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DISCRETE SCHRODINGER OPERATORS AND TOPOLOGY*

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0. Introduction. During the last 3 years the present author made a series of works [1, 2, 3, 4, 5, 8, 6, 7, 9] dedicated to the study of the unusual spectral properties of low-dimensional continuous and discrete (difference) Schrodinger Operators. Some of these works were done in collaboration with A. Veselov, I. Taimanov and I. Dynnikov. First, let me briefly describe the list of problems discussed in these works.

1. Euler-Darboux-Backlund (EDB)-Transformations as nonstandard spectral symmetries for the 1-dimensional Schrodinger Operators and its discrete analogs on the lattice Z. Problem of cyclic chains, its solutions for the special cases. Exactly solvable spectral problems for some operators. EDB Transformations for the nonstationary Schrodinger Equation and the problem of cyclic chains (see [12, 13, 14, 15, 16, 2, 8, 17]).

2. Laplace Transformations for the 2D stationary Schrodinger operators in the double-periodic magnetic field and potential, acting on the space of eigenfunctions of one energy level. Problems of cyclic, semicyclic and quasicyclic chains. The possibility to have two exactly solvable highly degenerate energy levels as a maximal possible solvability for the spectral theory in the Hilbert space $L_2(R^2)$ (except the Landau case in constant magnetic field and trivial potential)? Discretization of the Laplace transformations: square lattice is compatible with hyperbolic equations; equilateral triangle lattice is compatible with elliptic selfadjoint operators. Exactly solvable operators. See [1, 2, 3, 4, 6].

3. The second order operators on simplicial complexes. Factorizations and Laplace Transformations. The cases of 2-manifolds with 2-colored triangulation and multidimensional equilateral lattices. Zero modes problem. First order equations in the simplicial complexes and nonstandard discretization of connections, combinatorial curvature. See [6, 7].

4. Schrodinger Operators on simplicial complexes. The combinatorial analog of Wronskians–the **Symplectic Wronskians or SWronskians** in our terminology; their topological properties. Special case of graphs with finite number of tails. Scattering Theory and Symplectic Geometry. See [5, 9].

This work is a direct continuation of [5, 9] (the idea was quoted in these papers communicated to the present author by I. Gelfand in 1971 as a reaction to the authors works [10, 11], where Symplectic Algebra was used for the needs of Differential Topology). We extend here the definition and topological properties of the Wronskians (Symplectic Wronskians or SWronskians) to the broad class of operators on the simplicial complexes.

1. Finite order selfadjoint combinatorial operators – symplectic wronskians and topology. Let us consider any locally finite simplicial complex K where any simplex belongs to the finite number of simplices only.

By definition, the **Distance between two simplices** of any dimensions $d(\sigma, \sigma')$ is equal to zero if and only if they coinside. It is equal to $1/2$ if and only if one of them

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belongs to the boundary of the other one. It is equal to $s/2$ if s is such a minimal number s that there exists a **simplicial path**, i. e. sequence of simplices

$$\sigma = \sigma_0, \sigma_1, \ldots, \sigma_s = \sigma'$$

with $d(\sigma_j, \sigma_{j+1}) = 1/2, j = 0, 1, \ldots, s-1$.

The Operators L of the order less or equal to k we define by the formula

$$L\psi(\sigma) = \sum_{\substack{\sigma' \\ d(\sigma,\sigma') \leq k/2}} b_{\sigma:\sigma'}\psi(\sigma') \tag{1}$$

Here $\psi(\sigma)$ belongs to some space of real or complex scalar-valued or vector-valued functions on the set of simplices.

The operator L is **Symmetric** iff $b^{ij}_{\sigma:\sigma'} = \bar{b}^{ji}_{\sigma':\sigma}$.

The operator is **Real** iff all coefficients b are real.

For the **Second Order Operators** we have exactly $d(\sigma, \sigma') \leq 1$ for the nontrivial coefficients $b_{\sigma:\sigma'}$. Some nontrivial coefficients should be such that $d = 1$ exactly. For the **Homogeneous Operators** of some order k we have $d = k/2$ for all nontrivial coefficients.

In the previous works we restricted our attention to the case where all nonzero coefficients $b_{\sigma_1^p:\sigma_2^s}$ are concentrated on the simplices of some specific dimensions p, s. In this case the operator maps the space of functions (or vector–functions) on the set of p-simplices into the space of functions (vector–functions) on the set of s-simplices:

$$L : C^p \to C^s \tag{2}$$

We call them **the operators of the type** p, s . The most interesting classes are as follows:

1. The second order selfadjoint (i. e. Schrodinger) Operators for $p = s$.

2. The first order operators of the type p, s. Especially interesting is the case $p \pm 1 = s$, but other cases also appeared before (see [6]).

The symmetric (hermitian) matrix-function $V(\sigma) = b_{\sigma:\sigma}$ will be called **Potential.** Let us consider the real operators acting on the l-component vector-valued functions $\psi(\sigma)$, where $\sigma \in K, \psi = (\psi^j) \in C^l, j = 1, \ldots, l$, and σ is a simplex of any dimension. The operator L acts on the space $C^* = \oplus_p C^p(K)$ where summation is extended to all dimensions (it is a full set of vector-valued cochains).

