ASIAN J. MATH.
Vol. 2, No. 4, pp. 779–832, December 1998

008

KAZHDAN-LUSZTIG CONJECTURE FOR SYMMETRIZABLE KAC-MOODY LIE ALGEBRAS. III – POSITIVE RATIONAL CASE*

MASAKI KASHIWARA† AND TOSHIYUKI TANISAKI‡

1. Introduction. The aim of this paper is to prove the Kazhdan-Lusztig type character formula for irreducible highest weight modules with positive rational highest weights over symmetrizable Kac-Moody Lie algebras.

Let us formulate our results precisely. Let $\mathfrak{g}$ be a symmetrizable Kac-Moody Lie algebra over the complex number field $\mathbb{C}$ with Cartan subalgebra $\mathfrak{h}$. We denote by W the Weyl group and by $\{\alpha_i\}_{i\in I}$ the set of simple roots. For a real root α, we define the corresponding coroot by $\alpha^\vee = 2\alpha/(\alpha,\alpha)$, where $(\ ,\ )$ denotes a standard non-degenerate symmetric bilinear form on $\mathfrak{h}^*$. For $\lambda \in \mathfrak{h}^*$, let $\Delta^+(\lambda)$ denote the set of positive real roots α satisfying $(\alpha^\vee, \lambda) \in \mathbb{Z}$, and let $\Pi(\lambda)$ denote the set of $\alpha \in \Delta^+(\lambda)$ such that $s_\alpha(\Delta^+(\lambda) \setminus \{\alpha\}) = \Delta^+(\lambda) \setminus \{\alpha\}$. Here $s_\alpha \in W$ denotes the reflection with respect to α. Then the subgroup $W(\lambda)$ of W generated by $\{s_\alpha\ ;\ \alpha \in \Delta^+(\lambda)\}$ is a Coxeter group with the canonical generator system $\{s_\alpha\ ;\ \alpha \in \Pi(\lambda)\}$. Fix $\rho \in \mathfrak{h}^*$ satisfying $(\rho, \alpha_i^\vee) = 1$ for any $i \in I$ and define a shifted action of W on $\mathfrak{h}^*$ by

$$w \circ \lambda = w(\lambda + \rho) - \rho \quad \text{for } w \in W \text{ and } \lambda \in \mathfrak{h}^*.$$

For $\lambda \in \mathfrak{h}^*$ let $M(\lambda)$ (resp. $M^*(\lambda)$, $L(\lambda)$) be the Verma module (dual Verma module, irreducible module) with highest weight λ. We denote their characters by $\mathrm{ch}(M(\lambda))$, $\mathrm{ch}(M^*(\lambda))$, $\mathrm{ch}(L(\lambda))$ respectively. We have $\mathrm{ch}(M(\lambda)) = \mathrm{ch}(M^*(\lambda))$, and $\mathrm{ch}(M(\lambda))$ is easily described.

The main result of this paper is the following.

THEOREM 1.1. *Assume that $\lambda \in \mathfrak{h}^*$ satisfies the following conditions.*

$$2(\alpha, \lambda + \rho) \neq (\alpha, \alpha) \text{ for any positive imaginary root } \alpha. \tag{1.1}$$

$$(\alpha^\vee, \lambda + \rho) \notin \mathbb{Z}_{\leq 0} \text{ for any positive real root } \alpha. \tag{1.2}$$

$$\text{If } w \in W \text{ satisfies } w \circ \lambda = \lambda, \text{ then } w = 1. \tag{1.3}$$

$$(\alpha^\vee, \lambda) \in \mathbb{Q} \text{ for any real root } \alpha. \tag{1.4}$$

Then for any $w \in W(\lambda)$ we have

$$\mathrm{ch}(M(w \circ \lambda)) = \sum_{y \geq_\lambda w} P^\lambda_{w,y}(1)\, \mathrm{ch}(L(y \circ \lambda)), \tag{1.5}$$

$$\mathrm{ch}(L(w \circ \lambda)) = \sum_{y \geq_\lambda w} (-1)^{\ell_\lambda(y) - \ell_\lambda(w)} Q^\lambda_{w,y}(1)\, \mathrm{ch}(M(y \circ \lambda)). \tag{1.6}$$

Here, $\geq_\lambda$, $P^\lambda_{w,y}$, ℓ_λ, $Q^\lambda_{w,y}$ denote the Bruhat ordering, the Kazhdan-Lusztig polynomial, the length function, and the inverse Kazhdan-Lusztig polynomial for the Coxeter group $W(\lambda)$, respectively.

---

*Received November 9, 1998; accepted for publication December 21, 1998.

†Research Institute for Mathematical Sciences, Kyoto University, Kyoto, 606–8502, Japan (masaki@kurims.kyoto-u.ac.jp).

‡Department of Mathematics, Hiroshima University, Higashi-Hiroshima, 739–8526, Japan (tanisaki@math.sci.hiroshima-u.ac.jp).

When $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra, this result for integral weights was conjectured by Kazhdan-Lusztig [19], and proved by Beilinson-Bernstein [1] and Brylinski-Kashiwara [2] independently. Later its generalization to rational weights was obtained by combining the results by Beilinson-Bernstein (unpublished) and Lusztig [21].

As for the symmetrizable Kac-Moody Lie algebra, Theorem 1.1 for integral weights was obtained by Kashiwara(-Tanisaki) in [14] and [16] (see also Casian [3]). We note that a generalization of the original Kazhdan-Lusztig conjecture to affine Lie algebras in the negative level case was obtained by Kashiwara-Tanisaki [17], Casian [4] (integral weights), Kashiwara-Tanisaki [18] (rational weights). We finally point out that (1.6) for $w = 1$ was proved by Kac-Wakimoto [11].

Let us give a sketch of the proof of our theorem.

Let $X = G/B^-$ be the flag manifold introduced in Kashiwara [13], which is an infinite-dimensional scheme. We have a stratification $X = \bigsqcup_{w\in W} X_w$ by finite-codimensional Schubert cells $X_w = BwB^-/B^-$. For $\lambda \in \mathfrak{h}^*$ let D_λ be the TDO-ring (ring of twisted differential operators) on X corresponding to the parameter λ. For $w \in W$ define D_λ-modules $\mathcal{B}_w(\lambda)$ (resp. $\mathcal{M}_w(\lambda)$, $\mathcal{L}_w(\lambda)$) as the meromorphic extension (resp. dual meromorphic extension, minimal extension) of the D_{X_w}-module $\mathcal{O}_{X_w}$ to a D_λ-module. They are objects of the category $\mathbb{H}(\lambda)$ consisting of N^+-equivariant holonomic D_λ-modules.

For $\lambda \in \mathfrak{h}^*$ satisfying the conditions (1.1), (1.2) and (1.3), we define a modified global section functor $\tilde{\Gamma}$ from $\mathbb{H}(\lambda)$ to the category $\mathbb{M}(\mathfrak{g})$ of $\mathfrak{g}$-modules. Then Theorem 1.1 is a consequence of the following results.

THEOREM 1.2. *Assume that $\lambda \in \mathfrak{h}^*$ satisfies the conditions* (1.1), (1.2) and (1.3).

(i) *The functor $\tilde{\Gamma} : \mathbb{H}(\lambda) \to \mathbb{M}(\mathfrak{g})$ is exact.*

(ii) $\tilde{\Gamma}(\mathcal{B}_w(\lambda)) = M^*(w \circ \lambda)$ *for any $w \in W$.*

(iii) $\tilde{\Gamma}(\mathcal{M}_w(\lambda)) = M(w \circ \lambda)$ *for any $w \in W$.*

(iv) $\tilde{\Gamma}(\mathcal{L}_w(\lambda)) = L(w \circ \lambda)$ *for any $w \in W$.*

THEOREM 1.3. *Assume that $\lambda \in \mathfrak{h}^*$ satisfies the condition* (1.4). *Then for any $w \in W$ which is the smallest element of $wW(\lambda)$ and any $x \in W(\lambda)$, we have*

$$[\mathcal{L}_{wx}(\lambda)] = \sum_{y \geq_\lambda x} (-1)^{\ell_\lambda(y)-\ell_\lambda(x)} Q^\lambda_{x,y}(1)[\mathcal{M}_{wy}(\lambda)] \tag{1.7}$$

in the (*modified*) *Grothendieck group $K(\mathbb{H}(\lambda))$ of $\mathbb{H}(\lambda)$.*

The proof of Theorem 1.2 is similar to the one in [14]. In the course of the proof we also use the modified localization functor $D_\lambda \hat{\otimes} \bullet$ from a category of certain $\mathfrak{g}$-modules to a category of certain D_λ-modules as in [14], and we prove simultaneously that $D_\lambda \hat{\otimes} \tilde{\Gamma}(\mathcal{M}) \simeq \mathcal{M}$ for any $\mathcal{M} \in \mathrm{Ob}(\mathbb{H}(\lambda))$. Aside from the technical complexity in dealing with non-integral weights, the main new ingredients compared with the integral case [14] are the embeddings of Verma modules (Theorem 2.5.3) and the proof of the injectivity of the canonical morphism $M(w \circ \lambda) \to \tilde{\Gamma}(\mathcal{M}_w(\lambda))$ (Proposition 4.7.2).

The proof of Theorem 1.3 is based on the theory of Hodge modules by M. Saito [23] as in [16]. As for the combinatorics concerning the Kazhdan-Lusztig polynomials we use the dual version of the result in [22].

In the affine case, we can deduce the non-regular highest weight case from the above result by using the translation functors.

THEOREM 1.4. *Let $\mathfrak{g}$ be an affine Lie algebra, and assume that $\lambda \in \mathfrak{h}^*$ satisfies*

$$(\delta, \lambda + \rho) \neq 0, \text{ where } \delta \text{ is the imaginary root.} \tag{1.8}$$

(1.9) $$(\alpha^\vee, \lambda+\rho) \in \mathbb{Q} \setminus \mathbb{Z}_{<0} \text{ for any positive real root } \alpha.$$

Then $W_0(\lambda) = \{w \in W; w \circ \lambda = \lambda\}$ *is a finite group. Let* w *be an element of* $W(\lambda)$ *which is the longest element of* $wW_0(\lambda)$. *Then we have*

$$\mathrm{ch}(L(w \circ \lambda)) = \sum_{y \geq_\lambda w} (-1)^{\ell_\lambda(y)-\ell_\lambda(w)} Q^\lambda_{w,y}(1)\,\mathrm{ch}(M(y \circ \lambda)).$$

A motivation of our study comes from a recent work of W. Soergel [24] concerning tilting modules over affine Lie algebras. We would like to thank H. H. Andersen for leading our attention to this problem.

2. Highest weight modules.

2.1. Kac-Moody Lie algebras. In this section, we shall review the definition of Kac-Moody Lie algebras, and fix notations employed in this paper.

Let $\mathfrak{h}$ be a finite-dimensional vector space over $\mathbb{C}$, and let $\Pi = \{\alpha_i\}_{i\in I}$ and $\Pi^\vee = \{h_i\}_{i\in I}$ be subsets of $\mathfrak{h}^*$ and $\mathfrak{h}$ respectively indexed by the same finite set I subject to

(2.1.1) Π and $\Pi^\vee$ are linearly independent subsets of $\mathfrak{h}^*$ and $\mathfrak{h}$ respectively,

(2.1.2) $(\langle h_i, \alpha_j\rangle)_{i,j\in I}$ is a symmetrizable generalized Cartan matrix.

Here $\langle\ ,\ \rangle : \mathfrak{h} \times \mathfrak{h}^* \to \mathbb{C}$ denotes the natural paring. The elements of Π and $\Pi^\vee$ are called simple roots and simple coroots respectively. We fix a non-degenerate symmetric bilinear form $(\ ,\ )$ on $\mathfrak{h}^*$ such that

(2.1.3) $$(\alpha_i, \alpha_i) \in \mathbb{Q}_{>0} \quad \text{for any } i \in I,$$

(2.1.4) $$\langle h_i, \lambda\rangle = 2(\lambda, \alpha_i)/(\alpha_i, \alpha_i) \quad \text{for any } \lambda \in \mathfrak{h}^* \text{ and } i \in I.$$

We denote the corresponding Kac-Moody Lie algebra by $\mathfrak{g}$. Recall that $\mathfrak{g}$ is the Lie algebra over $\mathbb{C}$ generated by elements e_i, f_i $(i \in I)$ and the vector space $\mathfrak{h}$ satisfying the following defining relations (see Kac [9]):

(2.1.5) $$\begin{aligned} &[h, h'] = 0 \quad \text{for } h, h' \in \mathfrak{h}, \\ &[h, e_i] = \langle h, \alpha_i\rangle e_i, \quad [h, f_i] = -\langle h, \alpha_i\rangle f_i \quad \text{for } h \in \mathfrak{h} \text{ and } i \in I, \\ &[e_i, f_j] = \delta_{ij} h_i \quad \text{for } i, j \in I, \\ &\mathrm{ad}(e_i)^{1-\langle h_i,\alpha_j\rangle}(e_j) = \mathrm{ad}(f_i)^{1-\langle h_i,\alpha_j\rangle}(f_j) = 0 \quad \text{for } i, j \in I \text{ with } i \neq j. \end{aligned}$$

Define the subalgebras $\mathfrak{n}^+, \mathfrak{n}^-, \mathfrak{b}, \mathfrak{b}^-$ of $\mathfrak{g}$ by

(2.1.6) $$\begin{aligned} &\mathfrak{n}^+ = \langle e_i\,;\, i \in I\rangle, && \mathfrak{n}^- = \langle f_i\,;\, i \in I\rangle, \\ &\mathfrak{b} = \langle h, e_i\,;\, h \in \mathfrak{h}, i \in I\rangle, && \mathfrak{b}^- = \langle h, f_i\,;\, h \in \mathfrak{h}, i \in I\rangle. \end{aligned}$$

The vector space $\mathfrak{h}$ is naturally regarded as an abelian subalgebra of $\mathfrak{g}$, and we have the decompositions

(2.1.7) $$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+, \quad \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+, \quad \mathfrak{b}^- = \mathfrak{h} \oplus \mathfrak{n}^-.$$

For $\lambda \in \mathfrak{h}^*$ set $\mathfrak{g}_\lambda = \{x \in \mathfrak{g}\,;\, [h, x] = \langle h, \lambda\rangle x \text{ for } h \in \mathfrak{h}\}$, and define the root system Δ of $\mathfrak{g}$ by

(2.1.8) $$\Delta = \{\lambda \in \mathfrak{h}^*\,;\, \mathfrak{g}_\lambda \neq 0\} \setminus \{0\}.$$

Set

$$Q=\sum_{i\in I}\mathbb{Z}\alpha_i,\quad Q^{\pm}=\pm\sum_{i\in I}\mathbb{Z}_{\geq 0}\alpha_i, \tag{2.1.9}$$

$$\Delta^{\pm}=\Delta\cap Q^{\pm}. \tag{2.1.10}$$

We have $\Pi\subset\Delta^+$ and $\Delta=\Delta^+\sqcup\Delta^-$. The elements of Δ^+ and Δ^- are called positive and negative roots respectively.

For a subset Θ of Δ such that $(\Theta+\Theta)\cap(\Delta\cup\{0\})\subset\Theta$ we define the subalgebra $\mathfrak{n}(\Theta)$ of $\mathfrak{g}$ by

$$\mathfrak{n}(\Theta)=\sum_{\alpha\in\Theta}\mathfrak{g}_\alpha. \tag{2.1.11}$$

For $\alpha=\sum_{i\in I}m_i\alpha_i\in Q$, its height $\mathrm{ht}(\alpha)$ is defined by

$$\mathrm{ht}(\alpha)=\sum_{i\in I}m_i. \tag{2.1.12}$$

For $i\in I$ define the simple reflection $s_i\in GL(\mathfrak{h}^*)$ by

$$s_i(\lambda)=\lambda-\langle h_i,\lambda\rangle\alpha_i. \tag{2.1.13}$$

The subgroup W of $GL(\mathfrak{h}^*)$ generated by $S=\{s_i\,;\,i\in I\}$ is called the Weyl group.

It is a Coxeter group with the canonical generator system S. The length function $\ell: W\to\mathbb{Z}_{\geq 0}$ of the Coxeter group W satisfies

$$\ell(w)=\sharp(\Delta^-\cap w\Delta^+)\qquad\text{for any } w\in W. \tag{2.1.14}$$

We denote the Bruhat ordering on W by $\geq$. Note that we have

$$(w\lambda,w\mu)=(\lambda,\mu)\qquad\text{for any }\lambda,\mu\in\mathfrak{h}^*\text{ and }w\in W. \tag{2.1.15}$$

Set

$$\Delta_{\mathrm{re}}=W\Pi\,,\quad \Delta_{\mathrm{im}}=\Delta\setminus\Delta_{\mathrm{re}}\,,\quad \Delta^{\pm}_{\mathrm{re}}=\Delta_{\mathrm{re}}\cap\Delta^{\pm}\,,\quad \Delta^{\pm}_{\mathrm{im}}=\Delta_{\mathrm{im}}\cap\Delta^{\pm}\,. \tag{2.1.16}$$

The elements of Δ_{re} and Δ_{im} are called real and imaginary roots, respectively. For $\alpha\in\Delta_{\mathrm{re}}$ set

$$\alpha^\vee=2\alpha/(\alpha,\alpha), \tag{2.1.17}$$

and define the reflection $s_\alpha\in GL(\mathfrak{h}^*)$ by

$$s_\alpha(\lambda)=\lambda-(\lambda,\alpha^\vee)\alpha. \tag{2.1.18}$$

Then we have $s_\alpha\in W$ for any $\alpha\in\Delta_{\mathrm{re}}$.

We fix a vector $\rho\in\mathfrak{h}^*$ such that $\langle h_i,\rho\rangle=1$ for any $i\in I$. Then the shifted action of W on $\mathfrak{h}^*$ is defined by

$$w\circ\lambda=w(\lambda+\rho)-\rho\,. \tag{2.1.19}$$

Note that $\rho-w\rho\in Q^+$ for any $w\in W$ and it does not depend on the choice of ρ.

2.2. Integral Weyl groups. In this section, we study the properties of integral Weyl groups. We start the study in a more general setting [1].

Let Δ_1 be a subset of $\Delta_{\rm re}$ satisfying the following condition:

$$s_\alpha\beta \in \Delta_1 \text{ for any } \alpha, \beta \in \Delta_1. \tag{2.2.1}$$

In particular, we have $-\Delta_1 = \Delta_1$. We set

$$\Delta_1^\pm = \Delta_1 \cap \Delta^\pm, \tag{2.2.2}$$

$$\Pi_1 = \{\alpha \in \Delta_1^+ \,;\, s_\alpha(\Delta_1^+ \setminus \{\alpha\}) \subset \Delta_1^+\}, \tag{2.2.3}$$

$$W_1 = \langle s_\alpha ; \alpha \in \Pi_1 \rangle \subset W. \tag{2.2.4}$$

We call the elements of Δ_1^+ (resp. Δ_1^-, Π_1) positive roots (resp. negative roots, simple roots) for Δ_1, and W_1 the Weyl group for Δ_1.

Note that if Δ_1 satisfies the condition (2.2.1), then $w\Delta_1$ also satisfies (2.2.1) for any $w \in W$.

LEMMA 2.2.1. *If Δ_1 contains a simple root α_i, then α_i is in Π_1.*

LEMMA 2.2.2. *Assume that $\alpha_i \notin \Delta_1$. Set $\Delta_1' = s_i\Delta_1$. Then Δ_1' satisfies the condition* (2.2.1). *Moreover $\Delta_1' \bigcap \Delta^+ = s_i\Delta_1^+$, the set of simple roots for Δ_1' is $s_i\Pi_1$, and the Weyl group for Δ_1' is $s_iW_1s_i$.*

The above two lemmas immediately follow from $s_i\Delta^+ = (\Delta^+ \setminus \{\alpha_i\}) \cup \{-\alpha_i\}$.

LEMMA 2.2.3. *If $\alpha \in \Pi_1$ and $i \in I$ satisfy $(\alpha_i, \alpha) > 0$, then either $\alpha = \alpha_i$ or $\alpha_i \notin \Delta_1$.*

Proof. Assume $\alpha \neq \alpha_i$ and $\alpha_i \in \Delta_1$. Then $\beta = s_\alpha\alpha_i = \alpha_i - (\alpha^\vee, \alpha_i)\alpha$ is a positive root. Then $\alpha_i = \beta + (\alpha^\vee, \alpha_i)\alpha$ contradicts $(\alpha^\vee, \alpha_i) \in \mathbb{Z}_{>0}$. □

LEMMA 2.2.4. *For any $\alpha \in \Pi_1$ there exist $w \in W$ and $i \in I$ such that $w\alpha = \alpha_i$ and $w\Delta_1^+ = w\Delta_1 \bigcap \Delta^+$.*

Proof. We shall show this by induction on $\mathrm{ht}(\alpha)$. If $\mathrm{ht}(\alpha) = 1$, then there is nothing to prove. Assume $\mathrm{ht}(\alpha) > 1$. Write $\alpha = \sum_{j\in I} m_j\alpha_j$ with $m_j \geq 0$. Then we have

$$0 < (\alpha, \alpha) = \sum_{j\in I} m_j(\alpha, \alpha_j),$$

and hence there exists some $j \in I$ such that $(\alpha, \alpha_j) > 0$. Since $\mathrm{ht}(\alpha) > 1$, we have $\alpha \neq \alpha_j$ and hence $\alpha_j \notin \Delta_1$ by Lemma 2.2.3. Set $\Delta_1' = s_j\Delta_1, (\Delta_1')^+ = s_j\Delta_1^+, \Pi_1' = s_j\Pi_1$. Then $(\Delta_1')^+$ and Π_1' are the set of positive and simple roots for Δ_1' respectively by Lemma 2.2.2. Set $\alpha' = s_j\alpha \in \Pi_1'$. Since $\mathrm{ht}(\alpha') < \mathrm{ht}(\alpha)$, there exist some $w' \in W$ and $i \in I$ such that $w'\alpha' = \alpha_i$ and $w'(\Delta_1')^+ \subset \Delta^+$ by the hypothesis of induction. Then setting $w = w's_j$ we have $w\alpha = \alpha_i$ and $w\Delta_1^+ = w'(\Delta_1')^+ \subset \Delta^+$. □

The following lemma follows from the above lemma by reducing to the case $\alpha = \alpha_i$ for $i \in I$.

[1] After writing up this paper, the authors were informed by S. Naito the existence of two papers, R. Moody–A. Pianzola, Lie Algebras with Triangular Decompositions, Canadian Mathematical Society series of monographs and advanced texts, A Wiley-Interscience Publication, John Wiley & Sons, 1995, and Jong-Min Ku, On the uniqueness of embeddings of Verma modules defined by the Shapovalov elements, J. Algebra, Vol. 118, (1988) 85–101. They showed results similar to those in this subsection by a different formulation and method. In the last paper, Ku also obtained a result weaker than Theorem 2.5.3.

LEMMA 2.2.5. *For any positive integer n and $\alpha \in \Pi_1$, we have*

$$n\alpha \notin \sum_{\beta \in (\Delta_1^+ \setminus \{\alpha\}) \cup \Delta_{\mathrm{im}}^+} \mathbb{Z}_{\geq 0}\beta.$$

LEMMA 2.2.6. *For any $\beta \in \Delta_1^+$, there exists $\alpha \in \Pi_1$ such that $(\alpha, \beta) > 0$.*

Proof. We shall prove this by the induction on $\mathrm{ht}(\beta)$. If $\mathrm{ht}(\beta) = 1$, then Lemma 2.2.1 implies $\beta \in \Pi_1$, and we can take β as α. Assume that $\mathrm{ht}(\beta) > 1$. Take i such that $(\alpha_i, \beta) > 0$. If $\alpha_i \in \Delta_1$, then it is enough to take α_i as α. Now assume that $\alpha_i \notin \Delta_1$. Set $\Delta_1' = s_i\Delta_1, (\Delta_1')^+ = s_i\Delta_1^+, \Pi_1' = s_i\Pi_1$. Then $(\Delta_1')^+$ and Π_1' are the set of positive and simple roots for Δ_1' respectively by Lemma 2.2.2. Set $\beta' = s_i\beta \in (\Delta_1')^+$. We have $\mathrm{ht}(\beta') < \mathrm{ht}(\beta)$ by $(\beta, \alpha_i) > 0$. Hence by the hypothesis of induction there exists some $\alpha' \in \Pi_1'$ such that $(\alpha', \beta') > 0$. Then $\alpha = s_i\alpha' \in \Pi_1$ satisfies $(\alpha, \beta) = (\alpha', \beta') > 0$. □

LEMMA 2.2.7.

(i) $\Delta_1 = W_1\Pi_1$.

(ii) $\Delta_1^+ \subset \sum_{\alpha \in \Pi_1} \mathbb{Z}_{\geq 0}\alpha$.

(iii) *W_1 contains s_α for any $\alpha \in \Delta_1$.*

Proof. Since (iii) follows from (i), it is enough to show that any $\beta \in \Delta_1^+$ is contained in $W_1\Pi_1 \bigcap \sum_{\alpha \in \Pi_1} \mathbb{Z}_{\geq 0}\alpha$. We shall prove this by the induction on $\mathrm{ht}(\beta)$. By Lemma 2.2.6 there exists $\alpha_0 \in \Pi_1$ such that $(\alpha_0, \beta) > 0$. If $\beta = \alpha_0$, then there is nothing to prove. If $\beta \neq \alpha_0$, then $\gamma = s_{\alpha_0}\beta \in \Delta_1^+$ by the definition of Π_1 and $\mathrm{ht}(\gamma) < \mathrm{ht}(\beta)$. Now we can apply the hypothesis of induction to conclude $\gamma \in W_1\Pi_1 \bigcap \sum_{\alpha \in \Pi_1} \mathbb{Z}_{\geq 0}\alpha$, which implies the desired result. □

LEMMA 2.2.8. *For $\alpha \in \Delta_1^+$, the following conditions are equivalent.*

(i) $\alpha \in \Pi_1$.

(ii) *$s_\beta\alpha \in \Delta_1^-$ for any $\beta \in \Delta_1^+$ such that $(\alpha, \beta) > 0$.*

(iii) *α cannot be written as $\alpha = m_1\beta_1 + m_2\beta_2$ for $\beta_\nu \in \Delta_1^+$ and $m_\nu \in \mathbb{Z}_{>0}$ ($\nu = 1$, 2).*

(iv) *α cannot be written as $\alpha = \sum_{\nu=1}^k \beta_\nu$ for $k > 1$, $\beta_\nu \in \Delta_1^+$ ($1 \leq \nu \leq k$).*

Proof. (i)⇒(iv) follows from Lemma 2.2.5. (iv)⇒(iii) is trivial. (iii)⇒(ii) is also immediate. Let us prove (ii)⇒(i). By Lemma 2.2.6, there exists $\beta \in \Pi_1$ such that $(\beta, \alpha) > 0$. Hence $\gamma = -s_\beta\alpha \in \Delta_1^+$. Rewriting this, we have $(\beta^\vee, \alpha)\beta = \alpha + \gamma$. Then Lemma 2.2.5 implies $\alpha = \beta$ or $\gamma = \beta$. It is now enough to remark that $\gamma = \beta$ implies $\alpha = \beta$. □

The following proposition is proved by a standard argument (see e.g. [18, §3.2]).

PROPOSITION 2.2.9.

(i) *W_1 is a Coxeter group with a generator system $S_1 = \{s_\alpha\,;\, \alpha \in \Pi_1\}$.*

(ii) *Its length function $\ell_1 : W_1 \to \mathbb{Z}_{\geq 0}$ is given by $\ell_1(w) = \sharp(\Delta_1^- \cap w\Delta_1^+)$.*

(iii) *For $x, y \in W$, $x \geq_1 y$ with respect to the Bruhat order $\geq_1$ for (W_1, S_1) if and only if there exist $\beta_1, \ldots, \beta_r \in \Delta_1^+$ ($r \geq 0$) such that $x = ys_{\beta_1} \cdots s_{\beta_r}$ and $ys_{\beta_1} \cdots s_{\beta_{j-1}}\beta_j \in \Delta_1^+$ for $j = 1, \ldots, r$.*

LEMMA 2.2.10. *For $\alpha, \beta \in \Pi_1$ such that $\alpha \neq \beta$ we have $(\alpha, \beta) \leq 0$.*

Proof. We have $s_\alpha\beta \in \Delta_1^+$ by the definition of Π_1. Since $\beta \in \Pi_1$, Lemma 2.2.8 implies the desired result. □

By this lemma, $\Big((\beta, \alpha^\vee)\Big)_{\alpha,\beta \in \Pi_1}$ is a symmetrizable generalized Cartan matrix. Hence W_1 is isomorphic to the Weyl group for the Kac-Moody Lie algebra with $\Big((\beta, \alpha^\vee)\Big)_{\alpha,\beta \in \Pi_1}$ as a generalized Cartan matrix.

PROPOSITION 2.2.11. *For $w \in W$ the following conditions are equivalent.*

(i) $l(x) \geq l(w)$ *for any* $x \in wW_1$.
(ii) $wx \geq wy$ *for any* $x, y \in W_1$ *such that* $x \geq_1 y$.
(iii) $w\Delta_1^+ \subset \Delta^+$.

Proof. Let us first prove (iii)$\Rightarrow$(ii). We may assume without loss of generality that $x = ys_\beta$ for some $\beta \in \Delta_1^+$. Then $y\beta \in \Delta_1^+$ and hence $wy\beta \in \Delta^+$. This implies $wx = wys_\beta \geq wy$.

(ii) implies (i) by taking $y = 1$ in (ii). (i) implies (iii) because, for any $\alpha \in \Delta_1^+$, $l(ws_\alpha) \geq l(w)$ implies $w\alpha \in \Delta^+$. □

For $\lambda \in \mathfrak{h}^*$ set

$$\Delta(\lambda) = \{\beta \in \Delta_{\mathrm{re}};\ (\beta^\vee, \lambda + \rho) \in \mathbb{Z}\} = \{\beta \in \Delta_{\mathrm{re}};\ (\beta^\vee, \lambda) \in \mathbb{Z}\}. \tag{2.2.5}$$

This satisfies the condition (2.2.1). We set $\Delta^\pm(\lambda) = \Delta(\lambda) \cap \Delta^\pm$. Let $\Pi(\lambda)$ and $W(\lambda)$ be the set of simple roots and the Weyl group for $\Delta(\lambda)$, respectively. We call $W(\lambda)$ the *integral Weyl group* for λ. We denote by $\ell_\lambda : W(\lambda) \to \mathbb{Z}_{\geq 0}$ and $\geq_\lambda$ the length function and the Bruhat order of the Coxeter group $W(\lambda)$, respectively.

REMARK 2.2.12.
(i) In [18], we introduced $W(\lambda)$ and $W'(\lambda)$. The integral Weyl group introduced here is equal to $W'(\lambda)$ loc.cit. As a matter of fact, $W(\lambda)$ and $W'(\lambda)$ loc.cit. coincide. The opposite statement in [18, Remark 3.3.2] should be corrected.
(ii) The set $\Pi(\lambda)$ is linearly independent when $\mathfrak{g}$ is finite-dimensional. But it is not necessarily linearly independent in the affine case, although we have assumed the linear independence of $\{\alpha_i\}_{i \in I}$. For example, for $\mathfrak{g} = A_3^{(1)}$ and $\lambda = (\Lambda_1 + \Lambda_3)/2$, we have $\Pi(\lambda) = \{\alpha_0,\ \alpha_2,\ -\alpha_0 + \delta,\ -\alpha_2 + \delta\}$.
(iii) For $x, y \in W_1$, $x \geq_1 y$ implies $x \geq y$ (Lemma 2.2.11). However the converse is false in general. For example for $\mathfrak{g} = A_3$ and $\lambda = (\Lambda_1 + \Lambda_3)/2$, we have $\Pi(\lambda) = \{\alpha_2,\ \alpha_1 + \alpha_2 + \alpha_3\}$ and $s_{\alpha_1+\alpha_2+\alpha_3} \geq s_{\alpha_2}$.

2.3. Category of highest weight modules. In this subsection we shall recall some properties of the category $\mathbb{O}$ of highest weight $\mathfrak{g}$-modules.

In general, for a Lie algebra $\mathfrak{a}$ we denote its enveloping algebra by $U(\mathfrak{a})$ and the category of (left) $U(\mathfrak{a})$-modules by $\mathbb{M}(\mathfrak{a})$.

For $k \in \mathbb{Z}_{\geq 0}$ set

$$\mathfrak{n}_k^\pm = \mathfrak{n}(\pm\Delta_k^+) \ \text{ with } \Delta_k^+ = \{\alpha \in \Delta^+\,;\ \mathrm{ht}(\alpha) \geq k\} \tag{2.3.1}$$

(see (2.1.11) and (2.1.12) for the notation). A $U(\mathfrak{g})$-module M is called *admissible* if, for any $m \in M$, there exists some k such that $\mathfrak{n}_k^+ m = 0$. We denote by $\mathbb{M}_{adm}(\mathfrak{g})$ the full subcategory of $\mathbb{M}(\mathfrak{g})$ consisting of admissible $U(\mathfrak{g})$-modules. It is obviously an abelian category.

