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A THEOREM OF DENSITY FOR KLOOSTERMAN INTEGRALS*

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1. Introduction. Let F be a local non-Archimedean field of characteristic zero. We denote by $\mathcal{O}_F$ the ring of integers of F, by $\mathcal{P}_F$ or simply $\mathcal{P}$ its maximal ideal and by ϖ a uniformizer. We let ψ be a non-trivial additive character of F. The Haar measure on F is the self-dual Haar measure and vol$(\bullet)$ denotes the volume of a set for this measure. We denote by $(\bullet, \bullet)$ the quadratic residue symbol on $F^\times \times F^\times$.

We regard the group $G = GL(n)$ as an algebraic group over F. We use the following notations. We denote by w_n or $w_{GL(n)}$ the $n \times n$ permutation matrix whose entries are one on the second diagonal and whose other entries are 0. We denote by $N(n, \bullet)$ or simply N the group of upper-triangular matrices with unit diagonal, by $A(n, \bullet)$ or simply A the group of diagonal matrices and by $W(n)$ or W the group of permutation matrices. We let ψ be a non-trivial additive character of F and we denote by θ the character of $N(F)$ defined by

$$\theta(n) = \psi(\sum n_{i+1,i}) \,.$$

The group $N(F) \times N(F)$ operates on $GL(n, F)$ by:

$$g \overset{(n_1,n_2)}{\mapsto} {}^t n_1 g n_2 \,.$$

It follows from the Bruhat decomposition that the elements of the form wa, $w \in W$, $a \in A(F)$, form a system of representatives for the orbits of $N(F) \times N(F)$. We let Φ be a smooth function of compact support on $G(F)$ and consider **orbital (Kloosterman) integrals** of the form:

$$I(wa; \Phi) := \int \Phi \left[{}^t n_2 w a n_1\right] \theta(n_1 n_2) dn_1 dn_2 \,. \tag{1.1}$$

The element wa is assumed to be **relevant**; this means that the character $\theta(n_1n_2)$ is trivial on the stabilizer of wa in $N(F) \times N(F)$. Then the above integral is over the quotient of $N(F) \times N(F)$ by the stabilizer of wa in $N(F) \times N(F)$. The measure is an invariant measure on the quotient. The exact normalization of the measure is described in [7].

Relevant elements can be described as follows. Consider the standard Levi-subgroup M of G of type $(n_1, n_2, \ldots, n_m)$. Thus M is the group of matrices of the form:

$$\begin{pmatrix} g_1 & 0 & 0 & \cdots & 0 \\ 0 & g_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & g_m \end{pmatrix}$$

with $g_i \in GL(n_i)$. Let $w_M \in M$ be the permutation matrix defined by $g_i = w_{n_i}$. Let A_M be the center of M. Then any element of the form $w_M a$ with $a \in A_M(F)$

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is relevant. All relevant elements are obtained in this way. We denote by $W_R(n)$ or W_R the set of relevant elements in $W(n)$. If $w \in W_R$ then the unique M such that $w = w_M$ is denoted by M_w. We also write A_w for A_M. For instance, if $w = e$ then $A_e = A$; if M is the standard Levi-subgroup of type $(n-1, 1)$, then:

$$w_M = \begin{pmatrix} w_{GL(n-1)} & 0 \\ 0 & 1 \end{pmatrix} . \tag{1.2}$$

The integrals $I(w_M a; \Phi)$ are relatively easy to compute and closely related to the hyper-Kloosterman sums:

$$\sum_{x_1 x_2 \cdots x_n = 1} \psi(x_1 + x_2 + \cdots + x_n) ,$$

where ψ is a non-trivial additive character of a finite field $\mathbb{F}$ and the x_i are in $\mathbb{F}$.

The motivation for studying this kind of orbital integrals is as follows. Let us go to a global situation where F is a number field. Let Φ be a smooth function of compact support on $G(F_{\mathbb{A}})$. We assume it is a product of local functions Φ_v. We define as usual a kernel function:

$$K(x, y) = \sum_{\xi \in G(F)} \Phi(x^{-1} \xi y) .$$

We consider the integral

$$I(\Phi) = \int_{N(F)\backslash N(F_{\mathbb{A}}) \times N(F)\backslash N(F_{\mathbb{A}})} K({}^t n_2^{-1}, n_1) \theta(n_2 n_1) dn_2 dn_1 .$$

On the one hand, it can expressed in terms of the orbital integrals $I(wa, \Phi_v)$. On the other hand, it can expressed in terms of the automorphic spectrum of G. Hopefully, the resulting identity can be used to establish various properties of the cuspidal spectrum (See for instance [1], [2], [14].).

Our first result is a theorem of density, asserting that the knowledge of the integrals of the form $I(a; \Phi)$ determine the other orbital integrals:

THEOREM 1.1. *Suppose that Φ is a smooth function of compact support on $GL(n, F)$ such that $I(a; \Phi) = 0$ for all $a \in A_e(F)$. Then all integrals of the form $I(wa; \Phi)$ with wa relevant vanish.*

To state the second result, we introduce more notations. Let E be a quadratic extension of F. We denote by $x \mapsto \overline{x}$ the Galois conjugation in E. We denote by $S(F)$ the variety of invertible Hermitian matrices in $GL(n, E)$:

$$S(F) = \{ s \in GL(n, E) \mid {}^t\overline{s} = s \} .$$

The group $GL(n, E)$ operates on $S(F)$:

$$s \overset{g}{\mapsto} {}^t\overline{g} s g .$$

Recall that the elements of the form wa with $w \in W$, $w^2 = 1$, $a \in A(E)$ with $waw = \overline{a}$, form a set of representatives for the orbits of $N(E)$ on $S(F)$ ([12]). We assume that E is unramified and write $E = F(\sqrt{\tau})$ where τ is a unit of F. Thus the quadratic character η of F associated to E is given by $\eta(a) = (\tau, a)$. We assume further that the residual characteristic of F is larger than $2n+1$, and that the character ψ has for conductor the ring of integers $\mathcal{O}_F$. The Haar measure on E is self-dual for the character $\psi \circ tr$. Since the element $\overline{n}n$ is in $N(F)$ times an element of the derived group of $N(E)$ we can define a character $n \mapsto \theta(\overline{n}n)$ of $N(E)$. If Ψ is a smooth function of compact support on $S(F)$ we consider **relative orbital (Kloosterman) integrals** of the form

$$J(wa;\Psi) = \int \Psi({}^t\overline{n}wan)\theta(\overline{n}n)dn \tag{1.3}$$

with $w \in W$, $a \in A(E)$ and $wa \in S(F)$. We assume wa is relevant, that is, the character $\theta(\overline{n}n)$ is trivial on the stabilizer of wa in $N(E)$. The element wa is relevant if and only if wa is in $GL(n,F)$ and relevant there. The integral is over the quotient of $N(E)$ by the stabilizer of wa in $N(E)$. The measure is an invariant measure on the quotient, normalized as explained in [7].

