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AUTOMORPHIC INDUCTION AND LEOPOLDT TYPE CONJECTURES FOR $GL(N)$*

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1. Introduction. Let F be a number field and G_0 be a non-split inner form of $GL(n)_{/F}$ split at a rational prime p and ∞. We define $G = Res_{F/\mathbb{Q}}G_0$. We start with cohomological study of automorphic forms on the reductive algebraic group $G_{/\mathbb{Q}}$ which is isomorphic to a product of copies of $GL(n)$ over a number field. In particular, under not so restrictive conditions on G, we shall show that the nearly p–ordinary (cuspidal) cohomology group

$$H = H^q_{n.ord}(X_1(Np^\infty), \mathbb{Q}_p/\mathbb{Z}_p) = \varinjlim_r H^q_{n.ord}(X_1(Np^r), \mathbb{Q}_p/\mathbb{Z}_p)$$

is of co-finite type over $\mathbf{\Lambda} = \mathbb{Z}_p[[T(\mathfrak{r}_p)]]$ for a maximal (split) torus $T_{/\mathbb{Z}}$ of $GL(n)_{/\mathbb{Z}}$ (Theorem 6.2), where the injective limit is taken with respect to the restriction maps, $\mathfrak{r}_p = \mathfrak{r} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ for the integer ring $\mathfrak{r}$ of F and $\mathbf{\Lambda}$ is the completed group algebra: $\mathbf{\Lambda} = \varprojlim_n \mathbb{Z}_p[T(\mathbb{Z}/p^n\mathbb{Z})]$. Thus for a suitable $q = r$ (that is the degree, called the bottom degree; see (BD) in 6.3 for the exact formula of r), we will also show the finiteness over $\mathbf{\Lambda}$ of the universal p–adic Hecke algebra $\mathbf{h}^{n.ord}$ defined as a subalgebra of $\mathrm{End}_{\mathbf{\Lambda}}(H)$ generated by the standard Hecke operators (Corollary 6.3). Actually we prove, under some assumptions, that the Hecke algebra of weight $\chi \in X(T)$ is obtained, up to finite error, as the specialization of the universal Hecke algebra along the algebra homomorphism $\chi : \mathbf{\Lambda} \to \mathcal{O}$ induced by χ (Control theorem: Theorem 6.5).

Here we do not disregard the torsion part of finite level cohomology groups (although in my earlier works for $GL(2)$, we restricted our study to their torsion-free part), because the Hecke algebra might be too small for non totally real or non CM fields if we kill torsion of each cohomology group.

Although the algebra $\mathbf{h}^{n.ord}$ satisfies various good properties as listed above, basic invariants in ring theory, for example, its Krull dimension, are still difficult to determine in general. The structure of $\mathbf{h}^{n.ord}$ is known to our satisfaction only for $GL(1)$ and $GL(2)$ over $\mathbb{Q}$, for which $\mathbf{h}^{n.ord}$ and the Pontryagin dual H^* of H are projective $\mathbf{\Lambda}$–modules of finite rank. More generally, we know the expected value of the Krull dimension for $GL(1)$ over an arbitrary number field and for all inner forms of $GL(2)$ over totally real fields F, assuming the Leopoldt conjecture for (F, p) (see the description after Conjecture 7.1 for known information). However, we can make a reasonable guess for the dimension of $\mathbf{h}^{n.ord}$ for general G. Here we shall make the following conjecture predicting the Krull dimension of $\mathbf{h}^{n.ord}$:

CONJECTURE 1.1. *Suppose that G_0 is an inner form of $GL(n)_{/F}$. Then we have*

$$\dim(\mathbf{h}^{n.ord} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \leq \begin{cases} m[F:\mathbb{Q}] + 1 & \text{if } n = 2m, \\ m[F:\mathbb{Q}] + r_2 + 1 & \text{if } n = 2m+1, \end{cases}$$

where r_2 is the number of complex places of F.

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It might look odd to have inequality, but depending on the prime-to-p level N, the Hecke algebra may be just trivial. Also the algebra $\mathbf{h}^{n.ord}$ might be very small if F is neither a totally real field nor a CM field. In other words, we conjecture the equality in the above formula if N is sufficiently small (so its norm sufficiently large) and if F is either a totally real field or a CM field. This conjecture is a $GL(n)$-version of the Leopoldt conjecture. In fact, when $m = 0$ ($\Leftrightarrow n = 1$), the conjecture is just the Leopoldt conjecture for p and F. We will check in 7.2 the compatibility of the conjecture with various Langlands functoriality; in particular, it behaves well under automorphic induction. If we admit the automorphic induction functoriality (see [Cl] 1.1) relative to finite extensions L/F' for real abelian extensions $F'/\mathbb{Q}$ and Conjecture 1.1 for $GL(n)_{/F'}$, the above conjecture implies the original Leopoldt conjecture for an arbitrary totally real field F (see Corollary 7.4). Note here that the Leopoldt conjecture is known to be true for abelian extensions $F'/\mathbb{Q}$.