For $M \in \mathbb{M}(\mathfrak{h})$ and $\xi \in \mathfrak{h}^*$ we set

$$M_\xi = \{u \in M\,;\ (h - \langle h, \xi\rangle)^n u = 0 \text{ for any } h \in \mathfrak{h} \text{ and } n \gg 0\}.$$

It is called the generalized weight space of M with weight ξ. We denote by $\mathbb{O}$ the full subcategory of $\mathbb{M}(\mathfrak{g})$ consisting of $U(\mathfrak{g})$-modules M satisfying

(2.3.2) $M = \bigoplus_{\xi \in \mathfrak{h}^*} M_\xi$,

(2.3.3) $\dim M_\xi < \infty$ for any $\xi \in \mathfrak{h}^*$,

(2.3.4) for any $\xi \in \mathfrak{h}^*$ there exist only finitely many $\mu \in \xi + Q^+$ such that $M_\mu \neq 0$.

It is an abelian subcategory of $\mathbb{M}_{adm}(\mathfrak{g})$.

For $M \in \mathrm{Ob}(\mathbb{O})$, or more generally for an $\mathfrak{h}$-module M satisfying (2.3.2) and (2.3.3), we define its character as the formal infinite sum

$$\mathrm{ch}(M) = \sum_{\xi \in \mathfrak{h}^*} (\dim M_\xi)\, e^\xi .$$

For a $U(\mathfrak{g})$-module M, the dual space $\mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$ is endowed with a $U(\mathfrak{g})$-module structure by

$$\langle xm^*, m\rangle = \langle m^*, a(x)m\rangle \quad \text{for } m^* \in \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C}),\ m \in M,\ x \in \mathfrak{g},$$

where $a : \mathfrak{g} \to \mathfrak{g}$ is the anti-automorphism of the Lie algebra $\mathfrak{g}$ given by

$$a(h) = h \quad \text{for } h \in \mathfrak{h}, \qquad a(e_i) = f_i, \quad a(f_i) = e_i \quad \text{for } i \in I.$$

If $M \in \mathrm{Ob}(\mathbb{O})$, then

$$M^* := \bigoplus_{\xi \in \mathfrak{h}^*} (M_\xi)^* \subset \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$$

is a $U(\mathfrak{g})$-submodule of $\mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$ belonging to $\mathrm{Ob}(\mathbb{O})$. Indeed we have

$$(M^*)_\xi = (M_\xi)^* .$$

Moreover, it defines a contravariant exact functor $(\bullet)^* : \mathbb{O} \to \mathbb{O}$ such that $(\bullet)^{**}$ is naturally isomorphic to the identity functor on $\mathbb{O}$. In particular, we have

$$\text{(2.3.5)} \qquad \mathrm{Hom}_{\mathfrak{g}}(M, N) \simeq \mathrm{Hom}_{\mathfrak{g}}(N^*, M^*) \quad \text{for } M, N \in \mathrm{Ob}(\mathbb{O}).$$

We also note

$$\text{(2.3.6)} \qquad \mathrm{ch}(M^*) = \mathrm{ch}(M) \text{ for any } M \in \mathrm{Ob}(\mathbb{O}).$$

An element m of a $U(\mathfrak{g})$-module M is called a highest weight vector with weight λ if $m \in M_\lambda$ and $e_i m = 0$ for any $i \in I$. A $U(\mathfrak{g})$-module M is called a highest weight module with highest weight λ if it is generated by a highest weight vector with weight λ. Highest weight modules belong to the category $\mathbb{O}$.

For $\lambda \in \mathfrak{h}^*$ define a highest weight module $M(\lambda)$ with highest weight λ, called a Verma module, by

$$M(\lambda) = U(\mathfrak{g}) / \Big(\sum_{h \in \mathfrak{h}} U(\mathfrak{g})(h - \lambda(h)) + \sum_{i \in I} U(\mathfrak{g}) e_i\Big).$$

The element of $M(\lambda)$ corresponding to $1 \in U(\mathfrak{g})$ will be denoted by u_λ. Set $M^*(\lambda) = (M(\lambda))^*$. There exists a unique (up to a constant multiple) non-zero homomorphism $M(\lambda) \to M^*(\lambda)$. Its image $L(\lambda)$ is a unique irreducible quotient of $M(\lambda)$ and a unique irreducible submodule of $M^*(\lambda)$. In particular, we have $(L(\lambda))^* \simeq L(\lambda)$.

We have the following lemma (see Lemma 9.6 of Kac [9]).

LEMMA 2.3.1. *For any $M \in \mathrm{Ob}(\mathbb{O})$ and $\mu \in \mathfrak{h}^*$, there exists a finite filtration*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

of M by $U(\mathfrak{g})$-modules M_k (k=0,..., r) such that for any k we have either $(M_k/M_{k-1})_\mu = 0$ *or* $M_k/M_{k-1} \simeq L(\xi)$ *for some* $\xi \in \mathfrak{h}^*$.

For $M \in \mathrm{Ob}(\mathbb{O})$ and $\mu \in \mathfrak{h}^*$ we set

$$[M : L(\mu)] = \sharp\{k\,;\, M_k/M_{k-1} \simeq L(\mu)\},$$

for a filtration of M as in Lemma 2.3.1. It does not depend on the choice of a filtration. Then we have the equalities

$$[M : L(\mu)] = [M^* : L(\mu)]\,,$$
$$\mathrm{ch}(M) = \sum_{\mu \in \mathfrak{h}^*} [M : L(\mu)]\,\mathrm{ch}(L(\mu)).$$

We frequently use the following lemma later.

LEMMA 2.3.2. *Let $M \in \mathrm{Ob}(\mathbb{O})$ and $\mu \in \mathfrak{h}^*$.*

(i) $\dim \mathrm{Hom}_{\mathfrak{g}}(M(\mu), M)$ *and* $\dim \mathrm{Hom}_{\mathfrak{g}}(M, M^*(\mu))$ *are less than or equal to* $[M : L(\mu)]$.

(ii) *Assume that if $\xi \in \mathfrak{h}^*$ satisfies $[M : L(\xi)] \neq 0$ and $[M(\xi) : L(\mu)] \neq 0$, then $\xi = \mu$. Assume further $[M : L(\mu)] \neq 0$. Then neither* $\mathrm{Hom}_{\mathfrak{g}}(M(\mu), M)$ *nor* $\mathrm{Hom}_{\mathfrak{g}}(M, M^*(\mu))$ *vanishes.*

Proof. (i) is obvious. Let us prove (ii). Consider the set $\mathcal{A}$ of submodules R of M satisfying $[R : L(\mu)] = 0$. There exists the largest element K of $\mathcal{A}$ with respect to the inclusion relation. Set $N = M/K$.

We shall prove $N_{\mu+\gamma} = 0$ for any $\gamma \in Q^+ \setminus \{0\}$. Assume that there exist some $\gamma \in Q^+ \setminus \{0\}$ such that $N_{\mu+\gamma} \neq 0$. Since N is an object of $\mathbb{O}$, there exists finitely many such γ. Take $\gamma \in Q^+ \setminus \{0\}$ such that $N_{\mu+\gamma} \neq 0$ and $N_{\mu+\gamma+\delta} = 0$ for any $\delta \in Q^+ \setminus \{0\}$. Then we have $[N : L(\mu+\gamma)] > 0$. Let N' be the $\mathfrak{g}$-submodule of N generated by $N_{\mu+\gamma}$. By the maximality of K we have $[N' : L(\mu)] \neq 0$. Hence, $[M(\mu+\gamma) : L(\mu)] \neq 0$ by the construction of N'. This contradicts

$$[M : L(\mu+\gamma)] \geq [N : L(\mu+\gamma)] > 0.$$

Hence $N_{\mu+\gamma} = 0$ for any $\gamma \in Q^+ \setminus \{0\}$, which implies

$$\mathrm{Hom}(M(\mu), N^*) = \{u \in N^*_\mu ; hu = \mu(h)u \quad \text{for any } h \in \mathfrak{h}\}.$$

Since $\dim N^*_\mu \geq [N : L(\mu)] = [M : L(\mu)] > 0$, $\mathrm{Hom}(M(\mu), N^*)$ does not vanish. Hence $\mathrm{Hom}(M(\mu), M^*)$ which contains $\mathrm{Hom}(M(\mu), N^*)$, does not vanish either. By applying the same argument to M^* we have $\mathrm{Hom}(M(\mu), M) \neq 0$. □

2.4. Enright functor for non-integral weights. In order to obtain some results on Verma modules (Proposition 2.4.8), we construct a version of Enright functor with non-integral weights (see Enright [8], Deodhar [5]).

Since the action of $\mathrm{ad}(f_i)$ on $U(\mathfrak{g})$ is locally nilpotent, the ring $U(\mathfrak{g})[f_i^{-1}]$, a localization of $U(\mathfrak{g})$ by f_i, is well-defined. It contains $U(\mathfrak{g})$ as a subring. Similarly we can consider a $U(\mathfrak{g})$-bimodule $U(\mathfrak{g})f_i^{a+\mathbb{Z}}$ for any scalar $a \in \mathbb{C}$. As a left $U(\mathfrak{g})$-module it is given by

(2.4.1) $$U(\mathfrak{g})f_i^{a+\mathbb{Z}} = \varinjlim_n U(\mathfrak{g})f_i^{a-n},$$

where $U(\mathfrak{g})f_i^{a-n}$ is a rank one free $U(\mathfrak{g})$-module generated by the symbol f_i^{a-n} and the homomorphism $U(\mathfrak{g})f_i^{a-n} \to U(\mathfrak{g})f_i^{a-n-1}$ is given by $f_i^{a-n} \mapsto f_i f_i^{a-n-1}$. The left module $U(\mathfrak{g})f_i^{a-n}$ is naturally identified with a submodule of $U(\mathfrak{g})f_i^{a+\mathbb{Z}}$ and we have $U(\mathfrak{g})f_i^{a+\mathbb{Z}} = \bigcup_{n\in\mathbb{Z}} U(\mathfrak{g})f_i^{a-n}$. Its right module structure is given by

$$(2.4.2)\quad f_i^{a+m}P = \sum_{k=0}^{\infty} \binom{a+m}{k} (\mathrm{ad}(f_i)^k P) f_i^{a+m-k} \text{ for any } m \in \mathbb{Z} \text{ and any } P \in U(\mathfrak{g}).$$

As a right $U(\mathfrak{g})$-module, we also have

$$U(\mathfrak{g})f_i^{a+\mathbb{Z}} = \varinjlim_n f_i^{a-n}U(\mathfrak{g}) = \bigcup_n f_i^{a-n}U(\mathfrak{g}).$$

By (2.4.2) we have

$$(2.4.3)\qquad Pf_i^{a+m} = \sum_{k=0}^{\infty}(-1)^k \binom{a+m}{k} f_i^{a+m-k}(\mathrm{ad}(f_i)^k P).$$

In particular, we have

$$(2.4.4)\qquad e_i^n f_i^a = \sum_{k=0}^{n} f_i^{a-k} e_i^{n-k} (k!)^2 \binom{n}{k}\binom{a}{k}\binom{h_i+n-a}{k}.$$

The $U(\mathfrak{g})$-bimodule $U(\mathfrak{g})f_i^{a+\mathbb{Z}}$ depends only on a modulo $\mathbb{Z}$.

LEMMA 2.4.1. *For a, $b \in \mathbb{C}$, the map $f_i^{a+b+n} \mapsto f_i^{a+n} \otimes f_i^b = f_i^a \otimes f_i^{b+n}$ $(n \in \mathbb{Z})$ defines an isomorphism of $U(\mathfrak{g})$-bimodules*

$$U(\mathfrak{g})f_i^{a+b+\mathbb{Z}} \to U(\mathfrak{g})f_i^{a+\mathbb{Z}} \otimes_{U(\mathfrak{g})} U(\mathfrak{g})f_i^{b+\mathbb{Z}}.$$

Since the proof is straightforward, we omit it. Hence $\bigoplus_{a\in\mathbb{C}/\mathbb{Z}} U(\mathfrak{g})f_i^{a+\mathbb{Z}}$ has a structure of a ring containing $U(\mathfrak{g})$.

For any $\mathfrak{g}$-module M, $U(\mathfrak{g})f_i^{a+\mathbb{Z}} \otimes_{U(\mathfrak{g})} M$ is isomorphic to the inductive limit

$$M \xrightarrow{f_i} M \xrightarrow{f_i} M \xrightarrow{f_i} \cdots$$

as a vector space. Hence we obtain the following result.

PROPOSITION 2.4.2. *The functor $M \to U(\mathfrak{g})f_i^{a+\mathbb{Z}} \otimes_{U(\mathfrak{g})} M$ is an exact functor from $\mathbb{M}(\mathfrak{g})$ into itself.*

Let $\mathfrak{a}$ be a Lie algebra, and let $\mathfrak{k}$ be its subalgebra such that $\mathfrak{a}$ is locally $\mathfrak{k}$-finite with respect to the adjoint action. Then for any $\mathfrak{a}$-module M, the subspace $\{m \in M\,;\, \dim U(\mathfrak{k})m < \infty\}$ is an $\mathfrak{a}$-submodule of M. In particular, for a $\mathfrak{g}$-module M its subspace $\{m \in M\,;\, \dim \mathbb{C}[e_i]m < \infty\}$ is a $\mathfrak{g}$-submodule of M. For $a \in \mathbb{C}$, we define a functor

$$(2.4.5)\qquad T_i(a) : \mathbb{M}(\mathfrak{g}) \to \mathbb{M}(\mathfrak{g})$$

by

$$T_i(a)(M) = \{u \in U(\mathfrak{g})f_i^{a+\mathbb{Z}} \otimes_{U(\mathfrak{g})} M\,;\, \dim \mathbb{C}[e_i]u < \infty\}$$

for $M \in \mathrm{Ob}(\mathbb{M}(\mathfrak{g}))$. It is obviously a left exact functor.

For $a \in \mathbb{C}$, let $\mathbb{M}_i^a(\mathfrak{g})$ be the category of locally $\mathbb{C}[e_i]$-finite $U(\mathfrak{g})$-modules M such that M has a weight decomposition

$$M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda,$$

the action of $\mathfrak{h}$ on M is semisimple, and $M_\lambda = 0$ unless $\langle h_i, \lambda\rangle - a \in \mathbb{Z}$.

LEMMA 2.4.3. *For $a \in \mathbb{C}$, $M \in \operatorname{Ob}(\mathbb{M}_i^a(\mathfrak{g}))$ and $u \in M$, we have $f_i^{a+n} \otimes u \in T_i(a)(M)$ for $n \gg 0$.*

Proof. We may assume that u has weight λ such that $\langle h_i, \lambda\rangle = a$ without loss of generality. We have $e_i^{m_0}u = 0$ for some $m_0 > 0$. We have

$$\begin{aligned} e_i^m(f_i^{a+n} \otimes u) &= \sum_{k=0}^{m} f_i^{a+n-k} e_i^{m-k}(k!)^2 \binom{m}{k}\binom{a+n}{k}\binom{h_i+m-n-a}{k} \otimes u \\ &= \sum_{k=0}^{m} f_i^{a+n-k}(k!)^2 \binom{m}{k}\binom{a+n}{k}\binom{m-n}{k} \otimes e_i^{m-k}u. \end{aligned}$$

Assume $m \geq n \geq m_0$. Then each term survives only when $k \leq m-n$ and $m-k < m_0$, or equivalently $m - m_0 < k \leq m-n$, and there is no such k. Hence $e_i^m(f_i^{a+n} \otimes u) = 0$ for $m \geq n \geq m_0$. □

For $a \in \mathbb{C}$, the functor $T_i(a)$ sends $\mathbb{M}_i^a(\mathfrak{g})$ to $\mathbb{M}_i^{-a}(\mathfrak{g})$.

The morphism of $U(\mathfrak{g})$-bimodules

$$U(\mathfrak{g}) \to U(\mathfrak{g})f_i^{-a+\mathbb{Z}} \otimes_{U(\mathfrak{g})} U(\mathfrak{g})f_i^{a+\mathbb{Z}} \qquad (1 \mapsto f_i^{-a} \otimes f_i^a) \tag{2.4.6}$$

(see Lemma 2.4.1) induces a morphism of functors

$$\mathrm{id}_{\mathbb{M}_i^a(\mathfrak{g})} \to T_i(-a) \circ T_i(a). \tag{2.4.7}$$

Indeed, for $M \in \operatorname{Ob}(\mathbb{M}_i^a(\mathfrak{g}))$, (2.4.6) gives a morphism

$$M \to U(\mathfrak{g})f_i^{-a+\mathbb{Z}} \otimes_{U(\mathfrak{g})} U(\mathfrak{g})f_i^{a+\mathbb{Z}} \otimes_{U(\mathfrak{g})} M.$$

For any $u \in M$, the image of u by the above homomorphism is equal to $f_i^{-a-n} \otimes f_i^{a+n} \otimes u$, and Lemma 2.4.3 implies that $f_i^{a+n} \otimes u$ belongs to $T_i(a)(M)$ for $n \gg 0$. Hence the image of the above homomorphism is contained in $T_i(-a) \circ T_i(a)(M) \subset U(\mathfrak{g})f_i^{-a+\mathbb{Z}} \otimes_{U(\mathfrak{g})} T_i(a)(M)$.

Define the ideal $\mathfrak{n}^-(i)$ of $\mathfrak{n}^-$ by

$$\mathfrak{n}^-(i) = \mathfrak{n}(\Delta^- \setminus \{-\alpha_i\}). \tag{2.4.8}$$

By the PBW theorem, we have $U(\mathfrak{n}^-) = U(\mathfrak{n}^-(i)) \otimes \mathbb{C}[f_i]$, which implies

$$U(\mathfrak{n}^-(i)) \otimes \mathbb{C}[f_i, f_i^{-1}]f_i^a \otimes U(\mathfrak{b}) \xrightarrow{\sim} U(\mathfrak{g})f_i^{a+\mathbb{Z}}.$$

The following lemma follows immediately from this isomorphism.

LEMMA 2.4.4. *For any $\lambda \in \mathfrak{h}^*$, we have an isomorphism*

$$U(\mathfrak{n}^-(i)) \otimes \mathbb{C}[f_i, f_i^{-1}]f_i^a \otimes \mathbb{C}u_\lambda \xrightarrow{\sim} U(\mathfrak{g})f_i^{a+\mathbb{Z}} \otimes_{U(\mathfrak{g})} M(\lambda).$$

LEMMA 2.4.5. *For any $\lambda \in \mathfrak{h}^*$, the element $f_i^{\langle h_i,\lambda\rangle+1} \otimes u_\lambda$ of $U(\mathfrak{g})f_i^{\langle h_i,\lambda\rangle+\mathbb{Z}} \otimes_{U(\mathfrak{g})} M(\lambda)$ is a highest weight vector with weight $s_i \circ \lambda$. Here $\circ$ is the shifted action defined in* (2.1.19).

Proof. Set $\lambda_i = \langle h_i, \lambda\rangle$. We have for $j \neq i$

$$e_j(f_i^{\lambda_i+1} \otimes u_\lambda) = f_i^{\lambda_i+1} e_j \otimes u_\lambda = f_i^{\lambda_i+1} \otimes e_j u_\lambda = 0.$$

If $j = i$, then

$$e_i(f_i^{\lambda_i+1} \otimes u_\lambda) = (f_i^{\lambda_i+1} e_i + (\lambda_i+1) f_i^{\lambda_i}(h_i - \lambda_i)) \otimes u_\lambda = 0.$$

□

PROPOSITION 2.4.6. *Assume that $\lambda \in \mathfrak{h}^*$ satisfies $a = \langle h_i, \lambda+\rho\rangle \notin \mathbb{Z}_{>0}$. Then we have*

$$T_i(a)(M(\lambda)) = U(\mathfrak{g})(f_i^{\langle h_i,\lambda+\rho\rangle} \otimes u_\lambda) \cong M(s_i \circ \lambda).$$

Proof. We have

$$T_i(a)(M(\lambda)) = \{u \in U(\mathfrak{g})f_i^{a+\mathbb{Z}} \otimes_{U(\mathfrak{g})} M\,;\ e_i^m u = 0 \text{ for a sufficiently large } m\}.$$

By the preceding lemma, $f_i^a \otimes u_\lambda$ is a highest weight vector of $T_i(a)(M(\lambda))$. It is enough to show that $T_i(a)(M(\lambda))$ is generated by this vector.

By Lemma 2.4.4, any $v \in T_i(a)(M(\lambda))$ can be written in a unique way

$$v = \sum_{n\in\mathbb{Z}} P_n f_i^{a+n} \otimes u_\lambda$$

for $P_n \in U(\mathfrak{n}^-(i))$. Here P_n vanishes except for finitely many n.
Take a positive integer m such that $e_i^m v = 0$. Then we have

$$\begin{aligned}
0 &= e_i^m v\\
&= \sum_n \sum_{k=0}^m \binom{m}{k} (\mathrm{ad}(e_i)^{m-k} P_n) e_i^k f_i^{a+n} \otimes u_\lambda\\
&= \sum_n \sum_{k=0}^m \binom{m}{k} (\mathrm{ad}(e_i)^{m-k} P_n) \sum_{\nu=0}^k f_i^{a+n-\nu} e_i^{k-\nu}\\
&\qquad\qquad (\nu!)^2 \binom{k}{\nu}\binom{a+n}{\nu}\binom{h_i+k-a-n}{\nu} \otimes u_\lambda\\
&= \sum_n \sum_{k=0}^m \binom{m}{k} (\mathrm{ad}(e_i)^{m-k} P_n) f_i^{a+n-k} \binom{k-1-n}{k} (k!)^2 \binom{a+n}{k} \otimes u_\lambda.
\end{aligned}$$

Rewriting this equality, we have

$$\begin{aligned}
0 &= \sum_n \sum_{k=0}^m \binom{m}{k} (\mathrm{ad}(e_i)^{m-k} P_n) f_i^{n-k} \binom{k-1-n}{k} (k!)^2 \binom{a+n}{k}\\
&= \sum_n \sum_{k=0}^m \binom{m}{k} (\mathrm{ad}(e_i)^{m-k} P_{n+k}) f_i^{n} \binom{-1-n}{k} (k!)^2 \binom{a+n+k}{k}.
\end{aligned}$$

The vanishing of the coefficient of f_i^n implies

$$\sum_{k=0}^{m}\binom{m}{k}\binom{-1-n}{k}(k!)^2\binom{a+n+k}{k}(\operatorname{ad}(e_i)^{m-k}P_{n+k})=0 \tag{2.4.9}$$

for any n.

Now we shall prove that $P_n=0$ for $n<0$. Assuming the contrary we take the largest $c>0$ such that $P_{-c}\neq 0$. By taking $n=-c-m$ in (2.4.9), only $k=m$ survives, and we obtain

$$\binom{-1-n}{m}(m!)^2\binom{a+n+m}{m}P_{-c}=0.$$

Hence we obtain

$$\binom{-1+c+m}{m}\binom{a-c}{m}=0.$$

Since $-1+c+m\geq m$, $\binom{-1+c+m}{m}$ does not vanish, and $\binom{a-c}{m}$ must vanish. This means that $a-c$ is an integer and satisfies $0\leq a-c<m$. This leads to the contradiction $a\geq c>0$. Hence we have P_n=0 for $n<0$, and we conclude $v\in U(\mathfrak{n}^-)f_i^a\otimes u_\lambda=U(\mathfrak{n}^-)(f_i^a\otimes u_\lambda)$. □

Proposition 2.4.6 implies the following proposition.

PROPOSITION 2.4.7. *Assume* $a\equiv\langle h_i,\lambda\rangle\not\equiv 0 \bmod \mathbb{Z}$. *Then the morphism* (2.4.7) *induces an isomorphism*

$$M(\lambda)\xrightarrow{\sim}T_i(-a)\circ T_i(a)(M(\lambda)).$$

Now we are ready to prove the following proposition used later.

PROPOSITION 2.4.8. *Assume that* λ, $\mu\in\mathfrak{h}^*$ *satisfy* $\langle h_i,\lambda\rangle\notin\mathbb{Z}$. *Then we have*

$$\operatorname{Hom}(M(s_i\circ\mu),M(s_i\circ\lambda))\simeq\operatorname{Hom}(M(\mu),M(\lambda)).$$

Proof. If $\lambda-\mu$ is not in the root lattice Q, then the both sides vanish. If $\lambda-\mu\in Q$, then $\langle h_i,\mu\rangle\equiv\langle h_i,\lambda\rangle\not\equiv 0 \bmod \mathbb{Z}$. Hence the assertion follows from the preceding proposition. □

2.5. Embeddings of Verma modules. We shall use the following result of Kac-Kazhdan.

THEOREM 2.5.1 ([10]). *Let* $\lambda,\mu\in\mathfrak{h}^*$. *Then the following three conditions are equivalent.*

(i) *The irreducible highest weight module* $L(\mu)$ *with highest weight* μ *appears as a subquotient of* $M(\lambda)$.

(ii) *There exist a sequence of positive roots* $\{\beta_k\}_{k=1}^l$, *a sequence of positive integers* $\{n_k\}_{k=1}^l$ *and a sequence of weights* $\{\lambda_k\}_{k=0}^l$ *such that* $\lambda_0=\lambda$, $\lambda_l=\mu$ *and* $\lambda_k=\lambda_{k-1}-n_k\beta_k$, $2(\beta_k,\lambda_{k-1}+\rho)=n_k(\beta_k,\beta_k)$ *for* $k=1,\dots,l$.

(iii) *There exists a non-zero homomorphism* $M(\mu)\to M(\lambda)$.

Note that any non-zero homomorphism from a Verma module to another Verma module must be a monomorphism. The implication (ii)⇒(iii) is not explicitly stated in Kac-Kazhdan [10]. But it easily follows from Lemma 3.3 (b) in Kac-Kazhdan [10] and (i)⇔(ii).

We use also the following result in Kashiwara [14].

PROPOSITION 2.5.2. *For $\lambda, \mu \in \mathfrak{h}^*$ and $i \in I$, we assume $\langle h_i, \mu+\rho\rangle \in \mathbb{Z}_{\geq 0}$ (which implies $M(s_i \circ \mu) \subset M(\mu)$) and $\langle h_i, \lambda+\rho\rangle \notin \mathbb{Z}_{<0}$. Then we have*

$$\mathrm{Ext}^1_{\mathfrak{g}}(M(\mu)/M(s_i \circ \mu), M(\lambda)) = 0.$$

Let $\mathcal{K}$ denote the set of $\lambda \in \mathfrak{h}^*$ satisfying the following two conditions.

$$2(\beta, \lambda+\rho) \neq (\beta, \beta) \text{ for any positive imaginary root } \beta, \tag{2.5.1}$$

$$\{\beta \in \Delta^+_{\mathrm{re}}; (\beta^\vee, \lambda+\rho) \in \mathbb{Z}_{<0}\} \text{ is a finite set.} \tag{2.5.2}$$

The condition (2.5.2) implies that there exists $w \in W(\lambda)$ such that $w \circ \lambda + \rho$ is integrally dominant (i.e. $(\beta^\vee, w \circ \lambda + \rho) \notin \mathbb{Z}_{<0}$ for any $\beta \in \Delta^+_{\mathrm{re}}$).

If λ in Theorem 2.5.1 satisfies the condition (2.5.1), then β_k in (ii) must be a real positive root. This easily follows from the fact that $n\beta$ is an imaginary root for any positive integer n and any imaginary root β.

Note that $\mathcal{K}$ is invariant by the shifted action of W.

THEOREM 2.5.3. *For $\lambda \in \mathcal{K}$ we have*

$$\dim \mathrm{Hom}_{\mathfrak{g}}(M(\mu), M(\lambda)) \leq 1$$

for any $\mu \in \mathfrak{h}^*$.

Proof. There exists an embedding $M(\lambda) \hookrightarrow M(\lambda')$ for some $\lambda' \in W(\lambda) \circ \lambda$ such that $\lambda' + \rho$ is integrally dominant. Hence we may assume that $\lambda + \rho$ is integrally dominant from the beginning.

We assume that $\mathrm{Hom}(M(\mu), M(\lambda))$ is not zero. Then by Theorem 2.5.1, there exists $w \in W(\lambda)$ such that $\mu = w \circ \lambda$.

We shall argue by the induction on the lenght of w.

If $w = 1$, then it is evident. Assuming $w \neq 1$, let us take $\alpha \in \Pi(\lambda)$ such that $l_\lambda(s_\alpha w) < l_\lambda(w)$, which is equivalent to $w^{-1}\alpha \in \Delta^-(\lambda)$. Since $\lambda + \rho$ is integrally dominant, $(w^{-1}\alpha^\vee, \lambda+\rho) \leq 0$. Since we may assume $s_\alpha \circ \mu \neq \mu$, we have

$$(\alpha^\vee, \mu+\rho) = (w^{-1}\alpha^\vee, \lambda+\rho) \in \mathbb{Z}_{<0}. \tag{2.5.3}$$

Now we shall argue by the induction on $\mathrm{ht}(\alpha)$.

(1) Case $\mathrm{ht}(\alpha) = 1$. In this case, $\alpha = \alpha_i$ for some $i \in I$. Then we have $M(s_i \circ \mu) \supset M(\mu)$. Since $\langle h_i, \lambda+\rho\rangle \in \mathbb{Z}_{\geq 0}$, Proposition 2.5.2 implies

$$\mathrm{Ext}^1(M(s_i \circ \mu)/M(\mu), M(\lambda)) = 0.$$

Therefore the following sequence is exact.

$$\mathrm{Hom}(M(s_i \circ \mu), M(\lambda)) \to \mathrm{Hom}(M(\mu), M(\lambda)) \to 0.$$

Since $\dim \mathrm{Hom}(M(s_i \circ \mu), M(\lambda)) \leq 1$ by the induction hypothesis on the length of w, we obtain $\dim \mathrm{Hom}(M(\mu), M(\lambda)) \leq 1$.

(2) Case $\mathrm{ht}(\alpha) > 1$. Take i such that $\langle h_i, \alpha\rangle > 0$. Then $\alpha_i \notin \Delta(\lambda)$ by Lemma 2.2.3. Hence we have

$$\langle h_i, \lambda\rangle \notin \mathbb{Z}. \tag{2.5.4}$$

Set $\lambda' = s_i \circ \lambda$. Then $s_i\Delta(\lambda) = \Delta(\lambda')$, and $s_i\Pi(\lambda) = \Pi(\lambda')$ by Lemma 2.2.2. Moreover $\lambda' + \rho$ is also integrally dominant. Then $w' = s_i w s_i \in W(\lambda')$ and $l_{\lambda'}(w') = l_\lambda(w)$. Set $\alpha' = s_i\alpha$. Then $\alpha' \in \Pi(\lambda')$, $\mathrm{ht}(\alpha') < \mathrm{ht}(\alpha)$ and $l_{\lambda'}(s_{\alpha'}w') < l_{\lambda'}(w')$. We have also $\mu' = s_i \circ \mu = w' \circ \lambda'$. Hence the induction hypothesis on $\mathrm{ht}(\alpha)$ implies

$$\dim \mathrm{Hom}(M(\mu'), M(\lambda')) \leq 1.$$

By (2.5.4) we can apply Proposition 2.4.8 to deduce

$$\mathrm{Hom}(M(\mu'), M(\lambda')) \simeq \mathrm{Hom}(M(\mu), M(\lambda)).$$

Thus we obtain the desired result $\dim \mathrm{Hom}(M(\mu), M(\lambda)) \leq 1$. □

We denote by $\mathcal{K}_{\mathrm{reg}}$ the set of $\lambda \in \mathcal{K}$ subject to the following condition:

(2.5.5) If $w \in W$ satisfies $w \circ \lambda = \lambda$, then $w = 1$.

In particular, this condition implies

$$(\lambda + \rho, \alpha^\vee) \neq 0 \text{ for any } \alpha \in \Delta_{\mathrm{re}}.$$

Define a subset $\mathcal{K}^+_{\mathrm{reg}}$ of $\mathcal{K}_{\mathrm{reg}}$ by

$$\mathcal{K}^+_{\mathrm{reg}} = \{\lambda \in \mathcal{K}_{\mathrm{reg}}\,;\ (\lambda + \rho, \alpha^\vee) > 0 \quad \text{for any } \alpha \in \Delta^+(\lambda)\}. \tag{2.5.6}$$

LEMMA 2.5.4. *We have* $W \circ \mathcal{K}_{\mathrm{reg}} = \mathcal{K}_{\mathrm{reg}}$ *and* $\mathcal{K}_{\mathrm{reg}} = \bigsqcup_{\lambda \in \mathcal{K}^+_{\mathrm{reg}}} W(\lambda) \circ \lambda$.

The proof is standard by using the results in §2.2 and omitted.

By Theorem 2.5.1 and Theorem 2.5.3 we have the following proposition.

PROPOSITION 2.5.5. *Let* $\lambda \in \mathcal{K}^+_{\mathrm{reg}}$.

