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BREAKDOWN OF A SHALLOW WATER EQUATION*

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1. Introduction. The 1-dimensional shallow water equation:

1) $$\frac{\partial v}{\partial t} + v\,\frac{\partial v}{\partial x} + \frac{\partial p}{\partial x} = 0 \quad \text{with "pressure"}$$

2) $$p(x) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|}\left(v^2 + \frac{1}{2}v'^2\right)dy$$

was noted by FOKAS-FUCHSTEINER [1981] as being formally integrable. CAMASSA-HOLM [1993] rediscovered it from a hydrodynamical point of view and developed it to a large degree, both there and in CAMASSA-HOLM-HYMAN [1994]. Their most striking discovery is that it has (peaked) solitons $v(t,x) = \frac{1}{2}\,p\,e^{-|x-q-pt/2|}$ with constant q and p, capable of being superposed in an elementary way, as sums $v(t,x) = \frac{1}{2}\sum_{i=1}^{n} p_i(t)\,e^{|x-q_i(t)|}$, where now the q's and p's obey the equations $q^\bullet = \partial H/\partial p$, $p^\bullet = -\partial H/\partial q$ of the geodesic flow in the co-tangent bundle of $\mathbb{R}^n$, equipped with the inverse metric tensor $\frac{1}{2}e^{|q_i-q_j|} : 1 \le i,\ j \le n$, *i.e.*, with Hamiltonian $H = \frac{1}{2}|p|^2 + \frac{1}{4}\sum p_i\,p_j\,e^{-|q_i-q_j|}$. From this, it is only a little jump to the realization that the full shallow water flow in, *e.g.*, $C_\downarrow^\infty(\mathbb{R})$ is just a transcription of the geodesic flow in the group $D(\mathbb{R})$ of (smooth proper) diffeomorphisms of the line, equipped with an analogous Riemannian geometry; see MISIOLEK [1998] for the more general case $v \in \mathbb{R} + C_\downarrow^\infty(\mathbb{R})$ with the attendant BOTT-VISASORO group, and compare ARNOLD-KHESIN [1998] for background and other illustrations of the general principle involved. Not much of that is needed here: it will suffice to observe a) if $v(t)$ obeys the shallow water flow in, *e.g.*, $C^\infty(\mathbb{R})$ and if $Q(t,x)$ solves $Q^\bullet = v(t,Q)$ with $Q_0(x) \equiv x$, then Q is a diffeomorphism of the line[1], imitating the identity near $x = \pm\infty$, and obeying the geodesic flow in $D(\mathbb{R})$, and b) the shallow water flow, expressed by means of $m = v - v''$ as in $*$) $m^\bullet + (mD + Dm)\,v = 0$, has first integrals in the form $m(Q)\,Q'^2(x) \equiv m_0(x)$.[2] $*$) shows that $m(Q)$ retains the general shape of $m_0(x)$, independently of $t \ge 0$, illustrating the utility of the diffeomorphism for the study of the flow; indeed, it is the principal tool for the question posed here: *When, and if so how, does the shallow water flow breakdown?*

Now, unlike KdV, breakdown is common enough as already noted in CAMASSA-HOLM [1993]: for example, if $\int_{-\infty}^{\infty}(v^2 + v'^2)\,dx \equiv H < \infty$, this being a constant of motion, then, from 5) below in the form $\frac{d}{dt}\,v'(Q) \le \frac{1}{2}v^2(Q) - \frac{1}{2}\,v'^2(Q)$, you see that $v_0'(x_0) < -\sqrt{H}$ at any place $x_0 \in \mathbb{R}$ implies $v_0'(t,x_0) \downarrow -\infty$ at some finite time.

To fix ideas, let us suppose that either $v_0 \in C^\infty(\mathbb{R})$ is of period 1 or else $v_0 \in \mathbb{R} + C_\downarrow^\infty(\mathbb{R})$, the latter being the most interesting case physically, and let us also simplify life by requiring that $m \equiv v_0 - v_0''$ has only a finite number of (proper) changes of sign (per period). *Then the solution exists in its function class for all time* $0 \le t < \infty$ *only if the points* x_- *where* $m_0(x_-) < 0$ *lie wholly to the left of the points*

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[1] $Q'^\bullet = v'(t,Q)\,Q'$ and $Q_0' \equiv 1$ forces $Q'(x) > 0$.