In the standard way we define a "baricentrical" subdivision of the simplicial complex K. We put new vertices (0-simplices) in the centers of all original simplices of all dimensions $k \geq 0$. After that new edges connect the centrum of every simplex with all new vertices located on its boundary. The new k-simplices of any dimension are exactly the cones looking from the centers of the old simplices into the new $k-1$-simplices already constructed by the induction on the boundary. We denote the baricentrical subdivision of the simplicial complex K by K'.

Consider now any real symmetric operator $L : C(K) \to C(K)$ of the order k, acting on the space of all vector-valued cochains.

Any such operator can be treated as an Operator $L' = L$ of the type (0,0) acting on the zero-dimensional cochains in the baricentrical subdivision K' :

$$L' : C^0(K') \to C^0(K')$$

Take any pair of solutions for the equation

$$L'\psi = \lambda\psi, L'\phi = \lambda\phi$$

For every pair of vertices $\sigma\sigma' \in K'$ fix a unique naturally oriented path $l(\sigma, \sigma')$ (i. e. 1-chain $[l]$) such that $\partial[l(\sigma, \sigma')] = \sigma' - \sigma$. Let for conveniency this path be the one of the minimal lengh. For the cases $d(\sigma\sigma') \leq 1$ such a path is unique. It is always unique for any pair of vertices in every simply-connected Graph (tree). It is also unique for the pairs of vertices if the distance between them is small enough: $d(\sigma, \sigma') < 1/2d_0$ where d_0 is a size of the smallest 1-cycle, $d_0/2$ is the number of edges in it.

DEFINITION 1. *The Symplectic Wronskian (SWronskian) for the pair of solutions for the operator L' of the type $(0,0)$ in any simplicial complex K' is a one-dimensional (possibly infinite) simplicial chain $W(\psi, \phi)$ in the complex K' defined by the formulas below:*

$$W(\psi, \phi) = \sum_{\sigma\sigma'} W_{\sigma\sigma'}(\psi, \phi) \tag{3}$$

$$W_{\sigma\sigma'} = \sum_{ij} b^{ii}_{\sigma:\sigma'}\{\psi^i(\sigma)\phi^j(\sigma') - \phi^j(\sigma')\psi^i(\sigma)\}[l(\sigma\sigma')]$$

For the locally finite complex and finite order operator L' this sum makes sense as an infinite chain in this complex. **We consider the operators L acting on the simplices of any dimension in the complex K as the operators L' acting on the vertices of the baricentrical subdivision K'. Therefore we defined the SWronskians for all selfadjoint real operators of any finite order $k \geq 1$ acting on the spaces of vector–valued functions on the set of simplices of all dimensions.**

THEOREM 1. *The Symplectic Wronskian (SWronskian) defined above as a C-valued finite or infinite 1-chain in K' is in fact an open cycle, i. e. $\partial W = 0$. This cycle is a bilinear skew-symmetric functional of the pair of solutions for the equation $L\psi = \lambda\psi, L\phi = \lambda\phi$.*

REMARK 1. *Let any solution $L\psi = \lambda\psi$ be given describing in the sense of Quantum Mechanics the stationary state of electron, living in the simplicial complex K with Hamiltonian L and energy λ. This state defines a* **Quantum Current** *$J(\psi) = W(\psi, \bar{\psi})$ along the arcs in K' satisfying to the* **Kirchhof Law** *in every vertex.*

Proof of the theorem. Consider the expression $\sum_i \phi^i(\sigma)(L\psi)^i(\sigma) - \psi^i(L\phi)^i(\sigma)$ for the pair of vector-functions. We can easily see that all zero order terms containing $b_{\sigma:\sigma}$ disappear from this expression obviously for the real operators.
For any vertex σ of the complex K' we should consider all 1-simplices of K' meeting each other in the vertex σ. By definition of the Wronskian, we have

$$(\partial W)_\sigma = \sum_{\sigma''} W_{\sigma\sigma''}$$

where either σ is a nontrivial face of σ'' or vise versa, i. e. $d(\sigma, \sigma'') = 1/2$. At the same time, $(L\psi)_\sigma = \sum_{\sigma'} b_{\sigma:\sigma'}\psi(\sigma'$. Canceling from the expression $\phi(\sigma)L\psi(\sigma) - \psi(\sigma)L\phi(\sigma)$ all zero order terms, we group others in such a way that our expression looks as a sum of the "elementary Wronskians" $\sum_{\sigma''} W(\psi, \phi)_{\sigma\sigma'}$.

After that we memorize that ψ, ϕ are in fact the solutions for the equation $L\phi = \lambda\phi, L\psi = \lambda\psi$, so our expresssion is equal to zero. Theorem is proved.