(i) *For* $x \in W(\lambda)$ *and* $\mu \in \mathfrak{h}^*$ *we have* $[M(x \circ \lambda) : L(\mu)] \neq 0$ *if and only if* $\mu = y \circ \lambda$ *for some* $y \in W(\lambda)$ *satisfying* $y \geq_\lambda x$.

(ii) *For* $x, y \in W(\lambda)$ *we have* $\dim \mathrm{Hom}(M(y \circ \lambda), M(x \circ \lambda)) = 1$ *or* 0 *according to whether* $y \geq_\lambda x$ *or not.*

COROLLARY 2.5.6. *Let* $\lambda \in \mathcal{K}^+_{\mathrm{reg}}$, $x \in W$ *and* $\mu \in \mathfrak{h}^*$. *Then* $[M(x \circ \lambda) : L(\mu)] \neq 0$ *implies* $\mu = y \circ \lambda$ *for some* $y \in xW(\lambda)$ *satisfying* $y \geq x$.

Proof. Assume $[M(x\circ\lambda) : L(\mu)] \neq 0$. Take $z_1 \in W(x\circ\lambda)$ such that $\lambda' = z_1^{-1}x\circ\lambda \in \mathcal{K}^+_{\mathrm{reg}}$. Then we have $z_1 \in W(\lambda')$ and $x \circ \lambda = z_1 \circ \lambda'$. By Proposition 2.5.5 there exists some $z_2 \in W(\lambda')$ such that $\mu = z_2 \circ \lambda'$ and $z_2 \geq_{\lambda'} z_1$. Setting $w = z_1^{-1}x$, $y = z_2 w$ we have $x = z_1 w$ and $\mu = y \circ \lambda$. Since $y = z_2 z_1^{-1} x \in W(\lambda')x = xW(\lambda)$, the assertion follows from the following lemma.

LEMMA 2.5.7. *Assume that* $\lambda, \lambda' \in \mathcal{K}^+_{\mathrm{reg}}$ *and* $w \in W$ *satisfy* $\lambda' = w \circ \lambda$. *Then for* $z_1, z_2 \in W(\lambda')$ *such that* $z_2 \geq_{\lambda'} z_1$ *we have* $z_2 w \geq z_1 w$.

Proof. For $\lambda \in \mathcal{K}^+_{\mathrm{reg}}$, we have $\Delta^+(\lambda) = \{\alpha \in \Delta_{\mathrm{re}}; (\alpha^\vee, \lambda + \rho) \in \mathbb{Z}_{>0}\}$. This implies $w^{-1}\Delta^+(\lambda') = \Delta^+(\lambda) \subset \Delta^+$. Then it is enough to apply Lemma 2.2.11. □

For a subset Ω of $\mathcal{K}_{\mathrm{reg}}$ we denote by $\mathbb{O}\{\Omega\}$ the full subcategory of $\mathbb{O}$ consisting of $M \in \mathrm{Ob}(\mathbb{O})$ such that any irreducible subquotient of M is isomorphic to $L(\lambda)$ for some $\lambda \in \Omega$. For $\lambda \in \mathcal{K}_{\mathrm{reg}}$ we set

$$\mathbb{O}[\lambda] = \mathbb{O}\{W(\lambda) \circ \lambda\}, \qquad \mathbb{O}(\lambda) = \mathbb{O}\{W \circ \lambda\}. \tag{2.5.7}$$

By the definition, for any $\lambda \in \mathcal{K}_{\mathrm{reg}}$, we have

$$\mathbb{O}[\lambda] = \mathbb{O}[w \circ \lambda] \quad \text{for any } w \in W(\lambda), \tag{2.5.8}$$

$$\mathbb{O}(\lambda) = \mathbb{O}(w \circ \lambda) \text{ for any } w \in W. \tag{2.5.9}$$

By Proposition 2.5.5 we have

$$M(\lambda) \in \mathrm{Ob}(\mathbb{O}[\lambda]) \quad \text{for any } \lambda \in \mathcal{K}_{\mathrm{reg}}. \tag{2.5.10}$$

By Lemma 2.3.2 and Corollary 2.5.6 we have the following lemma.

LEMMA 2.5.8. *Let $\lambda \in \mathcal{K}^+_{\mathrm{reg}}$ and $w \in W$. Assume that $M \in \mathrm{Ob}(\mathbb{O})$ satisfies the conditions*

$$[M : L(w \circ \lambda)] \neq 0,$$
$$[M : L(y \circ \lambda)] = 0 \textit{ for any } y \in wW(\lambda) \textit{ such that } y < w.$$

Then neither $\mathrm{Hom}_{\mathfrak{g}}(M(w \circ \lambda), M)$ *nor* $\mathrm{Hom}_{\mathfrak{g}}(M, M^*(w \circ \lambda))$ *vanishes.*

We shall use later the following result of S. Kumar [20] (a generalization of a result in Deodhar-Gabber-Kac [6]).

THEOREM 2.5.9. *Any object M of $\mathbb{O}\{\mathcal{K}_{\mathrm{reg}}\}$ decomposes uniquely into*

$$M = \bigoplus_{\lambda \in \mathcal{K}^+_{\mathrm{reg}}} M^\lambda \quad (M^\lambda \in \mathrm{Ob}(\mathbb{O}[\lambda])).$$

In [20], the theorem is proved for M with a semisimple action of $\mathfrak{h}$. However the same arguments can be applied in our situation.

For $\lambda \in \mathcal{K}_{\mathrm{reg}}$ we denote by

$$P_\lambda : \mathbb{O}\{\mathcal{K}_{\mathrm{reg}}\} \to \mathbb{O}[\lambda] \tag{2.5.11}$$

the projection functor.

We define a new abelian category $\widetilde{\mathbb{O}}$ by

$$\widetilde{\mathbb{O}} = \prod_{\lambda \in \mathcal{K}^+_{\mathrm{reg}}} \mathbb{O}[\lambda]. \tag{2.5.12}$$

We denote by the same symbol P_λ the projection functor $P_\lambda : \widetilde{\mathbb{O}} \to \mathbb{O}[\lambda]$. It is an exact functor. By the definition we have

$$\mathrm{Hom}_{\widetilde{\mathbb{O}}}(M, N) = \prod_{\lambda \in \mathcal{K}^+_{\mathrm{reg}}} \mathrm{Hom}_{\mathfrak{g}}(P_\lambda(M), P_\lambda(N)) \quad \text{for } M,\, N \in \widetilde{\mathbb{O}}.$$

The category $\mathbb{O}\{\mathcal{K}_{\mathrm{reg}}\}$ can be regarded as a full subcategory of $\widetilde{\mathbb{O}}$. For $M \in \widetilde{\mathbb{O}}$ and $\lambda \in \mathcal{K}_{\mathrm{reg}}$ we set

$$[M : L(\lambda)] = [P_\lambda(M) : L(\lambda)].$$

For a subset Ω of $\mathcal{K}_{\mathrm{reg}}$, we set

$$\widetilde{\mathbb{O}}\{\Omega\} = \prod_{\lambda \in \mathcal{K}^+_{\mathrm{reg}}} \mathbb{O}\{\Omega \cap (W(\lambda) \circ \lambda)\}, \tag{2.5.13}$$

and for $\lambda \in \mathcal{K}_{\mathrm{reg}}$,

$$\widetilde{\mathbb{O}}(\lambda) = \widetilde{\mathbb{O}}\{W \circ \lambda\}. \tag{2.5.14}$$

3. Twisted D-modules. We shall give a generalization of the theory of D-modules on infinite-dimensional schemes developped in [14] and [16] to that of twisted left D-modules (modules over a TDO-ring). Since the arguments are analogous to the original non-twisted case, we only state the results and omit proofs.

3.1. Finite-dimensional case. For a scheme X we denote by $\mathcal{O}_X$ the structure sheaf. For a scheme X smooth (in particular quasi-compact and separated) over $\mathbb{C}$, we denote by Ω_X, Θ_X and D_X the canonical sheaf, the sheaf of vector fields, and the sheaf of rings of differential operators on X, respectively.

Let X be a scheme smooth over $\mathbb{C}$. A TDO-ring on X is by definition a sheaf A of rings on X containing $\mathcal{O}_X$ as a subring such that there exists an increasing filtration $F = \{F_nA\}_{n\in\mathbb{Z}}$ of the abelian sheaf A satisfying the following conditions.

(3.1.1) $F_nA = 0$ for $n < 0$.

(3.1.2) $F_nA \cdot F_mA \subset F_{n+m}A$.

(3.1.3) $[F_nA, F_mA] \subset F_{n+m-1}A$.

(3.1.4) $F_0A = \mathcal{O}_X$.

(3.1.5) The homomorphism $\mathrm{gr}_1 A \to \Theta_X$ ($P \bmod F_0A \mapsto (\mathcal{O}_X \ni a \mapsto [P,a] \in \mathcal{O}_X)$) of $\mathcal{O}_X$-modules induced by (3.1.3), (3.1.4) is an isomorphism.

(3.1.6) The homomorphism $S_{\mathcal{O}_X}(\mathrm{gr}_1 A) \to \mathrm{gr}A$ of commutative $\mathcal{O}_X$-algebras is an isomorphism.

Here we set $\mathrm{gr}_nA = F_nA/F_{n-1}A$, $\mathrm{gr}A = \bigoplus_n \mathrm{gr}_nA$, and $S_{\mathcal{O}_X}(\mathrm{gr}_1A)$ denotes the symmetric algebra of the locally free $\mathcal{O}_X$-module gr_1A. The filtration F is uniquely determined by the above conditions, and it is called the order filtration. A TDO-ring is quasi-coherent over $\mathcal{O}_X$ with respect to its left and right $\mathcal{O}_X$-module structures.

Let A be a TDO-ring on a scheme X smooth over $\mathbb{C}$. For a coherent (left) A-module $\mathcal{M}$ we can define its characteristic variety $\mathrm{Ch}(M)$ as a subvariety of the cotangent bundle T^*X as in the case $A = D_X$. A coherent A-module $\mathcal{M}$ is called holonomic if $\dim \mathrm{Ch}(\mathcal{M}) \le \dim X$. We denote by $\mathbb{M}_h(A)$ the category of holonomic A-modules, and by $D^b_h(A)$ the derived category consisting of bounded complexes of quasi-coherent A-modules with holonomic cohomologies. Set

$$A^{-\sharp} = \Omega_X^{\otimes -1} \otimes_{\mathcal{O}_X} A^{\mathrm{op}} \otimes_{\mathcal{O}_X} \Omega_X,$$

where A^{op} denotes the opposite ring of A. Then $A^{-\sharp}$ is also a TDO-ring. We define the duality functor

$$\mathbf{D} : D^b_h(A) \to D^b_h(A^{-\sharp})^{\mathrm{op}}$$

by

$$\mathbf{D}\mathcal{M} = \mathbb{R}\mathcal{H}om_A(\mathcal{M}, A) \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1}[\dim X].$$

Let $f : X \to Y$ be a morphism of smooth schemes over $\mathbb{C}$, and let A be a TDO-ring on Y. Set

$$A_{X\to Y} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}A, \qquad A_{Y\leftarrow X} = f^{-1}A \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\Omega_Y^{\otimes -1} \otimes_{f^{-1}\mathcal{O}_Y} \Omega_X.$$

Define the subring $f^\sharp A$ of $\mathcal{E}nd_{f^{-1}A}(A_{X\to Y})$ by

(3.1.7) $f^\sharp A = \bigcup_{n\in\mathbb{N}} F_n(f^\sharp A)$,

(3.1.8) $F_n(f^\sharp A) = 0$ for $n < 0$,
(3.1.9) $F_n(f^\sharp A) = \{P \in \mathcal{E}nd_{f^{-1}A}(A_{X\to Y})\,;\, [P, \mathcal{O}_X] \subset F_{n-1}(f^\sharp A)\}$ for $n \geq 0$.

Then $f^\sharp A$ is a TDO-ring on X. Moreover, $A_{X\to Y}$ has a structure of an $(f^\sharp A, f^{-1}A)$-bimodule, and $A_{Y\leftarrow X}$ has a structure of an $(f^{-1}A, f^\sharp A)$-bimodule. We have

$$f^\sharp(A^{-\sharp}) = (f^\sharp A)^{-\sharp} \tag{3.1.10}$$

for any TDO-ring A. We define functors

$$\mathbb{D}f^\bullet : D_h^b(A) \to D_h^b(f^\sharp A), \tag{3.1.11}$$

$$\int_f : D_h^b(f^\sharp A) \to D_h^b(A), \qquad \int_{f!} : D_h^b(f^\sharp A) \to D_h^b(A) \tag{3.1.12}$$

by

$$\begin{aligned} &\mathbb{D}f^\bullet(\mathcal{M}) = A_{X\to Y} \otimes^{\mathbb{L}}_{f^{-1}A} f^{-1}M, \quad \textstyle\int_f \mathcal{M} = \mathbb{R}f_*(A_{Y\leftarrow X} \otimes^{\mathbb{L}}_{f^\sharp A} \mathcal{M}), \\ &\textstyle\int_{f!} = \mathbf{D} \circ \int_f \circ \mathbf{D}. \end{aligned} \tag{3.1.13}$$

We shall also use their variants

$$\mathbb{D}f^*,\ \mathbb{D}f^! : D_h^b(A) \to D_h^b(f^\sharp A), \qquad \mathbb{D}f_*,\ \mathbb{D}f_! : D_h^b(f^\sharp A) \to D_h^b(A) \tag{3.1.14}$$

given by

$$\begin{aligned} &\mathbb{D}f^* = \mathbf{D} \circ \mathbb{D}f^\bullet \circ \mathbf{D}, && \mathbb{D}f^! = \mathbb{D}f^\bullet\ [2(\dim X - \dim Y)], \\ &\mathbb{D}f_* = \textstyle\int_f\ [\dim Y - \dim X], && \mathbb{D}f_! = \textstyle\int_{f!}\ [\dim Y - \dim X]. \end{aligned}$$

3.2. Infinite-dimensional case. Now we shall study TDO-rings on infinite-dimensional varieties. We say that a scheme X over $\mathbb{C}$ satisfies the property (S) if $X \simeq \varprojlim_n S_n$ for some projective system $\{S_n\}_{n\in\mathbb{N}}$ satisfying the following conditions.

(3.2.1) The scheme S_n is smooth (in particular quasi-compact and separated) over $\mathbb{C}$ for any n.

(3.2.2) The morphism $p_{nm} : S_m \to S_n$ is affine and smooth for any $m \geq n$.

We call such $\{S_n\}_{n\in\mathbb{N}}$ a *smooth projective system* for X. For example, the infinite-dimensional affine space

$$\mathbb{A}^\infty = \varprojlim_n \mathbb{A}^n = \operatorname{Spec} \mathbb{C}[x_i\,; i \in \mathbb{N}]$$

satisfies the property (S).

Let $\mathcal{S}$ denote the category whose objects are smooth $\mathbb{C}$-schemes and whose morphisms are affine and smooth morphisms. Then the pro-object "$\varprojlim_n$"S_n in $\mathcal{S}$ depends only on X and does not depend on the choice of a smooth projective system $\{S_n\}_{n\in\mathbb{N}}$ ([7]). This follows from the fact

$$\operatorname{Hom}(X, S) = \varinjlim_n \operatorname{Hom}(S_n, S)$$

for any scheme S.

A $\mathbb{C}$-scheme X is called *pro-smooth* if it is covered by open subsets satisfying (S).

Let $f : X \to Y$ be a morphism of $\mathbb{C}$-schemes such that X is pro-smooth and Y is smooth over $\mathbb{C}$, and let A be a TDO-ring on Y. Set $A_{X\to Y} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}A$, and define the subring $f^\sharp A$ of $\mathcal{E}nd_{f^{-1}A}(A_{X\to Y})$ by (3.1.7)–(3.1.9). Then $A_{X\to Y}$ is an $(f^\sharp A, f^{-1}A)$-bimodule. For an A-module $\mathcal{M}$ we define the $f^\sharp A$-module $f^\bullet\mathcal{M}$ by

$$f^\bullet\mathcal{M} = A_{X\to Y} \otimes_{f^{-1}A} f^{-1}\mathcal{M}. \tag{3.2.3}$$

For a pro-smooth scheme X, a TDO-ring on X is by definition a sheaf A of rings on X containing $\mathcal{O}_X$ as a subring satisfying the following condition.

(3.2.4) For any $x \in X$, there exist a morphism $f : U \to Y$ from an open neighborhood U of x to a smooth $\mathbb{C}$-scheme Y and a TDO-ring B on Y such that $A|_U = f^\sharp B$.

By the definition, a TDO-ring A on a pro-smooth scheme X is locally of the form $A = f^\sharp B$ where B is a TDO-ring on a smooth $\mathbb{C}$-scheme. We can patch together $f^\sharp B^{-\sharp}$ and obtain a TDO-ring $A^{-\sharp}$ on X.

For an invertible $\mathcal{O}_X$-module $\mathcal{L}$ on a pro-smooth scheme X, we have a TDO-ring $D_X(\mathcal{L})$ given as follows.

(3.2.5) $D_X(\mathcal{L}) = \bigcup_n F_n D_X(\mathcal{L}) \subset \mathcal{E}nd_{\mathbb{C}}\,\mathcal{L}$.

(3.2.6) $F_n D_X(\mathcal{L}) = 0$ for $n < 0$.

(3.2.7) $F_n D_X(\mathcal{L}) = \{P \in \mathcal{E}nd_{\mathbb{C}}\,\mathcal{L}\,;\ [P, a] \in F_{n-1}D_X(\mathcal{L})$ for any $a \in \mathcal{O}_X\}$ for $n \geq 0$.

We set $D_X = D_X(\mathcal{O}_X)$. Then we have $D_X(\mathcal{L}) \simeq \mathcal{L} \otimes_{\mathcal{O}_X} D_X \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1}$. More generally, for an invertible $\mathcal{O}_X$-module $\mathcal{L}$ and a scalar $a \in \mathbb{C}$ we can define a TDO-ring $D_X(\mathcal{L}^a) = \mathcal{L}^a \otimes_{\mathcal{O}_X} D_X \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -a}$ by the following patching procedure although $\mathcal{L}^a$ does not necessarily exist. A section of $D_X(\mathcal{L}^a)$ is locally of the form $s^a \otimes P \otimes s^{-a}$, where s is a nowhere vanishing section of $\mathcal{L}$ and P is a section of D_X, and we have $s_1^a \otimes P_1 \otimes s_1^{-a} = s_2^a \otimes P_2 \otimes s_2^{-a}$ if and only if $P_1 = (s_2/s_1)^a P_2 (s_2/s_1)^{-a}$ as sections of D_X.

Let A be a TDO-ring on a pro-smooth scheme X. We call a (left) A-module $\mathcal{M}$ admissible if it satisfies the following conditions.

(3.2.8) $\mathcal{M}$ is quasi-coherent over $\mathcal{O}_X$.

(3.2.9) For any affine open subset U of X and any $s \in \Gamma(U; \mathcal{M})$, there exists a finitely generated $\mathbb{C}$-subalgebra B of $\Gamma(U; \mathcal{O}_X)$ such that $Ps = 0$ for any $P \in \Gamma(U; A)$ satisfying $P(B) = 0$.

We denote the category of admissible A-modules by $\mathbb{M}_{adm}(A)$. We call an admissible A-module $\mathcal{M}$ *holonomic* if it satisfies the following condition.

(3.2.10) For any $x \in X$ there exist a morphism $f : U \to Y$, a TDO-ring B on Y as in (3.2.4) and a holonomic B-module $\mathcal{M}'$ such that $\mathcal{M}|_U = f^\bullet\mathcal{M}'$.

We denote the category of holonomic A-modules by $\mathbb{M}_h(A)$.

Let A be a TDO-ring on a $\mathbb{C}$-scheme X satisfying the property (S). Then we can take a smooth projective system $\{S_n\}_{n\in\mathbb{N}}$ for X and TDO-rings A_n on S_n such that

$$p^{\sharp}_{nm}A_n \cong A_m \text{ for any } m \geq n, \qquad p^{\sharp}_n A_n \cong A, \tag{3.2.11}$$

where $p_n : X \to S_n$ is the projection. We call $\{(S_n, A_n)\}_{n\in\mathbb{Z}}$ a smooth projective system for (X, A). Since p_{nm} is smooth, the functor $p^{\bullet}_{nm} : \mathbb{M}_h(A_n) \to \mathbb{M}_h(A_m)$ is exact, and we have the equivalence of categories

$$\mathbb{M}_h(A) \cong \varinjlim_n \mathbb{M}_h(A_n).$$

Let X be a pro-smooth $\mathbb{C}$-scheme and A a TDO-ring on X. Let $D(\mathbb{M}_{adm}(A))$ be the derived category of $\mathbb{M}_{adm}(A)$. Let us denote by $D^b_h(A)$ the full subcategory of $D^b(\mathbb{M}_{adm}(A))$ consisting of bounded complexes whose cohomology groups are holonomic.

If X satisfies (S) and $\{(S_n, A_n)\}_{n\in\mathbb{Z}}$ is a smooth projective system for (X, A), then we have an equivalence of categories:

$$D^b_h(A) = \varinjlim_n D^b_h(A_n).$$

The duality functors $\mathbf{D} : D^b_h(A_n) \to D^b_h(A_n^{-\sharp})^{\mathrm{op}}$ induce the duality functor

$$\mathbf{D} : D^b_h(A) \to D^b_h(A^{-\sharp})^{\mathrm{op}}. \tag{3.2.12}$$

For a morphism $f : X \to Y$ of pro-smooth $\mathbb{C}$-schemes and a TDO-ring A on Y, we define a TDO-ring $f^{\sharp}A$ on X by the same formulas (3.1.7)–(3.1.9). The functor

$$\mathbb{D}f^{\bullet} : D^b_h(A) \to D^b_h(f^{\sharp}A) \tag{3.2.13}$$

is defined by the same formula as in (3.1.13). It is well-defined as seen in the following. The question being local, we may assume that X and Y satisfy (S). Then we can take a smooth projective system $\{X_n\}$ for X and a smooth projective system $\{(Y_n, A_n)\}_{n\in\mathbb{Z}}$ for (Y, A). Let $p_{X_n} : X \to X_n$ and $p_{Y_n} : Y \to Y_n$ be the projections. We may assume further that there exists $\{f_n\} : \{X_n\} \to \{Y_n\}$ such that $f = \varprojlim_n f_n$. For $\mathcal{M} \in \mathrm{Ob}(D^b_h(A))$ there exist some n and $\mathcal{M}_n \in \mathrm{Ob}(D^b_h(A_n))$ such that $\mathcal{M} = p^{\bullet}_{Y_n}\mathcal{M}_n$. Then we have

$$\mathbb{D}f^{\bullet}\mathcal{M} = p^{\bullet}_{X_n}\mathbb{D}f^{\bullet}_n\mathcal{M}_n.$$

Let $f : X \to Y$ be a morphism of pro-smooth schemes. We assume that f is of finite presentation type. Let A be a TDO-ring on Y. We define a functor

$$\int_f : D^b_h(f^{\sharp}A) \to D^b_h(A) \tag{3.2.14}$$

by the same formula as in (3.1.13). We can see that it is well-defined as follows. The question being local on Y, we can take a smooth projective system $\{X_n\}$ for X, a smooth projective system $\{(Y_n, A_n)\}_{n\in\mathbb{Z}}$ for (Y, A), and $\{f_n\} : \{X_n\} \to \{Y_n\}$ such that $f = \varprojlim_n f_n$ and the following diagram is Cartesian for any n ([7]).

$$\begin{array}{ccc} X_n & \xrightarrow{f_n} & Y_n \\ \downarrow & & \downarrow \\ X_0 & \xrightarrow{f_0} & Y_0\,. \end{array}$$

Let $p_{Xn} : X \to X_n$ and $p_{Yn} : Y \to Y_n$ be the projections. Let $\mathcal{M} \in \mathrm{Ob}(D_h^b(f^\sharp A))$. There exists some n and $\mathcal{M}_n \in \mathrm{Ob}(D_h^b(f_n^\sharp A_n))$ such that $\mathcal{M} = p_{Xn}^\bullet \mathcal{M}_n$. Then we have

$$\int_f \mathcal{M} = p_{Yn}^\bullet \int_{f_n} \mathcal{M}_n.$$

Under the same assumption, we define the relative dimension d_f by $\dim X_0 - \dim Y_0$.

We shall also use the following functors for a morphism $f : X \to Y$ of schemes satisfying (S):

$$\text{(3.2.15)} \quad \mathbb{D}f^*,\ \mathbb{D}f^! : D_h^b(A) \to D_h^b(f^\sharp A), \quad \int_{f!},\ \mathbb{D}f_*,\ \mathbb{D}f_! : D_h^b(f^\sharp A) \to D_h^b(A)$$

defined by

$$\mathbb{D}f^! = \mathbb{D}f^\bullet[2d_f], \quad \mathbb{D}f^* = \mathbf{D} \circ \mathbb{D}f^\bullet \circ \mathbf{D},$$

$$\int_{f!} = \mathbf{D} \circ \int_f \circ \mathbf{D}, \quad \mathbb{D}f_* = \int_f [-d_f], \quad \mathbb{D}f_! = \int_{f!} [-d_f].$$

Note that $\mathbb{D}f^\bullet$ and $\mathbb{D}f^*$ are defined for any morphism f of schemes satisfying (S), while other functors in (3.2.14), (3.2.15) are defined only when f is of finite presentation type.

For a morphism $f : X \to Y$ of pro-smooth schemes, a TDO-ring A on Y, and $k \in \mathbb{Z}$ we can define functors

$$\text{(3.2.16)} \quad H^k\mathbb{D}f^* : \mathbb{M}_h(A) \to \mathbb{M}_h(f^\sharp A)$$

by patching together the locally defined object $H^k(\mathbb{D}f^*\mathcal{M})$ for $\mathcal{M} \in \mathrm{Ob}(\mathbb{M}_h(A))$. Similarly, for a morphism $f : X \to Y$ of pro-smooth schemes which is of finite presentation type, a TDO-ring A on Y, and $k \in \mathbb{Z}$ we can define functors

$$\text{(3.2.17)} \quad H^k \textstyle\int_{f!},\ H^k\mathbb{D}f_! : \mathbb{M}_h(f^\sharp A) \to \mathbb{M}_h(A).$$

3.3. Equivariant D-modules. Let G be an affine group scheme over $\mathbb{C}$. We assume that $\mathcal{O}_G(G)$ is generated by countably many generators as a $\mathbb{C}$-algebra. Then $G \simeq \varprojlim_{n\in\mathbb{N}} G_n$ for a projective systems of affine algebraic group over $\mathbb{C}$, and hence G satisfies the condition (S). Let $\mathfrak{g}$ be the Lie algebra of G. Then $\mathfrak{g}$ is the projective limit of the Lie algebras $\mathfrak{g}_n$ of G_n.

Let X be a pro-smooth $\mathbb{C}$-scheme with an action of G. Then we have the diagram

$$\text{(3.3.1)} \quad G \times G \times X \ \substack{\xrightarrow{p_1} \\ \xrightarrow{p_2} \\ \xrightarrow{p_3}}\ G \times X \ \substack{\xrightarrow{\mu_X} \\ \xleftarrow{i} \\ \xrightarrow{\mathrm{pr}_X}}\ X.$$

Here $\mu_X : G \times X \to X$ is the action morphism, $\mathrm{pr}_X : G \times X \to X$ the second projection, and

$$\begin{aligned} i(x) &= (1, x), \\ p_1(g_1, g_2, x) &= (g_1, g_2 x), \\ p_2(g_1, g_2, x) &= (g_1 g_2, x), \\ p_3(g_1, g_2, x) &= (g_2, x). \end{aligned}$$

A G-equivariant TDO-ring A on X is a TDO-ring endowed with an isomorphism of TDO-rings

$$\alpha : \mu_X^\sharp A \xrightarrow{\sim} \mathrm{pr}_X^\sharp A$$

with the cocycle condition (see [12, §4.6]), i.e. the commutativity of the following diagram.

$$\begin{array}{ccc} p_2^\sharp \mu_X^\sharp A & \xrightarrow{\qquad\qquad p_2^\sharp\alpha \qquad\qquad} & p_2^\sharp \mathrm{pr}_X^\sharp A \\ \wr\| & & \wr\| \\ p_1^\sharp \mu_X^\sharp A & \xrightarrow{p_1^\sharp\alpha} p_1^\sharp \mathrm{pr}_X^\sharp A \cong p_3^\sharp \mu_X^\sharp A \xrightarrow{p_3^\sharp\alpha} & p_3^\sharp \mathrm{pr}_X^\sharp A \end{array} \tag{3.3.2}$$

Then the G-equivariance structure induces a ring homomorphism

$$U(\mathfrak{g}) \to \Gamma(X; A).$$

A G-equivariant module M over a G-equivariant TDO-ring A on X is an A-module endowed with an isomorphism of $\mathrm{pr}_X^\sharp A$-modules

$$\mu_X^\bullet M \xrightarrow{\sim} \mathrm{pr}_X^\bullet M$$

with a similar cocycle condition (see [12, §4.7] and (3.3.5) below).

We can generalize the notion of equivariance to that of twisted equivariance. Assume for the sake of simplicity that G is a finite-dimensional affine algebraic group with Lie algebra $\mathfrak{g}$. Let $q_i : G\times G \to G$ $(i = 1, 2)$ be the first and the second projection and $\mu_G : G \times G \to G$ the multiplication morphism. Let $i_G : \mathrm{pt} \to G$ be the identity.

Let $\lambda \in \mathfrak{g}^*$ be a G-invariant vector. Let $\mathcal{T}(\lambda)$ be the free $\mathcal{O}_G$-module generated by the symbol e^λ. We define its D_G-module structure by

$$R_A \mathrm{e}^\lambda = \lambda(A)\mathrm{e}^\lambda \quad \text{for any } A \in \mathfrak{g},$$

where R_A is the left invariant vector field on G corresponding to A. Then we have a canonical isomorphism of D_G-modules

$$\begin{aligned} &i_G^\bullet \mathcal{T}(\lambda) \cong \mathbb{C}, \\ &m_\lambda : \mu_G^\bullet \mathcal{T}(\lambda) \xrightarrow{\sim} q_1^\bullet \mathcal{T}(\lambda) \otimes q_2^\bullet \mathcal{T}(\lambda), \end{aligned} \tag{3.3.3}$$

sending e^λ to 1 and $\mathrm{e}^\lambda \otimes \mathrm{e}^\lambda$, respectively. A twisted G-equivariant A-module $\mathcal{M}$ with twist λ is an A-module with an isomorphism of $\mathrm{pr}_X^\sharp A$-modules

$$\beta : \mu_X^\bullet \mathcal{M} \xrightarrow{\sim} q^\bullet \mathcal{T}(\lambda) \otimes \mathrm{pr}_X^\bullet \mathcal{M}, \tag{3.3.4}$$

with the cocycle condition. Here $q : G \times X \to G$ is the first projection. The cocycle condition means the commutativity of the following diagram of $r_3^\sharp A$-modules on $G \times$

$G \times X$.

$$
\begin{tikzcd}
q_{12}^{\bullet}\mu_G^{\bullet}\mathcal{T}(\lambda) \otimes r_3^{\bullet}\mathcal{M} \arrow[r, "m_\lambda"] \arrow[d, equal] & q_{12}^{\bullet}(q_1^{\bullet}\mathcal{T}(\lambda) \otimes q_2^{\bullet}\mathcal{T}(\lambda)) \otimes r_3^{\bullet}\mathcal{M} \arrow[ddd, equal] \\
p_2^{\bullet}(q^{\bullet}\mathcal{T}(\lambda) \otimes \mathrm{pr}_X^{\bullet}\mathcal{M}) & \\
p_2^{\bullet}\mu_X^{\bullet}\mathcal{M} \arrow[u, "\beta"'] \arrow[d, equal] & \\
p_1^{\bullet}\mu_X^{\bullet}\mathcal{M} \arrow[d, "\beta"] & r_1^{\bullet}\mathcal{T}(\lambda) \otimes r_2^{\bullet}\mathcal{T}(\lambda) \otimes r_3^{\bullet}\mathcal{M} \arrow[dd, equal] \\
p_1^{\bullet}(q^{\bullet}\mathcal{T}(\lambda) \otimes \mathrm{pr}_X^{\bullet}\mathcal{M}) \arrow[d, equal] & \\
r_1^{\bullet}\mathcal{T}(\lambda) \otimes p_3^{\bullet}\mu_X^{\bullet}\mathcal{M} \arrow[r, "\beta"] & r_1^{\bullet}\mathcal{T}(\lambda) \otimes p_3^{\bullet}(q^{\bullet}\mathcal{T}(\lambda) \otimes \mathrm{pr}_X^{\bullet}\mathcal{M}) .
\end{tikzcd}
\tag{3.3.5}
$$

Here $q_{12} : G \times G \times X \to G \times G$ is the $(1,2)$-th projection, and r_i is the i-th projection from $G \times G \times X$.