We introduce a character μ_n of $A(F)$ with values ± 1 as follows:

$$\mu_n(\mathrm{diag}(a_1,a_2,\ldots,a_n)) = \eta(a_1)\eta(a_1a_2)\eta(a_1a_2a_3)\cdots\eta(a_1a_2\cdots a_n)\,. \tag{1.4}$$

If a has the form:

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \tag{1.5}$$

with $a_i \in A(n_i,F)$ then

$$\mu_n(a) = \mu_{n_1}(a_1)\eta(\det a_1)^{n_2}\mu_{n_2}(a_2)\,. \tag{1.6}$$

THEOREM 1.2. *Suppose that $\Phi \in C_c^\infty(G(F))$ and $\Psi \in C_c^\infty(S(F))$ are functions such that*

$$I(a;\Phi) = \mu_n(a)J(a;\Psi)$$

for all $a \in A(F)$. Then for all $w \in W_R$ and all $a \in A_w(F)$ we have

$$I(wa,\Phi) = \mu_n(a)J(wa,\Psi)\,.$$

If Φ and Ψ are the characteristic functions of $GL(n,\mathcal{O}_F)$ and $\mathcal{O}_E \cap S(F)$ respectively it is conjectured that the first relation holds (**fundamental lemma**). Our theorem asserts that the other relations holds as well. The fundamental lemma has been proven for $GL(3)$ ([6]) and $GL(4)$ ([18]) and, in the case of the positive characteristic, for $GL(n)$ ([10]). Similar identities are expected to be true for more general Hecke functions and have been proven in the context of $GL(n)$ in the case of positive characteristic ([11]).

The motivation for this result is as follows: we consider a quadratic extension of number fields E/F. Suppose that Φ is a smooth function of compact support on $S(F_{\mathbb{A}})$ which is a product of local functions. We construct a kernel function as follows:

$$H(g) = \sum_{\xi \in S(F)} \Phi({}^t\overline{g}\xi g)$$

and consider the integral

$$J(\Phi) := \int_{N(E)\backslash N(E_{\mathbb{A}})} H(n)\theta(n\overline{n})dn\,.$$

As before this can computed in terms of the local relative orbital integrals. There is also a spectral expression for the integral $J(\Phi)$. The cuspidal automorphic representations which enter the spectral expression are those which are **distinguished** by a unitary group H, that is, contain a vector ϕ such that

$$\int_{H(F)\backslash H(F_{\mathbb{A}})} \phi(h)dh \neq 0 .$$

Suppose now that Φ is a smooth function of compact support on the group $GL(n, F_{\mathbb{A}})$ which is also a product. If the local components of Φ and Ψ have orbital integrals related as in the theorem, we can expect to have the identity $I(\Phi) = J(\Psi)$. By equating the corresponding spectral expressions we can hope to prove that the distinguished cuspidal representations are exactly the representations which are base change of representations of $GL(n, F_{\mathbb{A}})$ (see [7]).

2. Preliminary result. The behavior at infinity of the orbital integrals $I(w_M \bullet, \Phi)$ is determined by the behavior of the following integral for $| a | \to 0$:

$$I(a; n) := \mathrm{vol}(\mathcal{P}_F^m)^{-1} \int \psi \left[\frac{x_1 + x_2 + \cdots + x_n}{a} \right] \otimes dx_i . \tag{2.1}$$

The integral is over the subset of F^n defined by:

$$x_i \equiv 1 \bmod \mathcal{P}^m , \; x_1 x_2 \cdots x_n \equiv 1 \bmod a\mathcal{P}^m .$$

Here m is an integer, fixed but large. In particular, m is so chosen that the character ψ is trivial on $\mathcal{P}^m$. Before stating our result we recall the definition of the Weil constant $\gamma(\bullet, \psi)$: given a compact open neighborhood Ω of 0 in F, for $| b |$ large enough, we have:

$$\int_\Omega \psi(\frac{bx^2}{2})dx = | b |^{-1/2} \gamma(b, \psi) . \tag{2.2}$$

PROPOSITION 2.1. *If the integer m is sufficiently large, then:*

$$I(a; n) = | a |^{\frac{n+1}{2}} \frac{1}{| n |^{1/2}} \psi \left[\frac{n}{a} \right] \prod_{1 \leq i \leq n-1} \gamma(\frac{i+1}{ia}, \psi) \tag{2.3}$$

if $| a |$ is small enough. In particular, if the residual characteristic of F is larger than n and the conductor of ψ is the ring of integers, then

$$I(a; n) := | a |^{\frac{n+1}{2}} \psi \left[\frac{n}{a} \right] (n, a) \gamma(a, \psi)^{n-1} \tag{2.4}$$

for $| a |$ small enough.

Proof. We change variables and set:

$$x_n = t(x_1 x_2 \cdots x_{n-1})^{-1},$$

where now the domain of integration is defined by:

$$x_i \equiv 1 \bmod \mathcal{P}^m , \; 1 \leq i \leq n-1 , \; t \equiv 1 \bmod a\mathcal{P}^m .$$

After integrating over t the integral becomes

$$| a | \int \psi \left[\frac{\phi}{a} \right] \otimes dx_i ,$$

where the phase function ϕ is given by:

$$\phi = x_1 + x_2 + \cdots x_{n-1} + \frac{1}{x_1 x_2 \cdots x_{n-1}}.$$

We set $x_i = 1 + u_i$ with $u_i \in \mathcal{P}^m$. The phase function takes the form

$$\phi = n - 1 + \sum_{1 \leq i \leq n-1} u_i + \prod \frac{1}{1 + u_i}.$$

The Taylor expansion of this function at the origin has the form:

$$n + \sum_{1 \leq i \leq n-1} u_i^2 + \sum_{1 \leq i < j \leq n-1} u_i u_j + \text{higher degree terms}.$$

We now appeal to the following lemma:

LEMMA 2.1. *Let $n \geq 2$ be an integer. Let F be a field of characteristic* 0.
(i) The quadratic form

$$\sum_{1 \leq i \leq n} X_i^2 + \sum_{1 \leq i < j \leq n} X_i X_j$$

is equivalent over F, by a unipotent linear transformation, to the quadratic form

$$\frac{1}{2} \sum_{1 \leq i \leq n} \frac{i+1}{i} Y_i^2 .$$

(ii) The quadratic form

$$\sum_{1 \leq i \leq n-1} X_i^2 + \sum_{1 \leq i < j \leq n-1} X_i X_j + X_{n-1} X_n + \frac{X_n^2}{2}$$

is equivalent over F, by a unipotent transformation, to the quadratic form:

$$\frac{1}{2} \sum_{1 \leq i \leq n-1} \frac{i+1}{i} Y_i^2 + \frac{1}{2n} Y_n^2 .$$

PROOF OF THE LEMMA: We prove the first assertion. It is trivial for $n = 1$. Thus we may assume $n > 1$ and our assertion proven for $n - 1$. Consider the quadratic linear form

$$\frac{1}{2} \sum_{1 \leq i \leq n} \frac{i+1}{i} Y_i^2 .$$

By the induction hypothesis it is equivalent by a unipotent transformation to the quadratic form:

$$\sum_{1 \leq i \leq n-1} U_i^2 + \sum_{1 \leq i < j \leq n} U_i U_j + \frac{1}{2} \frac{n+1}{n} Y^2 .$$

We change variables as follows:

$$U_i = X_i + \frac{X_n}{n}, \, 1 \leq i \leq n-1 \, , Y = X_n .$$

In terms of the these new variables the quadratic form has the required type.