COROLLARY 1. *Let K is a Graph, i. e. $dimK = 1$. For any second order operator L acting on the full space of vector-valued cochains $C = C^0 \oplus C^1$ and any pair of solutions ψ, ϕ for the spectral problem, their SWronskian is an open cycle (i. e. open homology class) in the same Graph*

$$W(\phi, \psi) \in H_1^{open}(K, C)$$

Proof. For graphs every simplicial 1-cycle in K' is in fact a simplicial 1-cycle in the original graph K.

REMARK 2. *Let us point out that we already proved and used this observation for the scattering theory on the graphs–see [5, 9]. However, in these works we considered strictly homogeneous second order operators only, acting on the spaces of vertices $C^0 \to C^0$ or edges $C^1 \to C^1$ separately. We also defined in [9] the SWronskians for the higher order operators acting on the space of vertices and SWronskians for the strictly homogeneous second order operators on the simplices of every fixed dimension. Here we extend the class of admissible operators. In particular we may work with operators $L : C^* \to C^*$ mixing cochains of the different dimensions.*

All previous authors' definitions of the Wronskians as a symplectic (skew-symmetric bilinear) vector–valued 2-forms are the partial cases of this one.

For any simplicial complex K there is a famous selfadjoint first order operator $L = d + d^* : C(K) \to C(K)$ where $d = \partial^* : C^k \to C^{k+1}$ and $d^* = \partial : C^k \to C^{k-1}$ for every value of the dimension k. Its square is a direct sum or the Laplace -Beltrami Operators $\Delta_k = dd^* + d^*d : C_k \to C_k$. For the finite complexes zero modes of the operators L, Δ give certain "Harmonic" basis for the Homology (Cohomology) Groups $H_k(K, R)$. Both these Operators are selfadjoint. They are the Euler-Lagrange operators for the quadratic functionals:

$$S_\Delta(\psi) =< \psi, \Delta\psi >=< d\psi, d\psi > + < d^*\psi, d^*\psi > \tag{4}$$
$$S_L(\psi) =< \psi, (d + d^*)\psi >$$

In the elasticity theory for the isotropic media the linear combinations appear $\lambda dd^* + \mu d^*d$ acting on 1-forms, where λ, μ are the Lame' parameters (in the continuous case).

REMARK 3. *For the zero modes of the Laplace-Beltrami Operators Δ_k on the finite simplicial complexes we can easily prove that their SWronskian is always identically equal to zero.*

EXAMPLE 1. *In the works [2, 4, 6, 7] factorizations and Laplace Transformations were considered on the 2-colored (black and white) triangulated two-manifolds M^2 for the different classes of Schrodinger Operators. In the case of vertices we consider the operators $L : \psi(P) = \sum_{P'} b_{P:P'}\psi_{P'}$ where P' is such that $d(P, P') = 1$. These real selfadjoint operators can be factorized in the Laplace-type ("weak") form $L = QQ^t + V$ where V is a "potential", i. e. multiplication by the real function, and $Q^t : C_0 \to C_2^{black}$. It means that this first order operator Q maps functions on the set of vertices into the functions on the set of the black triangles. Such an operator is defined by the set of all coefficients $c_{P:T}$ where T is a black triangle and P is one of its vertices. So* *the simplicial complex K in this case is M^2 minus white triangles. It has the same vertices and edges as M^2, but twice less number of triangles. In the case $V = 0$ the ground level (if it is equal to zero), can be found from the square integrable solutions for the first order* **Triangle Equation** *$Q^t\psi = 0$. Especially interesting is the classical case of the equilateral lattice Z^2 considered as a triangulation of R^2.*

For the Graphs $K = \Gamma$ several examples were considered in the work [9], especially for the graphs with finite a number of infinite tails. We shall come to this later in connection with the Scattering Theory.

Let us consider here the special case of the discretized line with vertices numerated by the even numbers $2n = \sigma_n^0, n \in Z$ and edges numerated by the odd integers $2n+1 = \sigma_n^1 = [2n, 2n+2], n \in Z$. So we have a lattice Z' of the integers as a complex K'. The operator L in K determines the operator L' in K' as a $(0,0)$ type one:

$$(L'\psi)^i(n) = \sum_{j,s} b^{ij}_{n,n+s}\psi^j(n+s), -k \leq s \leq k$$

We choose a basis C_m of the solutions

$$C^i_{m;p}, i = 1\ldots, l, m \in Z, p = -k+1, -k+2, \ldots, k-1, k$$

in the form:

$$(C^i_{m;p})^j(m+s) = \delta^{ij}\delta_{ps} \tag{5}$$

Let us compute the Symplectic Wronskian form in this important case. This form is a scalar-valued skew-symmetric bilinear form because there is only one basic geometrical cycle, the oriented line itself.