[2] $\frac{d}{dt}\,m(Q)\,Q'^2 = [m^\bullet(Q) + m'(Q)\,Q^\bullet]\,Q'^2 + 2mQ^\bullet Q'^\bullet = 0$, as you will check.

x_+ *where* $m_0(x_+) > 0$ (which never happens in the periodic case unless m_0 is of one sign). *Contrariwise, if* m_0 *is not of such a shape, then the solution breaks down at some time* $T < \infty$*: in fact, as* $t \uparrow T$*,* $v'(t,Q) \downarrow -\infty$ *at one or more of the (time independent) roots* x_0 *of* $m(Q) = 0$*; simultaneously,* $Q'(t,x_0) \downarrow 0$*, and more—*$Q(t,x)$ *flattens out in the largest interval* $a \le x \le b$ *containing* x_0 *in which* $m_0(x) \equiv 0$. This is how the diffeomorphism comes to the "edge" of the group; in particular, the geodesic flow in $D(\mathbb{R})$ is incomplete. The existence of $v(t,x) : t < T,\ x \in \mathbb{R}$ is not discussed here: That is covered by the methods of CONSTANTIN-ESCHER [1998] and CONSTANTIN-McKEAN [1998]. The present paper deals only with breakdown; it is a technical amplification of McKEAN [1999] which reviewed the whole connection with the diffeomorphism group and explained the breaking in a simple case.

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2. Preliminaries. Keep in mind the constants of motion $m(Q)\,Q'^2 = m_0(x)$ and the moral they convey: that the points x_0 where $m(Q)$ changes sign are fixed in time. Note also the duality: that if $v(t,x)$ is a solution, then so is $-v(t,-x)$. This will cut down the number of cases to be treated. Now come some little tricks which will be continually in use. They obtain up to the breaking time $T \le \infty$.

3) $$\frac{d}{dt}\, e^{Q}\,(v'-v)(Q) = -\frac{1}{2}\, e^{Q}\,(v'-v)^2(Q) - \frac{1}{2}\int_{-\infty}^{Q} e^{y}\,(v'-v)^2(y)\,dy.$$

4) $$\frac{d}{dt}\, e^{-Q}\,(v'+v)(Q) = -\frac{1}{2}\, e^{-Q}\,(v'+v)^2(Q) - \frac{1}{2}\int_{Q}^{\infty} e^{-y}\,(v'+v)^2(y)\,dy.$$

5) $$\frac{d}{dt}\, v'(Q) = \frac{1}{2}\, v^2(Q) - \frac{1}{2}\, v'^2(Q) - \frac{1}{4}\int_{-\infty}^{\infty} e^{-|Q(x)-y|}\,(v' \pm v)^2(y)\,dy,$$

in which the ambiguous signature is that of $y - Q(x)$.

6) $|v|$ is bounded, independently of time, by a fixed constant C.

7) v' is bounded above by $2C + \max v_0'$.

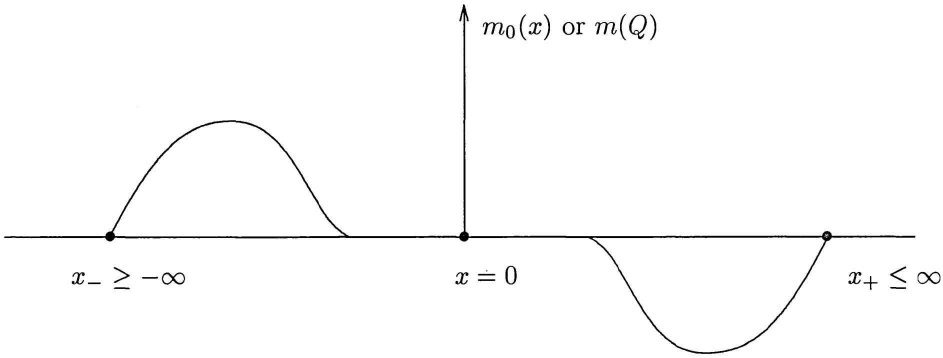
8) v' is bounded below by $-C$ if breakdown does not take place.