We now prove the second assertion. By the first assertion (or its proof) the quadratic form is equivalent by a unipotent transformation to the form:

$$\frac{1}{2}\sum_{1\leq i\leq n-1}\frac{i+1}{i}U_i^2+U_{n-1}Y+\frac{Y^2}{2}.$$

Now

$$\frac{n}{2(n-1)}U_{n-1}^2+U_{n-1}Y+\frac{Y^2}{2}=\frac{n}{2(n-1)}\left(U_{n-1}+\frac{n-1}{n}Y\right)^2+\frac{Y^2}{2n}.$$

Thus we obtain a quadratic form of the required type by setting:

$$Y_{n-1}=U_{n-1}+\frac{n-1}{n}Y\,,\ Y_n=Y\,,\ Y_i=U_i \text{ for } i\neq n-1,n\,.\square$$

The lemma being proven we see that after a unimodular change of coordinates the Taylor expansion of ϕ at the origin reads:

$$\phi=n+\frac{1}{2}\sum_{1\leq i\leq n-1}\frac{i+1}{i}y_i^2+\text{higher degree terms}.$$

Thus the origin is a regular critical point. Moreover, if we choose m sufficiently large, the origin is the only critical point of the phase function on the domain of integration. By the principle of the stationary phase there is a compact neighborhood Ω of 0 in F such that, for $\mid a\mid$ small enough, the integral is equal to:

$$I(a;n)=\mid a\mid\psi\left[\frac{n}{a}\right]\int\psi\left[\frac{\sum\frac{i+1}{i}y_i^2}{2a}\right]\otimes dy_i\,,$$

where each variable is integrated over Ω. Thus, for $\mid a\mid$ small enough, $I(a;n)$ is the product of the following factors:

$$\mid a\mid\psi\left[\frac{n}{a}\right],$$

and

$$\int_\Omega\psi\left[\frac{i+1}{2ai}y_i^2\right]dy_i\,,\ 1\leq i\leq n-1\,.$$

Moreover, by definition of the γ factor (see (2.2)), for $\mid a\mid$ small enough, the factor corresponding to the index i is equal to:

$$\left|\frac{ai}{i+1}\right|^{1/2}\gamma\left[\frac{i+1}{ai},\psi\right].$$

Collecting factors we arrive at our first assertion.

Under the assumptions of the second assertion we have $(b,c)=1$ and $\gamma(b,\psi)=1$ if b and c are units. In particular $\gamma(1,\psi)=1$. For an arbitrary pair (b,c)

$$\gamma(b,\psi)\gamma(c,\psi)=\gamma(bc,\psi)(b,c)\,. \tag{2.5}$$

It follows that

$$\gamma\left[\frac{i+1}{ai},\psi\right]=\gamma\left[\frac{i+1}{i},\psi\right]\gamma(a,\psi)\left(\frac{i+1}{i},a\right)=\gamma(a,\psi)\left(\frac{i+1}{i},a\right).$$

Taking the product of these factors we obtain the second assertion. $\square$

3. Computation of the germ. We let M be the standard Levi-subgroup of type $(n-1,1)$. The corresponding element w_M is given by (1.2). We recall the asymptotic properties of the integral $I(w_M a; f)$ ([7]). We denote by $A_{w_M}^{w_G}$ the set of matrices $a \in A_{w_M}(F)$ such that $\det(a) = det w_M w_G$. There exists a smooth function $K_{w_M}^{w_G}$ on $A_{w_M}^{w_G}$ with the following property: for any $\Phi \in \mathcal{C}_c(G)$ there is a smooth function of compact support ω_Φ on A_M such that

$$I(w_M a; \Phi) = \omega_\Phi(a) + \sum_{\alpha\beta=a} K_{w_M}^{w_G}(\alpha) I(w_G \beta; \Phi) . \tag{3.1}$$

The sum is over all pairs in $(\alpha, \beta) \in (A_{w_M}^{w_G}, A_G(F))$ such that $\alpha\beta = a$. The function $K_{w_M}^{w_G}$ is the **germ** (for the orbital integrals) along the subset $A_{w_M}^{w_G}$. It is not unique. However, let K_m be the principal congruence subgroup of $GL(m, F)$; denote by Φ the product of the characteristic function of $w_G K_m$ and the scalar $\mathrm{vol}(\mathcal{P}^m)^{-\frac{n(n-1)}{2}}$. Then $I(w_G, \Phi) = 1$. Moreover, for $z \in A_G(F)$ (i.e. z a scalar matrix) with $z^n = 1$ but $z \neq 1$ we have $I(w_G z, \Phi) = 0$ (if m is sufficiently large). It follows that

$$K_{w_M}^{w_G}(\alpha) = I(w_M \alpha; \Phi) ,$$

where

$$\alpha = \mathrm{diag}\left(a, a, \dots, a, a^{1-n} \det(w_M w_G)\right) ,$$

and $\mid a \mid$ is small enough (see [8]).

Our goal in this section is to compute the germ $K_{w_M}^{w_G}$, or, what amounts to the same, the orbital integral $I(w_M \alpha, \Phi)$ where Φ is the function defined above. Let P be the parabolic subgroup of type $(n-1, 1)$ and U its unipotent radical. Then $P = MU$. We set $N_M = N \cap M$. The stabilizer of $w_M \alpha$ in $N(F) \times N(F)$ is the set of pairs (n_1, n_2) with $n_i \in N_M$ and ${}^t n_1 w_M \alpha n_2 = w_M \alpha$. Then:

$$I(w_M \alpha, \Phi) = \int \Phi\left({}^t u_2 w_M \alpha u_1 n\right) \theta(u_1 u_2 n) du_1 du_2 dn , \tag{3.2}$$

where the integral is over $U(F) \times U(F) \times N_M(F)$. After a change of variables we find that the orbital integral is equal to:

$$\mid a \mid^{-n-\frac{n(n-1)}{2}+1} \int \Phi(x) \psi\left(\frac{\sum_{i+j=n+1} x_{ij}}{a}\right) \otimes dx_{ij} .$$

Here $x = (x_{i,j})$ denotes a matrix of the following form:

$$x_{ij} = 0 \text{ for } i+j < n \ , \ x_{ij} = a \text{ for } i+j = n ;$$

the variables are the entries x_{ij} with $i + j \geq n+1$, $(i,j) \neq (n,n)$; the entry $Z := x_{nn}$ is a dependent variable. The entry Z can be computed from the condition that the determinant of the matrix x be $\det w_G$. For instance in the case $n = 4$:

$$x = \begin{pmatrix} 0 & 0 & a & x_{41} \\ 0 & a & x_{32} & x_{42} \\ a & x_{23} & x_{33} & x_{43} \\ x_{14} & x_{24} & x_{34} & Z \end{pmatrix} ,$$

$$a^{n-1}\det(w_M)Z + \det\begin{pmatrix} 0 & 0 & a & x_{41} \\ 0 & a & x_{32} & x_{42} \\ a & x_{23} & x_{33} & x_{43} \\ x_{14} & x_{24} & x_{34} & 0 \end{pmatrix} = \det(w_G)\,.$$

We see that the integral is equal to

$$\mid a \mid^{-n-\frac{n(n-1)}{2}+1} \operatorname{vol}(\mathcal{P}^m)^{-\frac{n(n-1)}{2}} \int \psi\left(\frac{\sum_{i+j=n+1} x_{ij}}{a}\right) \otimes dx_{ij}$$

integrated over the domain defined by:

$$\begin{aligned} &x_{ij} \equiv 1 \bmod \mathcal{P}^m \text{ for } i+j=n+1\,, \\ &x_{ij} \equiv 0 \bmod \mathcal{P}^m \text{ for } i+j>n+1, (i,j)\neq(n,n)\,, \\ &Z \equiv 0 \bmod \mathcal{P}^m\,. \end{aligned} \tag{3.3}$$

The last condition may also be written as follows (we illustrate the case $n=4$):

$$\det\begin{pmatrix} 0 & 0 & a & x_{41} \\ 0 & a & x_{32} & x_{42} \\ a & x_{23} & x_{33} & x_{43} \\ x_{14} & x_{24} & x_{34} & 0 \end{pmatrix} \equiv \det(w_G) \bmod a^{n-1}\mathcal{P}^m\,.$$