THEOREM 2. *The Symplectic Wronskian form written in the basis C_m of the solutions $C^i_{m;p}$ for any given integer m admits two k-dimensional Lagrangian Planes $L_\pm$ (i. e. this form is equal to zero on these planes), with basises $C^i_{m;p} \in L_+$ for $p = -k+1, \ldots, 0$ and $C^i_{m;p} \in L_-$ for $p = 1, \ldots, k$. For the SWronskian scalar product between these two planes we have*

$$W(C^i_{m;p}, C^j_{m;q}) = 0, q - p > k, p \leq 0, q \geq 1$$

$$W(C^i_{m;p}, C^j_{m;q}) = b^{ij}_{m+p,m+q}, q - p \leq k$$

In particular, this form is nondegenerate if and only if the Operator L' has everywhere nondegenerate higher order terms

$$\det b^{ij}_{n,n+k} \neq 0$$

for every $n \in Z$.

Proof. We can easily verify the form of this matrix from the definition of the basis and SWronskian above. The matrix of SWronskians can be considered naturally as a number-valued one because there is only one canonical open geometrical 1-cycle on the line $R = K = K'$ oriented in the direction of $n \to +\infty$. We compute the value of SWronskians $W(\phi, \psi)$ for all solutions from our basis on the 1-edge [01]. Therefore only those pairs of vertices should be considered which contain the segment [01]. Otherwise the elementary SWronskian for the pair of vertices would not contain

[01]. It means in particular that we may have a nonzero SWronskian in our basis only between the subspaces L_+ and L_-. The value of the spectral parameter λ does not affect this matrix in the given basis. Let us point out that the matrix SW_m of SWronskians $W(L_+, L_-)$ in our basis is a block-triangle one, where the matrices $b_{m+p;m+p+k}, p = -k+1, \ldots, 0$ are positioned along the diagonal. We have zero SWronskians below this block-diagonal part. Therefore the determinant of this matrix is a product

$$\det SW_m = \prod_{p=-k+1}^{p=0} (\det b^{ij}_{m+p,m+p+k})^2$$

This implies the nondegeneracy of the Symplectic form given by the SWronskians. Therefore our theorem is proved.

The SWronskian form is equal to constant along the line according to the theorem 1. We have following

COROLLARY 2. *The Evolution Map $T_{m,m+1}(\lambda)$ from the basis C_m to the basis C_{m+1}, given by the equation $L\psi = \lambda\psi$ is a Linear Symplectic Transformation.*

This theorem, of course, is very clear. It is valid also for the nonlinear systems, as we shall see later. Its continuous analog has been known many years. However, even in the continuous case there was some difficulcy in finding the canonically adjoint ("Darboux") basis, following the "Ostrogradski Transformation" for the vector-valued higher order variational problems (B. Deconinck pointed this out to me). We don't try to find a canonical basis, but nondegeneracy of the Symplectic form is very easy in our case. The nonlinearity is unimportant in these problems. Probably, no one considered the discrete variational problems in classical mathematics. This business was used in the Theory of Solitons for the discrete linear second order systems in the theory of "Toda Lattice" and "Discrete KdV" since the works [18, 19, 20]. It was started for the second order nonlinear systems in [21, 22]. For the discrete systems of higher order we do not know any literature. Our main idea is that for the graphs and simplicial complexes instead of line as a time we have a Symplectic form taking values in the linear space of the open one-cycles $Z_1^{open}(K', C)$.

Consider now any Graph Γ which is presented as a regular Z-covering over the finite Graph Γ_1 with free simplicial action of the monodromy group Z generated by the map F :

$$P : \Gamma \to \Gamma_1, F : \Gamma \to \Gamma, FP = PF$$

THEOREM 3. *Any Operator of the finite order in the Graph Γ with free Z-action and finite factor can be presented as a higher order vector-valued operator on the discretized line-lattice Z. The operators with Z-invariant coefficients (i. e. operators whose coefficients are coming from the Graph Γ_1) will be presented as operators with constant coefficients on the discretized line.*

We call this presentation a **Direct Image** of the operator on the lattice Z.

Proof. For the proof, we construct a map $f : \Gamma \to R$ commuting wth free Z-action. It certainly exists. First of all we choose "initial vertices" in one-to-one correspondence with vertices of the factor-space Γ_1. It is good to choose them in the "Fundamental Domain" of the minimal size for the group Z in Γ, starting from any initial vertex. We map all these initial vertices into $0 \in Z$. After that we map all other vertices following the group action. The continuation to the 1-skeleton of Γ is easy: for any edge its boundary vertices already mapped into R. The linear continuation is unique. It might happen that image of the edge is an interval $n, n+k$ where $k > 1$. Therefore the original edge should be subdivided in k parts. After that we have a simplicial Z-invariant map. Any function on the vertices of the Graph Γ can be naturally and tautologically presented as a vector-valued function ψ_n on the vertices of the discretized line Z with the number of components of vector ψ_n numerated by the vertices from $f^{-1}(n) \in \Gamma$. After this presentation of the functional space, we can see that the same operator looks as an operator of finite order on the lattice Z. Theorem is proved.