Let $\psi : G \to \mathbb{C}^{\times}$ be a character, and let $\delta\psi \in \mathfrak{g}^*$ be its differential. Then $\mathrm{e}^{\lambda+\delta\psi} \leftrightarrow \psi\mathrm{e}^{\lambda}$ gives a canonical isomorphism

$$\mathcal{T}(\lambda + \delta\psi) \cong \mathcal{T}(\lambda)$$

compatible with the multiplicative structure (3.3.3). Hence the twisted equivariance with twist λ is equivalent to that with twist $\lambda + \delta\psi$.

4. D-modules on the flag manifold.

4.1. Flag manifolds. We recall basic properties of the flag manifold for the Kac-Moody Lie algebra $\mathfrak{g}$ (Kashiwara [13]).

Fix a $\mathbb{Z}$-lattice P of $\mathfrak{h}^*$ satisfying

$$\alpha_i \in P, \quad \langle P, h_i \rangle \subset \mathbb{Z} \quad \text{for any } i \in I. \tag{4.1.1}$$

We define affine group schemes as follows:

$$H = \operatorname{Spec} \mathbb{C}[P], \tag{4.1.2}$$

$$N^{\pm} = \varprojlim_k \exp(\mathfrak{n}^{\pm}/\mathfrak{n}_k^{\pm}), \tag{4.1.3}$$

$$B = (\text{the semi-direct product of } H \text{ and } N^{+}), \tag{4.1.4}$$

$$B^{-} = (\text{the semi-direct product of } H \text{ and } N^{-}) \tag{4.1.5}$$

(see (2.3.1) for the definition of $\mathfrak{n}_k^{\pm}$). Here, for a finite-dimensional nilpotent Lie algebra $\mathfrak{a}$ we denote the corresponding unipotent algebraic group by $\exp(\mathfrak{a})$. Then $N^{\pm}$ is an affine scheme isomorphic to $\operatorname{Spec}(S(\mathfrak{n}^{\mp}))$.

For a subset Θ of $\Delta^{\pm}$ such that $(\Theta+\Theta)\cap\Delta \subset \Theta$, we denote by $N(\Theta)$ the subgroup $\exp(\mathfrak{n}(\Theta))$ of $N^{\pm}$.

In Kashiwara [13], a separated scheme G is constructed with a free right action of B^{-} and a free left action of B. The flag manifold X is defined as the quotient scheme

$X = G/B^-$. The flag manifold is a separated scheme. For $w \in W$, $U_w = wBB^-/B^-$ is an open subset of X. A locally closed subscheme $X_w = BwB^-/B^-$ of X is called a Schubert cell.

PROPOSITION 4.1.1.

(i) $X = \bigcup_{w \in W} U_w = \bigsqcup_{w \in W} X_w$.

(ii) *For any* $w \in W$, *we have* $\overline{X}_w = \bigcup_{y \geq w} X_y$ *and* $X_w \subset U_w \subset \bigcup_{x \leq w} X_x$.

(iii) *We have an isomorphism*

$$N(w\Delta^+ \cap \Delta^-) \times N(w\Delta^+ \cap \Delta^+) \xrightarrow{\sim} U_w \quad ((x,y) \mapsto xywB^-)$$

of schemes. Moreover, the subscheme $\{1\} \times N(w\Delta^+ \cap \Delta^+)$ *is isomorphic to* X_w *by this isomorphism.*

In particular, U_w and X_w are isomorphic to

$$\mathbb{A}^\ell = \operatorname{Spec} \mathbb{C}[x_k \,;\, 0 \leq k < \ell]$$

for some $\ell \in \mathbb{Z}_{\geq 0} \sqcup \{\infty\}$, and the codimension of X_w in X is the length $\ell(w)$ of $w \in W$. Hence X is pro-smooth.

We call a subset Φ of W *admissible* if

$$w \in \Phi, y \leq w \Longrightarrow y \in \Phi.$$

For an admissible subset Φ of W we define an open subset X_Φ of X by $X_\Phi = \bigcup_{w \in \Phi} X_w$.

For a finite admissible subset Φ, X_Φ is a quasi-compact scheme with the condition (S). Indeed for $k \gg 0$, the subgroup $\exp(\mathfrak{n}_k^+) = \varprojlim_{l \geq k} \exp(\mathfrak{n}_k^+/\mathfrak{n}_l^+)$ acts freely on X_Φ and $\{X_\Phi/\exp(\mathfrak{n}_k^+)\}_k$ is a smooth projective system for X_Φ (see [18]). Note that, since X_Φ is separated over $\mathbb{C}$, $X_\Phi/\exp(\mathfrak{n}_k^+)$ is separated for $k \gg 0$ by the following lemma.

LEMMA 4.1.2. *Let* $\{X_n\}_{n \in \mathbb{N}}$ *be a projective system of quasi-compact and quasi-separated schemes. Assume that the morphism* $X_{n+1} \to X_n$ *is an affine morphism for any* n. *Let* X_∞ *be its projective limit. If* X_∞ *is separated, then* X_n *is separated for* $n \gg 0$.

Proof. Let $f_{nm} : X_m \to X_n$ be the canonical projection $(0 \leq n \leq m \leq \infty)$. Since X_0 is quasi-compact, X_0 is covered by finitely many affine open subsets U_j^0 $(j = 1, \dots, N)$. Then the inclusion $U_j^0 \to X_0$ is of finite presentation. Set $U_j^n = f_{0n}^{-1}(U_j^0)$ $(0 \leq n \leq \infty)$. Since X_∞ is separated, $U_j^\infty \to X_\infty$ is an affine morphism. Hence [7, Theorem (8.10.5)] implies that $U_j^n \to X_n$ is an affine morphism for $n \gg 0$. Hence we may assume from the beginning that $U_j^0 \cap U_k^0$ is affine for any $j, k = 1, \dots, N$.

The ring homomorphism $\mathcal{O}_X(U_j^\infty) \otimes \mathcal{O}_X(U_k^\infty) \to \mathcal{O}_X(U_j^\infty \cap U_k^\infty)$ is surjective by the assumption that X_∞ is separated. Since $\mathcal{O}_X(U_j^0 \cap U_k^0)$ is a finitely generated algebra over $\mathcal{O}_X(U_j^0)$, the image of $\mathcal{O}_X(U_j^0 \cap U_k^0) \to \mathcal{O}_X(U_j^n \cap U_k^n)$ is contained in the image of $\mathcal{O}_X(U_j^n) \otimes \mathcal{O}_X(U_k^n) \to \mathcal{O}_X(U_j^n \cap U_k^n)$ for $n \gg 0$. On the other hand, we have

$$\mathcal{O}_X(U_j^n \cap U_k^n) \cong \mathcal{O}_X(U_j^n) \otimes_{\mathcal{O}_X(U_j^0)} \mathcal{O}_X(U_j^0 \cap U_k^0).$$

Hence $\mathcal{O}_X(U_j^n) \otimes \mathcal{O}_X(U_k^n) \to \mathcal{O}_X(U_j^n \cap U_k^n)$ is surjective for $n \gg 0$. This shows that X_n is separated for $n \gg 0$. □

Set $\tilde{X} = G/N^-$. We denote by $\xi : \tilde{X} \to X$ the canonical projection. It is an H-principal bundle. For $w \in W$ we set $\tilde{X}_w = \xi^{-1}X_w = BwN^-/N^-$, and for an admissible subset Φ of W we set $\tilde{X}_\Phi = \xi^{-1}X_\Phi = \bigcup_{w \in \Phi} \tilde{X}_w$. The scheme $\tilde{X}$ is also pro-smooth.

4.2. Twisted D-modules on the flag manifold. Let $p : G \to X$ be the projection. For $\mu \in P$ we define the invertible $\mathcal{O}_X$-module $\mathcal{O}_X(\mu)$ as follows:

$$\begin{aligned}\Gamma(U;\mathcal{O}_X(\mu)) &= \{\varphi \in \Gamma(p^{-1}U;\mathcal{O}_G)\,;\, \varphi(xg) = g^{-\mu}\varphi(x) \text{ for } (x,g) \in p^{-1}U \times B^-\}\\ &\cong \{\varphi \in \Gamma(\xi^{-1}U;\mathcal{O}_{\tilde{X}})\,;\, \varphi(xh) = h^{-\mu}\varphi(x) \text{ for } (x,h) \in \xi^{-1}U \times H\}\end{aligned}$$

for any open subset U of X. Here $x \mapsto x^{-\mu}$ is the character of B^- corresponding to the weight $-\mu$. Twisting D_X by $\mathcal{O}_X(\mu)$ we obtain a TDO-ring

$$D_\mu = \mathcal{O}_X(\mu) \otimes_{\mathcal{O}_X} D_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(-\mu).$$

This definition can be generalized to any $\mu \in \mathfrak{h}^*$ and we can define an N^+-equivariant TDO-ring D_μ on X (see Kashiwara [12] and Kashiwara-Tanisaki [18]). Note that the pull-back $\xi^\sharp D_\mu$ of the TDO-ring D_μ under $\xi : \tilde{X} \to X$ is canonically isomorphic to $D_{\tilde{X}}$. Hence the pull-back $\xi^\bullet\mathcal{M}$ of an admissible D_μ-module $\mathcal{M}$ is naturally regarded as a $D_{\tilde{X}}$-module. Moreover, by the functor $\xi^\bullet$, the category of admissible D_μ-modules is equivalent to the category of admissible twisted H-equivariant $D_{\tilde{X}}$-modules with twist μ.

The infinitesimal action of $\mathfrak{g}$ on X lifts to an algebra homomorphism

$$U(\mathfrak{g}) \to \Gamma(X;D_\mu).$$

In particular, $H^n(U;\mathcal{M})$ has a $\mathfrak{g}$-module structure for any open subset U of X, any $D_\mu|_U$-module $\mathcal{M}$ and $n \in \mathbb{Z}$.

For $w \in W$, let $i_w : X_w \hookrightarrow X$ be the inclusion. Then for any $\mu \in \mathfrak{h}^*$, the N^+-equivariant TDO-ring $i_w^\sharp D_\mu$ is canonically isomorphic to the N^+-equivariant TDO-ring D_{X_w}. We define the N^+-equivariant holonomic D_μ-modules $\mathcal{B}_w(\mu), \mathcal{M}_w(\mu)$ by

$$\mathcal{B}_w(\mu) = H^0\int_{i_w} \mathcal{O}_{X_w}, \quad \mathcal{M}_w(\mu) = H^0\int_{i_w!} \mathcal{O}_{X_w}. \tag{4.2.1}$$

Note that $H^k\int_{i_w} \mathcal{O}_{X_w} = H^k\int_{i_w!} \mathcal{O}_{X_w} = 0$ for any $k \neq 0$ because i_w is an affine embedding. By the definition we have

$$\mathrm{Hom}_{D_\mu}(\mathcal{M}_w(\mu),\mathcal{M}) \xrightarrow{\sim} \mathrm{Hom}_{D_\mu|_U}(\mathcal{M}_w(\mu)|_U,\mathcal{M}|_U), \tag{4.2.2}$$

$$\mathrm{Hom}_{D_\mu}(\mathcal{M},\mathcal{B}_w(\mu)) \xrightarrow{\sim} \mathrm{Hom}_{D_\mu|_U}(\mathcal{M}|_U,\mathcal{B}_w(\mu)|_U) \tag{4.2.3}$$

for any open subset U of X containing X_w and any holonomic D_μ-module $\mathcal{M}$. The isomorphism $\mathcal{M}_w(\mu)|_U \cong \mathcal{B}_w(\mu)|_U$ extends to a canonical non-zero homomorphism $\mathcal{M}_w(\mu) \to \mathcal{B}_w(\mu)$. We denote its image by $\mathcal{L}_w(\mu)$. It is a unique irreducible quotient of $\mathcal{M}_w(\mu)$ and a unique irreducible submodule of $\mathcal{B}_w(\mu)$. Note that $\dim \mathrm{Hom}_{D_\mu}(\mathcal{M}_w(\mu),\mathcal{B}_w(\mu)) = 1$.

For $\mu \in \mathfrak{h}^*$ and a finite admissible subset Φ of W, we denote by $\mathbb{H}_\Phi(\mu)$ the category of N^+-equivariant holonomic $D_\mu|_{X_\Phi}$-modules. For $\mu \in \mathfrak{h}^*$ we denote by $\mathbb{H}(\mu)$ the category of N^+-equivariant holonomic D_μ-modules. Then we have obviously

$$\mathbb{H}(\mu) = \varprojlim_{\Phi} \mathbb{H}_\Phi(\mu),$$

where Φ ranges over the set of finite admissible subset of W.

For any $w \in W$ and $\mu \in \mathfrak{h}^*$ the D_μ-modules $\mathcal{B}_w(\mu)$, $\mathcal{M}_w(\mu)$ and $\mathcal{L}_w(\mu)$ are objects of $\mathbb{H}(\mu)$. Note that $\mathcal{L}_w(\mu)$ is a simple object of $\mathbb{H}(\mu)$.

For any N^+-equivariant admissible D_μ-module $\mathcal{M}$, we denote the support of $\mathcal{M}$ by $\mathrm{Supp}(\mathcal{M})$. It is an N^+-stable closed subset of X, and hence it is also B-stable.

LEMMA 4.2.1. *For $w \in W$, let U be an open subset of X which contains X_w as a closed subset. For any N^+-equivariant admissible D_μ-module $\mathcal{M}$ such that* $\mathrm{Supp}(\mathcal{M}) \cap U \subset X_w$, *there exist some index set J and isomorphisms*

$$\mathcal{M}_w(\mu)^{\oplus J}|_U \xrightarrow{\sim} \mathcal{M}|_U \xrightarrow{\sim} \mathcal{B}_w(\mu)^{\oplus J}|_U .$$

Furthermore if $\mathcal{M}$ is holonomic, then J is a finite set and the above isomorphisms can be extended to morphisms in $\mathbb{H}(\mu)$

$$\mathcal{M}_w(\mu)^{\oplus J} \to \mathcal{M} \to \mathcal{B}_w(\mu)^{\oplus J}.$$

Proof. Let $i : X_w \to U$ be the closed embedding. By the condition on $\mathcal{M}$ there exists an N^+-equivariant holonomic D_{X_w}-module $\mathcal{N}$ such that $\mathcal{M}|_U \simeq \int_i \mathcal{N}$. Since X_w is a homogeneous space of N^+ with a connected isotropy subgroup, we see that $\mathcal{N}$ is isomorphic to $\mathcal{O}_{X_w}^{\oplus J}$ for some J. If $\mathcal{M}$ is holonomic then $\mathcal{N}$ is holonomic, and hence J is a finite set. Thus we have

$$\mathcal{M}|_U \simeq \int_i \mathcal{O}_{X_w}^{\oplus J} \simeq (\mathcal{B}_w(\mu)|_U)^{\oplus J} \simeq (\mathcal{M}_w(\mu)|_U)^{\oplus J}.$$

To see the last statement, it is enough to apply (4.2.2) and (4.2.3). □

Let $\mathcal{M} \in \mathrm{Ob}(\mathbb{H}(\mu))$. By Lemma 4.2.1, for any finite admissible subset Φ of W, $\mathcal{M}|_{X_\Phi} \in \mathrm{Ob}(\mathbb{H}_\Phi(\mu))$ has finite length and it has finite composition series whose composition factors are isomorphic to $\mathcal{L}_w(\mu)|_{X_\Phi}$ for some $w \in \Phi$. For $w \in W$ the multiplicity of $\mathcal{L}_w(\mu)|_{X_\Phi}$ in the composition series of $\mathcal{M}|_{X_\Phi}$ does not depend on the choice of a finite admissible subset Φ of W such that $w \in \Phi$. We denote it by $[\mathcal{M} : \mathcal{L}_w(\mu)]$. Note that the multiplicity does not depend on the N^+-equivariance structure.

LEMMA 4.2.2. *We have $[\mathcal{M}_w(\mu) : \mathcal{L}_y(\mu)] = [\mathcal{B}_w(\mu) : \mathcal{L}_y(\mu)]$ for any $w, y \in W$.*
Replacing the modules $\mathcal{M}_w(\mu)$, $\mathcal{L}_y(\mu)$ and $\mathcal{B}_w(\mu)$ with their images by $\xi^\bullet$, this follows from the following general result.

PROPOSITION 4.2.3 ([15]). *Let $j : X \to Y$ be an embedding of smooth $\mathbb{C}$-schemes. Then for any holonomic D_X-module $\mathcal{M}$, we have the equality*

$$\sum_{i \in \mathbb{Z}} (-1)^i [H^i(\int_j \mathcal{M})] = \sum_{i \in \mathbb{Z}} (-1)^i [H^i(\int_{j!} \mathcal{M})]$$

in the Grothendieck group of the category of holonomic D_Y-modules.

Proof. We can decompose this proposition into the closed embedding case and the open embedding case. Since the first case is obvious, we may assume that j is an open embedding. Since the question is local on Y, we can easily reduce to the case where X is the complement of a hypersurface of Y. Then $\int_j \mathcal{M}$ and $\int_{j!} \mathcal{M}$ are concentrated at degree 0. We may assume further that $Y \setminus X$ is defined by $f = 0$ for some $f \in \Gamma(Y; \mathcal{O}_Y)$. Let $\psi_f(\mathcal{M})$ be the near-by cycle of $\mathcal{M}$, which is a holonomic D_Y-module with support in $Y \setminus X$. Let $\mathrm{var} : \psi_f(\mathcal{M}) \to \psi_f(\mathcal{M})$ be the variation. Then the kernel (resp. the cokernel) of $\int_{j!} \mathcal{M} \to \int_j \mathcal{M}$ is isomorphic to the kernel (resp. the cokernel) of $\mathrm{var} : \psi_f(\mathcal{M}) \to \psi_f(\mathcal{M})$. Therefore we have

$$[\int_j \mathcal{M}] - [\int_{j!} \mathcal{M}] = [\mathrm{Coker}(\mathrm{var})] - [\mathrm{Ker}(\mathrm{var})] = 0.$$

□

4.3. Cohomologies of $\mathcal{B}_w(\mu)$. We first study the cohomology groups of $\mathcal{B}_w(\mu)$.

PROPOSITION 4.3.1. *Let $\mu \in \mathfrak{h}^*$, $w \in W$ and let Φ be a finite admissible subset of W containing w. Then we have*

(i) $H^n(X_\Phi; \mathcal{B}_w(\mu)) = H^n(X; \mathcal{B}_w(\mu)) = 0$ *for any* $n \neq 0$.

(ii) *We have*

$$\Gamma(X_\Phi; \mathcal{B}_w(\mu)) = \Gamma(X; \mathcal{B}_w(\mu)) \simeq U({}^w\mathfrak{b}) \otimes_{U({}^w\mathfrak{b}\cap\mathfrak{b})} (\Gamma(X_w; \mathcal{O}_{X_w}) \otimes \mathbb{C}_{w\circ\mu})$$

as a $U({}^w\mathfrak{b})$-module. Here ${}^w\mathfrak{b} = \mathfrak{h} \oplus (\bigoplus_{\alpha\in w\Delta^+} \mathfrak{g}_\alpha)$ and $\mathbb{C}_{w\circ\mu}$ is the one-dimensional $U({}^w\mathfrak{b}\cap\mathfrak{b})$-module with $w\circ\mu$ as a weight.

Proof. By the definition of $\mathcal{B}_w(\mu)$ we have

$$H^n(X_\Phi; \mathcal{B}_w(\mu)) \simeq H^n(X; \mathcal{B}_w(\mu)) \simeq H^n(U_w; \mathcal{B}_w(\mu)).$$

Hence the assertion easily follows from Proposition 4.1.1 (iii) (cf. [14, 18]). □

Later, we shall see that $\Gamma(X; \mathcal{B}_w(\mu))$ is isomorphic to the dual Verma module under certain conditions. The following corollary is a key of its proof.

COROLLARY 4.3.2. *Let $\mu \in \mathfrak{h}^*$ and $w \in W$.*

(i) $\mathrm{ch}(\Gamma(X; \mathcal{B}_w(\mu))) = \mathrm{ch}(M(w\circ\mu))$.

(ii) *If $\zeta \in \mathfrak{h}^*$ and non-zero $m \in \Gamma(X; \mathcal{B}_w(\mu))_\zeta$ satisfy $\mathfrak{n}(w\Delta^+\cap\Delta^+)m = 0$, then we have $\zeta \in w\circ\mu - \sum_{\alpha\in\Delta^+\cap w\Delta^-} \mathbb{Z}_{\geq 0}\alpha$.*

(iii) *If $\zeta \in \mathfrak{h}^*$ and non-zero $m \in \Gamma(X; \mathcal{B}_w(\mu))^*_\zeta$ satisfy $\mathfrak{n}(w\Delta^-\cap\Delta^+)m = 0$, then we have $\zeta \in w\circ\mu - \sum_{\alpha\in\Delta^+\cap w\Delta^+} \mathbb{Z}_{\geq 0}\alpha$.*

Proof. By Proposition 4.3.1 we have

$$\begin{aligned}\Gamma(X; \mathcal{B}_w(\mu)) &\simeq U(\mathfrak{n}(w\Delta^+\cap\Delta^-)) \otimes \Gamma(X_w; \mathcal{O}_{X_w}) \otimes \mathbb{C}_{w\circ\mu} \\ &\simeq U(\mathfrak{n}(w\Delta^+\cap\Delta^-)) \otimes S(\mathfrak{n}(w\Delta^-\cap\Delta^-)) \otimes \mathbb{C}_{w\circ\mu},\end{aligned} \tag{4.3.1}$$

and hence we have

$$\begin{aligned}\mathrm{ch}(\Gamma(X; \mathcal{B}_w(\mu))) &= \mathrm{ch}(U(\mathfrak{n}(w\Delta^+\cap\Delta^-)))\,\mathrm{ch}(S(\mathfrak{n}(w\Delta^-\cap\Delta^-)))e^{w\circ\mu} \\ &= \prod_{\alpha\in w\Delta^+\cap\Delta^-} (1-e^\alpha)^{-\dim\mathfrak{g}_\alpha} \prod_{\alpha\in w\Delta^-\cap\Delta^-} (1-e^\alpha)^{-\dim\mathfrak{g}_\alpha} e^{w\circ\mu} \\ &= \prod_{\alpha\in\Delta^-} (1-e^\alpha)^{-\dim\mathfrak{g}_\alpha} e^{w\circ\mu} \\ &= \mathrm{ch}(M(w\circ\mu)).\end{aligned}$$

Thus (i) is proved.

Let us prove (iii). Assume that a non-zero vector $m \in \Gamma(X; \mathcal{B}_w(\mu))^*_\zeta$ with $\zeta \in \mathfrak{h}^*$ satisfies $\mathfrak{n}(w\Delta^-\cap\Delta^+)m=0$. Then we have $\Big(\Gamma(X; \mathcal{B}_w(\mu))/\mathfrak{n}(w\Delta^+\cap\Delta^-)\Gamma(X; \mathcal{B}_w(\mu))\Big)_\zeta \neq 0$. Then (4.3.1) implies

$$\begin{aligned}&\mathrm{ch}\Big(\Gamma(X; \mathcal{B}_w(\mu))/\mathfrak{n}(w\Delta^+\cap\Delta^-)\Gamma(X; \mathcal{B}_w(\mu))\Big) \\ &= \mathrm{ch}(S(\mathfrak{n}(w\Delta^-\cap\Delta^-)))e^{w\circ\mu} \\ &= \prod_{\alpha\in w\Delta^-\cap\Delta^-} (1-e^\alpha)^{-\dim\mathfrak{g}_\alpha} e^{w\circ\mu},\end{aligned}$$

and we obtain (iii).

Let us finally show (ii). Define a filtration $\{F_\ell\}_{\ell\in\mathbb{Z}}$ of $U(\mathfrak{n}(w\Delta^+\cap\Delta^-))$ by

$$F_\ell = \bigoplus_{\mathrm{ht}(\beta)\le\ell} U(\mathfrak{n}(w\Delta^+\cap\Delta^-))_{-\beta}.$$

Then the subspace $F_\ell\otimes\Gamma(X_w;\mathcal{O}_{X_w})\otimes\mathbb{C}_{w\circ\mu}$ of $\Gamma(X;\mathcal{B}_w(\mu))$ is stable under the action of $\mathfrak{n}(w\Delta^+\cap\Delta^+)$, and the action of $\mathfrak{n}(w\Delta^+\cap\Delta^+)$ on the quotient

$$\begin{aligned}&(F_\ell\otimes\Gamma(X_w;\mathcal{O}_{X_w})\otimes\mathbb{C}_{w\circ\mu})/(F_{\ell-1}\otimes\Gamma(X_w;\mathcal{O}_{X_w})\otimes\mathbb{C}_{w\circ\mu})\\ &=(F_\ell/F_{\ell-1})\otimes\Gamma(X_w;\mathcal{O}_{X_w})\otimes\mathbb{C}_{w\circ\mu}\end{aligned}$$

is given by

$$x(u\otimes n)=u\otimes xn\quad\text{for } x\in\mathfrak{n}(w\Delta^+\cap\Delta^+),\ u\in F_\ell/F_{\ell-1},\ n\in\Gamma(X_w;\mathcal{O}_{X_w})\otimes\mathbb{C}_{w\circ\mu}.$$

Take the smallest ℓ such that $m\in F_\ell\otimes\Gamma(X_w;\mathcal{O}_{X_w})\otimes\mathbb{C}_{w\circ\mu}$ and denote by $\overline{m}$ the corresponding element of $(F_\ell/F_{\ell-1})\otimes\Gamma(X_w;\mathcal{O}_{X_w})\otimes\mathbb{C}_{w\circ\mu}$. Write $\overline{m}$ as

$$\overline{m}=\sum_{j=1}^r\overline{u}_j\otimes n_j,$$

where $\overline{u}_j\ (j=1,\dots,r)$ are linearly independent elements of $F_\ell/F_{\ell-1}$ and $n_j\ (j=1,\dots,r)$ are elements of $\Gamma(X_w;\mathcal{O}_{X_w})\otimes\mathbb{C}_{w\circ\mu}$. By the assumption on m we have $\mathfrak{n}(w\Delta^+\cap\Delta^+)n_j=0$ for any j. Since X_w is a homogeneous space of $N(w\Delta^+\cap\Delta^+)$, we have $n_j\in\mathbb{C}\otimes\mathbb{C}_{w\circ\mu}\subset\Gamma(X_w;\mathcal{O}_{X_w})\otimes\mathbb{C}_{w\circ\mu}$. Thus

$$\overline{m}\in(F_\ell/F_{\ell-1})\otimes\mathbb{C}\otimes\mathbb{C}_{w\circ\mu},$$

and we obtain (ii). □

Proposition 4.3.3. *For $\lambda\in\mathcal{K}^+_{\mathrm{reg}}$ and $w\in W$, we have*

$$\Gamma(X;\mathcal{B}_w(\lambda))\simeq M^*(w\circ\lambda).$$

Proof. Since $\mathrm{ch}(\Gamma(X;\mathcal{B}_w(\lambda)))=\mathrm{ch}(M^*(w\circ\lambda))$, it is sufficient to show that if there exists $m\in\Gamma(X;\mathcal{B}_w(\lambda))_\zeta\setminus\{0\}$ such that $\mathfrak{n}^+m=0$, then $\zeta=w\circ\lambda$.

Since $[M(w\circ\lambda):L(\zeta)]\ne0$, Corollary 2.5.6 implies $\zeta=y\circ\lambda$ for $y\in wW(\lambda)$. Hence there exist some $\gamma_1,\cdots,\gamma_r\in\Delta^+(\lambda)$ such that

$$w^{-1}y=s_{\gamma_1}\cdots s_{\gamma_r},\qquad s_{\gamma_1}\cdots s_{\gamma_{j-1}}\gamma_j\in\Delta^+.$$

Then we have

$$\begin{aligned}\lambda+\rho-w^{-1}y(\lambda+\rho)&=\lambda+\rho-s_{\gamma_1}\cdots s_{\gamma_r}(\lambda+\rho)\\&=\sum_{j=1}^r(s_{\gamma_1}\cdots s_{\gamma_{j-1}}(\lambda+\rho)-s_{\gamma_1}\cdots s_{\gamma_j}(\lambda+\rho))\\&=\sum_{j=1}^r(\lambda+\rho,\gamma_j^\vee)s_{\gamma_1}\cdots s_{\gamma_{j-1}}(\gamma_j).\end{aligned}$$

Since $\lambda\in\mathcal{K}^+_{\mathrm{reg}}$, we have $(\lambda+\rho,\gamma_j^\vee)\in\mathbb{Z}_{>0}$, and hence $\lambda+\rho-w^{-1}y(\lambda+\rho)\in Q^+$.

On the other hand, Corollary 4.3.2 (ii) implies

$$\lambda + \rho - w^{-1}y(\lambda + \rho) = w^{-1}(w \circ \lambda - \zeta) \in w^{-1}\Big(\sum_{\alpha \in \Delta^+ \cap w\Delta^-} \mathbb{Z}_{\geq 0}\alpha\Big) \subset -Q^+ .$$

Thus we obtain $\lambda + \rho - w^{-1}y(\lambda + \rho) = 0$, and hence $\zeta = w \circ \lambda$. □

REMARK 4.3.4. This proposition 4.3.3 can be also proved by using the theory of the Radon transform in §4.6. Indeed, Theorem 4.6.2 implies $\Gamma(X; \mathcal{B}_w(\lambda)) \cong \Gamma(X; \mathcal{B}_e(w \circ \lambda))$. The last module is isomorphic to $\Gamma(X_e; \mathcal{O}_X) \otimes \mathbb{C}_{w\circ\lambda}$, and it has a unique highest weight vector, because X_e is a homogeneous space of N^+.

4.4. Modified cohomology groups. The cohomology group $H^n(X; \mathcal{M})$ itself may be too wild. We shall replace it with a modified one easier to manipulate.

LEMMA 4.4.1. *Let Φ be a finite admissible subset of W. For any $\mu \in \mathfrak{h}^*$, $n \in \mathbb{Z}$ and $\mathcal{M} \in \mathrm{Ob}(\mathbb{H}(\mu))$, we have the following.*

(i) *$H^n(X_\Phi; \mathcal{M})$ is an object of $\mathbb{O}$.*

(ii) *If $[H^n(X_\Phi; \mathcal{M}) : L(\zeta)] \neq 0$, then there exists some $w \in \Phi$ such that $X_w \subset \mathrm{Supp}(\mathcal{M})$ and that $[M(w \circ \mu) : L(\zeta)] \neq 0$.*

(iii) *For any admissible subset Ψ of W such that $\Psi \subset \Phi$, let N_1 (resp. N_2) be the kernel (resp. cokernel) of the natural homomorphism $H^n(X_\Phi; \mathcal{M}) \to H^n(X_\Psi; \mathcal{M})$. Then N_i belongs to $\mathrm{Ob}(\mathbb{O})$ for $i = 1, 2$. Moreover, if $[N_i : L(\zeta)] \neq 0$ for $i = 1$ or 2, then $[M(x \circ \mu) : L(\zeta)] \neq 0$ for some $x \in \Phi \setminus \Psi$.*

Proof. We first show (iii) by the induction of $\sharp(\Phi \setminus \Psi)$.

If $\sharp(\Phi \setminus \Psi) = 0$, it is trivial. In the case $\sharp(\Phi \setminus \Psi) = 1$, set $\Phi \setminus \Psi = \{x\}$. Let $i : X_x \to X_\Phi$ and $j : X_\Psi \to X_\Phi$ be the inclusion. By the assumption i is a closed embedding and j is an open embedding. The distinguished triangle

$$\mathbb{D}i_*\mathbb{D}i^!\mathcal{M} \to \mathcal{M} \to \mathbb{D}j_*\mathbb{D}j^\bullet M \xrightarrow{+1}$$

induces an exact sequence

$$(4.4.1)\quad H^n(X_\Phi; \mathbb{D}i_*\mathbb{D}i^!\mathcal{M}) \to H^n(X_\Phi; \mathcal{M}) \to H^n(X_\Psi; \mathcal{M}) \to H^{n+1}(X_\Phi; \mathbb{D}i_*\mathbb{D}i^!\mathcal{M}).$$

Therefore the kernel N_1 is a quotient of $H^n(X_\Phi; \mathbb{D}i_*\mathbb{D}i^!\mathcal{M})$, and the cokernel N_2 is a submodule of $H^{n+1}(X_\Phi; \mathbb{D}i_*\mathbb{D}i^!\mathcal{M})$. By Lemma 4.2.1, the object $H^k(\mathbb{D}i_*\mathbb{D}i^!\mathcal{M})$ in $\mathbb{H}_\Phi(\mu)$ is isomorphic to a direct sum of finitely many copies of $\mathcal{B}_x(\mu)|_{X_\Phi}$. Hence Proposition 4.3.1 (i) implies

$$H^k(X_\Phi; \mathbb{D}i_*\mathbb{D}i^!\mathcal{M}) = \Gamma(X_\Phi; H^k(\mathbb{D}i_*\mathbb{D}i^!\mathcal{M})) ,$$

and its character is a constant multiple of the character of the Verma module $M(x \circ \mu)$ for any k by Corollary 4.3.2. This shows (iii) in the case $\sharp(\Phi \setminus \Psi) = 1$.