In the last condition we single out the variable $x_{2,n}$. We write the condition in the form:

$$x_{24}\det\begin{pmatrix} 0 & a & x_{41} \\ 0 & x_{32} & x_{42} \\ a & x_{33} & x_{43} \end{pmatrix} + \det\begin{pmatrix} 0 & 0 & a & x_{41} \\ 0 & a & x_{32} & x_{42} \\ a & x_{23} & x_{33} & x_{43} \\ x_{14} & 0 & x_{34} & 0 \end{pmatrix} \equiv \det(w_G) \bmod a^{n-1}\mathcal{P}^m\,.$$

The coefficient of $x_{2,n}$ has the form $a\epsilon$ where ϵ is a unit, which depends only on the variables $x_{i,j}$ with $(i,j)\neq(2,n)$. After changing $x_{2,n}$ to $x_{2,n}\epsilon^{-1}$ we may rewrite the last condition in the form:

$$x_{2,n}a + T \equiv 0 \bmod a^{n-1}\mathcal{P}^m\,. \tag{3.4}$$

where we have set

$$T := \det\begin{pmatrix} 0 & 0 & a & x_{41} \\ 0 & a & x_{32} & x_{42} \\ a & x_{23} & x_{33} & x_{43} \\ x_{14} & 0 & x_{34} & 0 \end{pmatrix} - \det(w_G)\,.$$

Since $x_{2,n} \equiv 0 \bmod \mathcal{P}^m$ we see that the previous condition implies $T \equiv 0 \bmod a\mathcal{P}^m$. Thus the conditions on $x_{2,n}$ can be written as:

$$T \equiv 0 \bmod a\mathcal{P}^m\,,\ x_{2,n} \equiv a^{-1}T \bmod a^{n-2}\mathcal{P}^m\,. \tag{3.5}$$

We can integrate over $x_{2,n}$ to obtain the scalar factor $\mid a \mid^{n-2} \operatorname{vol}(\mathcal{P}^m)$. We have thus eliminated the variable $x_{2,n}$. At this point the integrand is the same as before but the domain of integration is now defined by:

$$\begin{aligned} &x_{i,j} \equiv 1 \bmod \mathcal{P}^m \text{ for } i+j=n+1\,, \\ &x_{i,j} \equiv 0 \bmod \mathcal{P}^m \text{ for } i+j>n+1, (i,j)\neq(2,n),(n,n)\,, \\ &T \equiv 0 \bmod a\mathcal{P}^m\,. \end{aligned} \tag{3.6}$$

In the above condition the determinant of the matrix entering T has the form

$$x_{1,n}x_{2,n-1}\cdots x_{1,n}\det w_G + ay$$

with $y \in \mathcal{P}^m$. Thus the last condition reads

$$x_{1,n}x_{2,n-1}\cdots x_{1,n} \equiv 1 \bmod a\mathcal{P}^m. \tag{3.7}$$

At this point we integrate over the variables $x_{i,j}$ with $i+j > n+1$ and we get the following proposition:

PROPOSITION 3.1. *Set*

$$\alpha = \operatorname{diag}\left(a, a, \ldots, a, a^{1-n}\det(w_M w_G)\right) .$$

Then, for $|\, a \,|$ *sufficiently small,*

$$K^{w_G}_{w_M}(\alpha) = |\, a \,|^{-1-\frac{n(n-1)}{2}} I(a;n), \tag{3.8}$$

$$K^{w_G}_{w_M}(\alpha) = |\, a \,|^{-\frac{(n-1)^2}{2}} \frac{1}{|\, n \,|^{1/2}}\psi\left[\frac{n}{a}\right] \prod_{1\le i\le n-1} \gamma\left(\frac{i+1}{ia}, \psi\right). \tag{3.9}$$

In particular, if the residual characteristic of F *is larger than* n*, we have, for* $|\, a \,|$ *small enough:*

$$K^{w_G}_{w_M}(\alpha) = |\, a \,|^{-\frac{(n-1)^2}{2}} \psi\left[\frac{n}{a}\right](n,a)\gamma(a,\psi)^{n-1} . \tag{3.10}$$

4. The theorem of density. We now prove the density Theorem 1.1. Our key step is the following result:

LEMMA 4.1. *Suppose* Φ *is a smooth function of compact support such that* $I(w_M a;\Phi) = 0$ *for all* $a \in A_M(F)$*. Then* $I(w_G a;\Phi) = 0$ *for all* $a \in A_G(F)$.

Proof. From the germ relation (3.1) we get for $a \in A_{w_M}(F)$

$$0 = \omega_\Phi(a) + \sum_{\alpha\beta=a} K^{w_G}_{w_M}(\alpha) I(w_G\beta;\Phi)$$

where ω_Φ has compact support. Given α and β the pairs (α',β') such that $\alpha'\beta' = \alpha\beta$ have all the form $\alpha' = \alpha z$ and $\beta' = \beta z^{-1}$ where z is an n-th root of unity, as follows from the fact that α' and α have the same determinant. Given $\beta \in A_G(F)$ we choose α of the form

$$\alpha = \operatorname{diag}(a, a, \ldots, a, \det w_M w_G a^{1-n})$$

with a so small that $\omega_\Phi(\alpha\beta) = 0$. We get then

$$\sum_{z^n=1} K^{w_G}_{w_M}(\alpha z^{-1}) I(w_G\beta z, \Phi) = 0 .$$

We have to see that this condition implies that $I(w_G\beta z,\Phi) = 0$ for all z. If we set

$$m(z) := I(w_G\beta z, \Phi),$$

we see that the above relation reads:

$$\sum_{z^n=1} \psi\left[\frac{nz}{a}\right] \prod_{1\le i\le n-1} \gamma\left(\frac{(i+1)z}{ia}, \psi\right) m(z) = 0\,,$$

for $\mid a \mid$ small enough. We have to see that $m(z) = 0$ for all z. Now the γ-factors are non-zero. Viewed as functions of a, they depend only on the class of a modulo the squares. Hence we may write:

$$\prod_{1\le i\le n-1} \gamma\left(\frac{(i+1)z}{ia}, \psi\right) = \sum_{\chi^2=1} \chi(a)c_\chi(z)\,,$$

the sum over all quadratic (or trivial) characters of $F^\times$. Moreover, for each z, there is at least a χ such that $c_\chi(z) \neq 0$. We have then

$$\sum_{\chi,z} \chi(a)\psi\left[\frac{nz}{a}\right] c_\chi(z) = 0$$

for all a with $\mid a \mid$ small enough. Thus our assertion will follow from the following lemma. □

LEMMA 4.2. *Suppose distinct points x_i are given in F and, for each index i and each quadratic character χ of $F^\times$, there is a constant $m_{i,\chi}$ such that*

$$\sum_{i,\chi^2=1} m_{i,\chi}\psi(x_i x)\chi(x) = 0$$

for all x with $\mid x \mid$ large enough. Then $m_{i,\chi} = 0$ for all i and all χ.