2. Scattering and Symplectic Geometry. As it was pointed out in paragraph 1, any combinatorial Schrodinger Operator L of the order k, acting on the functions on the set of simplices of any dimension, can be considered as an operator L' of the order $2k$ acting on the vertices of the baricentrical subdivision K'. Therefore it depends on the 1-skeleton of the complex K' only. So we shall consider any higher order real selfajoint operator L acting on the vertices of the Graph Γ. For the Scattering Theory we need to consider the following picture:

1. Our Graph Γ has a finite number N of "tails" (i. e. subgraphs $z_j, j = 1, \ldots, N$, isomorphic to the "half" of the special "line-like" graphs K_j–graphs with free action of the group Z generated by the map F_j, and finite factor K_j/Z). In particular, the map F_j is well defined far enough into the tail. After removal of the tails, what remains is a finite subgraph Γ'.

2. All coefficients of the Operator L rapidly enough tend to constants (i. e. F_j-independent) in every tail $z_j, j = 1, \ldots, N$. So, far enough in every tail z_j we have an operator L_j^{as} with asymptotically constant coefficients. The vertices in every tail z_j are numerated by the positive integers $n \geq 0$ and by the finite number of vertices of the factor-graph K_j/Z. The map $f_j : K_j \to R$ is given of the tail into the discretized line (see the end of paragraph 1), commuting with the action of Z. Therefore our operator far enough in the tail is presented as an operator on the discretized line.

DEFINITION 2. *The solution ψ for the equation $L\psi = \lambda\psi$ belongs to the* **spectrum** *of the operator L in the Hilbert Space $L_2(\Gamma)$ of the square integrable complex vector-functions on the Graph iff its growth in the tails is less than exponential, i. e. there exists a number s such that $|\psi_{j,n}| < n^s$ for all tails z_j and $n \to +\infty, n \in Z$.*

The solution ψ belongs to the **discrete spectrum** *of the operator L iff $\sum_{\sigma \in \Gamma} |\psi(\sigma)|^2 < \infty$. In particular, it is sufficient to require that $\sum_{j,n} |\psi_{j,n}|^2 < \infty$ for all tails z_j. The eigenfunction is* **singular** *iff it is equal to zero in all tails.*

For the operators L_j^{as} with constant coefficients we describe all solutions through the one symplectic matrix $T_j = T_{j;n,n+1}^{as}$ defined in paragraph 1. This matrix expresses the basis C_n through the basis C_{n+1} in the neighboring point. We obviously have the discrete evolution for any $n < m$ where $T_{j;n,m}^{as}$ depends on $m - n$ only for the operators with constant coefficients:

$$T_{j;m,n}^{as} = T_{j;n,n+1}^{as} T_{j;n+1,n+2}^{as} \cdots T_{j;m-1,m}^{as} = T_j^{m-n}$$

Therefore the eigenvalues $\mu_{j;r}(\lambda)$ of the matrices $T_j(\lambda)$ in the tail z_j determine the asymptotic properties of the eigenfunctions in the tails except of the "singular part" nonvisible from the tails. The structure of operator L inside of the graph leads to the algebraic relations between the tails.

According to the modern textbook literature (see [23],paragraph 4), the eigenvalues of the **generic** real one-parametric λ-family of the symplectic matrices $T_j(\lambda)$ are crossing in the isolated points $\lambda_* \in R$ the so-called "codimension 1 degeneracies":

Path 1. It may have a pair of Jordan blocks of length 2 corresponding to the pair of real eigenvalues $\mu_1 = (\mu_2)^{-1} \neq \pm 1$;

Path 2. It may have a pair of Jordan blocks of length 2 corresponding to the pair of unimodular complex eigenvalues $\mu_1 = \bar{\mu}_2 \neq \pm 1$;

Path 3. It may have a unique Jordan block of length 2 corresponding to the eigenvalue $\mu_1 = \pm 1$.

All other eigenvalues remain simple during these processes.

Let us remind that the eigenvalues of any symplectic matrix are invariant under the complex conjugation $\mu \to \bar{\mu}$ and inversion $\mu \to \mu^{-1}$.

Therefore the symplectic $2M \times 2M$-matrix $T_j(\lambda)$ has in the generic point λ_* of the λ-line:

I. some number s of the nonmultiple unimodular eigenvalues $|\mu_i| = 1$ in the upper halfplane $Im(\mu_i) > 0$ (and their complex adjoint), not equal to the ± 1.

II. $2p$ nonmultiple nonreal eigenvalues inside of the unit circle $|\mu| < 1$ and the same number outside of the unit circle.

III. q nonmultiple real eigevalues inside of the unit circle and the same number outside of the unit circle.

So we have the total dimension $2M$ equal to $2M = 2s + 4p + 2q$.

