use an argument of Hecke's. Let $\zeta_{\mathbb{Q}(\omega)}$ denote the Dedekind zeta function of $\mathbb{Q}(\omega)$ and let $Z_{\mathbb{Q}(\omega)}(s) = (2\pi)^{-s}\Gamma(s)\zeta_{\mathbb{Q}(\omega)}(s)$. We see from [17], §5 that the constant term has the form $v^s + \frac{Z_{\mathbb{Q}(\omega)}(3s-3)}{Z_{\mathbb{Q}(\omega)}(3s-2)} \cdot A(s)v^{2-s}$ where A is a finite Laurent series in 3^{-3s}. The functional equation implies that $A(s)\cdot A(2-s) = 3$ and so we deduce that $A(s) = \pm\frac{1}{\sqrt{3}}$. On using the formula [17] (3.21) one can see that only the positive sign is possible. Thus we see that the constant term of $\tilde{E}(w,s)$ is

$$v^s + \frac{1}{\sqrt{3}}\frac{Z_{\mathbb{Q}(\omega)}(3s-3)}{Z_{\mathbb{Q}(\omega)}(3s-2)}v^{2-s}.$$

It follows that

$$3^{\frac{3s}{2}} Z_{\mathbb{Q}(\omega)}(3s-2)\tilde{E}(w,s)$$

is invariant under $s \mapsto 2-s$. We also note that the function $s \mapsto \tilde{E}(w,s)$ has a pole at $s = \frac{4}{3}$ with residue

$$\frac{1}{3^3}\frac{1}{Z_{\mathbb{Q}(\omega)}(2)}\theta(w),$$

where θ represents the cubic theta function of [17], Theorem 8.1 but normalized to have constant term $\frac{1}{2}v(w)^{\frac{2}{3}}$. The same argument as before shows that if $s : 0 < \mathrm{Re}(s) < 2$ then

$$\int_{\Gamma_2\backslash\mathbb{H}^3} K(w,w';w_0)\tilde{E}(w',s)d\sigma(w')$$

is equal to $(v_0 = v(w_0))$

$$\left\{3^{\frac{1}{2}+\frac{s_1}{2}} Z_{\mathbb{Q}(\omega)}(s_1)\cdot E(w_0,s_1) - (3^{\frac{1}{2}+\frac{s_1}{2}} Z_{\mathbb{Q}(\omega)}(s_1)v_0^{s_1} + 3^{\frac{3}{2}-\frac{s_1}{2}} Z_{\mathbb{Q}(\omega)}(2-s_1)v_0^{2-s_1})\right\} \times \tilde{E}(w,s),$$

where $s_1 = 3s-2$. Likewise

$$\int_{\Gamma_2\backslash\mathbb{H}^3} K(w,w';w_0)\theta(w')d\sigma(w')$$

is equal to

$$\left\{3^{\frac{1}{2}+\frac{s_1}{2}} Z_{\mathbb{Q}(\omega)}(s_1)E(w_0,s_1) - (3^{\frac{1}{2}+\frac{s_1}{2}} Z_{\mathbb{Q}(\omega)}(s_1)v_0^{s_1} + 3^{\frac{3}{2}-\frac{s_1}{2}} Z_{\mathbb{Q}(\omega)}(2-s_1)v_0^{2-s_1})\right\}\Bigg|_{s_1=2} \times \theta(w).$$

These results will allow us to determine the spectral decomposition of $K(*,*;w_0)$. For this purpose we need a lemma, and, to avoid having to interrupt the argument later we shall state and prove it here. First of all we note that $\tilde{E}(w,1)=0$ as the constant term of $\tilde{E}(w,s)$ vanishes at $s=1$. Then we have:

LEMMA 4.4. *We have*

$$\frac{1}{2\pi i}\int_{(1)} \tilde{E}(w,s)\tilde{\tilde{E}}(w',2-s)Z_{\mathbb{Q}(\omega)}(3s-2)y^s ds$$

is equal to

$$\frac{1}{54}y^{\frac{4}{3}}\sum_{\gamma_1,\gamma_2}\sum_{\mu\in\mathbb{Z}[\omega]-\{0\}} \overline{\chi}(\gamma_1)\chi(\gamma_2)v(\gamma_1(w))^{\frac{2}{3}}v(\gamma_2(w'))^{\frac{2}{3}} \exp\left(-\frac{2\pi}{3}\frac{N(\mu)y^{\frac{1}{3}}}{v(\gamma_1(w))^{\frac{1}{3}}v(\gamma_2(w'))^{\frac{1}{3}}}\right)$$
$$-\operatorname*{Res}_{s=\frac{4}{3}}\{\tilde{E}(w,s)\tilde{\tilde{E}}(w',s)\ Z_{\mathbb{Q}(\omega)}(3s-2)3^{3s-3}y^{2-s}\},$$

where (σ) *denotes the path* $\mathbb{R}\to\mathbb{C};\ t\mapsto\sigma+it$ *and* γ_1,γ_2 *are summed over* $\Gamma_{2,\infty}\backslash\Gamma_2$.

Proof. We recall that $\tilde{E}(w,s)\cdot Z_{\mathbb{Q}(\omega)}(3s-2)3^{\frac{3s}{2}}$ is invariant under $s\mapsto 2-s$; using this we see that the integral is

$$\frac{1}{2\pi i}\int_{(1)} \tilde{E}(w,2-s)\tilde{\tilde{E}}(w',2-s)3^{3(1-s)}\cdot y^s Z_{\mathbb{Q}(\omega)}(4-3s)ds.$$

We now replace s by $2-s$ and obtain

$$\frac{1}{2\pi i}\int_{(1)} \tilde{E}(w,s)\tilde{\tilde{E}}(w',s)3^{3(s-1)}y^{2-s}Z_{\mathbb{Q}(\omega)}(3s-2)ds.$$

(Note that this does not follow from the functional equation for $\tilde{\tilde{E}}$.) We recall that for $\operatorname{Re}(s)>1$

$$\zeta_{\mathbb{Q}(\omega)}(s)=\frac{1}{6}\sum_{\mu\in\mathbb{Z}[\omega]-\{0\}}N(\mu)^{-s}.$$

If we now move the line of integration above to (σ) with $\sigma>2$ and apply the definitions of $\tilde{E}$, $\tilde{\tilde{E}}$ and that of $\zeta_{\mathbb{Q}(\omega)}(3s-2)$ as above then the assertion follows from the formula

$$\frac{1}{2\pi i}\int_{(\sigma)}\Gamma(s)u^{-s}ds=\exp(-u). \qquad \square$$

Now that we have proved this lemma we can now determine the spectral decomposition of $K(w,w';w_0)$. This is given in the following proposition which is a raw form of the main result of this paper.

PROPOSITION 4.5. *Let* $\tilde{f}_j(j\in J)$ *run through the set of cusp forms for* Γ_2 *and* χ, *where* J *is a countable index set, and we include both even and odd forms. Let* f_j *be* *the corresponding (Hecke normalized) cusp form for* Γ. *Then we have*

$$\frac{1}{2\sqrt{3}}K(w,w';w_0) = \sum_{j\in J}\frac{\tilde{f}_j(w)\tilde{f}_j(w')}{\|\tilde{f}_j\|^2}f_j(w_0)$$

$$+\frac{1}{4\pi i}\int_{(1)}\tilde{E}(w,s)\tilde{\tilde{E}}(w',2-s)E(w_0,3s-2)\cdot\frac{1}{2}\cdot 3^{\frac{3s}{2}-1}Z_{\mathbb{Q}(\omega)}(3s-2)ds$$

$$+\frac{1}{2}\frac{\theta(w)\overline{\theta(w')}}{\|\theta\|^2}\left\{3^{\frac{3s}{2}-1}E(w_0,3s-2)Z_{\mathbb{Q}(\omega)}(3s-2)-3^{2-\frac{3s}{2}}Z_{\mathbb{Q}(\omega)}(4-3s)\right\}\Big|_{s=\frac{4}{3}}$$