Proof. Suppose that $m_{i_0,\chi_0} \neq 0$. At the cost of multiplying by $\psi(-x_{i_0}x)$ and $\chi_0(x)$ we may assume that our relation takes the form:

$$1 = \sum \psi(x_i x)\chi(x)m_{i,\chi},$$

where now the pair with $x_i = 0, \chi = 1$ does not appear on the right. We choose an a with $\mid a \mid$ large and integrate this identity over the set $\mid a \mid\le\mid x \mid\le\mid a\varpi^{-1} \mid$ against the multiplicative Haar measure. The left hand side gives a positive value. On the other hand we have

$$\int_{|a|\le|x|\le|a\varpi^{-1}|} \chi(x)d^\times x = \int_{|x=|a|} \chi(x)d^\times x + \int_{|x|=|a\varpi^{-1}|} \chi(x)d^\times x\,.$$

If χ is ramified, each term is 0. If χ is unramified but non-trivial the two terms are opposite. Thus the terms with $x_i = 0$ contribute zero to the integral of the right hand side. For fixed χ and fixed $x_i \neq 0$ and $\mid b \mid$ large the integral

$$\int_{|x|=|b|} \psi(x_i x)\chi(x)d^\times x$$

vanishes. Thus the terms with $x_i \neq 0$ contribute zero as well if $\mid a \mid$ is sufficiently large and we get a contradiction. □

We go back to the proof of Theorem 1.1. Lemma 4.1 already implies our assertion for $n = 2$. Thus we may assume $n > 2$ and our assertion established for all groups $GL(m)$ with $1 \le m \le n - 1$. It is then true for a product $GL(n_1) \times GL(n_2)$ in the following sense. Let h be a smooth function on the product $GL(n_1, F) \times GL(n_2, F)$. Suppose that h is supported on a set Ω with the following two properties: the image of Ω under the map $(g_1, g_2) \mapsto \det g_1 \det g_2$ is a compact set of $F^\times$; the map $(g_1, g_2) \mapsto \det g_1$ from Ω to $F^\times$ is proper (Note that the map $(g_1, g_2) \mapsto \det g_2$ is then also proper.). If $w_1 a_1$ and $w_2 a_2$ are relevant in $GL(n_1, F)$ and $GL(n_2, F)$ respectively, we can define the **double** orbital integral $I(w_1 a_1, w_2 a_2; h)$:

$$I(w_1a_1, w_2a_2; h) = \int h\left({}^t u_1 w_1 a_1 v_1,\ {}^t u_2 w_2 a_2 v_2\right) \theta(u_1 v_1) du_1 dv_1 \theta(u_2 v_2) du_2 dv_2,$$

where (u_i, v_i) is integrated over the product $N(n_i, F) \times N(n_i, F)$ divided by the stabilizer of $w_i a_i$. We can also define the **partial** orbital integrals $I_1(w_1 a_1, g_2; h)$ and $I_2(g_1, w_2 a_2; h)$. For instance:

$$I_1(w_1 a_1, g_2; h) = \int h({}^t n_1 w_1 a_1 n_2, g_2) \theta(n_1 n_2) dn_1 dn_2\,;$$

the integral is over the product $N(n_1, F) \times N(n_1, F)$ divided by the stabilizer of the point $w_1 a_1$. Moreover, if we fix $w_1 a_1$ and denote by $f_2(g_2)$ the above function, then f_2 is a smooth function of compact support on $GL(n_2, F)$ and

$$I(w_1 a_1, w_2 a_2; h) = I(w_2 a_2; f_2)\,.$$

This being so assume that $I(a_1, a_2; h) = 0$ for all pairs (a_1, a_2). Applying the induction hypothesis to the function

$$g_1 \mapsto I_2(g_1, a_2; h),$$

we find that $I(w_1 a_1, a_2; h) = 0$ for all relevant elements $w_1 a_1$ in $GL(n_1, F)$. Now we fix $w_1 a_1$ and apply the induction hypothesis to the function $g_2 \mapsto I_1(w_1 a_1, g_2; h)$; we conclude that $I(w_1 a_1, w_2 a_2; h) = 0$ for all $w_2 a_2$.

Now suppose that $I(a; f) = 0$ for all $a \in A(F)$. Let us prove first that $I(wa; f) = 0$ for all $w \in W_R$, $w \neq w_G, e$ and $a \in A_w(F)$. Indeed, we can find two integers (n_1, n_2) such that $n = n_1 + n_2$ and wa has the form

$$wa = \begin{pmatrix} w_1 a_1 & 0 \\ 0 & w_2 a_2 \end{pmatrix}$$

with $w_i a_i$ relevant in $GL(n_i, F)$. Let U_1 be the group of matrices of the form

$$u = \begin{pmatrix} 1_{n_1} & X \\ 0 & 1_{n_2} \end{pmatrix} \tag{4.1}$$

Define a function

$$h(g_1, g_2) := \int_{U_1(F) \times U_1(F)} f\left[{}^t u_2 \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} u_1 \right] \theta(u_2 u_1) du_1 du_2 \tag{4.2}$$

More explicitely:

$$h(g_1, g_2) := \int f\left[\begin{pmatrix} g_1 & g_1 X_1 \\ X_2 g_1 & g_2 + X_2 g_1 X_1 \end{pmatrix} \right] \theta(u_2 u_1) du_1 du_2\,.$$

The determinant of the matrix in the integrand is $\det g_1 \det g_2$. Hence the image of the support Ω of h under the map $(g_1, g_2) \mapsto \det g_1 \det g_2$ is a compact set of $F^\times$.

Let us impose in addition the condition that $\det g_1$ be in a compact set of $F^\times$; then $\det g_2$ is in a compact set of $F^\times$. If the integrand is non-zero then g_1 is in a compact of $M(n \times n, F)$ hence in fact in a compact set of $GL(n_1, F)$. Moreover $g_1 X_1$ is in a compact set of F^{n_1}. Hence X_1 is in a compact set. Likewise X_2 is in a compact set. Thus $X_2 g_1 X_1$ is in a compact set of $M(n_2 \times n_2, F)$. Since $g_2 + X_2 g_1 X_1$ is in a compact set of $M(n_2 \times n_2, F)$ the same is true of g_2. Finally g_2 is in a compact set of $GL(n_2, F)$. Thus the map $(g_1, g_2) \mapsto \det g - 1$ from Ω to $F^\times$ is proper and we may apply the above considerations to the function h. We have then

$$I(wa; f) = I(w_1 a_1, w_2 a_2; h).$$

On the other hand, if $b \in A(n, F)$ is a diagonal matrix we can write it as a bloc of diagonal matrices (b_1, b_2), $b_i \in A(n_i, F)$, and then

$$I(b_1, b_2; h) = I(b, f) = 0.$$

As explained above, the induction hypothesis implies then that

$$I(w_1 a_1, w_2 a_2; h) = 0,$$

that is, $I(wa; f) = 0$. We have now established that $I(w\bullet, f) = 0$ for all $w \in W_R$, $w \neq w_G$. In particular $I(w_M \bullet, f) = 0$. By Lemma 4.1, $I(w_G \bullet, f) = 0$ and we are done.

5. The relative situation: preliminary results. We now consider a quadratic extension E of F. We assume (for simplicity) that E is unramified and write $E = F(\sqrt{\tau})$ where τ is a unit of F. Thus the quadratic character η of F associated to E is given by $\eta(a) = (\tau, a)$. We assume further that the residual characteristic of F is larger than $2n+1$ (in particular n odd), and that the character ψ has for conductor the ring of integers $\mathcal{O}_F$. The Haar measure on E is self-dual for the character $\psi \circ tr$.

We define a function $J(a; n)$ as follows.

If $n = 2r$ then

$$(5.1)\quad J(a;n) := \mathrm{vol}(\mathcal{P}_F^m)^{-1} \int \psi\left(\frac{x_1 + \overline{x}_1 + x_2 + \overline{x}_2 + \cdots + x_r + \overline{x}_r}{a}\right) \otimes dx_i,$$

where each x_i is in E and the measure dx_i is self-dual for the character $\psi \circ \mathrm{tr}$. The domain of integration is defined by

$$x_i \equiv 1 \bmod \mathcal{P}_E^m,$$

$$x_1 \overline{x}_1 x_2 \overline{x}_2 \cdots + x_r \overline{x}_r \equiv 1 \bmod a\mathcal{P}_F^m.$$

If $n = 2r+1$ then

$$(5.2)\ J(a;n) := \mathrm{vol}(\mathcal{P}_F^m)^{-1} \int \psi\left(\frac{x_1 + \overline{x}_1 + x_2 + \overline{x}_2 + \cdots x_r + \overline{x}_r + x_{r+1}}{a}\right) \otimes dx_i,$$

where each x_i, $1 \leq i \leq r$, is in E and x_{r+1} is in F. The Haar measures are again self dual. The domain of integration is defined by

$$x_i \equiv 1 \bmod \mathcal{P}_E^m, 1 \leq i \leq r,\ x_{r+1} \equiv 1 \bmod \mathcal{P}_F^m,$$

$$x_1\overline{x}_1x_2\overline{x}_2\cdots x_r\overline{x}_rx_{r+1}\equiv 1 \bmod a\mathcal{P}_F^m\,.$$

As before the behavior at infinity of the relative orbital integrals $J(w_Ma;\Psi)$ is determined by the behavior of these integrals for $|\,a\,|\to 0$.