EXAMPLE 2. *a) For the second order scalar operators or first order 2-vector-valued operators we have $2M = 2$. Therefore $p = 0$. We have either $s = 1$ or $q = 1$. In the isolated points λ_* our generic family is passing through the Jordan block of length 2 with eigenvalue ± 1.*

b) For the case $2M = 4$ we may have for p, q, s the following possibilities:

$$(p,q,s) = (1,0,0); (p,q,s) = (0,2,0); (p,q,s) = (0,1,1); (p,q,s) = (0,0,2)$$

In the isolated points this family may pass through the Jordan blocks of the types and multiplicities described above corresponding to the multiple eigenvalues on the unit circle, on the real line or in the special points ± 1. For the Paths 1-3 we have:

Path 1 transforms (p,q,s) into $(p+1, q-2, s)$ or vice-versa; two real eigenvalues collide with each other inside of the unit circle and transform into the complex adjoint pair or inverse process. **This process is unimportant for the Scattering Theory**. Only Paths 2 and 3 where the number s changes are important for the Spectral theory in the Hilbert Space $L_2(\Gamma)$.

We do not see any spectral singularity in the point λ_* critical for Path 1.

Path 2 transforms (p,q,s) into $(p+1, q, s-2)$ or vice-versa; two unimodular eigenvalues collide with each other in the upper part of the circle and transform into the pair inside and outside of this circle or inverse process. The structure of the Continuum Spectrum may be drastically changed in this point.

Path 3 transforms (p,q,s) into $(p, q+1, s-1)$ or vice-versa; two real eigenvalues collide with each other in the point ± 1 and transform into the unimodular complex adjoint pair or inverse process. This path also changes the structure of the spectrum.

We assume that the family $T_j(\lambda)$ is generic in the sense described here, for all tails z_j. Let us point out that our families $T_j(\lambda)$ have very special λ-dependence. Therefore this assumption in fact should be verified in the future for the generic operators with constant coefficients. It is certainly true for the second order scalar operators–it is almost obvious and was used in the literature many times. For the higher order operators and matrix operators we shall return to this in the later publications.

DEFINITION 3. *The solution ψ for the equation $L\psi = \lambda\psi$ is a point of the* **regular discrete spectrum** *iff in every tail z_j it belongs asymptotically to the linear span of the eigenspaces corresponding to the eigenvalues of $T_j(\lambda)$ inside of the unit circle for every j.*

Let us consider now the special important case:

All asymptotic operators L_j^{as} coinside with each other $L_j^{as} = L^{as}$;

For the asymptotic operator L^{as} there is a nonempty interval $[\lambda_0, \lambda_1] \subset R$ on the λ-line such that for all $\lambda \in [\lambda_0, \lambda_1]$ the corresponding matrix $T_j = T(\lambda)$ belongs to the case where $s > 0$. Here $M = kl$, where $2k$ is an order of L and l is its vector dimension.

Let H^{as} be a direct sum of the Hamiltonian (Symplectic) Spaces corresponding to the different tails:

$$H^{as} = \oplus_{j=1}^{j=N} H_j^{2kl}$$

with natural skew-symmetric scalar-valued nondegenerate product defined by the SWronskians in every tail z_j. For every solution ψ for the equation $L\psi = \lambda\psi$ on the whole graph Γ we have its **asymptotic value**:

$$\psi \to \psi^{as} \in H^{as}$$

where $\psi_j^{as} \in H_j^{2kl}$ is this solution in the tail z_j.

THEOREM 4. *The subspace $L^{as} \subset H^{as}$ of the asymptotic values for all solutions with given value of the spectral parameter λ, is a Lagrangian subspace of the half dimension equal to Nkl (i. e. the SWronskian scalar product is identically equal to zero in it, $< L^{as}, L^{as} >= 0$.)*

Proof. This Theorem appeared the first time in the work [5] for the special cases. The general proof is more or less the same as in this special case. Essentially the property of the asymptotic plane to be Lagrangian is a Topological Phenomenon, following directly from the fact that the SWronskian is a cycle. For any pair of solutions for the equation $L\psi = \lambda\psi, L\phi = \lambda\phi$ on the whole graph Γ we have a cycle of the form:

$$W(\phi, \psi) = \sum_{j=1}^{j=N} a_j z_j + (finite)$$

where z_j is a tail as a geometric cycle near infinity. Let me remind that far enough in the tail our operator is presented as one on the discretized line. However, only differences can be extended to the cycles on the whole graph. Therefore we can express our SWronskian through the differences only:

$$W(\psi, \phi) = \sum_{t=2}^{t=N} b_t(z_1 - z_t) + (finite)$$

Comparing these formulas, we see that

$$< \psi^{as}, \phi^{as} >= \sum_j a_j = 0$$

by the definition of the scalar product $<, >$ in the space H^{as}.

It is easy to see that the plane L^{as} of the asymptotic value of the solutions extended to the whole graph Γ is given by the number of equations equal to the half of the dimension of the space H^{as}. At the same time, we established the fact that this plane is Lagrangian (i. e. the scalar product in it is equal to zero). The dimension of Lagrangian plane is always less or equal to the half. Therefore it is equal to half exactly. Our theorem is proved.