$$-\frac{1}{36}\sum_{\gamma_1,\gamma_2,\mu}\overline{\chi}(\gamma_1)\chi(\gamma_2)v(\gamma_1(w))^{\frac{2}{3}}v(\gamma_2(w'))^{\frac{2}{3}}v(w_0)^2\exp\left(-\frac{2\pi}{\sqrt{3}}\frac{v(w_0)N(\mu)}{v(\gamma_1(w))^{\frac{1}{3}}v(\gamma_2(w'))^{\frac{1}{3}}}\right)$$

$$+\frac{1}{2}\operatorname*{Res}_{s=\frac{4}{3}}\{\tilde{E}(w,s)\tilde{\tilde{E}}(w',s)Z_{\mathbb{Q}(\omega)}(3s-2)3^{3s/2}\}$$

$$+\frac{\theta(w)\overline{\theta(w')}}{\|\theta\|^2}\left(\frac{3}{2}Z_{\mathbb{Q}(\omega)}(2)v_0^2+\left(\frac{1}{12}-\frac{1}{8\sqrt{3}}\right)\log v_0\right),$$

where γ_1 *and* γ_2 *are summed over* $\Gamma_{2,\infty}\setminus\Gamma$ *and* μ *over* $\mathbb{Z}[\omega]-\{0\}$. *We have set* $v_0 = v(w_0)$.

Proof. The analysis of Selberg (cf. [2],p.265 and the references given there) shows that $\frac{1}{2\sqrt{3}}K(w,w':w_0)$ has a decomposition of the following form

$$\sum_j\frac{\tilde{f}_j(w)\overline{\tilde{f}_j(w')}}{\|\tilde{f}_j\|^2}f_j(w_0)$$
$$+\frac{1}{4\pi i}\int_{(1)}\tilde{E}(w,s)\tilde{\tilde{E}}(w',2-s)\cdot\frac{1}{2}\cdot\Big(3^{\frac{3s}{2}-1}E(w_0,3s-2)Z_{\mathbb{Q}(\omega)}(3s-2)$$
$$-3^{\frac{3s}{2}-1}Z_{\mathbb{Q}(\omega)}(3s-2)v_0^{3s-2}-3^{2-\frac{3s}{2}}Z_{\mathbb{Q}(\omega)}(4-3s)v_0^{4-3s}\Big)\,ds$$
$$+\frac{1}{2}\frac{\theta(w)\overline{\theta(w')}}{\|\theta\|^2}\Big\{3^{\frac{3s}{2}-1}E(w_0,3s-2)Z_{\mathbb{Q}(\omega)}(3s-2)$$
$$-3^{\frac{3s}{2}-1}Z_{\mathbb{Q}(\omega)}(3s-2)v_0^{3s-2}-3^{2-\frac{3s}{2}}Z_{\mathbb{Q}(\omega)}(4-3s)v_0^{4-3s}\Big\}\Big|_{s=\frac{4}{3}}.$$

We call these terms the discrete (or cuspidal), continuous (or Eisenstein) and residual contribution to $\frac{1}{2\sqrt{3}}K(w,w';w_0)$. We retain the entire first term, the first part of the continuous term and, from the residual term

$$\frac{\theta(w)\overline{\theta(w')}}{\|\theta\|^2}\cdot\frac{1}{2}\cdot\left\{3^{\frac{3s}{2}-1}E(w_0,3s-2)Z_{\mathbb{Q}(\omega)}(3s-2)-3^{2-\frac{3s}{2}}Z_{\mathbb{Q}(\omega)}(4-3s)\right\}\Big|_{s=\frac{4}{3}}.$$