The above conjecture was first made in 1994 when I started writing this paper (after having finished an earlier work [H95]) and was also stated in a series of lectures I gave at the Galilée Institute of Université de Paris-Nord in June 1994. Earlier than this work, the lower bound of the Krull dimension of the full and p-ordinary deformation ring of a Galois representation had been computed by Mazur [M], and the formula of Mazur for p-ordinary case is the same as the one presented above for modular two dimensional Galois representations over $F = \mathbb{Q}$. As conjectured by Mazur, for modular two dimensional representations, the Hecke algebra is isomorphic to the deformation ring (at least in the p-ordinary cases); so, the identity of the formulas for $n = 2$ is a natural consequence of Mazur's conjecture, which is now a theorem of Taylor-Wiles [W] and [TW] (see also [HM] Section 4.3). In the meantime, generalizing Mazur's perspective, J. Tilouine [Ti] has made a conjecture predicting the Krull dimension of the (nearly p-ordinary) universal deformation ring deforming a fixed p-adic Galois representation having values in a smooth reductive group (over $\mathbb{Z}_p$). Of course, if the deformation ring for $GL(n)$ is isomorphic to the Hecke algebra, Conjecture 1.1 is a special case of Tilouine's conjecture. One of the purposes of this paper is to describe evidences for the conjecture from the automorphic side. Related to this, we will prove the following fact, as a special case of Theorem 8.1:

If p splits completely in F and π is nearly p-ordinary, then the Newton polygon of the Hecke polynomial of π at p coincides with the Hodge polygon of the motive (conjecturally) associated to π.

Thus if such a motive exists, the near p-ordinarity of automorphic representations implies the near p-ordinarity of the Galois representation (in the sense of [Ti]) of the motive. In particular, from this, we can deduce that the Newton polygon of any cohomological automorphic representation is located on or above the Hodge polygon. The corresponding fact for motives is a well known result of Mazur and Fontaine. We have said that the above fact is a special case of Theorem 8.1, because we actually prove a precise result without assuming the splitting of p in $F/\mathbb{Q}$ (see also Remark 8.1).

Since automorphic forms are invariant under the left translation by rational elements in $G(\mathbb{Q})$, $\mathbf{h}^{n.ord}$ is actually an algebra over $\Lambda = \mathbb{Z}_p[[T(\mathfrak{r}_p)/\overline{\mathfrak{r}^\times}]]$ for the integer ring $\mathfrak{r}$ of F. If we admit the Leopoldt conjecture for F and p, we see that

$$\dim(\Lambda \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) = (n-1)[F : \mathbb{Q}] + r_2 + 1.$$

Thus $\dim(\Lambda \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$ is possibly equal to $\dim(\mathbf{h}^{n.ord} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$ if and only if one of the following conditions holds:

(i) F is totally real and $n = 2$;
(ii) $n = 1$.

Admitting the conjecture, we see

(*)
$$\dim(\Lambda \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) - \dim(\mathbf{h}^{n.ord} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) = \begin{cases} (m-1)[F:\mathbb{Q}] + r_2 & \text{if } n = 2m, \\ m[F:\mathbb{Q}] & \text{if } n = 2m+1. \end{cases}$$

We write Δ for the right-hand side of the above formula. The left-hand side of (*) measures the non-abelian (or non-Leopoldt) part of the above conjecture. It seems plausible that if $\Delta \leq 1$, one can prove (*) without assuming the Leopoldt conjecture. This has been basically done in [H88] and [H89] when $\Delta = 0$, that is, the two cases (i) and (ii). It is remarkable that the non-abelian part of the conjecture is proven in Case (i) without assuming any particular condition on the structure of the Galois group of F over $\mathbb{Q}$, because only known cases of the Leopoldt conjecture assume that F is abelian either over $\mathbb{Q}$ or an imaginary quadratic field. This is one of the reason why we believe that the above conjecture (or at least its non-abelian part) might be more accessible than the original Leopoldt conjecture. Thus the next target of our investigation would be the case where $\Delta = 1$. We see that $\Delta = 1$ if and only if one of the following conditions is satisfied:

(a) F has only one complex place and $n = 2$;
(b) $3 \leq n \leq 4$ and $F = \mathbb{Q}$.

The formula (*) has been proven in [H94b] Sections 5 and 6 for all inner forms of $GL(2)$ in Case (a). We shall prove in this paper the conjecture for some inner forms of $GL(3)_{/\mathbb{Q}}$ and $GL(4)_{/\mathbb{Q}}$ (see Corollary 6.3 and the explanation after Conjecture 7.1).