Assume $\sharp(\Phi \setminus \Psi) > 1$. Taking a maximal element x of $\Phi \setminus \Psi$, set $\Phi' = \Phi \setminus \{x\}$. Then Φ' is an admissible subset such that $\Psi \subset \Phi' \subset \Phi$. Consider the diagram

$$H^n(X_\Phi; \mathcal{M}) \xrightarrow{\alpha} H^n(X_{\Phi'}; \mathcal{M}) \xrightarrow{\beta} H^n(X_\Psi; \mathcal{M}).$$

Then we have exact sequences

$$0 \to \mathrm{Ker}\,\alpha \to \mathrm{Ker}(\beta \circ \alpha) \to \mathrm{Ker}\,\beta$$
$$\mathrm{Coker}\,\alpha \to \mathrm{Coker}(\beta \circ \alpha) \to \mathrm{Coker}\,\beta \to 0.$$

By the induction hypothesis, $\operatorname{Ker}\alpha$ and $\operatorname{Ker}\beta$ belong to $\mathbb{O}$. Hence $N_1 = \operatorname{Ker}(\beta\circ\alpha)$ belongs to $\mathbb{O}$. If $[N_1 : L(\zeta)] \neq 0$, then $[\operatorname{Ker}\alpha : L(\zeta)] \neq 0$ or $[\operatorname{Ker}\beta : L(\zeta)] \neq 0$. The induction hypothesis implies $[M(x\circ\mu) : L(\zeta)] \neq 0$ in the first case and $[M(w\circ\mu) : L(\zeta)] \neq 0$ for some $w \in \Phi' \setminus \Psi$ in the second case. This shows the assertion for N_1. The assertion for N_2 is similarly proved.

We obtain (i) and (ii) from (iii) by taking $\Psi = \emptyset$ or $\Psi = \{w \in \Phi;\, X_w \cap \operatorname{Supp}\mathcal{M} = \emptyset\}$. □

By Lemma 4.4.1, Proposition 2.5.5 and the W-invariance of $\mathcal{K}_{\mathrm{reg}}$, we have the following corollary.

COROLLARY 4.4.2. *For $\lambda \in \mathcal{K}_{\mathrm{reg}}$ and $\mathcal{M} \in \operatorname{Ob}(\mathbb{H}(\lambda))$, we have*

$$H^n(X_\Phi;\mathcal{M}) \in \operatorname{Ob}(\mathbb{O}(\lambda))$$

for any finite admissible subset Φ of W and any $n \in \mathbb{Z}$.

LEMMA 4.4.3. *Let $\lambda, \mu \in \mathcal{K}_{\mathrm{reg}}$. Then for any $\zeta \in \mathfrak{h}^*$ there exists a finite admissible subset Φ of W such that the restriction homomorphism*

$$(P_\mu(H^n(X_{\Phi'};\mathcal{M})))_{\zeta'} \to (P_\mu(H^n(X_\Phi;\mathcal{M})))_{\zeta'}$$

is bijective for any finite admissible subset Φ' of W containing Φ, $\zeta' \in \zeta + Q^+$, $\mathcal{M} \in \operatorname{Ob}(\mathbb{H}(\lambda))$ and $n \geq 0$ (see (2.5.11) for the definition of P_μ).

Proof. We may assume $\mu \in \mathcal{K}^+_{\mathrm{reg}}$. Then $W(\mu)\circ\mu \cap (\zeta + Q^+)$ is a finite set. Since $\{w \in W; w\circ\lambda = \lambda\} = \{1\}$, there exist only finitely many $w \in W$ satisfying $w\circ\lambda \in W(\mu)\circ\mu$ and $w\circ\lambda - \zeta \in Q^+$. Thus we conclude that there exists a finite admissible subset Φ of W satisfying

$$w \in W,\ w\circ\lambda \in W(\mu)\circ\mu,\ w\circ\lambda - \zeta \in Q^+ \Longrightarrow w \in \Phi.$$

Then the assertion follows from Lemma 4.4.1 (iii) and the assumption on Φ. □

For $\lambda, \mu \in \mathcal{K}_{\mathrm{reg}}$, $\mathcal{M} \in \mathbb{H}(\lambda)$ and $n \in \mathbb{Z}_{\geq 0}$ we set

$$\tilde{H}^n_\mu(\mathcal{M}) = \bigoplus_{\xi\in\mathfrak{h}^*} \varprojlim_{\Phi} (P_\mu(H^n(X_\Phi;\mathcal{M})))_\xi\,, \tag{4.4.2}$$

where Φ is running over finite admissible subsets of W.

LEMMA 4.4.4. *For λ, $\mu \in \mathcal{K}_{\mathrm{reg}}$, $\tilde{H}^n_\mu$ is an additive functor from $\mathbb{H}(\lambda)$ to $\mathbb{O}[\mu]$ for any integer n.*

Proof. Let $\mathcal{M} \in \operatorname{Ob}(\mathbb{H}(\lambda))$. Take any $\zeta \in \mathfrak{h}^*$. Let Φ be as in Lemma 4.4.3. Then we have

$$\sum_{\xi\in\zeta+Q^+} \dim \tilde{H}^n_\mu(\mathcal{M})_\xi = \sum_{\xi\in\zeta+Q^+} \dim(P_\mu(H^n(X_\Phi;\mathcal{M})))_\xi < \infty$$

by Lemma 4.4.1 (i). Thus $\tilde{H}^n_\mu(\mathcal{M}) \in \operatorname{Ob}(\mathbb{O})$. □

Now for $\lambda \in \mathcal{K}_{\mathrm{reg}}$ and $\mathcal{M} \in \mathbb{H}(\lambda)$ we define the object of $\tilde{\mathbb{O}}$ (see (2.5.12)) as follows:

$$\tilde{H}^n(\mathcal{M}) = \prod_{\mu\in\mathcal{K}^+_{\mathrm{reg}}} \tilde{H}^n_\mu(\mathcal{M}). \tag{4.4.3}$$

Then $\tilde{H}^n(\mathcal{M})$ is the projective limit of $H^n(X_\Phi;\mathcal{M})$ in the category $\tilde{\mathbb{O}}$. We write $\tilde{\Gamma}(\mathcal{M})$ for $\tilde{H}^0(\mathcal{M})$.

PROPOSITION 4.4.5. *Let $\lambda \in \mathcal{K}_{\mathrm{reg}}$.*

(i) *$\tilde{H}^n$ is a functor from $\mathbb{H}(\lambda)$ to $\widetilde{\mathbb{O}}(\lambda)$.*

(ii) *For any short exact sequence*

$$0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to 0$$

in $\mathbb{H}(\lambda)$, we have a functorial long exact sequence

$$\begin{array}{ccccccc} 0 & \to & \tilde{H}^0(\mathcal{M}_1) & \to & \tilde{H}^0(\mathcal{M}_2) & \to & \tilde{H}^0(\mathcal{M}_3) \\ & \to & \tilde{H}^1(\mathcal{M}_1) & \to & \tilde{H}^1(\mathcal{M}_2) & \to & \tilde{H}^1(\mathcal{M}_3) \\ & \to & \cdots & & & & \end{array}$$

in $\widetilde{\mathbb{O}}$.

(iii) *Assume that $[\tilde{H}^n(\mathcal{M}) : L(\zeta)] \neq 0$ for $\mathcal{M} \in \mathrm{Ob}(\mathbb{H}(\lambda))$, $\zeta \in \mathfrak{h}^*$, $n \in \mathbb{Z}_{\geq 0}$. Then there exists some $w \in \Phi$ such that $X_w \subset \mathrm{Supp}(\mathcal{M})$ and that $[M(w \circ \lambda) : L(\zeta)] \neq 0$.*

(iv) *If $\lambda \in \mathcal{K}^+_{\mathrm{reg}}$ and $\mathcal{M} \in \mathbb{H}(\lambda)$, then $\tilde{H}^n(\mathcal{M})$ belongs to $\widetilde{\mathbb{O}}\{S(\mathcal{M}) \circ \lambda\}$, where $S(\mathcal{M}) = \{w \in W\,;\, X_w \subset \mathrm{Supp}(\mathcal{M})\}$.*

Proof. (i), (ii) and (iii) easily follow from the definition along with Lemma 4.4.1 and Corollary 4.4.2. Let us prove (iv). Assume $[\tilde{H}^n(\mathcal{M}) : L(\zeta)] \neq 0$. By (iii) there exists some $x \in \Phi$ such that $X_x \subset \mathrm{Supp}(\mathcal{M})$ and that $[M(x \circ \mu) : L(\zeta)] \neq 0$. Then by Corollary 2.5.6, we have $\zeta = w \circ \lambda$ for some $w \in W$ with $w \geq x$. Hence $X_w \subset \overline{X}_x \subset \mathrm{Supp}(\mathcal{M})$. Therefore $w \in S(\mathcal{M})$. □

LEMMA 4.4.6. *Let $\lambda \in \mathcal{K}_{\mathrm{reg}}$. For $M \in \mathrm{Ob}(\mathbb{O}\{\mathcal{K}_{\mathrm{reg}}\})$ and $\mathcal{M} \in \mathrm{Ob}(\mathbb{H}(\lambda))$ we have*

$$\mathrm{Hom}_{\widetilde{\mathbb{O}}}(M, \tilde{\Gamma}(\mathcal{M})) \simeq \mathrm{Hom}_{\mathfrak{g}}(M, \Gamma(X; \mathcal{M})).$$

Proof. Note that $\tilde{\Gamma}(\mathcal{M})$ is the projective limit of $\{\Gamma(X_\Phi; \mathcal{M})\}_\Phi$ in $\widetilde{\mathbb{O}}$, where Φ ranges over the set of finite admissible subsets of W, while $\Gamma(X; \mathcal{M})$ is the projective limit of $\{\Gamma(X_\Phi; \mathcal{M})\}_\Phi$ in the category of $\mathfrak{g}$-modules. Hence we have

$$\begin{aligned} \mathrm{Hom}_{\widetilde{\mathbb{O}}}(M, \tilde{\Gamma}(\mathcal{M})) &\simeq \varprojlim_{\Phi} \mathrm{Hom}_{\widetilde{\mathbb{O}}}(M, \Gamma(X_\Phi; \mathcal{M})) \\ &\simeq \varprojlim_{\Phi} \mathrm{Hom}_{\mathfrak{g}}(M, \Gamma(X_\Phi; \mathcal{M})) \\ &\simeq \mathrm{Hom}_{\mathfrak{g}}(M, \Gamma(X; \mathcal{M})). \end{aligned}$$

□

By Corollary 4.3.2 and Proposition 4.3.3 we have the following proposition.

PROPOSITION 4.4.7. *Let $\lambda \in \mathcal{K}_{\mathrm{reg}}$.*

(i) *$\tilde{H}^n(\mathcal{B}_w(\lambda)) = 0$ for $n \neq 0$.*

(ii) *$\tilde{\Gamma}(\mathcal{B}_w(\lambda)) = \Gamma(X; \mathcal{B}_w(\lambda))$ and $\mathrm{ch}(\tilde{\Gamma}(\mathcal{B}_w(\lambda))) = \mathrm{ch}(M(w \circ \lambda))$.*

(iii) *$\tilde{\Gamma}(\mathcal{B}_w(\lambda)) \simeq M^*(w \circ \lambda)$ if $\lambda \in \mathcal{K}^+_{\mathrm{reg}}$.*

PROPOSITION 4.4.8. *For $\lambda \in \mathcal{K}^+_{\mathrm{reg}}$ and $w \in W$ we have*

$$\mathrm{Hom}_{\widetilde{\mathbb{O}}}(M(w \circ \lambda), \tilde{\Gamma}(\mathcal{M}_w(\lambda))) \xrightarrow{\sim} \mathrm{Hom}_{\widetilde{\mathbb{O}}}(M(w \circ \lambda), \tilde{\Gamma}(\mathcal{B}_w(\lambda))) \cong \mathbb{C}. \tag{4.4.4}$$

Proof. Set $\mathcal{N}_1 = \mathcal{B}_w(\lambda)/\mathcal{L}_w(\lambda), \mathcal{N}_2 = \mathrm{Ker}(\mathcal{M}_w(\lambda) \to \mathcal{L}_w(\lambda))$. Then we have $\mathrm{Supp}(\mathcal{N}_i) \subset \overline{X}_w \setminus X_w$ for $i = 1, 2$. Hence Proposition 4.4.5 (iv) implies $[\tilde{H}^n(\mathcal{N}_i) : L(w \circ \lambda)] = 0$. Taking the cohomologies of the short exact sequences

$$0 \to \mathcal{N}_2 \to \mathcal{M}_w(\lambda) \to \mathcal{L}_w(\lambda) \to 0,$$
$$0 \to \mathcal{L}_w(\lambda) \to \mathcal{B}_w(\lambda) \to \mathcal{N}_1 \to 0,$$

we obtain exact sequences

$$0 \to \tilde{\Gamma}(\mathcal{N}_2) \to \tilde{\Gamma}(\mathcal{M}_w(\lambda)) \to \tilde{\Gamma}(\mathcal{L}_w(\lambda)) \to \tilde{H}^1(\mathcal{N}_2),$$
$$0 \to \tilde{\Gamma}(\mathcal{L}_w(\lambda)) \to \tilde{\Gamma}(\mathcal{B}_w(\lambda)) \to \tilde{\Gamma}(\mathcal{N}_1).$$

Therefor we have

$$\begin{aligned}[\tilde{\Gamma}(\mathcal{M}_w(\lambda)) : L(w \circ \lambda)] &= [\tilde{\Gamma}(\mathcal{L}_w(\lambda)) : L(w \circ \lambda)] \\ &= [\tilde{\Gamma}(\mathcal{B}_w(\lambda)) : L(w \circ \lambda)] \\ &= [M^*(w \circ \lambda) : L(w \circ \lambda)] \\ &= 1.\end{aligned}$$

Here the third equality follows from (iii) in the preceding proposition. By Proposition 4.4.5 (iv), we have

$$[\tilde{\Gamma}(\mathcal{M}_w(\lambda)) : L(y \circ \lambda)] = 0 \quad \text{for } y < w.$$

Hence Lemma 2.5.8 implies that $\mathrm{Hom}_{\tilde{\mathbb{O}}}(M(w \circ \lambda), \tilde{\Gamma}(\mathcal{M}_w(\lambda)))$ does not vanish. By the exact sequence

$$0 \to \tilde{\Gamma}(\mathcal{N}_2) \to \tilde{\Gamma}(\mathcal{M}_w(\lambda)) \to \tilde{\Gamma}(\mathcal{B}_w(\lambda))$$

and $\mathrm{Hom}_{\tilde{\mathbb{O}}}(M(w \circ \lambda), \tilde{\Gamma}(\mathcal{N}_2)) = 0$, the homomorphism

$$\mathrm{Hom}_{\tilde{\mathbb{O}}}(M(w \circ \lambda), \tilde{\Gamma}(\mathcal{M}_w(\lambda))) \to \mathrm{Hom}_{\tilde{\mathbb{O}}}(M(w \circ \lambda), \tilde{\Gamma}(\mathcal{B}_w(\lambda))) \cong \mathbb{C}$$

is injective, which implies the desired result. □

4.5. Modified localization functor. For $\mu \in \mathfrak{h}^*$ there exists a unique additive functor

$$D_\mu \hat{\otimes} \bullet : \mathbb{M}_{adm}(\mathfrak{g}) \to \mathbb{M}_{adm}(D_\mu) \qquad (M \mapsto D_\mu \hat{\otimes} M), \tag{4.5.1}$$

called the modified localization functor, such that

$$\begin{aligned}&\mathrm{Hom}_{\mathfrak{g}}(M, \Gamma(X; \mathcal{M})) = \mathrm{Hom}_{D_\mu}(D_\mu \hat{\otimes} M, \mathcal{M}) \\ &\qquad \text{for } M \in \mathrm{Ob}(\mathbb{M}_{adm}(\mathfrak{g})),\ \mathcal{M} \in \mathrm{Ob}(\mathbb{M}_{adm}(D_\mu)).\end{aligned} \tag{4.5.2}$$

In [14] it is constructed in the case where μ is integral. Since the construction in the general case is completely similar, we do not repeat it here. As in [14] we have the following proposition.

PROPOSITION 4.5.1. *Let $\mu \in \mathfrak{h}^*$.*

(i) *The functor* (4.5.1) *is right exact, and commutes with the inductive limit.*

(ii) *For any $M \in \mathbb{O}$, $D_\mu \hat{\otimes} M$ is an N^+-equivariant admissible D_μ-module.*

In particular, for $M \in \mathrm{Ob}(\mathbb{O})$, the support $\mathrm{Supp}(D_\mu \hat{\otimes} M)$ of $D_\mu \hat{\otimes} M$ is a B-stable closed subset of X.

By Lemma 4.4.6 we have the following lemma.

LEMMA 4.5.2. *For $\lambda \in \mathcal{K}_{\mathrm{reg}}$, $M \in \mathrm{Ob}(\mathbb{O}\{\mathcal{K}_{\mathrm{reg}}\})$ and $\mathcal{M} \in \mathrm{Ob}(\mathbb{H}(\lambda))$ we have*

$$\mathrm{Hom}_{\tilde{\mathbb{O}}}(M, \tilde{\Gamma}(\mathcal{M})) = \mathrm{Hom}_{D_\lambda}(D_\lambda \hat{\otimes} M, \mathcal{M}).$$

PROPOSITION 4.5.3. *Let $\lambda \in \mathcal{K}^+_{\mathrm{reg}}$, $M \in \mathrm{Ob}(\mathbb{O})$.*

(i) *Assume that X_w is open in $\overline{X}_w \bigcup \mathrm{Supp}(D_\lambda \hat{\otimes} M)$. Then $D_\lambda \hat{\otimes} M|_{U_w}$ is isomorphic to the direct sum of* $\dim \mathrm{Hom}_{\mathfrak{g}}(M, M^*(w \circ \lambda))$ *copies of $\mathcal{B}_w(\lambda)|_{U_w}$.*

(ii) $\mathrm{Supp}(D_\lambda \hat{\otimes} M)$ *is the union of $\overline{X}_w$ such that* $\mathrm{Hom}_{\mathfrak{g}}(M, M^*(w \circ \lambda)) \neq 0$.

(iii) $\mathrm{Supp}(D_\lambda \hat{\otimes} M)$ *is the union of $\overline{X}_w$ such that* $[M : L(w \circ \lambda)] \neq 0$.

Proof. Let us first show (i). Assume that X_w is open in $\overline{X}_w \cup \mathrm{Supp}(D_\lambda \hat{\otimes} M)$. Lemma 4.2.1 implies that $D_\lambda \hat{\otimes} M|_{U_w}$ is isomorphic to $\mathcal{B}_w(\lambda)^{\oplus J}|_{U_w}$ for some index set J. Hence by (4.2.3), (4.5.2), and Proposition 4.3.3 we have

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{g}}(M, M^*(w \circ \lambda)) &\simeq \mathrm{Hom}_{D_\lambda}(D_\lambda \hat{\otimes} M, \mathcal{B}_w(\lambda)) \\ &\simeq \mathrm{Hom}_{D_\lambda|_{U_w}}(D_\lambda \hat{\otimes} M|_{U_w}, \mathcal{B}_w(\lambda)|_{U_w}) \\ &\simeq \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\oplus J}, \mathbb{C}).\end{aligned}$$

Hence $\sharp J = \dim \mathrm{Hom}_{\mathfrak{g}}(M, M^*(w \circ \lambda)) < \infty$. Let us show (ii). By (i) it is obvious that $\mathrm{Supp}(D_\lambda \hat{\otimes} M)$ in contained in the union. Conversely assume $\mathrm{Hom}_{\mathfrak{g}}(M, M^*(w \circ \lambda)) \neq 0$. If X_w is not contained in $\mathrm{Supp}(D_\lambda \hat{\otimes} M)$, then X_w is open in $X_w \cup \mathrm{Supp}(D_\lambda \hat{\otimes} M)$, and hence (i) implies that $\mathrm{Supp}(D_\lambda \hat{\otimes} M)$ contains X_w.

Let us show (iii). By (ii), it is enough to show that $[M : L(w \circ \lambda)] \neq 0$ implies $X_w \subset \mathrm{Supp}(D_\lambda \hat{\otimes} M)$. Let us take $x \in W$ such that $x \leq w$, $[M : L(x \circ \lambda)] \neq 0$ and $[M : L(y \circ \lambda)] = 0$ for any $y < x$. Then Lemma 2.5.8 implies $\mathrm{Hom}_{\mathfrak{g}}(M, M^*(x \circ \lambda)) \neq 0$. Hence (ii) implies $\mathrm{Supp}(D_\lambda \hat{\otimes} M) \supset \overline{X}_x \supset X_w$. □

PROPOSITION 4.5.4. *For $\lambda \in \mathcal{K}^+_{\mathrm{reg}}$ the functor* (4.5.1) *induces the functor*

$$D_\lambda \hat{\otimes} \bullet : \mathbb{O} \to \mathbb{H}(\lambda). \tag{4.5.3}$$

Proof. Set $M_0 = U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{n}^+$. We shall first show that $D_\lambda \hat{\otimes} M_0$ is holonomic. Let Φ be a finite admissible subset of W. Set $Y_k = X_\Phi / \exp(\mathfrak{n}^+_k)$, and let $p_k : X_\Phi \to Y_k$ be the projection. Then Y_k is a smooth $\mathbb{C}$-scheme for $k \gg 0$. For $k_0 >> 0$, let $\{(Y_k, A_k)\}_{k \geq k_0}$ be a smooth projective system of $(X_\Phi, D_\lambda|_{X_\Phi})$. Since Y_k has finitely many orbits by the action of $\exp(\mathfrak{n}^+/\mathfrak{n}^+_k)$, the A_k-module $A_k \otimes_{U(\mathfrak{n}^+/\mathfrak{n}^+_k)} \mathbb{C}$ is a holonomic A_k-module. Since $D_\lambda \hat{\otimes} M_0|_{X_\Phi} = p_k^\bullet(A_k \otimes_{U(\mathfrak{n}^+/\mathfrak{n}^+_k)} \mathbb{C})$ for $k \gg 0$ (see [14]), it is also holonomic. Thus we have proved that $D_\lambda \hat{\otimes} M_0$ is holonomic. Then we see that $D_\lambda \hat{\otimes} (U(\mathfrak{g}) \otimes_{U(\mathfrak{n}^+)} V)$ is also holonomic for any finite-dimensional $\mathfrak{b}$-module V. Indeed V has a finite filtration whose graduation is isomorphic to $\mathbb{C}$ as an $\mathfrak{n}^+$-module. More generally for any $\mathfrak{g}$-module M which is generated by a finite-dimensional $\mathfrak{b}$-module V, $D_\lambda \hat{\otimes} M$ is holonomic.

Now let us show that $D_\lambda \hat{\otimes} M$ is holonomic for any $M \in \mathbb{O}$. Let Φ be a finite admissible subset of W. Let M_1 be the $\mathfrak{g}$-submodule of M generated by $\bigoplus_{w \in \Phi} M_{w \circ \lambda}$, and set $M_2 = M/M_1$. Then we have $[M_2 : L(w \circ \lambda)] = 0$ for any $w \in \Phi$. Hence Proposition 4.5.3 (iii) implies that $D_\lambda \hat{\otimes} M_2|_{X_\Phi} = 0$. By the exact sequence

$$D_\lambda \hat{\otimes} M_1 \to D_\lambda \hat{\otimes} M \to D_\lambda \hat{\otimes} M_2 \to 0$$

$D_\lambda\hat{\otimes}M_1|_{X_\Phi} \to D_\lambda\hat{\otimes}M|_{X_\Phi}$ is surjective. Since $D_\lambda\hat{\otimes}M_1$ is holonomic, $D_\lambda\hat{\otimes}M|_{X_\Phi}$ is holonomic. □

COROLLARY 4.5.5. *For $\lambda \in \mathcal{K}^+_{\mathrm{reg}}$ and $\mu = w\circ\lambda \in \mathcal{K}^+_{\mathrm{reg}}$ with $w \in W$,* $\mathrm{Supp}(D_\lambda\hat{\otimes}M) \subset \overline{X}_w$ *for any $M \in \mathbb{O}[\mu]$.*

Proof. For $x \in W$, if $[M : L(x\circ\lambda)] \neq 0$ then $x\circ\lambda \in W(\mu)\circ\mu$. Hence $x \in W(\mu)w$ and Lemma 2.5.7 implies $x \geq w$. Then the desired result follows from Proposition 4.5.3 (iii). □

COROLLARY 4.5.6. *Let $\lambda \in \mathcal{K}^+_{\mathrm{reg}}$ and $\mu \in \mathcal{K}_{\mathrm{reg}}$. For $M \in \mathrm{Ob}(\mathbb{O}[\mu])$ we have $D_\lambda\hat{\otimes}M = 0$ unless $\mu \in W\circ\lambda$.*

Proof. Assume $D_\lambda\hat{\otimes}M \neq 0$. Take $w \in W$ such that X_w is open in $\mathrm{Supp}(D_\lambda\hat{\otimes}M)$. Then Proposition 4.5.3 implies $[M : L(w\circ\lambda)] \neq 0$. Hence $w\circ\lambda \in W\circ\mu$. □

We define $D_\lambda\hat{\otimes}M$ for $\lambda \in \mathcal{K}^+_{\mathrm{reg}}$ and $M \in \widetilde{\mathbb{O}}$ by

$$D_\lambda\hat{\otimes}M = \bigoplus_{\mu\in\mathcal{K}^+_{\mathrm{reg}}} D_\lambda\hat{\otimes}P_\mu(M) = \bigoplus_{\mu\in W\circ\lambda\cap\mathcal{K}^+_{\mathrm{reg}}} D_\lambda\hat{\otimes}P_\mu(M).$$

Here the last equality follows from Corollary 4.5.6. Then Corollary 4.5.5 implies that, for any finite admissible subset Φ, $D_\lambda\hat{\otimes}P_\mu(M)|_{X_\Phi} = 0$ except finitely many $\mu \in W\circ\lambda\cap\mathcal{K}^+_{\mathrm{reg}}$. Hence $D_\lambda\hat{\otimes}\bullet$ is a right exact functor from $\widetilde{\mathbb{O}}$ to $\mathbb{H}(\lambda)$. The composition of functors

$$\mathbb{O}\{\mathcal{K}_{\mathrm{reg}}\} \longrightarrow \widetilde{\mathbb{O}} \xrightarrow{D_\lambda\hat{\otimes}} \mathbb{H}(\lambda)$$

coincides with the restriction of the original functor $D_\lambda\hat{\otimes}\bullet$ by Theorem 2.5.9. Moreover we have the following property analogous to (4.5.2).

PROPOSITION 4.5.7. *For $\lambda \in \mathcal{K}^+_{\mathrm{reg}}$, the functor $D_\lambda\hat{\otimes}\bullet : \widetilde{\mathbb{O}} \to \mathbb{H}(\lambda)$ is a left adjoint functor of $\tilde{\Gamma} : \mathbb{H}(\lambda) \to \widetilde{\mathbb{O}}$. Namely we have an isomorphism functorial in $\mathcal{M} \in \mathbb{H}(\lambda)$ and $M \in \widetilde{\mathbb{O}}$:*

$$\mathrm{Hom}_{\mathbb{H}(\lambda)}(D_\lambda\hat{\otimes}M, \mathcal{M}) \cong \mathrm{Hom}_{\widetilde{\mathbb{O}}}(M, \tilde{\Gamma}(\mathcal{M})).$$

In particular we have a morphism functorial in $\mathcal{M} \in \mathbb{H}(\lambda)$

$$D_\lambda\hat{\otimes}\tilde{\Gamma}(\mathcal{M}) \to \mathcal{M}.$$

PROPOSITION 4.5.8. *Let $\lambda \in \mathcal{K}^+_{\mathrm{reg}}$, and Φ a finite admissible subset of W. For $M \in \mathrm{Ob}(\widetilde{\mathbb{O}})$ we have $D_\lambda\hat{\otimes}M|_{X_\Phi} = 0$ if and only if $M \in \mathrm{Ob}(\widetilde{\mathbb{O}}\{(\mathcal{K}_{\mathrm{reg}} \setminus (\Phi\circ\lambda)\})$.*

Proof. We may assume $M \in \mathrm{Ob}(\mathbb{O})$. Then it is an immediate consequence of Proposition 4.5.3. □

THEOREM 4.5.9. *For any $\lambda \in \mathcal{K}^+_{\mathrm{reg}}$ and $w \in W$, there is a canonical isomorphism*

$$D_\lambda\hat{\otimes}M(w\circ\lambda) \xrightarrow{\sim} \mathcal{M}_w(\lambda).$$

Proof. We have $\dim \mathrm{Hom}_{\mathfrak{g}}(M(w\circ\lambda), M^*(y\circ\lambda)) = 1$ or 0 according to whether $w = y$ or $w \neq y$. Hence by Proposition 4.5.3 we have $\mathrm{Supp}(D_\lambda\hat{\otimes}M(w\circ\lambda)) = \overline{X}_w$, and

$$D_\lambda\hat{\otimes}M(w\circ\lambda)|_{U_w} \simeq \mathcal{B}_w(\lambda)|_{U_w} \simeq \mathcal{M}_w(\lambda)|_{U_w}.$$

By (4.2.2) the isomorphism $\mathcal{M}_w(\lambda)|_{U_w} \simeq D_\lambda \hat{\otimes} M(w \circ \lambda)|_{U_w}$ is uniquely extended to a homomorphism $\varphi : \mathcal{M}_w(\lambda) \to D_\lambda \hat{\otimes} M(w \circ \lambda)$.

On the other hand, by Proposition 4.4.8 we have a unique non-zero homomorphism $M(w \circ \lambda) \to \tilde{\Gamma}(\mathcal{M}_w(\lambda))$. Let $\psi : D_\lambda \hat{\otimes} M(w \circ \lambda) \to \mathcal{M}_w(\lambda)$ be the corresponding homomorphism. By the same proposition, the composition $D_\lambda \hat{\otimes} M(w \circ \lambda) \to \mathcal{M}_w(\lambda) \to \mathcal{B}_w(\lambda)$ is non-zero. Hence $\varphi|_{U_w}$ and $\psi|_{U_w}$ are inverse to each other up to a non-zero constant multiple. In particular $\psi \circ \varphi|_{U_w}$ is the identity endomorphism of $\mathcal{M}_w(\lambda)|_{U_w}$. Thus $\psi \circ \varphi = \mathrm{id}$ by (4.2.2).

It remains to show that φ is an epimorphism. Note that $\mathrm{Supp}(\mathrm{Coker}(\varphi)) \subset \overline{X}_w \setminus X_w$. Assume that $\mathrm{Coker}(\varphi) \neq 0$. By Lemma 4.2.1, there exists some $y \in W$ such that $y > w$ and that $\mathrm{Hom}_{D_\lambda}(\mathrm{Coker}(\varphi), \mathcal{B}_y(\lambda)) \neq 0$. Since $\mathrm{Coker}(\varphi)$ is a quotient of $D_\lambda \hat{\otimes} M(w \circ \lambda)$, we obtain

$$\mathrm{Hom}_{\mathfrak{g}}(M(w \circ \lambda), M^*(y \circ \lambda)) \simeq \mathrm{Hom}_{D_\lambda}(D_\lambda \hat{\otimes} M(w \circ \lambda), \mathcal{B}_y(\lambda)) \neq 0.$$

This implies $w = y$, which contradicts $y > w$. Thus $\mathrm{Coker}(\varphi) = 0$ and hence φ is an epimorphism. □

For $\lambda \in \mathcal{K}^+_{\mathrm{reg}}$, a canonical morphism $D_\lambda \hat{\otimes} M^*(w \circ \lambda) \to \mathcal{B}_w(\lambda)$ is defined by Proposition 4.3.3.

LEMMA 4.5.10. *For any $\lambda \in \mathcal{K}^+_{\mathrm{reg}}$ and $w \in W$, the canonical morphism $D_\lambda \hat{\otimes} M^*(w \circ \lambda) \to \mathcal{B}_w(\lambda)$ is surjective.*

Proof. By Proposition 4.3.3, we have $\Gamma(X; \mathcal{B}_w(\lambda)) \cong M^*(w \circ \lambda)$. Since the open embedding $i : U_w \to X$ is an affine morphism and $\mathcal{B}_w(\lambda)$ is a quasi-coherent $\mathcal{O}_X$-module, the natural homomorphism $\mathcal{O}_X \otimes M^*(w \circ \lambda) \cong \mathcal{O}_X \otimes \Gamma(U_w; \mathcal{B}_w(\lambda)) \to i_* i^{-1} \mathcal{B}_w(\lambda) \cong \mathcal{B}_w(\lambda)$ is surjective. Hence the assertion follows from the fact that $\mathcal{O}_X \otimes M^*(w \circ \lambda) \to \mathcal{B}_w(\lambda)$ decomposes into $\mathcal{O}_X \otimes M^*(w \circ \lambda) \to D_\lambda \hat{\otimes} M^*(w \circ \lambda) \to \mathcal{B}_w(\lambda)$. □

4.6. Radon transforms. For $i \in I$, $\mu \in \mathfrak{h}^*$ and $n \in \mathbb{Z}$, we shall construct functors

$$S^n_{i*} : \mathbb{H}(\mu) \to \mathbb{H}(s_i \circ \mu), \qquad S^n_{i!} : \mathbb{H}(\mu) \to \mathbb{H}(s_i \circ \mu), \tag{4.6.1}$$

called the Radon transforms, and investigate their properties. We use results in [18] without giving proofs.