The remain two terms from the continuous spectrum. In fact the substitution $s\mapsto 2-s$ shows that they correspond under $w\leftrightarrow w'$. We can compute these using Lemma 4.4;

we obtain

$$-\frac{1}{36}\cdot v_0^2\sum_{\gamma_1,\gamma_2,\mu}\overline{\chi}(\gamma_1)\chi(\gamma_2)v(\gamma_1(w))^{\frac{2}{3}}v(\gamma_2(w'))^{\frac{2}{3}}\exp\left(-\frac{2\pi\sqrt{3}}{3}\frac{N(\mu)v(w_0)}{v(\gamma_1(w))^{\frac{1}{3}}v(\gamma_2(w'))^{\frac{1}{3}}}\right)$$
$$+\frac{1}{2}\operatorname*{Res}_{s=\frac{4}{3}}\left\{\tilde{E}(w,s)\tilde{\tilde{E}}(w',s)Z_{\mathbb{Q}(\omega)}(3s-2)3^{3s/2}v_0^{4-3s}\right\}.$$

The first of these two terms appears in the formula given in the proposition. If we now take the second term along with what remains of the residual term we find

$$\frac{1}{2}\operatorname*{Res}_{s=\frac{4}{3}}\left\{\tilde{E}(w,s)\tilde{\tilde{E}}(w',s)Z_{\mathbb{Q}(\omega)}(3s-2)3^{3s/2}v_0^{4-3s}\right\}$$

$$-\frac{\theta(w)\overline{\theta(w')}}{\|\theta\|^2}\cdot\frac{3}{2}\cdot Z_{\mathbb{Q}(\omega)}(2)v_0^2+\frac{\theta(w)\overline{\theta(w')}}{\|\theta\|^2}\cdot\frac{1}{12}\log v_0.$$

We now observe that

$$\frac{1}{2}\operatorname*{Res}_{s=\frac{4}{3}}\left\{\tilde{E}(w,s)Z_{\mathbb{Q}(\omega)}(3s-2)3^{3s/2}(v_0^{4-3s}-1)\right\}$$

$$=-\frac{1}{2}\operatorname*{Res}_{s=\frac{4}{3}}\tilde{E}(w,s)\cdot\operatorname*{Res}_{s=\frac{4}{3}}\tilde{\tilde{E}}(w',s)\cdot Z_{\mathbb{Q}(\omega)}(2)3^2\log v_0.$$

However using [17], p. 159, l. 8 we see that this is

$$-\frac{1}{2}\frac{\theta(w)\overline{\theta(w')}}{\|\theta\|^2}\cdot\left\{\frac{1}{4\cdot 3^{\frac{3}{2}}Z_{\mathbb{Q}(\omega)}(2)}\right\}Z_{\mathbb{Q}(\omega)}(2)\log v_0$$

$$=-\frac{1}{2^3\cdot 3^{\frac{1}{2}}}\frac{\theta(w)\overline{\theta(w')}}{\|\theta\|^2}\log v_0.$$

We still have to discuss the convergence of the sum over J and the integral. Since, in the integral the factor $Z_{\mathbb{Q}(\omega)}(3s-2)$ decays exponentially, the integral converges.

In the case of the sum over J one knows that the L^2-norms of the f_j decay exponentially in $|\mathrm{Im}(s_j)|$; see, for example [8], §10. In order to prove the same pointwise (and locally uniformly) one observes that

$$R(t_1)R(t_2)f_j=(s_j(2-s_j)-t_1(2-t_1))^{-1}\cdot(s_j(2-s_j)-t_2(2-t_2))^{-1}f_j,$$

where $R(t)$ is the resolvent operators $(-\Delta-t(2-t))^{-1}$ on $L^2(\Gamma\backslash\mathbb{H}^3)$. Since $R(t_1)R(t_2)$ is represented by a smooth kernel we obtain an estimate of the type needed. We leave the relatively standard details to the reader. □