Let us point out that the complexified asymptotic space H^{as} in any noncritical real point λ has natural direct decomposition (with scalar product of different parts equal to zero):

$$H^{as} = H_+ \oplus H_- \oplus H_{bounded}$$

where the subspaces are defined in the following way:

Subspace H_- has dimension $(2p+q)N$. It contains all asymptotic solutions with decay in every tail $z_j, j = 1, \ldots, N$ for $n \to +\infty$;

Subspace H_+ contains the solutions corresponding to the eigenvalues of the asymptotic monodromy matrix $T(\lambda)$ outside of the unit circle, $|\mu| > 1$; they are increasing for $n \to \infty$ in every tail. The dimension of this subspace is also $(2p+q)N$;

Subspace $H_{bounded}$ of the dimension $2sN$ corresponds to the unimodular eigenvalues; after complexification there is a natural decomposition

$$H_{bounded} = H_{in} \oplus H_{out}$$

on the waves $\psi_{j,in}$ and $\psi_{j,out}$ coming inside and outside correspondingly in the tail z_j. It means precisely that the **in**-part corresponds to the eigenvalues μ of the monodromy matrix T with positive real parts and the **out**-part is complex adjoint. We have vector-functions $\psi^i_{j,in} = \bar{\psi}^i_{j,out}$ for the real λ, such that for the different indices i they correspond to the different eigenvalues in the same tail z_j, and have zero symplectic scalar product,

$$< \psi^i_{j,in}, \psi^t_{j,out} >= a(\lambda)\delta^{it}, a \neq 0$$

DEFINITION 4. *We call the interval on the real line* **generic and nonsingular** *if the following requirements are satisfied:*

it does not contain critical points (i. e. the numbers (p, q, s) are not changing in it, and all eigenvalues are nonmultiple);

the intersection of the Lagrangian Plane $L^{as}(\lambda)$ with the subspace $H_- \oplus H_{bounded}$ has the dimension exactly equal to sN;

the projection of this intersection on the subspace generated by the vectors $\psi^i_{j,in}$ for all j, i should be 'onto' (after complexification).

For the **Generic Operators** the spectrum consists of such intervals separated by the isolated points which should be passed transversally in the natural sense (see above their Jordan structure, but we require transversality for the interaction of the different tails also). **Obviously, for the real big enough $|\lambda|$ we always have $s = 0$.**

Therefore there is a finite number of finite intervals with nonzero values of $s > 0$ only.

DEFINITION 5. *Let the Scattering Matrix $S^{j',i}_{j,i'}(\lambda)$ for any generic nonsingular interval on the λ-line be defined using the complex basis of the intersection*

$$L^{as}(\lambda) \cap (H_- \oplus H_{bounded})$$

taken in the form

$$e_j^i = \psi_{j;in}^i + \sum_{j',i'} s_{j,i'}^{j',i} \psi_{j';out}^{i'} modulo(H_-)$$

THEOREM 5. *The Scattering Matrix S defined above is a Unitary Symmetric Matrix for the real generic nonsingular values of λ.*

The proof of this theorem is parallel to the special case of second order operators (see [9]).

As it was written already in [5], it follows directly from the Lagrangian property of the plane $L^{as} \in H^{as}$. Take the basis $\psi_{in} + S\psi_{out}$ in the complexification of this plane for real λ. Different vectors of this basis have a zero scalar product with each other. This property implies that the matrix S is symmetric $S^t = S$. From the reality we have $\bar{\psi}_{out} = \psi_{in}$ and

$$\phi = \psi_{out} + \bar{S}\psi_{in}$$

is complex adjoint to the previous basis. The basis

$$\bar{S}^{-1}\phi = \psi_{in} + \bar{S}^{-1}\psi_{out}$$

is coinside with the first one. Therefore we have $\bar{S}^{-1} = S$ and $S^t = S$. One may think that we took a real basis on the Lagrangian plane in the form

$$A\psi_{in} + \bar{A}\psi_{out}, S = A^{-1}\bar{A}$$

It follows from the Lagrangian property that A can be taken as a unitary matrix $A \in U_{klN}$. By unitarity, we have $\bar{A}^t = A^{-1}$ and $S = BB^t, B = A^{-1} \in U_{Nkl}$. Multiplying the matrix B from the right by the arbitrary real orthogonal matrix $B' = BO$, we see that

$$B'(B')^t = BOO^tB^t = BB^t$$

Therefore the Scattering Matrix S depends on the Lagrangian Plane only. This plane may be identified with a point in the space U/O.

So, the proof is exactly the same as in [9] for the Strongly Stable Case where $s = M, p = q = 0$. For the general case with $s > 0$ we have to use the fact that the SWroskians of any vector in the subspace H_- with themselves and with any vector from the subspace $H_{bounded}$ are identically equal to zero. It is completely obvious because any eigenfunction from the subspace H_- is exponentially decreasing far enough in the tail. Therefore this additional term in the definition of the basis above for $S - matrix$ is completely negligible. Theorem is proved.