PROPOSITION 5.1. *Suppose that the residual characteristic of E is larger than $2n+1$. Then, if the integer m is sufficiently large:*

$$J(a;n)=|\,a\,|^{\frac{n+1}{2}}\,\psi\left[\frac{n}{a}\right]\eta^{[n/2]}(a)(n,a)\gamma(a,\psi)^{n-1}\,, \tag{5.3}$$

for $|\,a\,|$ small enough. In particular:

$$J(a;n)=\eta(a)^{[n/2]}I(a;n)\,, \tag{5.4}$$

for $|\,a\,|$ small enough.

Proof. We first consider the case $n=2r+1$. We change variables setting:

$$x_{r+1}=\left(x_1\overline{x}_1x_2\overline{x}_2\cdots x_r\overline{x}_r\right)^{-1}t$$

with $t\equiv 1 \bmod a\mathcal{P}_F^m$ and integrate over t. The integral becomes

$$J(a;2r+1)=|\,a\,|\int\psi\left(\frac{\phi}{a}\right)\otimes dx_i$$

where the phase function ϕ is given by:

$$\phi=x_1+\overline{x}_1+x_2+\overline{x}_2+\cdots+x_r+\overline{x}_r+\frac{1}{x_1\overline{x}_1x_2\overline{x}_2\cdots x_r\overline{x}_r}\,.$$

We set $x_i=1+p_i+q_i\tau$, with $p_i,q_i\in\mathcal{P}_F^m$. Then the Taylor expansion of ϕ at the origin reads:

$$\phi=n+3\sum_{1\le i\le r}p_i^2+4\sum_{1\le i<j\le r}p_ip_j+\sum_{1\le i\le r}q_i^2\tau+\text{higher degree terms}\,.$$

We use the following lemma:

LEMMA 5.1. *Let $n\ge 2$ be an integer. Let F be a field of characteristic 0. The quadratic form*

$$3\sum_{1\le i\le n}X_i^2+4\sum_{1\le i<j\le n}X_iX_j$$

is equivalent over F, by a unipotent transformation, to the quadratic form

$$\sum_{1\le i\le n}\frac{2i+1}{2i-1}Y_i^2\,.$$

PROOF OF THE LEMMA: Our assertion is trivial for $n=1$. Hence we may assume $n>1$ and our assertion proven for $n-1$. Thus the quadratic form

$$\sum_{1\le i\le n}\frac{2i+1}{2i-1}Y_i^2\,.$$

is equivalent to the following form by a unipotent change of variables:

$$3\sum_{1\le i\le n-1}U_i^2+4\sum_{1\le i<j\le n-1}U_iU_j+\frac{2n+1}{2n-1}Y^2\,.$$

We now set

$$U_i = X_i + \frac{2Y}{2n-1}, \, X_n = Y$$

to obtain a form of the required type.

Thus, after a unimodular change of variables, the Taylor expansion of the phase function at the origin may be written in the form:

$$\phi = n + \sum_{1 \le i \le r} \frac{2i+1}{2i-1} x_i^2 + \sum_{1 \le i \le r} y_i^2 \tau + \text{higher degree terms}.$$

If m is large enough, the origin is then the only critical point in the domain of integration. By the principle of stationary phase, there is a neighborhood Ω of 0 in F such that for $\mid a \mid$ small enough $J(a; n)$ is the product of the following factors:

$$\mid a \mid \psi \left[\frac{a}{n} \right]$$

$$\int_\Omega \psi \left[\frac{2i+1}{(2i-1)a} x_i^2 \right] dx_i \, , \; 1 \le i \le r$$

$$\int_\Omega \psi \left[\frac{\tau}{a} y_i^2 \right] dy_i \, , \; 1 \le i \le r \, .$$

Taking $\mid a \mid$ small enough still we see that

$$J(a;n) = \mid a \mid \psi \left[\frac{a}{n} \right] \left(\left| \frac{a}{2\tau} \right|^{1/2} \gamma \left(\frac{2\tau}{a}, \psi \right) \right)^r \prod_{1 \le i \le r} \left| \frac{(2i-1)a}{2(2i+1)} \right|^{1/2} \gamma\left(\frac{2(2i+1)}{(2i-1)a}, \psi \right) .$$

Since $\tau, 2$ and $2i+1, 2i-1$ are units we find that

$$J(a;n) = \mid a \mid^{\frac{n+1}{2}} \psi \left[\frac{a}{n} \right] \left(\gamma \left(2\frac{\tau}{a}, \psi \right) \right)^r \prod_{1 \le i \le r} \gamma \left(2 \frac{2i+1}{(2i-1)a}, \psi \right) .$$

Now (see (2.5)):

$$\gamma \left(2 \frac{2i+1}{(2i-1)a}, \psi \right) = (2,a)\left(\frac{2i+1}{2i-1}, a\right) \gamma \left(2 \frac{2i+1}{(2i-1)}, \psi \right) \gamma(a, \psi),$$

$$\gamma \left(2\frac{\tau}{a}, \psi \right) = (2,a)(\tau, a) \gamma(2\tau, \psi) \gamma(a, \psi).$$

Since $2, \tau$ and $2i+1, 2i-1$ are units, this simplifies to:

$$\gamma \left(2 \frac{2i+1}{(2i-1)a}, \psi \right) = (2,a)\left(\frac{2i+1}{2i-1}, a\right) \gamma(a, \psi),$$

$$\gamma \left(2\frac{\tau}{a}, \psi \right) = (2,a)(\tau, a) \gamma(a, \psi).$$

Multiplying the factors together we obtain our result for $n = 2r+1$ (and $[n/2] = r$).

We next consider the case $n = 2r$. We change variables as follows: we set

$$x_r = vu(x_1 x_2 \cdots x_{r-1})^{-1}$$

with $u \in 1 + a\mathcal{P}_F^m$, $v \in \mathcal{P}_E^m$ with $v\overline{v} = 1$. More precisely,

$$v = \sqrt{1 + t^2\tau} + t\sqrt{\tau}$$

with $t \in \mathcal{P}_F^m$ and $dx_r = dudt$. We can integrate over u to get

$$J(a;2r) = \mid a \mid \int \psi \left(\frac{\phi}{a}\right) \otimes dx_i dv \,,$$

where the phase function ϕ is given by:

$$\phi = x_1 + \overline{x}_1 + x_2 + \overline{x}_2 + \cdots x_{r-1} + \overline{x}_{r-1} + \frac{v}{x_1 x_2 \cdots x_{r-1}} + \frac{\overline{v}}{\overline{x}_1 \overline{x}_2 \cdots \overline{x}_{r-1}} \,.$$

It is convenient to change x_{r-1} into vx_{r-1}. The phase function has now the form:

$$\begin{aligned}\phi = x_1 + \overline{x}_1 + x_2 + \overline{x}_2 + \cdots + x_{r-2} + \overline{x}_{r-2} + vx_{r-1} + \overline{v}\overline{x}_{r-1} \\ + \frac{1}{x_1 x_2 \cdots x_{r-1}} + \frac{1}{\overline{x}_1 \overline{x}_2 \cdots \overline{x}_{r-1}} .\end{aligned}$$