Proof. $e^Q(v'-v) = -\int_{-\infty}^{Q} e^y\, m(y)\,dy$. Now differentiate by t, using $Q^\bullet = v(Q)$ and $-m^\bullet = (mv)' + \frac{1}{2}\left(v^2 - v'^2\right)'$, and integrate by parts to obtain 3). 4) follows by duality. 5) is obtained by combining 3) and 4). 6) is plain if $v \in C_\downarrow^\infty(\mathbb{R})$: then $H = \int_{-\infty}^{\infty}(v^2 + v'^2)\,dx$ is a constant of motion and $v^2 = 2\int_{-\infty}^{x} vv' \le H$. The periodic case is similar with constant of motion $H = \int_0^1 (v^2 + v'^2)\,dx$ and $v^2 \le 2H$. Likewise, if $v = c + w \in \mathbb{R} + C_\downarrow^\infty(\mathbb{R})$, then c and $H = \int_{-\infty}^{\infty}(w^2 + w'^2)\,dx$ are constants of motion and $|v| \le |c| + \sqrt{H}$. 7) now follows from 3), 4), and 6): $e^{\pm Q}\,(v' \mp v)(Q)$ decreases with time and one of e^Q are e^{-Q} exceeds 1, so $v'(Q) \le e^{\mp Q}\,(v_0' \mp v_0)(Q) \pm v(Q)$, *etc.* 8)follows from 5) and 6) by the reasoning of sect. 1:

$$\frac{d}{dt}\, v'(Q) \le \frac{1}{2}\, v^2(Q) - \frac{1}{2}\, v'^2(Q) \le \frac{1}{2}\, C^2 - \frac{1}{2}\, v'^2(Q)$$

drives $v'(Q)$ down to $-\infty$ in finite time at any place where $v_0'(x)$ falls below $-C$. □

3. Proof of Breakdown. It is to be proved that breakdown takes place if there is any positive "mass" m_0 to the left of some negative mass. Let $x = 0$ be chosen as in the figure so that $m_0(x)$ and so also $m(Q)$ is positive near $x_- < 0$ and negative



near $x_+ > 0$, a single interval containing $x = 0$ being permitted for $m_0(x)$ to vanish. The picture is self-dual. Now if $(v' - v)(Q)$ is ever negative at $x = 0$, then 3) in the form

$$\frac{d}{dt} e^Q (v' - v)(Q) \leq -\frac{1}{2} \left[e^Q (v' - v)(Q) \right]^2 e^{-Q}$$

implies $\int_0^\infty e^{-Q}\, dt < \infty$ in the absence of breakdown; likewise, if $(v' + v)(Q)$ is ever negative at $x = 0$, then $\int_0^\infty e^Q\, dt < \infty$; and since $\int_0^\infty e^{\pm Q}\, dt$ cannot both converge, so you can assume $(v' - v)(Q) \geq 0$ at $x = 0$ at all times $t \geq 0$,by duality. Now distinguish two cases according as $(v' + v)(Q)$ is ever negative at $x = 0$ or not.

Case 1: $(v' + v)(Q) \geq 0$ at $x = 0$ at all times $t \geq 0$. $(d/dQ)\, e^Q (v' - v)(Q) = -e^Q\, m(Q)$ is ≤ 0 between x_- and $x = 0$, so $(v' - v)(Q) \geq 0$ there, and $(v' + v)(Q)$ likewise; in particular, $v'(Q) \geq |v(Q)|$ for $x_- \leq x \leq 0$, and, in the absence of breakdown, so that $v'(Q) \geq -C$ by 8), 5) implies

$$+\infty > \int_0^\infty dt\, e^{-Q(x)} \int_0^{Q(x)} e^y (v' - v)^2(y)\, dy \text{ between } x = x_- \text{ and } x = 0.$$