REMARK 4. *For the case $p + q > 0$ we may meet a new type of singularities where the projection of the intersection of the Lagrangian Plane $L^{as}(\lambda)$ with subspace $H_{bounded} \oplus H_-$ into the space $H_{bounded}$ has a rank smaller than ks (here k is a number of tails). This case corresponds to the discrete spectrum drawn in the continuous one.*

Appendix A. Nonlinear discrete systems on graphs (by S. Novikov and A. Schwarz). As already mentioned in paragraph 1, the Symplectic Geometry of Discrete Second Order Lagrangian Systems on the discretized line R (i. e. on the lattice Z) was started in work [21, 22] (the pioneering work of Aubrey is quoted in [21, 22] where specific important example, the so-called "Frenkel-Kontorova model", was investigated).

It was explained at the end of paragraph 1 and in paragraph 2 how to extend this construction to the higher order linear systems on the discretized line and on the general Graphs. Let us discuss here **Nonlinear Discrete Lagrangian Systems on Graphs**. Consider as before any locally finite Graph Γ presented as a 1-dimensional simplicial complex without ends (i. e. any vertex belongs to at least two edges). Suppose the following data are given:

Family of manifolds M_P^l numerated by the vertices $P \in \Gamma$;

Family X of the sets Q of vertices $P_j \in Q$ such that the maximal distance $d_{\max}(P_i, P_j)$between the vertices in any set Q is equal to D; normally this family contains exactly all "maximal" sets of the perimeter D containing all minimal paths between two vertices if the ends belong to Q; it should not contain any minimal paths longer than D, and any minimal path in it should be extendable to the path of the length D;

Family of C^∞-functions (the **Density of Lagrangian**)

$$\Lambda_Q : \prod_{P_j \in Q} M_{P_j} \to R$$

Using this data, we define an **Action** for any function ψ on the set of vertices such that $\psi(P) \in M_P$:

$$S\{\psi(P)\} = \sum_{Q \in X} \Lambda_Q(\psi(P_{i_1}, \ldots), P_i \in Q$$

For the infinite graphs this sum often does not exist, but we define the **Euler-Lagrange Equation** in the standard way:

$$\frac{\delta S}{\delta \psi(P)} = \frac{\partial S}{\partial \psi_P} = 0$$

Therefore only the sets Q containing the point P are involved in the calculation of the last variational derivative (which is an ordinary partial derivative for the discrete systems). We call the union of the sets Q containing the vertex P a **Combinatorial Neighborhood U_P of the Point P of the order D.**

There are different possibilities here:

I. The equation above is sufficient to express the function $\psi(P)$ through the values $\psi(P_j)$ in the neighboring points $P_j \in U_P$. This situation looks typical for **Elliptic-Type Problems** like the Dirichlet Boundary Problem and so on. For example, if the manifolds $M_{P_j}^l$ are compact for all vertices, we may take a minimum. We can do this also in many cases if all functions Λ_Q are nonnegative (or bounded from below).

II. The equation above is sufficient for the **Nondegenerate expression** of ψ_{P_j} in any point P_j on the boundary of the combinatorial neighborhood U_Q through other points in the combinatorial neighborhood U_P, where $d(P, P_j) = D$. This situation we call **Dynamical** . In some cases, beginning from the property of the last type, we may define also the Hyperbolic Type. Let me point out that the **Dynamical situation was considered in paragraph 1 for the linear systems**: in this case the nondegeneracy of the Symplectic Form, generated by the SWronskian, was proved.

THEOREM 6. *Let the nonlinear Discrete Euler-Lagrange System and its solution $\psi(P)$ be given. Consider the linearized self-adjoint operator L near the solution ψ and two solutions for the equation:*

$$L(\delta\psi_a) = 0, a = 1, 2$$

The SWronskian $W(\delta\psi_1, \delta\psi_2)$ defines a closed differential vector-valued 2-form SW with values in the space $H_1^{open}(\Gamma, C)$, on the space of solutions for the Nonlinear Discrete Euler-Lagrange System above[1]. *For the discretized line this form is nondegenerate for the nondegenerate Dynamical Type Systems.*

For the second order translation invariant systems (see below) on the discretized line our theorem follows from work [21, 22]. As A. Veselov pointed out to me, for the higher order translationally invariant systems on the discretized line this theorem also can be extracted from [21, 22]-see the article in Russian Math Syrveys, pp 6-7. We shall publish full proof of this theorem in separate paper.

DEFINITION 6. *We call the Discrete Action S and the Variational Problem above* **The Second Order Translation Invariant Problem** *in any Graph if all manifolds M_P^l are equal to the same manifold M, all sets Q contain the same number of points equal to two $D = 2$, and all functions Λ_Q are equal to the same function $\Lambda(P_1, P_2)$ of two variables (i. e. defined in $M \times M$).*

REMARK 5. *We can define the Translation Invariant Systems of any order for the discretized line–lattice Z. In the case of order four we can define them for the locally homogeneous Graphs, where all vertices meet the same number of edges equal to m. The function Λ_Q for every set Q has $m+1$ variables, i. e. it maps $M \times M \times \ldots M$ into R.*

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[1] More explicit definition of this form and proof of theorem can be found in the work published by the authors in Russian Math. Surveys, 54:1 (1999).