We set

$$x_i = 1 + p_i + q_i\sqrt{\tau} \,,\; v = \sqrt{1 + t^2\tau} + t\sqrt{\tau} \,,$$

where all the variables are in $\mathcal{P}_F^m$. In terms of these new variables, the Taylor expansion of the phase function at the origin can be written in the form:

$$n + 2\left(\sum_{1 \le i \le r-1} p_i^2 + \sum_{1 \le i < j \le r-1} p_i p_j + \sum_{1 \le i \le r-1} q_i^2\tau + \sum_{1 \le i < j \le r-1} q_i q_j \tau\right)$$

$$+ t^2\tau + 2q_{r-1}t\tau + \text{higher degree terms} .$$

After a unimodular change of variables (see Lemma 2.1), the quadratic form can be written:

$$\sum_{1 \le i \le r-1} \frac{i+1}{i} x_i^2 + \sum_{1 \le i \le r-1} \frac{i+1}{i} y_i^2 \tau + \frac{1}{r} y_r^2 \tau \,.$$

As before there is a neighborhood Ω of 0 in F such that, for $\mid a \mid$ small enough, the integral can be written as the product of the following factors:

$$\mid a \mid \psi \left[\frac{n}{a}\right]$$

$$\int_\Omega \psi \left[\frac{i+1}{ai} x_i^2\right] dx_i \;,\; 1 \le i \le r-1$$

$$\int_\Omega \psi \left[\frac{i+1}{ai} y_i^2 \tau\right] dy_i \;,\; 1 \le i \le r-1$$

$$\int_\Omega \psi \left[\frac{1}{ra} \tau y_r^2\right] dy_r$$

Since $\tau, 2$ and $i, i+1$ are units, for $\mid a \mid$ small enough this is equal to:

$$J(a;2r) =\mid a \mid^{\frac{n+1}{2}} \psi\left[\frac{n}{a}\right] \times \prod_{1\le i\le r-1} \gamma\left[\frac{2(i+1)}{ia},\psi\right] \prod_{1\le i\le r-1} \gamma\left[\frac{2(i+1)}{ia}\tau,\psi\right] \gamma\left[\frac{2\tau}{ra},\psi\right].$$

Now (see (2.5):

$$\gamma\left[\frac{2(i+1)}{ia},\psi\right] = \left(\frac{2(i+1)}{i},a\right) \, gamma(a,\psi),$$
$$\gamma\left[\frac{2(i+1)}{ia}\tau,\psi\right] = \left(\frac{2(i+1)}{i},a\right)(\tau,a)\gamma(a,\psi),$$
$$\gamma\left[\frac{2\tau}{ra},\psi\right] = (2r,a)(\tau,a)\gamma(a,\psi).$$

Taking the product of all the factors we arrive at our result. □

6. Computation of the relative germ. As before, we let M be the standard Levi-subgroup of type $(n-1,1)$. We recall the asymptotic properties of the integral $J(w_M a;\Psi)$. There exists a smooth function $K^{w_G}_{w_M}$ on $A^{w_G}_{w_M}$ such that for any $\Psi \in \mathcal{C}_c(S(F))$ there is a smooth function of compact support ω_Φ on A_M such that

$$I(w_M a;\Psi) = \omega_\Psi(a) + \sum_{\alpha\beta=a} K^{w_G}_{w_M}(\alpha) I(w_G\beta;\Psi)\,. \tag{6.1}$$

The sum is over all pairs in $(\alpha,\beta) \in (A^{w_G}_{w_M}, A_G(F))$ such that $\alpha\beta = a$. The function $L^{w_G}_{w_M}$ is the **germ** (for the relative orbital integrals) along the subset $A^{w_G}_{w_M}$. Let K_m be the principal congruence subgroup in $GL(n,E)$. We let Ψ be the product of the characteristic function of $w_G K_m \cap S$ and the scalar

$$\mathrm{vol}(\mathcal{P}^m_F)^{-[n/2]}\mathrm{vol}(\mathcal{P}^m_E)^{-\frac{n(n-1)}{2}+[n/2]}\,.$$

As before:

$$L^{w_G}_{w_M}(\alpha) = I(w_M\alpha,\Psi),$$

where

$$\alpha = \mathrm{diag}(a,a,\ldots,a,a^{1-n}\det w_M w_G)$$

and $\mid a \mid$ is small enough. After a unimodular change of variables we see that the orbital integral of Ψ has the form:

$$\mid a \mid^{-n-\frac{n(n-1)}{2}+1} \int \Psi(x)\psi\left(\frac{\sum_{i+j=n+1} x_{ij}}{a}\right) \otimes dx_{ij}\,.$$

Here $x = (x_{i,j})$ denotes a matrix of the following form:

$$x_{ij} = 0 \text{ for } i+j<n,$$
$$x_{ij} = a \text{ for } i+j=n,$$
$$x_{i,j} = \overline{x}_{j,i}.$$

The variables are the entries $x_{ij} \in E$ with $i+j \ge n+1$, $i<j$, the entries $x_{i,i} \in F$ with $2i \ge n+1$, except the entry $Z = x_{nn}$ which is a dependent variable. The entry Z

can be computed from the condition that the determinant of the matrix x be $\det w_G$. For instance in the case $n = 4$:

$$x = \begin{pmatrix} 0 & 0 & a & \overline{x}_{14} \\ 0 & a & x_{32} & \overline{x}_{24} \\ a & x_{23} & x_{33} & \overline{x}_{34} \\ x_{14} & x_{24} & x_{34} & 0 \end{pmatrix},$$

$$a^{n-1}\det(w_M)Z + \det \begin{pmatrix} 0 & 0 & a & \overline{x}_{14} \\ 0 & a & x_{32} & \overline{x}_{24} \\ a & x_{23} & x_{33} & \overline{x}_{34} \\ x_{14} & x_{24} & x_{34} & 0 \end{pmatrix} = \det(w_G)\,.$$

We see that the integral is equal to

$$\mid a \mid^{-n-\frac{n(n-1)}{2}+1} \operatorname{vol}(\mathcal{P}_F^m)^{-[n/2]}\operatorname{vol}(\mathcal{P}_E^m)^{-\frac{n(n-1)}{2}+[n/2]} \times \int \psi\left(\frac{\sum_{i+j=n+1} x_{ij}}{a}\right) \otimes dx_{ij}$$

integrated over the set:

$$\begin{aligned} &x_{ij} \equiv 1 \bmod \mathcal{P}_E^m \text{ for } i+j=n+1, \\ &x_{ij} \equiv 0 \bmod \mathcal{P}_E^m \text{ for } i+j>n+1, (i,j) \neq (n,n), \\ &Z \equiv 0 \bmod \mathcal{P}^m. \end{aligned} \tag{6.2}$$

The last condition may also be written as follows (we illustrate the case $n = 4$):

$$\det \begin{pmatrix} 0 & 0 & a & \overline{x}_{14} \\ 0 & a & \overline{x}_{23} & \overline{x}_{24} \\ a & x_{23} & x_{33} & \overline{x}_{34} \\ x_{14} & x_{24} & x_{34} & 0 \end{pmatrix} \equiv \det(w_G) \bmod a^{n-1}\mathcal{P}^m\,.$$