Fix $x_- < a < c < b < 0$ with $c = \frac{1}{2}(a + b)$ and $m_0(x) > 0$ between a and b. Then with $x = c$ in the last display,

$$\begin{aligned}
+\infty &> \int_0^\infty dt\, e^{-Q(c)} \int_{Q(a)}^{Q(c)} \left[e^y\, (v' - v) \right]^2 e^{-y}\, dy \\
&> \int_0^\infty dt\, e^{-Q(c)} \left[e^Q (v' - v) \text{ at } x = c \right]^2 \int_{Q(a)}^{Q(c)} e^{-y}\, dy \\
&= \int_0^\infty dt \left[(v' - v)(Q) \text{ at } x = c \right]^2 \times \left[e^{Q(c) - Q(a)} - 1 \right].
\end{aligned}$$

This contradicts the fact that $m_0(x) > 0$ for $a \leq x \leq b$, as will be seen in a few easy steps.

STEP 1 uses $(v'-v)(Q)\ge 0$ to obtain the simple bound:

$$\begin{aligned}\left[\int_c^b \sqrt{m_0(x)}\,dx\right]^2 &= \left[\int_c^b \sqrt{m(Q)}\,dQ\right]^2\\ &\le \int_c^b m(Q)\,dQ \times [\Delta Q \equiv Q(b)-Q(c)]\\ &= \int_c^b (v-v'+v'-v'')\,(Q)\,dQ\times\Delta Q\\ &\le [(v'-v)\,(Q) \text{ at } x=c]\times\Delta Q.\end{aligned}$$

STEP 2. Now let $v(Q)$ be negative at $x=b$. Then $v'(Q)\ge|v(Q)|$ implies $v(Q)\le 0$ down to $x=a$, so from $Q^\bullet=v(Q)$ and $Q'^\bullet=v'(Q)\,Q'$, it appears that

$$Q''=Q'\int_0^t v''(Q)\,Q'\,dt'=Q'\int_0^t (v-m)(Q)\,Q'\,dt'<0,$$

whence

$$\begin{aligned}\left[\int_c^b\sqrt{m_0}\right]^4 &\le \left[(v'-v)^2\,(Q)\text{ at }x=c\right]\times 2\left[e^{Q(b)-Q(c)}-1\right]\\ &\le \left[(v'-v)^2\,(Q)\text{ at }x=c\right]\times 2\left[e^{Q(c)-Q(a)}-1\right],\end{aligned}$$

and now the summability of the last line forces meas $(t\ge 0: v(Q)\le 0$ at $x=b)$ to be finite; in particular,[3] $\int_0^\infty v_-(Q)\,dt>-\infty$ at $x=b$, by 6). Obviously, there is nothing special about $x=b$: the same is true at $x=a$ by a self-evident reprise. This was the goal of Step 2.

STEP 3 will confirm that[4] $\int_0^\infty v_+(Q)\,dt$ is also finite at $x=a$: $(v'-v)\,(Q)$ is positive between x_- and 0 and decreases there in view of $v''-v'=-m+v-v'<0$, so

$$\exp\left[\int_0^t v'(Q)-\int_0^t v(Q)\right]=Q'\,e^{x-Q}\text{ is }\ge 1\text{ and decreases, too.}$$

Then

$$\begin{aligned}1\le Q'(b)\,e^{b-Q(b)} &\le \frac{1}{b-a}\int_a^b e^{x-Q(x)}\,dQ\\ &\le \frac{e^b}{b-a}\times e^{-Q(a)}\end{aligned}$$

and now this estimate, controlling $Q(a)=a+\int_0^t v_-(Q)+\int_0^t v_+(Q)$ from above, confirms $\int_0^\infty v_+(Q)\,dt<\infty$.

STEP 4. But this means that Q is bounded, not only if $x=a$, but at $x=c$ as well, by a self-evident reprise, and now $\left(\int_c^b\sqrt{m_0}\right)^4$ as seen in step 2, is over-estimated

[3] v_- is the negative part of v.