We now observe that all but the fourth and last terms on the right-hand side of the formula of Proposition 4.5 are Γ-automorphic in w_0. We therefore define

$$K^+(w,w';w_0) = K(w,w';w_0) + \frac{1}{6\sqrt{3}} \sum_{\gamma_1,\gamma_2,\mu} \overline{\chi}(\gamma_1)\chi(\gamma_2)v(\gamma_1(w))^{\frac{2}{3}}v(\gamma_2(w))^{\frac{2}{3}} \times$$

$$\times v(w_0)^2 \exp\left(-\frac{2\pi}{\sqrt{3}}\frac{v(w_0)N(\mu)}{v(\gamma_1(w))^{\frac{1}{3}}v(\gamma_2(w)^{\frac{1}{3}}}\right)$$

$$-\frac{\theta(w)\overline{\theta(w')}}{\| \theta \|^2}\left(3\sqrt{3}Z_{\mathbb{Q}(\omega)}(2)v_0^2 + \left(\frac{1}{4\sqrt{3}} - \frac{1}{4}\right)\log v_0\right).$$

We observe that $K^+(w,w',w_0)$ is explicit in the sense that it is given by an infinite series which can be computed to any degree of accuracy. Admittedly this series is not as elegant as one would wish, but nevertheless its genesis suggests that it should be taken seriously. We now have:

THEOREM 4.6. *The function $K^+(w,w';w_0)$ satisfies*

a) $K^+(\gamma(w),w')w_0) = \chi(\gamma)K^+(w,w';w_0) \qquad (\gamma \in \Gamma_2)$;

b) $K^+(w,\gamma(w');w_0) = \overline{\chi}(\gamma)K^+(w,w';w_0) \qquad (\gamma \in \Gamma_2)$;

c) $K^+(w,w';\gamma(w_0)) = K^+(w,w';w_0) \qquad (\gamma \in \Gamma)$;

d) $$K^+(w,w';w_0) = 2\sqrt{3}\sum_j \frac{\tilde{f}_j(w)\overline{\tilde{f}_j(w')}}{\| \tilde{f}_j \|^2}\cdot f_j(w_0)$$
$$+\frac{1}{4\pi i}\int \tilde{E}(w,s)\overline{\tilde{E}(w',2-s)}3^{\frac{3s-1}{2}}Z_{\mathbb{Q}(\omega)}(3s-2)E(w_0,3s-2)ds$$
$$+\frac{\theta(w)\overline{\theta(w')}}{\| \theta \|^2}\left\{3^{\frac{3s-1}{2}}E(w_0,3s-2)Z_{\mathbb{Q}(\omega)}(3s-2) - 3^{\frac{5-3s}{2}}Z_{\mathbb{Q}(\omega)}(4-3s)\right\}\Big|_{s=\frac{4}{3}}$$
$$+\frac{1}{2}\operatorname*{Res}_{s=\frac{4}{3}}\left\{\tilde{E}(w,s)\tilde{\tilde{E}}(w',s)Z_{\mathbb{Q}(\omega)}(3s-2)3^{-(3s+1)/2}\right\};$$

e) $$\int_{\Gamma_2\mathbb{H}^3} K^+(w,w';w_0)\tilde{f}_\mu(w')d\sigma(w') = 2\sqrt{3}\tilde{f}_\mu(w)f_\mu(w_0)$$

f) $$\int_{\Gamma\backslash\mathbb{H}^3} K^+(w,w';w_0)\overline{f_j(w_0)}d\sigma(w_0) = 2\sqrt{3}\tilde{f}_j(w)\overline{\tilde{f}_j(w')}\frac{\| f_j \|^2}{\| \tilde{f}_j \|^2}.$$

Proof. Properties a) and b) are obvious from the definition; d) follows from Proposition 4.5. Property c) follows from d). Likewise e) follows from the definition of K^+ in terms of K and the properties of K established above. Finally the same argument as was used at the end of the proof of Proposition 4.5 shows that $\frac{\tilde{f}_j(w)\overline{\tilde{f}_j(w')}}{\|\tilde{f}_j\|^2}$ is locally uniformly $\mathcal{O}(|s_j|^4)$. Since the f_j are orthogonal, and the L^2-norms decrease exponentially the convergence follows from the general theory. □

Theorem 4.6 represents the main result of this paper; the statements c) and f) are those which are most interesting.