In the last condition we single out the variable $x_{2,n}$. We write the condition in the form:

$$x_{24}\det \begin{pmatrix} 0 & a & \overline{x}_{14} \\ 0 & \overline{x}_{23} & 0 \\ a & x_{33} & \overline{x}_{34} \end{pmatrix} + \overline{x}_{2,4}\det \begin{pmatrix} 0 & 0 & a \\ a & x_{2,3} & x_{3,3} \\ x_{1,4} & 0 & x_{34} \end{pmatrix} + x_{2,4}\overline{x}_{2,4}a^2$$

$$+\det \begin{pmatrix} 0 & 0 & a & \overline{x}_{1,4} \\ 0 & a & \overline{x}_{2,3} & 0 \\ a & x_{2,3} & x_{33} & \overline{x}_{3,4} \\ x_{14} & 0 & x_{34} & 0 \end{pmatrix} \equiv \det(w_G) \bmod a^{n-1}\mathcal{P}^m\,.$$

The contribution of $x_{2,n}$ to the formula has the form

$$x_{2,n}au + \overline{x}_{2,n}a\overline{u} + x_{2,n}\overline{x}_{2,n}a^2v,$$

where u is a unit and v an integer. Both u and v depend only on the variables $x_{i,j}$ with $(i,j) \neq (2,n)$. We introduce a new variable:

$$y = x_{2,n}u + \frac{1}{2}ax_{2,n}\overline{x}_{2,n}v\,.$$

Then $y \in \mathcal{P}_E^m$ and the last condition reads

$$ay + a\overline{y} + T \equiv 0 \bmod a^{n-1}\mathcal{P}_F^m, \tag{6.3}$$

where we have set

$$T := \det \begin{pmatrix} 0 & 0 & a & \overline{x}_{1,4} \\ 0 & a & \overline{x}_{2,3} & 0 \\ a & x_{2,3} & x_{33} & \overline{x}_{3,4} \\ x_{14} & 0 & x_{34} & 0 \end{pmatrix} - \det(w_G)\,.$$

As before the above relation implies

$$T \equiv 0 \bmod a\mathcal{P}_F^m\,,$$

and then the condition on y reads:

$$y + \overline{y} \equiv a^{-1}T \bmod a^{m-2}\mathcal{P}_F^m,$$
$$y - \overline{y} \equiv a^{-1}T \bmod \sqrt{\tau}\mathcal{P}_F^m.$$

Thus we can integrate the variable y away. The rest of the computation is similar to the previous case. We obtain in this way:

PROPOSITION 6.1. *For*

$$\alpha = \mathrm{diag}(a, a, \ldots, a, a^{1-n} \det w_M w_G)$$

and $\mid a \mid$ *is small enough,*

$$L_{w_M}^{w_G}(\alpha) = \mid a \mid^{-1-\frac{n(n-1)}{2}} J(a; n). \tag{6.4}$$

In particular,

$$K_{w_M}^{w_G}(\alpha) = \mu_n(\alpha) L_{w_M}^{w_G}(\alpha). \tag{6.5}$$

The second relation follows from the first, Proposition 5.1 and the relation (see (1.4))

$$\eta(a)^{[n/2]} = \mu_n(a)\,.$$

7. Comparison of the orbital integrals. We now prove Theorem 1.2. The proof of the theorem is by induction on n. There is nothing to prove for $n = 1$. So we assume $n > 1$ and our assertion established for $m < n$. Consider a $w \in W_R(n)$, $\neq e, w_G$. Then we may write

$$w = \begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix}$$

with $w_i \in W_R(n_i)$. As before we associate to Φ a function h on $GL(n_1, F) \times GL(n_2, F)$ (see (4.2) and (4.1)) and we associate similarly to Ψ a function k on $S_{n_1}(F) \times S_{n_2}(F)$ by

$$k(s_1, s_2) := \int_{U_1(E)} \Psi \left[{}^t\overline{u} \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} u \right] \theta(u\overline{u}) du. \tag{7.1}$$

The image of the support Ω of k under the map $(s_1, s_2) \mapsto \det s_1 \det s_2$ is compact; the map $(s_1, s_2) \mapsto \det s_1$ from Ω to $E^\times$ is proper. We can then define the double orbital integrals $J(w_1a_1, w_2a_2; k)$ and the partial orbital integrals $J_2(g, w_2a_2; k)$ of the function k. For diagonal matrices $a_i \in GL(n_i, F)$ and a the corresponding diagonal matrix in $GL(n, F)$ we have

$$I(a; \Phi) = \mu_{n_1}(a_1)\mu_{n_2}(a_2)\eta(\det a_1)^{n_2} J(a; \Psi),$$

that is,

$$I(a_1, a_2; h) = J(a_1, a_2; k)\eta(\det a_1)^{n_2}.$$

Let us fix a_2 diagonal in $GL(n_2)$. Then the functions

$$g_1 \mapsto I_2(g_1, a_2; h)$$

and

$$s_1 \mapsto J_2(s_1, a_2; k)\eta(\det(s_1))^{n_2}$$

satisfy the conditions of the theorem for $GL(n_1)$. Thus we have for every relevant element w_1a_1 in $GL(n_1, F)$:

$$I(w_1a_1, a_2; h) = \mu_{n_1}(a_1)\eta(\det a_1)^{n_2}\mu_{n_2}(a_2)J(w_1a_1, a_2; k).$$

The functions

$$g_2 \mapsto I_1(w_1a_1, g_2; h)$$

$$s_2 \mapsto J_1(w_1a_1, s_2; k)\mu_{n_1}(a_1)\eta(\det a_1)^{n_2}$$

satisfy the conditions of the theorem for $GL(n_2)$. Thus we get:

$$I(w_1a_1, w_2a_2; h) = \mu_{n_1}(a_1)\eta(\det a_1)^{n_2}\mu_{n_2}(a_2)J(w_1a_1, w_2a_2; k)$$

or

$$I(wa; \Phi) = \mu_n(a)J(wa; \Psi).$$

In particular, this relation is true for w_M. Thus we get from the germ relations (3.1) and (6.1)

$$\omega(a) + \sum K_{w_M}^{w_G}(\alpha)I(w_G\beta; \Phi) = \left(\omega'(a) + \sum L_{w_M}^{w_G}(\alpha)J_{w_G}(\beta); \Psi)\right)\mu_n(a).$$

Here ω and ω' are suitable functions of compact support on $A_{w_M}(F)$. We fix $\beta \in A_{W_G}$ and choose α with a small enough. Then the above relation reads

$$\sum_{z^n=1} K_{w_M}^{w_G}(z\alpha)I(w_Gz^{-1}\beta, \Phi) = \sum_{z^n=1} L_{w_M}^{w_G}(z\alpha)J_{w_G}(z^{-1}\beta, \Psi)\mu_n(\alpha\beta).$$

By Proposition 6.1, this can be written as

$$\sum_{z^n=1} K_{w_M}^{w_G}(z\alpha)\left(I(w_Gz^{-1}\beta, \Phi) - \mu_n(z^{-1}\beta)J_{w_G}(z^{-1}\beta)\right) = 0.$$

Our conclusion follows as before.

CONCLUDING REMARKS: Theorem 1.2 suggests that the factors $\mu_n(a)$ are transfer factors; that is, given Φ there is Ψ (and conversely) such that the identities of the theorem are true. However, the combination of the result and the fundamental lemma does not imply that the factors are transfer factors. Indeed, to arrive at this conclusion we would need the following relations between the general germs defined in [7]:

$$K^w_{w'}(a) = \mu_n(a) L^w_{w'}(a)\,. \tag{7.2}$$

This relation is known for $n = 2, 3$ and, inductively for $n = 4$ and $w \neq w_G$. However the fundamental lemma for $GL(4)$ implies only the relation

$$\sum_{z^4=1} K_e^{w_G}(za) = \mu_n(a) \sum_{z^4=1} L_e^{w_G}(za),$$

but not the stronger relation (7.2).

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