Fix $i \in I$. Let P^- be the algebraic group containing B^- with Lie algebra $\mathfrak{b}^- \oplus \mathfrak{g}_{\alpha_i}$. We have a natural free right action of P^- on G (see [13]). Set $Y = G/P^-$, and let $\pi : X \to Y$ be the projection. Then Y is a separated pro-smooth scheme, and π is a $\mathbb{P}^1$-bundle. Set

$$Z = X \times_Y X, \qquad Z_0 = Z \setminus \Delta(X),$$

where $\Delta : X \hookrightarrow Z$ denotes the diagonal embedding. Let $p_r : Z_0 \to X$ $(r = 1, 2)$ be the first and the second projections.

For a holonomic D_μ-module $\mathcal{M}$ we set

$$S^n_{i*}\mathcal{M} = H^n\Big(\int_{p_1} p_2^{\bullet}\mathcal{M}\Big), \qquad S^n_{i!}\mathcal{M} = H^n\Big(\int_{p_1!} p_2^{\bullet}\mathcal{M}\Big). \tag{4.6.2}$$

Since $\Omega_\pi = \mathcal{O}_X(-\alpha_i)$, we have $p_1^\sharp D_{s_i \circ \mu} = p_2^\sharp D_\mu$ (see Lemma 1.3.3 of [18]), and hence S^n_{i*} and $S^n_{i!}$ are well-defined.

By Lemma 1.4.3 and Theorem 1.5.1 of [18], we have the following proposition.

PROPOSITION 4.6.1. *Let $\mu \in \mathfrak{h}^*$, and let $i \in I$ and $w \in W$ such that $ws_i < w$.*

(i) $S^0_{i*}\mathcal{B}_w(\mu) = \mathcal{B}_{ws_i}(s_i \circ \mu)$ *and* $S^n_{i*}\mathcal{B}_w(\mu) = 0$ *for* $n \neq 0$.
(ii) $S^0_{i!}\mathcal{M}_w(\mu) = \mathcal{M}_{ws_i}(s_i \circ \mu)$ *and* $S^n_{i!}\mathcal{M}_w(\mu) = 0$ *for* $n \neq 0$.
(iii) $S^0_{i!}\mathcal{B}_{ws_i}(\mu) = \mathcal{B}_w(s_i \circ \mu)$ *and* $S^n_{i!}\mathcal{B}_{ws_i}(\mu) = 0$ *for* $n \neq 0$.
(iv) $S^0_{i*}\mathcal{M}_{ws_i}(\mu) = \mathcal{M}_w(s_i \circ \mu)$ *and* $S^n_{i*}\mathcal{M}_{ws_i}(\mu) = 0$ *for* $n \neq 0$.

THEOREM 4.6.2. *Let* $\lambda \in \mathcal{K}_{\mathrm{reg}}, w \in W, i \in I$ *such that* $ws_i < w$ *and* $\langle h_i, \lambda + \rho\rangle \notin \mathbb{Z}_{>0}$. *Then we have*

$$\tilde{H}^n(\mathcal{B}_{ws_i}(\lambda)) \simeq \tilde{H}^n(\mathcal{B}_w(s_i \circ \lambda)), \qquad \tilde{H}^n(\mathcal{M}_w(\lambda)) \simeq \tilde{H}^n(\mathcal{M}_{ws_i}(s_i \circ \lambda))$$

for any $n \in \mathbb{Z}$.

Proof. Take a finite admissible subset Φ of W such that $\Phi s_i = \Phi$. By Proposition 4.6.1, Corollary 1.6.2 of [18], and by $\langle h_i, \lambda + \rho\rangle \notin \mathbb{Z}_{>0}$ we have

$$H^n(X_\Phi; \mathcal{B}_{ws_i}(\lambda)) \simeq H^n(X_\Phi; \mathcal{B}_w(s_i \circ \lambda)), \quad H^n(X_\Phi; \mathcal{M}_w(\lambda)) \simeq H^n(X_\Phi; \mathcal{M}_{ws_i}(s_i \circ \lambda))$$

for any $n \in \mathbb{Z}$. Hence the desired results follow by taking the projective limit with respect to Φ. □

COROLLARY 4.6.3. *Let* $\lambda \in \mathcal{K}^+_{\mathrm{reg}}, z \in W$. *Assume that* $z \circ \lambda \in \mathcal{K}^+_{\mathrm{reg}}$. *Then we have*

$$\tilde{H}^n(\mathcal{B}_w(\lambda)) \simeq \tilde{H}^n(\mathcal{B}_{wz^{-1}}(z \circ \lambda)), \qquad \tilde{H}^n(\mathcal{M}_w(\lambda)) \simeq \tilde{H}^n(\mathcal{M}_{wz^{-1}}(z \circ \lambda))$$

for any $w \in W$ *and* $n \in \mathbb{Z}$.

Proof. Take a reduced expression $z = s_{i_1}s_{i_2}\cdots s_{i_p}$ of $z \in W$. It is sufficient to show

$$\tilde{H}^n(\mathcal{B}_{ws_{i_p}\cdots s_{i_{j+1}}}(s_{i_{j+1}}\cdots s_{i_p} \circ \lambda)) \simeq \tilde{H}^n(\mathcal{B}_{ws_{i_p}\cdots s_{i_j}}(s_{i_j}\cdots s_{i_p} \circ \lambda))$$
$$\tilde{H}^n(\mathcal{M}_{ws_{i_p}\cdots s_{i_{j+1}}}(s_{i_{j+1}}\cdots s_{i_p} \circ \lambda)) \simeq \tilde{H}^n(\mathcal{M}_{ws_{i_p}\cdots s_{i_j}}(s_{i_j}\cdots s_{i_p} \circ \lambda))$$

for any j. By Theorem 4.6.2 we have only to show

$$(s_{i_{j+1}}\cdots s_{i_p} \circ \lambda + \rho, \alpha^\vee_{i_j}) \notin \mathbb{Z}_{>0} \cup \mathbb{Z}_{<0}.$$

Since $\lambda \in \mathcal{K}^+_{\mathrm{reg}}$ and $s_{i_p}\cdots s_{i_{j+1}}(\alpha_{i_j}) \in \Delta^+_{\mathrm{re}}$, we have

$$(s_{i_{j+1}}\cdots s_{i_p} \circ \lambda + \rho, \alpha^\vee_{i_j}) = (\lambda + \rho, s_{i_p}\cdots s_{i_{j+1}}(\alpha^\vee_{i_j})) \notin \mathbb{Z}_{<0}.$$

On the other hand, since $z \circ \lambda \in \mathcal{K}^+_{\mathrm{reg}}$ and $s_{i_1}\cdots s_{i_{j-1}}(\alpha_{i_j}) \in \Delta^+_{\mathrm{re}}$, we have

$$(s_{i_{j+1}}\cdots s_{i_p} \circ \lambda + \rho, \alpha^\vee_{i_j}) = -(z \circ \lambda + \rho, s_{i_1}\cdots s_{i_{j-1}}(\alpha^\vee_{i_j})) \notin \mathbb{Z}_{>0}.$$

The proof is completed. □

4.7. Global sections of $\mathcal{M}_w(\lambda)$. For $\lambda \in \mathcal{K}^+_{\mathrm{reg}}$ and $w \in W$, we denote by

$$\varphi^\lambda_w : M(w \circ \lambda) \to \tilde{\Gamma}(\mathcal{M}_w(\lambda)) \tag{4.7.1}$$

a non-zero morphism in $\widetilde{\mathbb{O}}$ (see Proposition 4.4.8). Note that φ^λ_w is unique up to a non-zero constant multiple. The aim of this section is to prove that it is a monomorphism.

Let $i \in I$. Let $\pi : X \to Y$ be the $\mathbb{P}^1$-bundle as in §4.6. Assume that $w \in W$ satisfy $ws_i > w$. Set $Y_w = \pi(X_w)$. Then Y_w is an affine scheme. $X_{iw} = \pi^{-1}(Y_w) = X_w \sqcup X_{ws_i}$ and $X_{ws_i} \to Y_w$ is an isomorphism. Hence X_{ws_i} is a closed hypersurface of X_{iw}. Let $j : X_{iw} \to X$ be the inclusion. Let λ be an element of $\mathfrak{h}^*$ satisfying $(\lambda+\rho, \alpha_i^\vee) \in \mathbb{Z}$. Then there exists an N^+-equivariant line bundle L on X_{iw} such that $j^\sharp D_\lambda \simeq D_{X_{iw}}(L)$. Let $j_0 : X_w \to X_{iw}$ be the open embedding and $i_0 : X_{ws_i} \to X_{iw}$ the closed embedding. We have the exact sequences of holonomic $D_{X_{iw}}$-modules

$$0 \to \int_{i_0\,!} \mathcal{O}_{X_{ws_i}} \to \int_{j_0\,!} \mathcal{O}_{X_w} \to \mathcal{O}_{X_{iw}} \to 0,$$
$$0 \to \mathcal{O}_{X_{iw}} \to \int_{j_0} \mathcal{O}_{X_w} \to \int_{i_0} \mathcal{O}_{X_{ws_i}} \to 0.$$

Since j is an affine morphism, $H^0 \int_{j\,!}$ and $H^0 \int_j$ are exact functors. Tensoring L to the exact sequences above, and applying the exact functors $H^0 \int_{j\,!}$ and $H^0 \int_j$, we obtain exact sequences in $\mathbb{H}(\lambda)$

$$\begin{aligned} 0 \longrightarrow \mathcal{M}_{ws_i}(\lambda) \xrightarrow{\iota} \mathcal{M}_w(\lambda) \longrightarrow \mathcal{L} \longrightarrow 0\,, \\ 0 \longrightarrow \mathcal{L}' \longrightarrow \mathcal{B}_w(\lambda) \longrightarrow \mathcal{B}_{ws_i}(\lambda) \longrightarrow 0\,. \end{aligned} \tag{4.7.2}$$

where $\mathcal{L} = H^0 \int_{j\,!} L$ and $\mathcal{L} = H^0 \int_j L$. We have $\mathcal{L}|_U \simeq \mathcal{L}'|_U$ for any open set containing X_{iw} as a closed subset.

LEMMA 4.7.1. *Assume that $\lambda \in \mathcal{K}^+_{\mathrm{reg}}$, $i \in I$ and $w \in W$ satisfy $ws_i > w$ and $\alpha_i \in \Delta(\lambda)$. Let $\iota_* : \tilde{\Gamma}(\mathcal{M}_{ws_i}(\lambda)) \to \tilde{\Gamma}(\mathcal{M}_w(\lambda))$ be the monomorphism in $\widetilde{\mathbb{O}}$ induced by the monomorphism $\iota : \mathcal{M}_{ws_i}(\lambda) \to \mathcal{M}_w(\lambda)$ in* (4.7.2). *Let $j : M(ws_i \circ \lambda) \to M(w \circ \lambda)$ be the injective homomorphism of $\mathfrak{g}$-modules* (*see* Proposition 2.5.5). *Then the following diagram in $\widetilde{\mathbb{O}}$ is commutative up to a non-zero constant multiple.*

$$\begin{tikzcd} M(ws_i \circ \lambda) \arrow[r, "\varphi^\lambda_{ws_i}"] \arrow[d, "j"'] & \tilde{\Gamma}(\mathcal{M}_{ws_i}(\lambda)) \arrow[d, "\iota_*"] \\ M(w \circ \lambda) \arrow[r, "\varphi^\lambda_w"] & \tilde{\Gamma}(\mathcal{M}_w(\lambda))\,. \end{tikzcd}$$

Proof. It is sufficient to show the following two statements.

$$\dim \operatorname{Hom}_{\widetilde{\mathbb{O}}}(M(ws_i \circ \lambda), \tilde{\Gamma}(\mathcal{M}_w(\lambda))) = 1. \tag{4.7.3}$$

$$\varphi^\lambda_w \circ j \neq 0. \tag{4.7.4}$$

We first show (4.7.3). The first exact sequence in (4.7.2)

$$0 \to \mathcal{M}_{ws_i}(\lambda) \to \mathcal{M}_w(\lambda) \to \mathcal{L} \to 0$$

induces an exact sequence

$$\begin{aligned} 0 \to \operatorname{Hom}_{\widetilde{\mathbb{O}}}(M(ws_i \circ \lambda), \tilde{\Gamma}(\mathcal{M}_{ws_i}(\lambda))) \to \operatorname{Hom}_{\widetilde{\mathbb{O}}}(M(ws_i \circ \lambda), \tilde{\Gamma}(\mathcal{M}_w(\lambda))) \\ \to \operatorname{Hom}_{\widetilde{\mathbb{O}}}(M(ws_i \circ \lambda), \tilde{\Gamma}(\mathcal{L})). \end{aligned} \tag{4.7.5}$$

Since $\dim \operatorname{Hom}_{\widetilde{\mathbb{O}}}(M(ws_i \circ \lambda), \tilde{\Gamma}(\mathcal{M}_{ws_i}(\lambda))) = 1$ by Proposition 4.4.8, it is sufficient to show $\operatorname{Hom}_{\widetilde{\mathbb{O}}}(M(ws_i \circ \lambda), \tilde{\Gamma}(\mathcal{L})) = 0$. Since $\mathcal{M}_{ws_i}(\lambda) \cong \mathcal{D}_\lambda \hat{\otimes} M(ws_i \circ \lambda)$ by Theorem 4.5.9, we reduce this to $\operatorname{Hom}(\mathcal{M}_{ws_i}(\lambda), \mathcal{L}) = 0$. Set $\Phi = \{x \in W\,;\, x \leq ws_i\}$ and $\Psi = \Phi \setminus \{ws_i\}$. They are finite admissible subsets of W, and $X_\Phi \cap \overline{X}_w = X_{iw} = X_w \sqcup X_{ws_i}$. Then (4.7.2) induces an exact sequence:

$$0 \to \mathcal{L}|_{X_\Phi} \to \mathcal{B}_w(\lambda)|_{X_\Phi} \to \mathcal{B}_{ws_i}(\lambda)|_{X_\Phi} \to 0.$$

Since $\mathcal{M}_{ws_i}(\lambda)|_{X_\Psi} = 0$, we have by (4.2.2) and (4.2.3)

$$\begin{aligned}\operatorname{Hom}(\mathcal{M}_{ws_i}(\lambda), \mathcal{L}) &\simeq \operatorname{Hom}(\mathcal{M}_{ws_i}(\lambda)|_{X_\Phi}, \mathcal{L}|_{X_\Phi}) \\ &\subset \operatorname{Hom}(\mathcal{M}_{ws_i}(\lambda)|_{X_\Phi}, \mathcal{B}_w(\lambda)|_{X_\Phi}) \\ &\simeq \operatorname{Hom}(\mathcal{M}_{ws_i}(\lambda)|_{X_\Psi}, \mathcal{B}_w(\lambda)|_{X_\Psi}) = 0.\end{aligned}$$

Thus the proof of (4.7.3) is completed.

Let us prove (4.7.4). Consider the chain of morphisms

$$\psi : M(w \circ \lambda) \to \tilde{\Gamma}(\mathcal{M}_w(\lambda)) \xrightarrow{\simeq} \tilde{\Gamma}(\mathcal{M}_{ws_i}(s_i \circ \lambda)) \to \tilde{\Gamma}(\mathcal{B}_{ws_i}(s_i \circ \lambda)).$$

Here the middle isomorphism follows from Theorem 4.6.2, because

$$(s_i \circ \lambda + \rho, \alpha_i^\vee) = -(\lambda + \rho, \alpha_i^\vee) \notin \mathbb{Z}_{>0}.$$

In order to prove (4.7.4), it is sufficient to show that the composition of

$$M(ws_i \circ \lambda) \xrightarrow{j} M(w \circ \lambda) \xrightarrow{\psi} \tilde{\Gamma}(\mathcal{B}_{ws_i}(s_i \circ \lambda))$$

is non-zero. Assuming that ψ is a non-zero homomorphism for a while, we shall finish the proof. Since $\operatorname{ch}(\tilde{\Gamma}(\mathcal{B}_{ws_i}(s_i \circ \lambda))) = \operatorname{ch}(M(w \circ \lambda))$, we have $[\operatorname{Ker}(\psi) : L(\zeta)] = [\operatorname{Coker}(\psi) : L(\zeta)]$ for any ζ. Assume that $\psi \circ j = 0$. Then $\operatorname{Ker}(\psi)$ contains $M(ws_i \circ \lambda)$ as a submodule. Hence we have

$$[\operatorname{Coker}(\psi) : L(ws_i \circ \lambda)] = [\operatorname{Ker}(\psi) : L(ws_i \circ \lambda)] > 0.$$

Since $\operatorname{Coker}(\psi)_{w\circ\lambda} = 0$, $[\operatorname{Coker}(\psi) : L(\zeta)] \neq 0$ only if $\zeta = y \circ \lambda$ for some $y \in W$ such that $y > w$. If $[M(\zeta) : L(ws_i \circ \lambda)] \neq 0$ and $[\operatorname{Coker}(\psi) : L(\zeta)] \neq 0$, then we have $\zeta = y \circ \lambda$ with $ws_i \geq y > w$ (by Corollary 2.5.6), and hence $\zeta = ws_i \circ \lambda$. Hence Lemma 2.3.2 implies

$$\dim \operatorname{Hom}_{\mathfrak{g}}(M(ws_i \circ \lambda), \operatorname{Coker}(\psi)^*) > 0.$$

Thus we obtain

$$\dim \operatorname{Hom}_{\mathfrak{g}}(M(ws_i \circ \lambda), \tilde{\Gamma}(\mathcal{B}_{ws_i}(s_i \circ \lambda))^*) \geq \dim \operatorname{Hom}_{\mathfrak{g}}(M(ws_i \circ \lambda), \operatorname{Coker}(\psi)^*) > 0.$$

Applying Corollary 4.3.2 (iii), we have

$$ws_i \circ \lambda \in w \circ \lambda - \sum_{\alpha \in \Delta^+ \cap ws_i\Delta^+} \mathbb{Z}_{\geq 0}\alpha,$$

and hence

$$\begin{aligned}(\lambda + \rho, \alpha_i^\vee)\alpha_i = w^{-1}(w \circ \lambda - ws_i \circ \lambda) &\in w^{-1} \sum_{\alpha \in \Delta^+ \cap ws_i\Delta^+} \mathbb{Z}_{\geq 0}\alpha \\ &= \sum_{\alpha \in w^{-1}\Delta^+ \cap s_i\Delta^+} \mathbb{Z}_{\geq 0}\alpha \subset \sum_{\alpha \in \Delta^+ \setminus \{\alpha_i\}} \mathbb{Z}_{\geq 0}\alpha.\end{aligned}$$

This is a contradiction. Hence $\psi \circ j \neq 0$.

It remains to prove that ψ does not vanish. In order to see this, it is sufficient to show the injectivity of the homomorphism

$$\operatorname{Hom}_{\tilde{\mathbb{O}}}(M(w \circ \lambda), \tilde{\Gamma}(\mathcal{M}_{ws_i}(s_i \circ \lambda))) \to \operatorname{Hom}_{\tilde{\mathbb{O}}}(M(w \circ \lambda), \tilde{\Gamma}(\mathcal{B}_{ws_i}(s_i \circ \lambda))).$$

Let $\mathcal{N}$ be the kernel of the morphism $\mathcal{M}_{ws_i}(s_i \circ \lambda) \to \mathcal{B}_{ws_i}(s_i \circ \lambda)$. By the exact sequence

$$0 \to \mathrm{Hom}_{\tilde{\mathbb{O}}}(M(w \circ \lambda), \tilde{\Gamma}(\mathcal{N})) \to \mathrm{Hom}_{\tilde{\mathbb{O}}}(M(w \circ \lambda), \tilde{\Gamma}(\mathcal{M}_{ws_i}(s_i \circ \lambda))) \\ \to \mathrm{Hom}_{\tilde{\mathbb{O}}}(M(w \circ \lambda), \tilde{\Gamma}(\mathcal{B}_{ws_i}(s_i \circ \lambda))),$$

we can reduce the assertion to $[\tilde{\Gamma}(\mathcal{N}) : L(w \circ \lambda)] = 0$. Assume the contrary. Then by Proposition 4.4.5 (iii), there exists some $x \in W$ such that

$$[M(xs_i \circ \lambda) : L(w \circ \lambda)] \neq 0 \quad \text{and} \tag{4.7.6}$$

$$X_x \subset \mathrm{Supp}\mathcal{N} \subset \overline{X_{ws_i}} \setminus X_{ws_i}. \tag{4.7.7}$$

Then (4.7.6) implies $xs_i \leq w$ by Corollary 2.5.6, and (4.7.7) implies $ws_i < x$. Hence we obtain $xs_i \leq w < ws_i < x$, which implies $l(xs_i) \leq l(w) < l(ws_i) < l(x) \leq l(xs_i) + 1$. This is a contradiction. Hence we have proved the non-vanishing of ψ. □

Now we are ready to prove the following result.

PROPOSITION 4.7.2. *Let $\lambda \in \mathcal{K}^+_{\mathrm{reg}}$, $w \in W$. Then the morphism*

$$\varphi_w^\lambda : M(w \circ \lambda) \to \tilde{\Gamma}(\mathcal{M}_w(\lambda))$$

is a monomorphism.

Proof. Take $x \in W(w \circ \lambda)$ such that $\lambda' = x^{-1}w \circ \lambda \in \mathcal{K}^+_{\mathrm{reg}}$. Then $w \circ \lambda = x \circ \lambda'$ and $x \in W(\lambda')$. By Corollary 4.6.3 we have $\tilde{\Gamma}(\mathcal{M}_w(\lambda)) \simeq \tilde{\Gamma}(\mathcal{M}_x(\lambda'))$. The composition of

$$M(w \circ \lambda) = M(x \circ \lambda') \xrightarrow{\varphi_x^{\lambda'}} \tilde{\Gamma}(\mathcal{M}_x(\lambda')) \simeq \tilde{\Gamma}(\mathcal{M}_w(\lambda))$$

coincides with φ_w^λ up to a non-zero scalar multiple by Proposition 4.4.8. Hence we may assume from the beginning that $w \in W(\lambda)$.

For $y, x \in W(\lambda)$ such that $y \geq_\lambda x$, we denote by $f_x^y : M(y \circ \lambda) \to M(x \circ \lambda)$ a non-zero homomorphism of $\mathfrak{g}$-modules.

Assume that for any $w \in W(\lambda)$ there exists a monomorphism

$$F_w : \tilde{\Gamma}(\mathcal{M}_w(\lambda)) \to \tilde{\Gamma}(\mathcal{M}_e(\lambda))$$

in $\tilde{\mathbb{O}}$ such that the diagram

$$\begin{array}{ccc} M(w \circ \lambda) & \xrightarrow{\varphi_w^\lambda} & \tilde{\Gamma}(\mathcal{M}_w(\lambda)) \\ {\scriptstyle f_e^w}\downarrow & & \downarrow{\scriptstyle F_w} \\ M(\lambda) & \xrightarrow[\varphi_e^\lambda]{} & \tilde{\Gamma}(\mathcal{M}_e(\lambda)) \end{array} \tag{4.7.8}$$

is commutative. If $\mathrm{Ker}\,\varphi_e^\lambda \neq 0$, then there exists some $w \in W$ such that $\mathrm{Im}\, f_e^w \subset \mathrm{Ker}\,\varphi_e^\lambda$. Since $\varphi_w^\lambda \neq 0$, this contradicts the injectivity of F_w. Hence φ_e^λ is injective. Thus φ_w^λ is a monomorphism by the commutativity of (4.7.8).

Therefore it remains to show that for any $w \in W(\lambda)$ there exists a monomorphism $F_w : \tilde{\Gamma}(\mathcal{M}_w(\lambda)) \to \tilde{\Gamma}(\mathcal{M}_e(\lambda))$ in $\tilde{\mathbb{O}}$ such that the diagram (4.7.8) is commutative.

For $w \in W(\lambda)$, take a reduced expression $w = s_{\beta_1} \cdots s_{\beta_r}$ ($\beta_k \in \Pi(\lambda)$). Set $w_k = s_{\beta_1} \cdots s_{\beta_k}$. Then $w_k = w_{k-1}s_{\beta_k}$, and $w_{k-1}\beta_k \in \Delta^+$.

Then it is sufficient to show that for any k there exists a monomorphism

$$F_k : \tilde{\Gamma}(\mathcal{M}_{w_k}(\lambda)) \to \tilde{\Gamma}(\mathcal{M}_{w_{k-1}}(\lambda))$$

in $\widetilde{\mathbb{O}}$ such that the diagram

$$
\begin{tikzcd}
M(w_k \circ \lambda) \arrow[r, "\varphi^{\lambda}_{w_k}"] \arrow[d, "f^{w_k}_{w_{k-1}}"'] & \tilde{\Gamma}(\mathcal{M}_{w_k}(\lambda)) \arrow[d, "F_k"] \\
M(w_{k-1} \circ \lambda) \arrow[r, "\varphi^{\lambda}_{w_{k-1}}"'] & \tilde{\Gamma}(\mathcal{M}_{w_{k-1}}(\lambda))
\end{tikzcd}
\tag{4.7.9}
$$

is commutative (see Theorem 2.5.3). By Lemma 2.2.4 we can take $x \in W$ such that $\lambda' = x \circ \lambda \in \mathcal{K}^+_{\mathrm{reg}}, x(\beta_k) = \alpha_i \in \Pi$. We have $\alpha_i \in \Delta(\lambda')$. Set $y = w_{k-1}x^{-1}$. Then $y\alpha_i = w_{k-1}\beta_k \in \Delta^+$ and hence $ys_i > y$. We have $w_{k-1} \circ \lambda = y \circ \lambda', w_k \circ \lambda = ys_i \circ \lambda'$, and hence $M(w_{k-1} \circ \lambda) = M(y \circ \lambda'), M(w_k \circ \lambda) = M(ys_i \circ \lambda')$. Moreover, by Corollary 4.6.3, we have $\tilde{\Gamma}(\mathcal{M}_{w_{k-1}}(\lambda)) = \tilde{\Gamma}(\mathcal{M}_y(\lambda')), \tilde{\Gamma}(\mathcal{M}_{w_k}(\lambda)) = \tilde{\Gamma}(\mathcal{M}_{ys_i}(\lambda'))$. Hence the desired result follows from Lemma 4.7.1. □

4.8. Correspondence of $\mathfrak{g}$-modules and D-modules. We shall prove the following theorem on a partial correspondence between $\mathbb{H}(\lambda)$ and $\widetilde{\mathbb{O}}(\lambda)$. This theorem says in particular that the composition $\mathbb{H}(\lambda) \xrightarrow{\tilde{\Gamma}} \widetilde{\mathbb{O}}(\lambda) \xrightarrow{D_\lambda \hat{\otimes}} \mathbb{H}(\lambda)$ is an equivalence of categories. Hence $\mathbb{H}(\lambda)$ is equivalent to a full subcategory of $\widetilde{\mathbb{O}}(\lambda)$. Moreover, $\mathcal{M}_w(\lambda)$, $\mathcal{B}_w(\lambda)$ and $\mathcal{L}_w(\lambda)$ in the category $\mathbb{H}(\lambda)$ correspond to $M(w \circ \lambda)$, $M^*(w \circ \lambda)$ and $L(w \circ \lambda)$ in $\widetilde{\mathbb{O}}(\lambda)$, respectively.

THEOREM 4.8.1. *Let $\lambda \in \mathcal{K}^+_{\mathrm{reg}}$.*

(i) $\tilde{H}^n(\mathcal{M}) = 0$ *for any* $\mathcal{M} \in \mathrm{Ob}(\mathbb{H}(\lambda))$ *and* $n \neq 0$.

(ii) $\tilde{\Gamma}(\mathcal{B}_w(\lambda)) \simeq M^*(w \circ \lambda)$ *and* $\tilde{\Gamma}(\mathcal{M}_w(\lambda)) \simeq M(w \circ \lambda)$ *for any* $w \in W$.

(iii) $\tilde{\Gamma}(\mathcal{L}_w(\lambda)) \simeq L(w \circ \lambda)$ *for any* $w \in W$.

(iv) $D_\lambda \hat{\otimes} \tilde{\Gamma}(\mathcal{M}) \xrightarrow{\sim} \mathcal{M}$ *for any* $\mathcal{M} \in \mathrm{Ob}(\mathbb{H}(\lambda))$.

Proof. Note that the first statement in (ii) is already proved (Proposition 4.4.7). We first show (i), (ii) and (iv). It is sufficient to show the following statements (a), (b) and (c) for any finite admissible subset Φ of W.

(a) For $\mathcal{M} \in \mathrm{Ob}(\mathbb{H}(\lambda))$, we have $\tilde{H}^n(\mathcal{M}) \in \mathrm{Ob}(\widetilde{\mathbb{O}}\{(W \setminus \Phi) \circ \lambda\})$ for any $n \neq 0$.

(b) For $w \in \Phi$, the cokernel of the monomorphism $\varphi^\lambda_w : M(w \circ \lambda) \to \tilde{\Gamma}(\mathcal{M}_w(\lambda))$ belongs to $\widetilde{\mathbb{O}}\{(W \setminus \Phi) \circ \lambda\}$.

(c) For $\mathcal{M} \in \mathrm{Ob}(\mathbb{H}(\lambda))$ and a morphism $\varphi : M \to \tilde{\Gamma}(\mathcal{M})$ in $\widetilde{\mathbb{O}}$, assume that $\mathrm{Ker}(\varphi)$ and $\mathrm{Coker}(\varphi)$ belong to $\widetilde{\mathbb{O}}\{(W \setminus \Phi) \circ \lambda\}$. Then the canonical homomorphism $D_\lambda \hat{\otimes} M|_{X_\Phi} \to \mathcal{M}|_{X_\Phi}$ is an isomorphism.

Fixing Φ, we shall show the following statements $(a)_\Psi$, $(b)_\Psi$, $(c)_\Psi$ for a finite admissible subset Ψ of W such that $\Psi \subset \Phi$ by induction on $\sharp(\Phi \setminus \Psi)$. Note that $(a)=(a)_\emptyset$, $(b)=(b)_\emptyset$ and $(c)=(c)_\emptyset$.

$(a)_\Psi$ Let $\mathcal{M} \in \mathrm{Ob}(\mathbb{H}(\lambda))$ such that $\mathrm{Supp}(\mathcal{M}) \cap X_\Psi = \emptyset$. Then we have $\tilde{H}^n(\mathcal{M}) \in \mathrm{Ob}(\widetilde{\mathbb{O}}\{(W \setminus \Phi) \circ \lambda\})$ for any $n > 0$.

$(b)_\Psi$ For $w \in \Phi \setminus \Psi$, the cokernel of the monomorphism $\varphi^\lambda_w : M(w \circ \lambda) \to \tilde{\Gamma}(\mathcal{M}_w(\lambda))$ belongs to $\widetilde{\mathbb{O}}\{(W \setminus \Phi) \circ \lambda\}$.

$(c)_\Psi$ Let $\mathcal{M} \in \mathrm{Ob}(\mathbb{H}(\lambda))$ such that $\mathrm{Supp}(\mathcal{M}) \cap X_\Psi = \emptyset$. Assume that a morphism $\varphi : M \to \tilde{\Gamma}(\mathcal{M})$ in $\widetilde{\mathbb{O}}$ satisfies $\mathrm{Ker}(\varphi), \mathrm{Coker}(\varphi) \in \mathrm{Ob}(\widetilde{\mathbb{O}}\{(W \setminus \Phi) \circ \lambda\})$. Then the canonical homomorphism $D_\lambda \hat{\otimes} M|_{X_\Phi} \to \mathcal{M}|_{X_\Phi}$ is an isomorphism.

In the case $\Phi = \Psi$, $(a)_\Phi$ follows from Proposition 4.4.5 (iv), $(b)_\Phi$ is trivial, and $(c)_\Phi$ is a consequence of Proposition 4.4.5 (iv) and Proposition 4.5.8.

Assume $\sharp(\Phi \setminus \Psi) > 0$. Take a minimal element y of the set $\Phi \setminus \Psi$, and set $\Psi' = \Psi \cup \{y\}$. Then Ψ' is a finite admissible subset of W satisfying $\Psi \subset \Psi' \subset \Phi$ and

$\sharp(\Phi \setminus \Psi') = \sharp(\Phi \setminus \Psi) - 1$. Hence we may assume the statements $(a)_{\Psi'}$, $(b)_{\Psi'}$, $(c)_{\Psi'}$ by the hypothesis of induction.