5. Discussion. In this section we shall discuss some aspects of Theorem 4.6. First of all one should remark that we have only used the fact that we are dealing with metaplectic forms of order 3 in rather weak ways. The same argument can be repeated in other contexts. If we apply it to non-metaplectic forms then we discover that the kernel is in fact a theta-function associated with an indefinite form – cf. [14], [29]. This is interesting as it gives an alternative method for proving the transformation property of certain Siegel theta-functions, to the usual methods using Fourier theory and, in particular, the Poisson Summation Formula.

Curiously, in the case of metaplectic forms of order 2 one does not find classical theta functions, since the function q_2 arises in the kernel. However there is, at present, no analogue of the discussion of §2 in this case.

In all cases the argument given above is restricted to highest level. In order to prove results concerning functions of arbitrary level the argument will have to augumented by some technique to describe both the behaviour of the level of a form under the generalized Shimura correspondence and a corresponding theory of newforms.

In the formula of Theorem 4.5 one can split K^{+} into even and odd parts. The odd part is then quite attractive as neither the Eisenstein series nor the theta function θ plays any role. A rather more interesting transformation of the same type. Let $C : \mathbb{H}^3 \to \mathbb{H}^3$ be defined by

$$C((z,v)) = (\overline{z}, v).$$

Let $g \in GL_2(\mathbb{C})$. Then one verifies that

$$\overline{g}(C(w)) = C(g(w)),$$

where $\overline{g}$ is the complex conjugate of g, entry by entry. One verifies easily that $\chi(\overline{g}) = \chi(g)^{-1}$ for $g \in \Gamma_2$. One therefore concludes that $K^{+}(w, C(w'); w_0)$ is χ-automorphic in both w and w'. This allows us to put the formulae into a rather more symmetric form. It is worth noting in this context that certain quadratic forms in the Fourier coefficients of metaplectic forms seem to be significant. Here one notes first and foremost Waldspurger's results in [30]. However, also in the case of biquadratic (exceptional) forms one has similar statements; see [1] for a full conjectural statement and [27], [28] for those results at present available. In the case of 6-fold covers again one has a similar structure, as was discovered empirically by G. Wellhausen [33]; this remains entirely conjectured.

The formula of f) of Theorem 4.6 represents in principle a method of computing $\tilde{f}_j$. The integral can be investigated using the Rankin-Selberg method. This leads one to evaluating the residue of a sum in which the Fourier coefficients of f_j appear. Although some simplification are possible the sum remains a complicated one, involving Legendre functions, cf. [31], p. 387, Eq. (2), and the author has not, as yet, been able to gain any illumination from it.

One should however note that it is also conceivable to use the formula, either in its original form, or in the transformed version as a method of numerically computing the $\tilde{f}_j$. These functions are not at all understood, and it would be of considerable interest to study the rules of formation of the Fourier coefficients. One does have a form of Hecke theory which leads to relations between the different coefficients but these do not suffice to determine them. In numerical studies one has first to determine the f_j. Studies of the spectrum of the Laplace-Beltrami operator can be found in [7] and [26], the latter giving the more accurate tables. Unfortunately, just as in the case of Maaß wave forms for $SL_2(\mathbb{Z})$, there do not seem to be any "arithmetic" automorphic forms of level 1, for example, forms associated with a Größencharakter of a suitable quadratic extension of $\mathbb{Q}(\omega)$, but there are at least forms lifted from $SL_2(\mathbb{Z})$. This has been experimentally observed in [26]; the theoretical basis lies in the converse theorems of Jacquet-Langlands and Weil - see [9],Th.12.2, [32],p.163, although there is no information here about the levels of the lifts. In view of the results of [26] it seems clear that one should be able to prove that forms on $SL_2(\mathbb{Z})$ lift to forms on $SL_2(\mathbb{Z}[\omega])$. This particular class of forms should be especial interest in connection with the Shimura correpondence as it indicates that one may be able to apply the techniques of metaplectic groups to questions concerning automorphic forms over $\mathbb{Q}$ and not merely some extension. The invesigation of the arithmetical properties of the Shimura lifts of automorphic forms is the most important goals of the theory of metaplectic forms.

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