[4] v_+ is the positive part of v.

by a constant multiple of $e^Q\,(v'-v)^2\,(Q)$ at $x=c$, this being summable by 3) and the fact that $(v'-v)\,(Q)\geq 0$. That is the contradiction.

Case 2: $(v'-v)\,(Q)\geq 0$ at $x=0$ for all $t\geq 0$ and $(v'+v)\,(Q)<0$ at $x=0$ for, *e.g.*, $t=0$. Then 4) shows that, in the absence of breakdown, $\int_0^\infty e^Q\,dt<\infty$ at $x=0$ and so also for $x\leq 0$, Q being increasing in x, in which case, $\int_0^\infty e^{-Q}\,dt=+\infty$ and $e^Q\,(v'-v)\geq 0$ for every $x\leq 0$, by 3). Now $\int_0^\infty e^Q\,dt<\infty$ for $x\leq 0$ only if

$$\infty>\int_0^\infty e^Q\,Q'\,dt=\int_0^\infty \exp\left[\int_0^t (v'+v)\,(Q)\,dt'\right]dt \quad a.e.,$$

from which you learn that $(v'-v)\,(Q)$ turns strictly negative, and stays that way, by 4), at sometime $T(x)<\infty$, for almost all $x\leq 0$; moreover, the fact that $e^{-Q}\,(v'+v)\,(Q)$ decreases for $x_-\leq x\leq 0$ implies that $T(x)$ is finite and decreasing in the half-open interval $x_-<x\leq 0$. Two subcases are now distinguished according to the behavior of $\Delta Q=Q(b)-Q(a)$ for $x_-<a<b<0$.

Case 2.1: ΔQ is bounded from above for every choice of $x_-<a<b<0$ (as it must be in the periodic case when $\Delta Q\leq Q(1)-Q(0)=1$). Now $\infty>T(a)=0$, say, in which case $(v'-v)\,(Q)>0$, $(v'+v)\,(Q)\leq 0$, and $v(Q)<0$ for all $a\leq x\leq b$ and $t\geq 0$. Now comes a familiar type of trick:

$$\begin{aligned}\left[\int_a^b \sqrt{m_0}\,dx\right]^2=\left[\int_a^b \sqrt{m}\,dQ\right]^2 &\leq \int_a^b m(Q)\,dQ\times\Delta Q\\ &\leq \int_a^b -v''(Q)\,dQ\times\Delta Q\\ &= -v'(Q)\big|_a^b\times\Delta Q\end{aligned}$$

from which it appears that, in the absence of breakdown, ΔQ is also bounded from below, by 7) and 8). But also $Q''<0$ as in Step 2 of Case 1, v being negative, from which you learn that $0<Q'(b)/Q'(a)<1$ and that

$$\frac{d}{dt}\frac{Q'(b)}{Q'(a)}=v'(Q)\big|_a^b\times\frac{Q'(b)}{Q'(a)}<0 \text{ is summable.}$$

$Q''<0$ and the boundedness of ΔQ for every choice of $x_-<a<b<0$ come into play once more to check that $Q'(b)/Q'(a)$ is bounded below, and now a contradiction is obtained: $v'(Q)|_a^b$ is summable, but

$$\left[\int_a^b \sqrt{m_0}\right]^2\leq v'(Q)|_a^b\times \text{ the upper bound of } \Delta Q \quad \text{is not.}$$

Case 2.2: $Q(b')-Q(a')$ is unbounded above for some $x_-<a'<b'<0$. Now pick $x_-<c<a<b<a'$ with $T(c)=0$ for simplicity and $m_0(x)>0$ between c and b. Note that $v(Q)$ is negative, as before, so that $v''(Q)<v(Q)<0$, $m(Q)$ being positive, and also that $Q''<0$. A variant of the familiar $\left[\int_a^b\sqrt{m_0}\,dx\right]^2\leq\int_a^b m(Q)\,dQ\times\Delta Q$ is now used. It reads