Set $\mathcal{L} = \mathcal{B}_y(\lambda)/\mathcal{L}_y(\lambda)$. Since $\mathrm{Supp}(\mathcal{L}) \cap X_{\Psi'} = \emptyset$, $(a)_{\Psi'}$ implies $\tilde{H}^n(\mathcal{L}) \in \mathrm{Ob}(\widetilde{\mathbb{O}}\{(W\setminus\Phi) \circ \lambda\})$ for $n \neq 0$. By considering the long exact sequence associated to the short exact sequence

$$0 \to \mathcal{L}_y(\lambda) \to \mathcal{B}_y(\lambda) \to \mathcal{L} \to 0\,,$$

we obtain $\tilde{H}^n(\mathcal{L}_y(\lambda)) \in \mathrm{Ob}(\widetilde{\mathbb{O}}\{(W \setminus \Phi) \circ \lambda\})$ for any $n \geq 2$ and an exact sequence

$$\tilde{\Gamma}(\mathcal{B}_y(\lambda)) \to \tilde{\Gamma}(\mathcal{L}) \to \tilde{H}^1(\mathcal{L}_y(\lambda)) \to 0. \tag{4.8.1}$$

Consider the following natural commutative diagram.

$$\begin{array}{ccc} D_\lambda \hat{\otimes} \tilde{\Gamma}(\mathcal{B}_y(\lambda))|_{X_\Phi} & \xrightarrow{f_1} & D_\lambda \hat{\otimes} \tilde{\Gamma}(\mathcal{L})|_{X_\Phi} \\ {\scriptstyle h_1}\downarrow & & \downarrow{\scriptstyle h_2} \\ \mathcal{B}_y(\lambda)|_{X_\Phi} & \xrightarrow{f_2} & \mathcal{L}|_{X_\Phi}\,. \end{array}$$

Then h_2 is an isomorphism by $(c)_{\Psi'}$, h_1 is surjective by Lemma 4.5.10, and f_2 is obviously surjective. Thus $D_\lambda \hat{\otimes} \tilde{\Gamma}(\mathcal{B}_y(\lambda))|_{X_\Phi} \to D_\lambda \hat{\otimes} \tilde{\Gamma}(\mathcal{L})|_{X_\Phi}$ is surjective. Hence we have $D_\lambda \hat{\otimes} \tilde{H}^1(\mathcal{L}_y(\lambda))|_{X_\Phi} = 0$ by (4.8.1). Then we obtain $\tilde{H}^1(\mathcal{L}_y(\lambda)) \in \mathrm{Ob}(\widetilde{\mathbb{O}}\{(W\setminus\Phi)\circ \lambda\})$ by Proposition 4.5.8. We have thus proved that $\tilde{H}^n(\mathcal{L}_y(\lambda)) \in \mathrm{Ob}(\widetilde{\mathbb{O}}\{(W\setminus\Phi)\circ\lambda\})$ for $n > 0$.

Set $\mathcal{L}' = \mathrm{Ker}(\mathcal{M}_y(\lambda) \to \mathcal{L}_y(\lambda))$. Since $\mathrm{Supp}(\mathcal{L}') \cap X_{\Psi'} = \emptyset$, $(a)_{\Psi'}$ implies $\tilde{H}^n(\mathcal{L}') \in \mathrm{Ob}(\widetilde{\mathbb{O}}\{(W \setminus \Phi) \circ \lambda\})$ for any $n > 0$. By considering the long exact sequence associated to the short exact sequence

$$0 \to \mathcal{L}' \to \mathcal{M}_y(\lambda) \to \mathcal{L}_y(\lambda) \to 0\,,$$

we obtain

$$\tilde{H}^n(\mathcal{M}_y(\lambda)) \in \mathrm{Ob}(\widetilde{\mathbb{O}}\{(W \setminus \Phi) \circ \lambda\}) \quad \text{for any } n > 0. \tag{4.8.2}$$

Let us show $(a)_\Psi$. By Lemma 4.2.1 there exists a morphism $f : \mathcal{M}_y(\lambda)^{\oplus r} \to \mathcal{M}$ whose restriction $\mathcal{M}_y(\lambda)^{\oplus r}|_{X_{\Psi'}} \xrightarrow{\sim} \mathcal{M}|_{X_{\Psi'}}$ is an isomorphism. Setting $\mathcal{N} = \mathrm{Im}(f)$, $\mathcal{N}_1 = \mathrm{Ker}(f)$ and $\mathcal{N}_2 = \mathrm{Coker}(f)$, we obtain exact sequences

$$0 \to \mathcal{N}_1 \to \mathcal{M}_y(\lambda)^{\oplus r} \to \mathcal{N} \to 0,$$
$$0 \to \mathcal{N} \to \mathcal{M} \to \mathcal{N}_2 \to 0.$$

Note that $\mathrm{Supp}(\mathcal{N}_i) \cap X_{\Psi'} = \emptyset$ for $i = 1, 2$. Let $n > 0$. By $(a)_{\Psi'}$ and (4.8.2), the objects $\tilde{H}^{n+1}(\mathcal{N}_1)$, $\tilde{H}^n(\mathcal{N}_2)$ and $\tilde{H}^n(\mathcal{M}_y(\lambda))$ belong to $\widetilde{\mathbb{O}}\{(W\setminus\Phi)\circ\lambda\}$. Hence $\tilde{H}^n(\mathcal{M})$ also belongs to $\widetilde{\mathbb{O}}\{(W \setminus \Phi) \circ \lambda\}$ by the exact sequences

$$\tilde{H}^n(\mathcal{M}_y(\lambda))^{\oplus r} \to \tilde{H}^n(\mathcal{N}) \to \tilde{H}^{n+1}(\mathcal{N}_1),$$
$$\tilde{H}^n(\mathcal{N}) \to \tilde{H}^n(\mathcal{M}) \to \tilde{H}^n(\mathcal{N}_2).$$

The statement $(a)_\Psi$ is proved.

We next show (b)$_\Psi$. By (b)$_{\Psi'}$ we have only to deal with the case $w = y$. By Lemma 4.2.2 we have, for any $\mu \in \mathcal{K}_{\mathrm{reg}}$,

$$\begin{aligned}\sum_n (-1)^n \operatorname{ch}(P_\mu \tilde{H}^n(\mathcal{M}_y(\lambda))) &= \sum_n (-1)^n \operatorname{ch}(P_\mu \tilde{H}^n(\mathcal{B}_y(\lambda))) = \operatorname{ch}(P_\mu \tilde{\Gamma}(\mathcal{B}_y(\lambda))) \\ &= \operatorname{ch}\left(P_\mu(M(y \circ \lambda))\right).\end{aligned}$$

Since $\tilde{H}^n(\mathcal{M}_y(\lambda))$ belongs to $\widetilde{\mathbb{O}}\{(W \setminus \Phi) \circ \lambda\}$ for any $n > 0$ by (a)$_\Psi$, we see that $\operatorname{ch}(P_\mu(\operatorname{Coker}\varphi_y^\lambda)) = \operatorname{ch}(P_\mu \tilde{\Gamma}(\mathcal{M}_y(\lambda))) - \operatorname{ch}(P_\mu(M(y \circ \lambda)))$ is in $\sum_{x \in W \setminus \Phi} \mathbb{Z} \operatorname{ch}(L(x \circ \lambda))$. The statement (b)$_\Psi$ is proved.

Let us show (c)$_\Psi$. Take $r, f, \mathcal{N}, \mathcal{N}_1, \mathcal{N}_2$ for $\mathcal{M}$ as in the proof of (a)$_\Psi$. Set $N = \varphi^{-1}(\operatorname{Im}(\tilde{\Gamma}(\mathcal{N}) \to \tilde{\Gamma}(\mathcal{M})))$ and $N_2 = M/N$. Then we obtain the following commutative diagrams whose rows are exact.

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ 0 & \to & N & \longrightarrow & M & \longrightarrow & N_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi'} & & \downarrow \\ 0 & \to & \tilde{\Gamma}(\mathcal{N}) & \longrightarrow & \tilde{\Gamma}(\mathcal{M}) & \xrightarrow{h} & \tilde{\Gamma}(\mathcal{N}_2) & \longrightarrow & \tilde{H}^1(\mathcal{N}) \end{array}$$

$$\begin{array}{ccccccccc} & & D_\lambda \hat{\otimes} N & \longrightarrow & D_\lambda \hat{\otimes} M & \longrightarrow & D_\lambda \hat{\otimes} N_2 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{N} & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{N}_2 & \to & 0. \end{array}$$

By the definition of N_2, φ' is a monomorphism. By (a)$_\Psi$, we have $\tilde{H}^1(\mathcal{N}) \in \operatorname{Ob}(\widetilde{\mathbb{O}}\{(W \setminus \Phi) \circ \lambda\})$. Hence by the exact sequence

$$0 \to \operatorname{Ker}\psi \to \operatorname{Ker}\varphi \to 0 \to \operatorname{Coker}\psi \to \operatorname{Coker}\varphi \to \operatorname{Coker}\varphi' \to \tilde{H}^1(\mathcal{N}),$$

and by the assumption on φ, we obtain

$$\operatorname{Ker}(\psi),\ \operatorname{Coker}(\psi),\ \operatorname{Coker}(\varphi') \in \operatorname{Ob}(\widetilde{\mathbb{O}}\{(W \setminus \Phi) \circ \lambda\}). \tag{4.8.3}$$

Since $\operatorname{Supp}(\mathcal{N}_2) \cap X_{\Psi'} = \emptyset$, the morphism $D_\lambda \hat{\otimes} N_2|_{X_\Phi} \to \mathcal{N}_2|_{X_\Phi}$ is an isomorphism by (c)$_{\Psi'}$ and (4.8.3). Hence it is sufficient to show that $D_\lambda \hat{\otimes} N|_{X_\Phi} \to \mathcal{N}|_{X_\Phi}$ is an isomorphism.

Set $N_0 = \psi^{-1}(\operatorname{Im}(\tilde{\Gamma}(\mathcal{M}_y(\lambda)^{\oplus r}) \to \tilde{\Gamma}(\mathcal{N})))$ and let $\psi_0 : N_0 \to \tilde{\Gamma}(\mathcal{N})$ be the restriction of ψ. Since $\tilde{H}^1(\mathcal{N}_1) \in \operatorname{Ob}(\widetilde{\mathbb{O}}\{(W \setminus \Phi) \circ \lambda\})$ by (a)$_{\Psi'}$, we have

$$N/N_0,\ \operatorname{Ker}\psi_0,\ \operatorname{Coker}\psi_0 \in \operatorname{Ob}(\widetilde{\mathbb{O}}\{(W \setminus \Phi) \circ \lambda\}) \tag{4.8.4}$$

by (4.8.3) and the exact sequences

$$\begin{gathered} 0 \to N/N_0 \to \tilde{H}^1(\mathcal{N}_1), \qquad 0 \to \operatorname{Ker}\psi_0 \to \operatorname{Ker}\psi, \\ N/N_0 \to \operatorname{Coker}\psi_0 \to \operatorname{Coker}\psi \to 0. \end{gathered}$$

Proposition 4.5.8 implies $D_\lambda \hat{\otimes}(N/N_0)|_{X_\Phi} = 0$, and hence $D_\lambda \hat{\otimes} N_0|_{X_\Phi} \to D_\lambda \hat{\otimes} N|_{X_\Phi}$ is surjective. Since $D_\lambda \hat{\otimes} N_0|_{X_\Phi} \to \mathcal{N}|_{X_\Phi}$ decomposes into $D_\lambda \hat{\otimes} N_0|_{X_\Phi} \to D_\lambda \hat{\otimes} N|_{X_\Phi} \to \mathcal{N}|_{X_\Phi}$, it is sufficient to show that $D_\lambda \hat{\otimes} N_0|_{X_\Phi} \to \mathcal{N}|_{X_\Phi}$ is an isomorphism.

Let R be the fiber product of $\tilde{\Gamma}(\mathcal{M}_y(\lambda)^{\oplus r})$ and N_0 over $\tilde{\Gamma}(\mathcal{N})$, and consider the following commutative diagrams whose rows are exact:

$$\begin{tikzcd}
0 \arrow[r] & \tilde{\Gamma}(\mathcal{N}_1) \arrow[r] \arrow[d, "\mathrm{id}"'] & R \arrow[r] \arrow[d, "\psi'"'] & N_0 \arrow[r] \arrow[d, "\psi_0"] & 0 \arrow[d] \\
0 \arrow[r] & \tilde{\Gamma}(\mathcal{N}_1) \arrow[r] & \tilde{\Gamma}(\mathcal{M}_y(\lambda)^{\oplus r}) \arrow[r, "k"] & \tilde{\Gamma}(\mathcal{N}) \arrow[r] & \tilde{H}^1(\mathcal{N}_1),
\end{tikzcd}$$

$$\begin{tikzcd}
 & D_\lambda\hat{\otimes}\tilde{\Gamma}(\mathcal{N}_1) \arrow[r] \arrow[d] & D_\lambda\hat{\otimes}R \arrow[r] \arrow[d] & D_\lambda\hat{\otimes}N_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{N}_1 \arrow[r] & \mathcal{M}_y(\lambda)^{\oplus r} \arrow[r] & \mathcal{N} \arrow[r] & 0 .
\end{tikzcd}$$

By $(c)_{\Psi'}$ the morphism $D_\lambda\hat{\otimes}\tilde{\Gamma}(\mathcal{N}_1)|_{X_\Phi} \to \mathcal{N}_1|_{X_\Phi}$ is an isomorphism. Hence it is sufficient to show that the morphism $D_\lambda\hat{\otimes}R|_{X_\Phi} \to \mathcal{M}_y(\lambda)^{\oplus r}|_{X_\Phi}$ is an isomorphism.

Since $\operatorname{Ker}\psi'$ and $\operatorname{Coker}\psi'$ are isomorphic to subobjects of $\operatorname{Ker}\psi_0$ and $\operatorname{Coker}\psi_0$ respectively, we have

$$\text{(4.8.5)} \qquad \operatorname{Ker}(\psi'),\ \operatorname{Coker}(\psi') \in \operatorname{Ob}(\tilde{\mathbb{O}}\{(W \setminus \Phi) \circ \lambda\})$$

by (4.8.4). Thus we have reduced $(c)_\Psi$ to the case $\mathcal{M} = \mathcal{M}_y(\lambda)^{\oplus r}$.

The statement $(b)_\Psi$ implies that $\varphi_y^\lambda : M(y \circ \lambda) \to \tilde{\Gamma}(\mathcal{M}_y(\lambda))$ is a monomorphism such that $\operatorname{Coker}\varphi_y^\lambda \in \operatorname{Ob}(\tilde{\mathbb{O}}\{(W \setminus \Phi) \circ \lambda\})$. Let R' be the Cartesian product of $M(y \circ \lambda)^{\oplus r}$ and R over $\tilde{\Gamma}(\mathcal{M}_y(\lambda)^{\oplus r})$:

$$\begin{tikzcd}
R' \arrow[r, hook, "\eta"] \arrow[d, "\psi''"'] & R \arrow[d, "\psi'"'] \\
M(y \circ \lambda)^{\oplus r} \arrow[r, hook, "\varphi_y^{\lambda \oplus r}"] & \tilde{\Gamma}(\mathcal{M}_y(\lambda)^{\oplus r}).
\end{tikzcd}$$

Then $\operatorname{Coker}\eta$ belongs to $\tilde{\mathbb{O}}\{(W \setminus \Phi) \circ \lambda\}$, and hence $D_\lambda\hat{\otimes}\operatorname{Coker}\eta|_{X_\Phi} = 0$ by Proposition 4.5.8. Hence we obtain

$$\text{(4.8.6)} \qquad D_\lambda\hat{\otimes}R'|_{X_\Phi} \to D_\lambda\hat{\otimes}R|_{X_\Phi} \text{ is surjective.}$$

On the other hand, the cokernel of ψ'' belongs to $\tilde{\mathbb{O}}\{(W\setminus\Phi)\circ\lambda\}$, which implies that ψ'' is surjective. Since the kernel of ψ'' also belongs to $\tilde{\mathbb{O}}\{(W\setminus\Phi)\circ\lambda\}$, $D_\lambda\hat{\otimes}\operatorname{Ker}\psi''|_{X_\Phi} = 0$. Hence we have

$$D_\lambda\hat{\otimes}R'|_{X_\Phi} \xrightarrow{\sim} D_\lambda\hat{\otimes}M(y \circ \lambda)^{\oplus r}|_{X_\Phi} \cong \mathcal{M}_y(\lambda)^{\oplus r}|_{X_\Phi}.$$

Since the isomorphism $D_\lambda\hat{\otimes}R'|_{X_\Phi} \xrightarrow{\sim} \mathcal{M}_y(\lambda)^{\oplus r}|_{X_\Phi}$ factors through $D_\lambda\hat{\otimes}R|_{X_\Phi}$, (4.8.6) implies that $D_\lambda\hat{\otimes}R|_{X_\Phi} \to \mathcal{M}_y(\lambda)^{\oplus r}|_{X_\Phi}$ is an isomorphism. The statement $(c)_\Psi$ is proved.

The proof of (i), (ii) and (iv) is now completed.

Let us finally show (iii). By (i) the functor $\tilde{\Gamma} : \mathbb{H}(\lambda) \to \tilde{\mathbb{O}}(\lambda)$ is exact. Hence the exact sequences

$$\mathcal{M}_w(\lambda) \to \mathcal{L}_w(\lambda) \to 0, \qquad 0 \to \mathcal{L}_w(\lambda) \to \mathcal{B}_w(\lambda)$$

induce exact sequences

$$\tilde{\Gamma}(\mathcal{M}_w(\lambda)) \to \tilde{\Gamma}(\mathcal{L}_w(\lambda)) \to 0, \qquad 0 \to \tilde{\Gamma}(\mathcal{L}_w(\lambda)) \to \tilde{\Gamma}(\mathcal{B}_w(\lambda)).$$

Since we have already seen $\tilde{\Gamma}(\mathcal{M}_w(\lambda)) \simeq M(w \circ \lambda)$ and $\tilde{\Gamma}(\mathcal{B}_w(\lambda)) \simeq M^*(w \circ \lambda)$, we have only to show that the morphism $\tilde{\Gamma}(\mathcal{M}_w(\lambda)) \to \tilde{\Gamma}(\mathcal{B}_w(\lambda))$ induced by the canonical morphism $\mathcal{M}_w(\lambda) \to \mathcal{B}_w(\lambda)$ is non-zero. This follows from (iv). The statement (iii) is proved. □

5. Twisted intersection cohomology groups.

5.1. Combinatorics. We first recall a result of Lusztig [22].
Set

$$\mathfrak{h}^*_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} P, \qquad \Gamma = \mathfrak{h}^*_{\mathbb{Q}}/P,$$

where P is as in §4.1. Note that the Weyl group W naturally acts on Γ. For $\lambda \in \Gamma$ let M^λ be the free $\mathbb{Z}[q, q^{-1}]$-module with basis $\{A^\lambda_w\}_{w \in W}$.

For $i \in I$ we define $\theta_{i*}, \theta_{i!} \in \mathrm{Hom}_{\mathbb{Z}[q,q^{-1}]}(M^\lambda, M^{s_i\lambda})$ by the following.

$$\theta_{i*}(A^\lambda_w) = \begin{cases} q^{-1}A^{s_i\lambda}_{ws_i} & \text{if } \langle\lambda, h_i\rangle \notin \mathbb{Z},\ ws_i > w, \\ A^{s_i\lambda}_{ws_i} & \text{if } \langle\lambda, h_i\rangle \notin \mathbb{Z},\ ws_i < w, \\ q^{-1}A^\lambda_{ws_i} + (q^{-1}-1)A^\lambda_w & \text{if } \langle\lambda, h_i\rangle \in \mathbb{Z},\ ws_i > w, \\ A^\lambda_{ws_i} & \text{if } \langle\lambda, h_i\rangle \in \mathbb{Z},\ ws_i < w, \end{cases}$$

$$\theta_{i!}(A^\lambda_w) = \begin{cases} A^{s_i\lambda}_{ws_i} & \text{if } \langle\lambda, h_i\rangle \notin \mathbb{Z},\ ws_i > w, \\ qA^{s_i\lambda}_{ws_i} & \text{if } \langle\lambda, h_i\rangle \notin \mathbb{Z},\ ws_i < w, \\ A^\lambda_{ws_i} & \text{if } \langle\lambda, h_i\rangle \in \mathbb{Z},\ ws_i > w, \\ qA^\lambda_{ws_i} + (q-1)A^\lambda_w & \text{if } \langle\lambda, h_i\rangle \in \mathbb{Z},\ ws_i < w. \end{cases}$$

Then $\theta_{i!} : M^\lambda \to M^{s_i\lambda}$ and $\theta_{i*} : M^{s_i\lambda} \to M^\lambda$ are inverse to each other.

LEMMA 5.1.1 (Lusztig [22]).

(i) *There exists a unique endomorphism $m \mapsto \overline{m}$ of the abelian group M^λ satisfying*

$$\overline{A^\lambda_e} = A^\lambda_e, \quad \overline{qm} = q^{-1}\overline{m}, \quad \overline{\theta_{i*}(m)} = \theta_{i!}(\overline{m})$$

for any $m \in M^\lambda$.

(ii) *We have $\overline{\overline{m}} = m$ for any $m \in M^\lambda$, and*

$$\overline{A^\lambda_w} \in q^{-\ell(w)}A^\lambda_w + \sum_{y<w} \mathbb{Z}[q, q^{-1}]A^\lambda_y$$

for any $w \in W$.

PROPOSITION 5.1.2 (Lusztig [22]).

(i) *For $w \in W$ and $\lambda \in \Gamma$ there exists a unique $C^\lambda_w \in M^\lambda$ satisfying*

$$\text{(5.1.1)} \quad C^\lambda_w \in A^\lambda_w + \sum_{y<w} \left(q^{(\ell(w)-\ell(y)-1)/2}\mathbb{Z}[q^{-1/2}] \cap \mathbb{Z}[q, q^{-1}]\right) A^\lambda_y,$$

$$\text{(5.1.2)} \quad \overline{C^\lambda_w} = q^{-\ell(w)}C^\lambda_w.$$

(ii) *If w is the element of $wW(\lambda)$ with minimal length, then for any $x \in W(\lambda)$ we have*

$$C^\lambda_{wx} = \sum_{y \in W(\lambda), y \le_\lambda x} (-1)^{\ell_\lambda(x)-\ell_\lambda(y)} q^{c(y,x)} P^\lambda_{y,x}(q) A^\lambda_{wy},$$

where $c(y,x) = \Big((\ell(x)-\ell(y)) - (\ell_\lambda(x)-\ell_\lambda(y))\Big)/2$, *and* $P^\lambda_{y,x}(q) \in \mathbb{Z}[q]$ *denotes the Kazhdan-Lusztig polynomial for the Coxeter group* $W(\lambda)$ *(Kazhdan-Lusztig* [19]*).*

We define $\kappa \in \operatorname{Hom}_{\mathbb{Z}[q,q^{-1}]}(M^\lambda, M^{-\lambda})$ by $\kappa(A^\lambda_w) = A^{-\lambda}_w$.

LEMMA 5.1.3.

(i) *For any $i \in I$ and $\lambda \in \Gamma$ we have $\theta_{i*} \circ \kappa = \kappa \circ \theta_{i*}$ and $\theta_{i!} \circ \kappa = \kappa \circ \theta_{i!}$ on M^λ.*

(ii) *We have $\kappa(\overline{m}) = \overline{\kappa(m)}$ for any $m \in M^\lambda$.*

(iii) *We have $\kappa(C^\lambda_w) = C^{-\lambda}_w$ for any $w \in W$.*

Proof. The statements (i) and (ii) follow from the definition of θ_{i*}, $\theta_{i!}$ and $\bar{\ }$. Applying κ to (5.1.1) we have

$$\kappa(C^\lambda_w) \in A^{-\lambda}_w + \sum_{y<w} (q^{(\ell(w)-\ell(y)-1)/2}\mathbb{Z}[q^{-1/2}] \cap \mathbb{Z}[q,q^{-1}]) A^{-\lambda}_y.$$

By (ii) and (5.1.2) we have

$$\overline{\kappa(C^\lambda_w)} = \kappa(\overline{C^\lambda_w}) = \kappa(q^{-\ell(w)} C^\lambda_w) = q^{-\ell(w)} \kappa(C^\lambda_w).$$

Thus we obtain (iii) by Proposition 5.1.2. □

Lusztig [22] used Proposition 5.1.2 to compute the twisted intersection cohomology groups of the finite-dimensional Schubert varieties. In order to compute that of the finite-*codimensional* Schubert varieties, we need its dual version.

Set

$$N^\lambda = \operatorname{Hom}_{\mathbb{Z}[q,q^{-1}]}(M^\lambda, \mathbb{Z}[q,q^{-1}]),$$

and define $B^\lambda_w \in N^\lambda$ for $w \in W$ by $\langle B^\lambda_w, A^\lambda_y \rangle = \delta_{y,w}$. Then any element of N^λ is uniquely written as a formal infinite sum $\sum_{w \in W} a_w B^\lambda_w$ with $a_w \in \mathbb{Z}[q,q^{-1}]$.

For $i \in I$ we define $\theta_{i*}, \theta_{i!} \in \operatorname{Hom}_{\mathbb{Z}[q,q^{-1}]}(N^\lambda, N^{s_i\lambda})$ by

$$\langle \theta_{i*} n, m \rangle = \langle n, \theta_{i*} m \rangle, \quad \langle \theta_{i!} n, m \rangle = \langle n, \theta_{i!} m \rangle \qquad \text{for } n \in N^\lambda, m \in M^\lambda.$$

By the definition we have the following.

$$\theta_{i*}(B^\lambda_w) = \begin{cases} B^{s_i\lambda}_{ws_i} & \text{if } \langle \lambda, h_i \rangle \notin \mathbb{Z},\ ws_i > w, \\ q^{-1} B^{s_i\lambda}_{ws_i} & \text{if } \langle \lambda, h_i \rangle \notin \mathbb{Z},\ ws_i < w, \\ B^\lambda_{ws_i} & \text{if } \langle \lambda, h_i \rangle \in \mathbb{Z},\ ws_i > w, \\ q^{-1} B^\lambda_{ws_i} + (q^{-1}-1) B^\lambda_w & \text{if } \langle \lambda, h_i \rangle \in \mathbb{Z},\ ws_i < w, \end{cases}$$

$$\theta_{i!}(B^\lambda_w) = \begin{cases} qB^{s_i\lambda}_{ws_i} & \text{if } \langle \lambda, h_i \rangle \notin \mathbb{Z},\ ws_i > w, \\ B^{s_i\lambda}_{ws_i} & \text{if } \langle \lambda, h_i \rangle \notin \mathbb{Z},\ ws_i < w, \\ qB^\lambda_{ws_i} + (q-1) B^\lambda_w & \text{if } \langle \lambda, h_i \rangle \in \mathbb{Z},\ ws_i > w, \\ B^\lambda_{ws_i} & \text{if } \langle \lambda, h_i \rangle \in \mathbb{Z},\ ws_i < w. \end{cases}$$

Let $a \mapsto \overline{a}$ be the endomorphism of the ring $\mathbb{Z}[q,q^{-1}]$ given by $\overline{q} = q^{-1}$. We define an endomorphism $n \mapsto \overline{n}$ of the abelian group N^λ by

$$\langle \overline{n}, m \rangle = \overline{\langle n, \overline{m} \rangle} \qquad \text{for } n \in N^\lambda, m \in M^\lambda.$$

By Lemma 5.1.1 we have $\overline{\overline{n}} = n$ for any $n \in N^\lambda$, and

$$\overline{B^\lambda_w} \in q^{\ell(w)} B^\lambda_w + \sum_{y>w} \mathbb{Z}[q, q^{-1}] B^\lambda_y$$

for any $w \in W$. Define $D^\lambda_w \in N^\lambda$ by $\langle D^\lambda_w, C^\lambda_y \rangle = \delta_{y,w}$. By Proposition 5.1.2 we have the following.

PROPOSITION 5.1.4.

(i) *For $w \in W$ and $\lambda \in \Gamma$ we have*

$$D^\lambda_w \in B^\lambda_w + \sum_{y>w} (q^{(\ell(y)-\ell(w)-1)/2}\mathbb{Z}[q^{-1/2}] \cap \mathbb{Z}[q, q^{-1}]) B^\lambda_y, \tag{5.1.3}$$

$$\overline{D^\lambda_w} = q^{\ell(w)} D^\lambda_w. \tag{5.1.4}$$

(ii) *If w is the element of $wW(\lambda)$ with minimal length, then for any $x \in W(\lambda)$ we have*

$$D^\lambda_{wx} = \sum_{y \in W(\lambda), y \geq_\lambda x} q^{c(x,y)} Q^\lambda_{x,y}(q) B^\lambda_{wy},$$

where $c(x, y) = \Big((\ell(y) - \ell(x)) - (\ell_\lambda(y) - \ell_\lambda(x))\Big)/2$ and $Q^\lambda_{x,y}(q) \in \mathbb{Z}[q]$ denotes the inverse Kazhdan-Lusztig polynomial for the Coxeter group $W(\lambda)$ given by

$$\sum_{x \leq_\lambda y \leq_\lambda z} (-1)^{\ell_\lambda(y)-\ell_\lambda(x)} Q^\lambda_{x,y}(q) P^\lambda_{y,z}(q) = \delta_{x,z}. \tag{5.1.5}$$

Define $\kappa \in \mathrm{Hom}_{\mathbb{Z}[q,q^{-1}]}(N^\lambda, N^{-\lambda})$ by

$$\langle \kappa(n), m \rangle = \langle n, \kappa(m) \rangle \qquad \text{for } n \in N^\lambda, m \in M^{-\lambda}.$$

By Lemma 5.1.3 we obtain the following.

LEMMA 5.1.5.

(i) *We have $\kappa(B^\lambda_w) = B^{-\lambda}_w$ for any $w \in W$.*

(ii) *For any $i \in I$ and $\lambda \in \Gamma$ we have $\theta_{i*} \circ \kappa = \kappa \circ \theta_{i*}$ and $\theta_{i!} \circ \kappa = \kappa \circ \theta_{i!}$ on N^λ.*

(iii) *We have $\kappa(\overline{n}) = \overline{\kappa(n)}$ for any $n \in N^\lambda$.*

(iv) *We have $\kappa(D^\lambda_w) = D^{-\lambda}_w$ for any $w \in W$.*

Let R be a ring containing $\mathbb{Z}[q, q^{-1}]$ as a subring. Assume that we are given an involutive automorphism $r \mapsto \overline{r}$ of the ring R and a family of $\mathbb{Z}$-submodules $\{R_i\}_{i\in\mathbb{Z}}$ of R such that

$$R = \bigoplus_{i\in\mathbb{Z}} R_i, \quad R_i R_j \subset R_{i+j}, \quad 1 \in R_0, \quad q \in R_2, \quad \overline{q} = q^{-1}, \quad \overline{R_i} = R_{-i}. \tag{5.1.6}$$

Define the endomorphism $m \mapsto \overline{m}$ of the abelian group $R \otimes_{\mathbb{Z}[q,q^{-1}]} M^\lambda$ by

$$\overline{a \otimes m} = \overline{a} \otimes \overline{m} \quad \text{for } a \in R \text{ and } m \in M^\lambda.$$

Set $N^\lambda_R = \mathrm{Hom}_R(R \otimes_{\mathbb{Z}[q,q^{-1}]} M^\lambda, R)$. We can naturally regard N^λ as a $\mathbb{Z}[q, q^{-1}]$-submodule of N^λ_R. Define an endomorphism $n \mapsto \overline{n}$ of the abelian group N^λ_R by

$$\langle \overline{n}, m \rangle = \overline{\langle n, \overline{m} \rangle} \qquad \text{for } n \in N^\lambda_R, m \in R \otimes_{\mathbb{Z}[q,q^{-1}]} M^\lambda,$$

and a homomorphism $\kappa : N_R^\lambda \to N_R^{-\lambda}$ of R-modules by

$$\langle \kappa(n), m\rangle = \langle n, \kappa(m)\rangle \qquad \text{for } n \in N_R^\lambda, m \in R \otimes_{\mathbb{Z}[q,q^{-1}]} M^\lambda.$$

Then we have the following characterization of D_w^λ.

PROPOSITION 5.1.6. *Let $w \in W$ and $\lambda \in \Gamma$. Assume that $D^+ \in N_R^\lambda$ and $D^- \in N_R^{-\lambda}$ satisfy the following properties:*

$$D^\pm \in B_w^{\pm\lambda} + \sum_{y>w} \Big(\bigoplus_{i \le \ell(y)-\ell(w)-1} R_i \Big) B_y^{\pm\lambda}, \tag{5.1.7}$$

$$\overline{\kappa(D^+)} = q^{\ell(w)} D^-. \tag{5.1.8}$$

Then we have $D^\pm = D_w^{\pm\lambda}$.

Proof. Since (5.1.7) and (5.1.8) are satisfied for $D^\pm = D_w^{\pm\lambda}$, it is sufficient to show that there exist unique $D^\pm \in N_R^\lambda$ satisfying (5.1.7) and (5.1.8).