$$\left[\int_a^b \sqrt{m_0(x)}\,dx\right]^2 \le \int_a^b m(Q)\,[Q(x)-Q(c)]\,dQ \int_a^b \frac{dQ}{Q(x)-Q(c)}$$
$$< \int_{Q(c)}^{Q(b)} -v''(x)\,[x-Q(c)]\,dx\ \ell g\,\frac{Q(b)-Q(a)}{Q(a)-Q(c)}$$
$$< \left.\begin{array}{l} -[v'(Q)\text{ at }x=b]\times Q(b)-Q(c) \\ -[v(Q)\text{ at }x=c] \end{array}\right\}\times \ell g\,\frac{b-a}{a-c},$$

$Q'' < 0$ being used to appraise the logarithm. I propose to show that $v'(Q) > 0$ at $x = b$ and that $v(Q) = o(1)$ at $x = c$ for suitable $t = t_1 < t_2 <$ *etc.* $\uparrow \infty$. This will be contradictory. Chose these "special" times so that $Q(b') - Q(a') \uparrow +\infty$. Then $Q(a') - Q(b) \uparrow +\infty$, too, since $Q'' < 0$, and you can also assume $(d/dt)\,[Q(a') - Q(b)] > 0$ at special times. But

$$\frac{d}{dt}\,[Q(a')-Q(b)] = \int_b^{a'} v'(Q)\,dQ \le [v'(Q)\text{ at }x=b]\times[Q(a')-Q(b)]$$

so $v'(Q)$ is positive at $x = b$ for special times and stays positive down to x_- in view of $v''(Q) < 0$. This is half the battle. Next, fix $d < c$. Then $-v(Q)$ decreases between d and c, so

$$0 < -[v(Q)\text{ at }x=c]\times[Q(c)-Q(d)] < -\int_d^c v''(Q)\,dQ,$$

in which the right hand side is bounded in the absence of breaking, and since $Q(c) - Q(d) \uparrow +\infty$ by reason of $Q'' < 0$, you may conclude that $v(Q) = o(1)$ at $x = c$ for special times. The proof is finished.

4. How It Breaks Down. The conditions for breaking are now established, but *how does it happen*?

Return to fig. 1 with $m(Q) > 0$ for $x_- < x < a$, $m(Q) = 0$ for $a \le x \le b$, and $m(Q) < 0$ for $b < x < x_+$; necessarily, $a \le 0 \le b$. $T < \infty$ is the breaking time, *i.e.*, the moment when $v'(Q)$ gets out of hand, assuming that this takes place between x_- and x_+. Breakdown can happen only in this way, as will be seen.

Item 1. $v'(Q) \equiv w$ is bounded above, by 7), and 5) in the form $m^\bullet \le \frac{1}{2}C^2 - \frac{1}{2}w^2$ shows that it can get out of hand only by an ultimate decrease to $-\infty$, in the style of $w =$ a negative constant $\times (T-t)^{-1}$ or worse, in which case $Q'(t,x) = \exp \int_0^t v'(Q) \downarrow 0$ as $t \uparrow T$, and conversely: if $Q' \downarrow 0$ as $t \uparrow T < \infty$, then $w = v'(Q)$ is ultimately very negative and 5) drives it down to $-\infty$.

Item 2. $Q(T-, x)$ exists everywhere, v being bounded, and $e^Q\,(v' - v)(Q)$ decreases/increases where $m(Q)$ is positive/negative, from which you see that breakdown occurs, if at all, at points where $m(Q)$ changes sign, from positive to negative: $x = 0$ is such a point.

Item 3. $v'(Q)$ at $x = 0$ now decreases to $-\infty$ as $t \uparrow T$ and carries with it the slope $v'(Q)$ at the general point x of the (maximal) interval $[a, b]$ where $m(Q) = 0$, $v'' = v$ being bounded there; in particular, $Q(x)$ flattens out in the whole interval $a \le x \le b$, by item 1.

Item 4 is the converse: if $Q(x)$ flattens out in any interval $a \le x \le b$, then $m_0(x)$ and so also $m(Q)$ vanishes there; simultaneously, $v'(Q) \downarrow -\infty$.

Proof. Let $Q(x)$ flatten out in $a \le x \le b$ with $x_- < a < b \le 0$, say, and take $c < d$ properly between a and b. Then

$$\left[\int_c^d \sqrt{m_0}\,dx\right]^2 \le \int_c^d m(Q)\,[Q(b)-Q(x)]\,dQ \int_c^d \frac{dQ}{Q(b)-Q(x)}$$

$$\le \int_c^b (v-v'')\,(Q)\,[Q(b)-Q(x)]\,dQ \times \ell g\,\frac{Q(b)-Q(c)}{Q(b)=Q(d)}$$

$$\left.\begin{array}{ll} \le & C\times\frac{1}{2}\,[Q(b)-Q(a)]^2 \\ + & v'(Q) \text{ at } x=a\times[Q(b)-Q(a)] \\ - & v(Q)\big|_a^b \end{array}\right\} \times \ell g\frac{Q(b)-Q(c)}{Q(b)-Q(d)}$$

$$\le \text{a constant multiple of } \sqrt{Q(b)-Q(a)} \times \ell g\frac{Q(b)-Q(a)}{Q(b)-Q(d)}$$

owing to the control of $v'(Q)$ above, by 7), and to the elementary estimate $|v(b)-v(a)|^2 \le \int_a^b (v')^2 \times (b-a)$, the integral being controlled by the proper constant of motion in all function classes. Now $Q'' = Q' \int_0^t v''(Q)Q'$ is bounded above by a multiple K of Q' for $a \le x \le b$ in view of $m = v - v'' > 0$ and 7); in particular, $Q' e^{-KQ}$ is decreasing in x, so the flattening out of Q in $[a,b]$ implies that, for $t \uparrow T$, $Q' \downarrow 0$ and so also $v'(Q) \downarrow -\infty$, uniformly in $[a,b]$.

Now use 5) for $w = v'(Q)$ in the form $w^\bullet = -\frac{1}{2}w^2 + 2V$, with $V = \frac{1}{4}v^2(Q)$ *etc.*, and write $w = 2\psi^\bullet/\psi$ with $\psi(0) = 1$, say, and $\psi^\bullet(0) < 0$, fixing the variable x between a and b. Then $\psi^{\bullet\bullet} = V\psi$, $\psi(T-) = 0$, and $\psi^\bullet(T-) < 0$[5], reflecting the fact that $w \downarrow -\infty$ as $t \uparrow T$. It follows that $v'(Q)$ behaves like $-2/(T-t)$ or nearly so as $t \uparrow T$, so Q' is no worse than $(T-t)^{3/2}$ for $a \le x \le b$. But now $\int_c^d \sqrt{m_0}$, as seen in the big display, is over-estimated by $(T-t)^{3/4}\,\ell g(T-t)$ and so vanishes, and $m_0(x) = 0$ in whole of $[a,b]$, the subinterval $[c,d]$ having been chosen as you will.

5. A Little Example. This falls outside the function classes admitted above but never mind. $v(t,x)$ is soliton/anti-soliton pair

$$\frac{1}{2}p\,e^{-|x+q|} - \frac{1}{2}p\,e^{-|x-q|},$$

symmetric about $x = 0$, with positive $q = q(t)$ and $p = p(t)$, $q^\bullet = -p(1-e^{-2q})$, $p^\bullet = p^2 e^{-2q}$, and the constant of motion $H = \frac{1}{2}p^2\left(1-e^{-2q}\right)$. With $\Theta = T - t$, you find $q = \ell g\,ch\,\Theta$, $p = ch\,\Theta/sh\,\Theta$ and, for $x > \ell g\,ch\,T = q(0)$, $e^Q = e^x + ch\,\Theta - ch\,T$, as you may check. In particular, Q flattens out and $v'(Q) \downarrow -\infty$ in the interval $|x| \le q(0) = \ell g\,ch\,T$, precisely.

[5] $\psi^\bullet(T-) = 0$ implies $\psi \equiv 0$ which is not the case.

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