By (5.1.7) we have

$$D^\pm = \sum_{y \ge w} F_y^\pm B_y^{\pm\lambda} \qquad \text{with } F_w^\pm = 1 \text{ and } F_y^\pm \in \textstyle\bigoplus_{i \le \ell(y)-\ell(w)-1} R_i \text{ for } y > w.$$

We have to show that $F_y^\pm$ are uniquely determined by the condition (5.1.8). Write

$$\overline{B_y^\lambda} = \sum_{z \ge y} G_{y,z} B_z^\lambda \qquad \text{with } G_{y,z} \in R, G_{y,y} = q^{\ell(w)}\ .$$

Then we have

$$q^{-\ell(w)}\overline{\kappa(D^+)} = q^{-\ell(w)} \sum_{y \ge w} \overline{F_y^+} \Big(\sum_{z \ge y} \overline{G_{y,z}} B_z^{-\lambda}\Big) = \sum_{z \ge w} \Big(\sum_{z \ge y \ge w} q^{-\ell(w)} \overline{F_y^+ G_{y,z}} \Big) B_z^{-\lambda},$$

and hence

$$\sum_{z \ge y \ge w} q^{-\ell(w)} \overline{F_y^+ G_{y,z}} = F_z^-$$

for any $z \ge w$. Thus

$$F_z^- - q^{-\ell(w)+\ell(z)} \overline{F_z^+} = \sum_{z > y \ge w} q^{-\ell(w)} \overline{F_y^+ G_{y,z}}$$

for any $z > w$. By the assumption we have

$$F_z^- \in \bigoplus_{i \le \ell(z)-\ell(w)-1} R_i, \qquad q^{-\ell(w)+\ell(z)} \overline{F_z^+} \in \bigoplus_{i \ge \ell(z)-\ell(w)+1} R_i.$$

Therefore $F_z^\pm$ are uniquely determined inductively. □

5.2. Character formula. For $\lambda \in \mathfrak{h}^*$ and a finite addmissible subset of W, let $K(\mathbb{H}_\Phi(\lambda))$ be the Grothendieck group of the category $\mathbb{H}_\Phi(\lambda)$. It is a module with $\big\{[\mathcal{M}_w(\lambda)]\big\}_{w \in \Phi}$ as a basis. Let $K(\mathbb{H}(\lambda))$ be the projective limit of $K(\mathbb{H}_\Phi(\lambda))$, where Φ ranges over the set of finite admissible subsets of W. Then $\{[\mathcal{M}_w(\lambda)]\}_{w \in W}$ as well as $\{[\mathcal{L}_w(\lambda)]\}_{w \in W}$ is a formal basis of $K(\mathbb{H}(\lambda))$.

The aim of this section is to prove the following result.

THEOREM 5.2.1. *Let $\lambda \in \mathfrak{h}^*_{\mathbb{Q}}$, and let $w \in W$ such that $\ell(z) > \ell(w)$ for any $z \in wW(\lambda) \setminus \{w\}$. Then for any $x \in W(\lambda)$ we have*

$$[\mathcal{L}_{wx}(\lambda)] = \sum_{y \geq_\lambda x} (-1)^{l(y)-l(x)} Q^\lambda_{x,y}(1)[\mathcal{M}_{wy}(\lambda)], \tag{5.2.1}$$

$$[\mathcal{M}_{wx}(\lambda)] = \sum_{y \geq_\lambda x} P^\lambda_{x,y}(1)[\mathcal{L}_{wy}(\lambda)]. \tag{5.2.2}$$

The proof of this theorem will be given in the next subsection. The corresponding result for finite-dimensional Schubert varieties was proved by Lusztig (see [21], [22]).

Note that (5.2.1) and (5.2.2) are equivalent by (5.1.5).

By Theorem 4.8.1, Proposition 4.4.5 and Theorem 5.2.1, we obtain the following main result of this paper.

THEOREM 5.2.2. *Assume that $\lambda \in \mathfrak{h}^*$ satisfies the following conditions.*

(5.2.3) $2(\alpha, \lambda + \rho) \neq (\alpha, \alpha)$ *for any positive imaginary root* α.

(5.2.4) $(\alpha^\vee, \lambda + \rho) \notin \mathbb{Z}_{\leq 0}$ *for any positive real root* α.

(5.2.5) *If $w \in W$ satisfies $w \circ \lambda = \lambda$, then $w = 1$.*

(5.2.6) $(\alpha^\vee, \lambda) \in \mathbb{Q}$ *for any real root* α.

Then for any $w \in W(\lambda)$ we have

$$\mathrm{ch}(M(w \circ \lambda)) = \sum_{y \geq_\lambda w} P^\lambda_{w,y}(1)\, \mathrm{ch}(L(y \circ \lambda)), \tag{5.2.7}$$

$$\mathrm{ch}(L(w \circ \lambda)) = \sum_{y \geq_\lambda w} (-1)^{\ell_\lambda(y)-\ell_\lambda(w)} Q^\lambda_{w,y}(1)\, \mathrm{ch}(M(y \circ \lambda)). \tag{5.2.8}$$

As a special case, we obtain the following result.

THEOREM 5.2.3. *Assume that $\mathfrak{g}$ is finite-dimensional or affine and $\lambda \in \mathfrak{h}^*$ satisfies*

(5.2.9) $(\beta^\vee, \lambda + \rho) \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ *for any* $\beta \in \Delta^+_{\mathrm{re}}$.

(5.2.10) $(\delta, \lambda + \rho) \neq 0$ *if $\mathfrak{g}$ is affine. Here δ is an imaginary root.*

Then (5.2.7) *and* (5.2.8) *hold for any $w \in W(\lambda)$.*

In the affine case, the condition (5.2.5) on the triviality of the isotropy subgroup of λ follows from the following well-known lemma.

LEMMA 5.2.4. *The isotropy subgroup $\{w \in W; w \circ \lambda = \lambda\}$ is generated by $\{s_\beta; \beta \in \Delta^+_{\mathrm{re}},\ (\beta, \lambda + \rho) = 0\}$ whenever $\mathfrak{g}$ is affine and $\lambda \in \mathfrak{h}^*$ satisfies* (5.2.10).

In the affine case, we can derive the following result on the non-regular highest weight case from the regular highest weight case above by using the translation functors (we omit the proof).

THEOREM 5.2.5. *Let $\mathfrak{g}$ be an affine Lie algebra, and assume that $\lambda \in \mathfrak{h}^*$ satisfies*

(5.2.11) $(\delta, \lambda + \rho) \neq 0$,

(5.2.12) $(\alpha^\vee, \lambda + \rho) \in \mathbb{Q} \setminus \mathbb{Z}_{<0}$ *for any positive real root* α.

Then $W_0(\lambda) = \{w \in W; w \circ \lambda = \lambda\}$ *is a finite group. Let* w *be an element of* $W(\lambda)$ *which is the longest element of* $wW_0(\lambda)$. *Then we have*

$$\mathrm{ch}(L(w \circ \lambda)) = \sum_{y \geq_\lambda w} (-1)^{\ell_\lambda(y) - \ell_\lambda(w)} Q^\lambda_{w,y}(1)\,\mathrm{ch}(M(y \circ \lambda)).$$

5.3. Hodge modules on flag manifolds. Let R (resp. R_i for $i \in \mathbb{Z}$) denote the Grothendick group of the category of mixed Hodge structures (resp. pure Hodge structures with weight i) over $\overline{\mathbb{Q}}$. Then we have $R = \bigoplus_{i \in \mathbb{Z}} R_i$. Let $\overline{\mathbb{Q}}^H(k)$ be the Hodge structure of Tate with weight $-2k$. Set $q = [\overline{\mathbb{Q}}^H(-1)] \in R_2$, and let $r \mapsto \overline{r}$ denote the endomorphism of the ring R induced by the duality operation on the mixed Hodge structures. Then the condition (5.1.6) is satisfied for the above R.

For a smooth $\mathbb{C}$-scheme S, let $\mathrm{MH}(S)$ denote the category of mixed Hodge modules on S (see Saito[23]). Here we use the convention that the perversity is stabel under the smooth inverse image.

For a scheme S satisfying (S) with a smooth projective system $\{S_n\}_{n \in \mathbb{N}}$, let us denote by $\mathrm{MH}(S)$ the inductive limit of $\mathrm{MH}(S_n)$. It is an abelian category and there is an exact functor

$$\mathrm{MH}(S) \to \mathbb{M}_h(\mathcal{D}_S)$$

We call an object of $\mathrm{MH}(S)$ a mixed Hodge module over S.

For $\lambda \in \Gamma = \mathfrak{h}^*_{\mathbb{Q}}/P$ we denote by $T^H(\lambda)$ the Hodge module on H corresponding to $\mathcal{T}(\lambda)$ (see §3.3) of weight 0. By the assumption on λ, the monodromies of the corresponding local system are roots of unity, and hence it has a structure of variation of polarizable Hodge structure. Hence $T^H(\lambda)$ is defined as a Hodge module on H of weight 0.

For a $\mathbb{C}$-scheme S satisfying (S) with an action of H, we can define the twisted H-equivariance of mixed Hodge module on S as in §3.3 by the aid of $T^H(\lambda)$. We denote by $\mathrm{MH}(S, \lambda)$ the category of twisted H-equivariant mixed Hodge modules on S with twist λ. It depends only on the image of λ in $\Gamma = \mathfrak{h}^*_{\mathbb{Q}}/P$.

Recall that $\tilde{X} = G/N^-$ and $\xi : \tilde{X} \to X$ is the natural projection. Then $B \times H$ acts on $\tilde{X}$ by $(b, h) \circ (gN^-) = bgh^{-1}N^-$. By the action of H on $\tilde{X}$, $\xi : \tilde{X} \to X$ is a principal H-bundle. For a finite admissible subset Φ of W and $\lambda \in \Gamma$ we denote by $\mathrm{MH}_\Phi(\lambda)$ the category of twisted $(N \times H)$-equivariant mixed Hodge modules with twist λ.

Set

$$\mathrm{MH}(\lambda) = \varprojlim_{\Phi} \mathrm{MH}_\Phi(\lambda).$$

Since $\xi^\sharp D_{X,\lambda} = D_{\tilde{X}}$ and since ξ is a smooth morphism, we have an equivalence $\xi^\bullet$ from the category of holonomic D_λ-modules to the category of H-equivariant holonomic $D_{\tilde{X}}$-modules with twist λ. Hence we have an exact functor

$$\mathrm{MH}(\lambda) \to \mathbb{H}(\lambda). \tag{5.3.1}$$

For $w \in W$, set $\tilde{X}_w = BwN^-/N^- = \xi^{-1}(X_w)$ and let $\tilde{\imath} : \tilde{X}_w \hookrightarrow \tilde{X}$ be the embedding. By the isomorphism of schemes

$$H \times N(\Delta^+ \cap w\Delta^+) \xrightarrow{\sim} \tilde{X}_w \qquad ((h, u) \mapsto uwh^{-1}N^-),$$

we can define a morphism $p_w : \tilde{X}_w \to H$ by $p_w(uwh^{-1}N^-) = h$ for $u \in N(\Delta^+ \cap w\Delta^+)$ and $h \in H$. We define a Hodge module $F_w^H(\lambda)$ on $\tilde{X}_w$ by $F_w^H(\lambda) = p_w^* T(\lambda)$. We denote by the same letter $F_w^H(\lambda)$ the object $\tilde{\imath}_! F_w^H(\lambda)$ in the derived category of the category of mixed Hodge modules. Let us denote by ${}^\pi F_w^H(\lambda)[-\ell(w)]$ the minimal extension of $F_w^H(\lambda)[-\ell(w)]$. Then $F_w^H(\lambda)[-\ell(w)]$ and ${}^\pi F_w^H(\lambda)[-\ell(w)]$ are objects of $\mathrm{MH}(\lambda)$. By the functor (5.3.1), $F_w(\lambda)[-\ell(w)]$ and ${}^\pi F_w(\lambda)[-\ell(w)]$ correspond to the objects $\mathcal{M}_w(\lambda)$ and $\mathcal{L}_w(\lambda)$ of $\mathbb{H}(\lambda)$.

For a finite admissible set Φ, $\tilde{X}_\Phi$ has the $N \times H$-orbit decomposition $\tilde{X}_\Phi = \bigsqcup_{w \in \Phi} \tilde{X}_w$. Hence the irreducible objects of $\mathrm{MH}_\Phi(\lambda)$ is of the form $H \otimes {}^\pi F_w^H(\lambda)[-\ell(w)]|\tilde{X}_\Phi$ for some $w \in \Phi$ and some irreducible Hodge structure H. We denote the Grothendieck group of $\mathrm{MH}_\Phi(\lambda)$ by $K(\mathrm{MH}_\Phi(\lambda))$. This has a structure of R-module. We set

$$K(\mathrm{MH}(\lambda)) = \varprojlim_{\Phi} K(\mathrm{MH}_\Phi(\lambda)).$$

For $F \in \mathrm{MH}(\lambda)$ we denote by $[F]$ the element of $K(\mathrm{MH}(\lambda))$ corresponding to F, and $\Big[F[n]\Big] = (-1)^n[F]$. Any $m \in K(\mathrm{MH}(\lambda))$ can be written uniquely as

$$m = \sum_{w \in W} a_w[{}^\pi F_w^H(\lambda)] = \sum_{w \in W} b_w[F_w^H(\lambda)] \qquad (a_w, b_w \in R).$$

Define an isomorphism

$$\varphi_\lambda : K(\mathrm{MH}(\lambda)) \xrightarrow{\sim} N_R^\lambda$$

of R-modules by $\varphi_\lambda([F_w^H(\lambda)]) = B_w^\lambda$.

For $i \in I$ we shall define

$$\tilde{S}_{i*},\ \tilde{S}_{i!} \in \mathrm{Hom}_R\Big(K(\mathrm{MH}(\lambda)), K(\mathrm{MH}(s_i\lambda))\Big).$$

The definition is analogous to §4.6, and we use the notations in §4.6.

Set $N_i^- = \exp(\mathfrak{n}(\Delta^- \setminus \{-\alpha_i\})) \subset N^-$, and $\tilde{Z}_0 = G/N_i^-$. The group $B \times H$ acts on $\tilde{Z}_0$ by $(b,h) \circ (gN_i^-) = gh^{-1}N_i^-$. Let $\tilde{p}_i : \tilde{Z}_0 \to \tilde{X}$ $(i = 1, 2)$ be the morphism defined by

$$p_1(gN_i^-) = gN^- \quad \text{and} \quad p_2(gN_i^-) = gs_iN^-.$$

We have the commutative diagram

$$\begin{array}{ccccc} \tilde{X} & \xleftarrow{\tilde{p}_1} & \tilde{Z}_0 & \xrightarrow{\tilde{p}_2} & \tilde{X} \\ {\scriptstyle \xi}\downarrow & & {\scriptstyle \xi'}\downarrow & & {\scriptstyle \xi}\downarrow \\ X & \xleftarrow{p_1} & Z_0 & \xrightarrow{p_2} & X. \end{array} \tag{5.3.2}$$

Then $\xi' : \tilde{Z}_0 \to Z$ is a principal H-bundle. The morphisms $\tilde{p}_1$ and $\tilde{p}_2$ are B-equivariant, and they satisfy the following relation with the action of H.

$$\begin{aligned} \tilde{p}_1(hz) &= h\tilde{p}_1(z) \\ \tilde{p}_2(hz) &= s_i(h)\tilde{p}_2(z) \end{aligned} \qquad \text{for } h \in H \text{ and } z \in \tilde{Z}_0. \tag{5.3.3}$$

Here s_i is the group automorphism of H corresponding to the simple reflection $s_i \in \mathrm{Aut}(\mathfrak{h})$.

For $F \in \mathrm{MH}(\tilde{X}, \lambda)$ we set

$$\tilde{S}_{i!}(F) = \mathbb{R}\tilde{p}_{2!}\tilde{p}_1{}^*F, \qquad \tilde{S}_{i*}(F) = \mathbb{R}\tilde{p}_{2*}\tilde{p}_1{}^!F. \tag{5.3.4}$$

Then $H^k(\tilde{S}_{i!}(F))$ and $H^k(\tilde{S}_{i!}(F))$ are objects of $\mathrm{MH}(s_i\lambda)$ by (5.3.3).

We define $\tilde{S}_{i*}$, $\tilde{S}_{i!} \in \mathrm{Hom}_R\Big(K(\mathrm{MH}(\lambda)), K(\mathrm{MH}(s_i\lambda))\Big)$ by

$$\tilde{S}_{i!}([F]) = \sum_k (-1)^k [H^k(\tilde{S}_{i!}\,F)], \qquad \tilde{S}_{i*}([F]) = \sum_k (-1)^k [H^k(\tilde{S}_{i*}\,F)].$$

PROPOSITION 5.3.1. *We have*

$$\varphi_{s_i\lambda} \circ \tilde{S}_{i*} = \theta_{i*} \circ \varphi_\lambda, \qquad \varphi_{s_i\lambda} \circ \tilde{S}_{i!} = \theta_{i!} \circ \varphi_\lambda.$$

Proof. Fix $w \in W$ such that $ws_i > w$. It is sufficient to show the following:

$$\tilde{S}_{i*}[F^H_w(\lambda)] = \begin{cases} [F^H_{ws_i}(s_i\lambda)] & \text{if } \langle\lambda, h_i\rangle \notin \mathbb{Z}, \\ [F^H_{ws_i}(\lambda)] & \text{if } \langle\lambda, h_i\rangle \in \mathbb{Z}, \end{cases} \tag{5.3.5}$$

$$\tilde{S}_{i*}[F^H_{ws_i}(\lambda)] = \begin{cases} q^{-1}[F^H_w(s_i\lambda)] & \text{if } \langle\lambda, h_i\rangle \notin \mathbb{Z}, \\ q^{-1}[F^H_w(\lambda)] + (q^{-1}-1)[F^H_{ws_i}(\lambda)] & \text{if } \langle\lambda, h_i\rangle \in \mathbb{Z}, \end{cases} \tag{5.3.6}$$

$$\tilde{S}_{i!}[F^H_w(\lambda)] = \begin{cases} q[F^H_{ws_i}(s_i\lambda)] & \text{if } \langle\lambda, h_i\rangle \notin \mathbb{Z}, \\ q[F^H_{ws_i}(\lambda)] + (q-1)[F^H_w(\lambda)] & \text{if } \langle\lambda, h_i\rangle \in \mathbb{Z}. \end{cases} \tag{5.3.7}$$

$$\tilde{S}_{i!}[F^H_{ws_i}(\lambda)] = \begin{cases} [F^H_w(s_i\lambda)] & \text{if } \langle\lambda, h_i\rangle \notin \mathbb{Z}, \\ [F^H_w(\lambda)] & \text{if } \langle\lambda, h_i\rangle \in \mathbb{Z}. \end{cases} \tag{5.3.8}$$

Set $Y = \tilde{X}_w \sqcup \tilde{X}_{ws_i}$ and let $j : Y \to \tilde{X}$ be the embedding.

As in Lemma 1.4.1 and Corollary 1.5.2 in [18], $\tilde{S}_{i!}$ and $\tilde{S}_{i*}$ commute with $j_!$. Hence we can reduce these statements to the case $\mathfrak{g} = \mathfrak{sl}_2$ where we can check them directly. Details are omitted. □

The duality functor for Hodge modules induces a contravariant exact functor

$$\mathbf{D} : \mathrm{MH}(\lambda) \to \mathrm{MH}(-\lambda).$$

We also denote by

$$\mathbf{D} : K(\mathrm{MH}(\lambda)) \to K(\mathrm{MH}(-\lambda)).$$

the induced homomorphism of abelian groups. By the definition we have the following result.

LEMMA 5.3.2.

(i) *We have*

$$\mathbf{D}(rn) = \bar{r}\mathbf{D}(n)$$

for any $r \in R$ and $n \in K(\mathrm{MH}(\lambda))$.

(ii) *We have*

$$\mathbf{D} \circ \tilde{S}_{i*} = \tilde{S}_{i!} \circ \mathbf{D}, \qquad \mathbf{D} \circ \tilde{S}_{i!} = \tilde{S}_{i*} \circ \mathbf{D}$$

on $K(\mathrm{MH}(\lambda))$.

PROPOSITION 5.3.3. *We have*

$$\varphi_{-\lambda}(\mathbf{D}n) = \kappa(\overline{\varphi_\lambda(n)})$$

for any $n \in K(\mathrm{MH}(\lambda))$.

Proof. It is sufficient to show

$$\langle \varphi_{-\lambda}(\mathbf{D}n), A_w^{-\lambda}\rangle = \langle \kappa(\overline{\varphi_\lambda(n)}), A_w^{-\lambda}\rangle$$

for any $w \in W$. By Lemma 5.1.5 the right side coincides with $\overline{\langle \varphi_\lambda(n), \overline{A_w^\lambda}\rangle}$ and hence we have to show

$$\langle \varphi_{-\lambda}(\mathbf{D}n), A_w^{-\lambda}\rangle = \overline{\langle \varphi_\lambda(n), \overline{A_w^\lambda}\rangle}. \tag{5.3.9}$$

We first consider the case $w = e$. In this case we have $\overline{A_e^\lambda} = A_e^\lambda$. We may assume that $n = [F_x^H(\lambda)]$ for $x \in W$. Since $\mathbf{D}F_x^H(\lambda) \cong F_x^H(\lambda)(-\ell(x))[-2\ell(x)]$ on a neighborhood of $\tilde{X}_w$, we have

$$\varphi_{-\lambda}(\mathbf{D}[F_x^H(\lambda)]) \in q^{\ell(x)} B_x^{-\lambda} + \sum_{y>x} RB_y^{-\lambda}.$$

Thus the both sides of (5.3.9) are equal to $\delta_{x,e}$.

For general $w \in W$, take a reduced expression $w = s_{i_1} \cdots s_{i_r}$. By the definition of $\theta_{i!}$ we have $A_w^{\pm\lambda} = \theta_{i_r!} \cdots \theta_{i_1!} A_e^{\pm w\lambda}$. Thus we have

$$\overline{A_w^\lambda} = \overline{\theta_{i_r!} \cdots \theta_{i_1!} A_e^{w\lambda}} = \theta_{i_r *} \cdots \theta_{i_1 *} \overline{A_e^{w\lambda}} = \theta_{i_r *} \cdots \theta_{i_1 *} A_e^{w\lambda}.$$

Hence by Lemma 5.3.2 we obtain

$$\begin{aligned}
\langle \varphi_{-\lambda}(\mathbf{D}n), A_w^{-\lambda}\rangle &= \langle \varphi_{-\lambda}(\mathbf{D}n), \theta_{i_r!} \cdots \theta_{i_1!} A_e^{-w\lambda}\rangle \\
&= \langle \theta_{i_1!} \cdots \theta_{i_r!} \varphi_{-\lambda}(\mathbf{D}n), A_e^{-w\lambda}\rangle \\
&= \langle \varphi_{-w\lambda}(\tilde{S}_{i_1!} \cdots \tilde{S}_{i_r!}\mathbf{D}n), A_e^{-w\lambda}\rangle, \\
&= \langle \varphi_{-w\lambda}(\mathbf{D}\,\tilde{S}_{i_1 *} \cdots \tilde{S}_{i_r *} n), A_e^{-w\lambda}\rangle, \\
&= \overline{\langle \varphi_{w\lambda}(\tilde{S}_{i_1 *} \cdots \tilde{S}_{i_r *} n), A_e^{w\lambda}\rangle} \\
&= \overline{\langle \theta_{i_1 *} \cdots \theta_{i_r *} \varphi_\lambda(n), A_e^{w\lambda}\rangle} \\
&= \overline{\langle \varphi_\lambda(n), \theta_{i_r *} \cdots \theta_{i_1 *} A_e^{w\lambda}\rangle} \\
&= \overline{\langle \varphi_\lambda(n), \overline{A_w^\lambda}\rangle}
\end{aligned}$$

□

THEOREM 5.3.4. *We have* $\varphi_\lambda([{}^\pi F_w^H(\lambda)]) = D_w^\lambda$ *for any* $w \in W$.

Note that Theorem 5.2.1 is a consequence of Theorem 5.3.4. In fact, if $w \in W$ satisfies $\ell(z) > \ell(w)$ for any $z \in wW(\lambda) \setminus \{w\}$, then we have

$$[{}^\pi F_{wx}^H(\lambda)] = \sum_{y \in W(\lambda), y \geq_\lambda x} q^{c(x,y)} Q_{x,y}^\lambda(q) [F_{wy}^H(\lambda)] \tag{5.3.10}$$

in $K(\mathrm{MH}(\lambda))$ for any $x \in W(\lambda)$ by Proposition 5.1.4 and Theorem 5.3.4. Applying the canonical homomorphism $K(\mathrm{MH}(\lambda)) \to K(\mathbb{H}(\lambda))$ to (5.3.10) we obtain (5.2.1).

Proof of Theorem 5.3.4. Set $D^\pm = \varphi_{\pm\lambda}([{}^\pi F_w^H(\pm\lambda)])$. It is sufficient to show that $D^\pm$ satisfy the conditions (5.1.7) and (5.1.8) in Proposition 5.1.6.

By Proposition 5.3.3 we have

$$\begin{aligned}\overline{\kappa(D^+)} &= \overline{\kappa\varphi_\lambda([{}^\pi F_w^H(\lambda)])}\\ &= \varphi_{-\lambda}(\mathbf{D}([{}^\pi F_w^H(\lambda)]))\\ &= \varphi_{-\lambda}([\mathbf{D}{}^\pi F_w^H(\lambda)])\\ &= \varphi_{-\lambda}(\left[{}^\pi F_w^H(-\lambda)[-2\ell(w)](-\ell(w))\right])\\ &= q^{\ell(w)}\varphi_{-\lambda}([{}^\pi F_w^H(-\lambda)])\\ &= q^{\ell(w)}D^-,\end{aligned}$$

and hence (5.1.8) holds.

For $x \in W$, let $\tilde{\imath}_x : \tilde{X}_x \hookrightarrow \tilde{X}$ be the embedding and let $\mathrm{MH}^x(\pm\lambda)$ denote the abelian category of twisted $N\times H$-equivariant mixed Hodge modules on $\tilde{X}_x$ with twist $\pm\lambda$. Since $\tilde{X}_w$ is an orbit of $N \times H$, any object of $\mathrm{MH}^x(\pm\lambda)$ has a form $H \otimes F_x^H(\pm\lambda)$ for some mixed Hodge module H. Hence its Grothendieck group $K(\mathrm{MH}^x(\pm\lambda))$ is a free R-module generated by $[F_x^H(\pm\lambda)]$. The inverse image functor $\tilde{\imath}_x^*$ induces a homomorphism

$$\iota_x : K(\mathrm{MH}(\pm\lambda)) \to R \qquad ([\tilde{\imath}_x^* F] = \iota_x([F])[F_x^H(\pm\lambda)]).$$

of R-modules. Then we have

$$\varphi_{\pm\lambda}(n) = \sum_{x\in W} \iota_x(n) B_w^{\pm\lambda}$$

for any $n \in \mathrm{MH}(\pm\lambda)$ because this formula obviously holds for $n = [F_y^H(\pm\lambda)]$ with $y \in W$. We have obviously $\iota_w([([{}^\pi F_w^H(\pm\lambda)]) = 1$. Let $y > w$. Since ${}^\pi F_w^H(\pm\lambda)$ is pure of weight 0, $H^j(\tilde{\imath}_y^*({}^\pi F_w^H(\pm\lambda)))$ is a mixed Hodge module of weight $\le j$ for any j. On the other hand the perversity property of ${}^\pi F_w^H(\pm\lambda)$ implies $H^j(\tilde{\imath}_y^*({}^\pi F_w^H(\pm\lambda))) = 0$ for $j \ge \ell(y) - \ell(w)$. Thus we obtain $\iota_y([([{}^\pi F_w^H(\pm\lambda)]) \in \sum_{j\le\ell(y)-\ell(w)-1} R_j$. Hence the condition (5.1.8) also holds. □

By using a $\mathbb{C}^\times$-action, we can prove that, for any j, $H^j(\tilde{\imath}_y^*({}^\pi F_w^H(\pm\lambda)))$ is a pure Hodge module of weight j as in Kazhdan-Lusztig [19] and Kashiwara-Tanisaki [16]. This gives the following stronger version of Theorem 5.2.1. Since this result is not used in this paper, the details are omitted.

THEOREM 5.3.5. *Let $\lambda \in \Gamma = \mathfrak{h}_{\mathbb{Q}}^*/P$, and let $w \in W$ such that $\ell(z) > \ell(w)$ for any $z \in wW(\lambda) \setminus \{w\}$. Let $x, y \in W(\lambda)$ such that $y \ge x$ and write $Q_{x,y}^\lambda(q) = \sum_j c_j q^j$.*

(i) $H^{2j+1}(\tilde{\imath}_y^*({}^\pi F_{wx}^H(\lambda)) = 0$ *for any* $j \in \mathbb{Z}$.

(ii) $H^{2j}(\tilde{\imath}_y^*({}^\pi F_{wx}^H(\lambda)) \simeq F_{wy}^H(\lambda)(-j)^{\oplus c_j}$ *for any* $j \in \mathbb{Z}$.

REFERENCES

[1] A. BEILINSON, J. BERNSTEIN, *Localisation de $\mathfrak{g}$-modules*, C. R. Acad. Sci. Paris Sér. I Math., 292 (1981), pp. 15–18.

[2] J.-L. BRYLINSKI, M. KASHIWARA, *Kazhdan-Lusztig conjecture and holonomic systems*, Invent. Math., 64 (1981), pp. 387–410.

[3] L. CASIAN, *Proof of the Kazhdan-Lusztig conjecture for Kac-Moody algebras* (*the characters* $ch\, L_{w\rho-\rho}$), Adv. Math., 119 (1996), pp. 207–281.

[4] ———, *Kazhdan-Lusztig conjecture in the negative level case (Kac-Moody algebras of affine type)*, preprint.

[5] V. V. Deodhar, *On a construction of representations and a problem of Enright*, Invent. Math., 57 (1980), pp. 101–118.
[6] V. V. Deodhar, O. Gabber, V. Kac, *Structure of some categories of representations of infinite-dimensional Lie algebras*, Adv. Math., 45 (1982), pp. 92–116.
[7] A. Grothendieck, *Éléments de Géométrie Algébrique, IV, Étude Locale des Schémas et des Morphismes de Schémas (troisième partie)*, Publications Mathématiques, Institut des Hautes Études Scientifiques, 28 (1966).
[8] T. J. Enright, *On the fundamental series of a real semisimple Lie algebra: their irreducibility, resolutions and multiplicity formulae*, Ann. Math., 73 (1994), pp. 383–413.
[9] V. Kac, *Infinite Dimensional Lie Algebras* (3rd edition), Cambridge Univ. Press, 1990.
[10] V. Kac, D. Kazhdan, *Structures of representations with highest weight of infinite dimensional Lie algebras*, Adv. in Math., 34 (1979), pp. 97–108.
[11] V. Kac, M. Wakimoto, *Modular invariant representations of infinite-dimensional Lie algebras and superalgebras*, Proc. Natl. Acad. Sci., USA, 85 (1988), pp. 4956–4960.
[12] M. Kashiwara, *Representation theory and D-modules on flag varieties*, Asterisque, 172–174 (1989), pp. 55–109.
[13] ———, *The flag manifold of Kac-Moody Lie algebra* in Algebraic Analysis, Geometry, and Number Theory, Johns Hopkins Univ. Press, Baltimore, 1989, pp. 161–190.
[14] ———, *Kazhdan-Lusztig conjecture for symmetrizable Kac-Moody Lie algebra* in The Grothendieck Festschrift, Vol. II, Progr. Math. 87 Birkhäuser, Boston, 1990, pp. 407–433.
[15] M. Kashiwara, T.Kawai, *On the characteristic variety of a holonomic system with regular singularities*, Advances in Math., 34 (1979), pp. 163–184.
[16] M. Kashiwara, T. Tanisaki, *Kazhdan-Lusztig conjecture for symmetrizable Kac-Moody Lie algebra II* in Operator Algebras, Unitary Representations, Enveloping Algebras, and Invariant Theory, Vol. II, Progr. Math., 92, Birkhäuser, Boston, 1990, pp. 159–195.
[17] ———, *Kazhdan-Lusztig conjecture for affine Lie algebras with negative level*, Duke Math. J., 77 (1995), pp. 21–62.
[18] ———, *Kazhdan-Lusztig conjecture for affine Lie algebras with negative level II:Nonintegral case*, Duke Math. J., 84 (1996), pp. 771–813.
[19] D. Kazhdan, G. Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math., 53 (1979), pp. 165–184.
[20] S. Kumar, *Extension of the category $\mathcal{O}^g$ and a vanishing theorem for the* Ext *functor for Kac-Moody algebras*, J. Alg., 108 (1987), pp. 472–491.
[21] G. Lusztig, *Characters of Reductive Groups over a Finite Field*, Ann. Math. Studies, 107, Princeton Univ. Press, 1984.
[22] ———, *Monodromic systems on affine flag varieties*, Proc. Roy. Soc. London Ser. A., 445 (1994), pp. 231–246; *Errata*, 450(1995), pp. 731–732.
[23] M. Saito, *Mixed Hodge modules*, Publ. RIMS, Kyoto Univ., 26 (1990), pp. 221–333.
[24] W. Soergel, *Kazhdan-Lusztig polynomials and a combinatoric for tilting modules*, Representation theory, A.M.S. 1 (1997), pp. 83–114.