The two key ingredients of the proof of the control theorem (Theorem 6.5) are strong multiplicity one theorem valid for cuspidal cohomology on $GL(n)$ and the semi-simplicity (Corollary 8.3) of the (cuspidal) nearly ordinary Hecke algebra. The proof of the semi-simplicity can be generalized to interior cohomology on any reductive group split at p (Corollary A.4). We prove the semi-simplicity in Sections 5 and 8 for $GL(n)$ and in Appendix A for general reductive groups. Since this paper is based on the results obtained in [H95], we have added in Appendix B a list of corrections to [H95].

1.1. Notation. Here is general notation we will use throughout the paper. First of all, we keep the notation introduced in Section 1 throughout the paper. For two algebraic groups $G \supset H$ and a polynomial representation ρ of H (that is, a morphism of algebraic groups from H to $GL(d)$ for some $d > 0$), $\mathrm{Ind}_H^G \rho$ indicates the induced representation in the category of polynomial representations of algebraic groups. Thus, for a representation space V of a polynomial representation $\rho : H \to GL(V)$,

$$\mathrm{Ind}_H^G(\rho) = \left\{ f : G \to V\text{: polynomial} \middle| f(gh) = \rho(h^{-1})f(g) \quad \text{for } h \in H \right\}.$$

Then the action of G is given by $gf(g') = f(g^{-1}g')$. For compact p–adic groups $G \supset H$ and a continuous representation ρ of H on a topological module V, $ind_H^G \rho$ is the representation of G on

$$ind_H^G V = \left\{ f : G \to V\text{: continuous} \middle| f(gh) = \rho(h^{\iota})f(g) \quad \text{for } h \in H \right\},$$

where ι is a suitable involution of G (usually $h^{\iota} = h^{-1}$) specified in the text. Then the action of G is given by $gf(g') = f(g^{\iota}g')$. For locally compact p–adic groups $G \supset H$

and a smooth representation ρ of H acting from the right on a vector space V over a field of characteristic 0 with the discrete topology, $Ind_H^G \rho$ is the right G–module of locally constant functions $f : G \to V$ compactly supported modulo H satisfying $f(gh) = f(g)\rho(h)$ for $h \in H$. The action of G on such functions f is given by $f(g')g = f(gg')$.

For an $n_i \times n_i$ matrix A_i for $i = 1, \dots, r$, we write $diag(A_1, \dots, A_r)$ for the $n \times n$ matrix ($n = n_1 + \cdots + n_r$) whose i–th diagonal block is A_i ($i = 1, \dots, r$) and all the other off-diagonal blocks are zero. To a partition $n = n_1 + \cdots + n_r$, we can associate a standard parabolic subgroup P of $GL(n)$ generated by $M_P = \{diag(A_1, \dots, A_r) | A_i \in GL(n_i)\}$ and upper unipotent matrices. The notion of near-ordinarity depends on a choice of a conjugacy class of a proper parabolic subgroup. In other words, for a fixed standard parabolic subgroup P, we define the nearly ordinary part to be the maximal quotient (or equivalently a maximal subspace) on which the Hecke operator associated to elements in the center of M_P is invertible (see 6.2 for the precise definition). Thus for a given open compact subgroup S, we can have different nearly ordinary cohomology groups depending on the choice of the standard parabolic subgroup. If it is necessary to emphasize the dependence on P, we write $H^q_{P-n.ord}$ for the nearly p–ordinary part with respect to the parabolic subgroup P. When no confusion is likely, we drop "P" from the subscript. There is one exception: When $P = B$, the standard Borel subgroup, we write $H^q_{n.ord}$ for $H^q_{B-n.ord}$ all the time. Then for the standard parabolic subgroup P, the standard Levi component is given by M_P as above. Then $P = M_P N_P$ for the unipotent radical N_P of P. We write M° for the derived group of M_P. Thus $M^\circ = SL(n_1) \times \cdots \times SL(n_r)$. We define a torus T_P by $P \cap SL(n)/M^\circ N_P$, which is isomorphic to $GL(1)^{r-1}$ via $diag(x_1, \dots, x_{r-1}, x_r) \mapsto (\det(x_1), \dots, \det(x_r))$. For each algebraic group H, we write $Z(H)$ for its center. In particular, we write $T_M = Z(M)$. By the above determinant map, we have $T_P \cong T_M^\circ = SL(n) \cap T_M$, and often we identify the two tori.

For the adele ring $\mathbb{A}$ of $\mathbb{Q}$ and a finite set Σ of places of $\mathbb{Q}$, we write $\mathbb{A}^{(\Sigma)} = \{x \in \mathbb{A} | x_v = 1, \ \forall v \in \Sigma\}$; in particular $\mathbb{A}^{(p\infty)} = \{x \in \mathbb{A} | x_p = x_\infty = 1\}$.

2. Preliminaries. Let F be a number field, that is, a finite extension of $\mathbb{Q}$. Let $G_{/\mathbb{Q}}$ be a reductive algebraic group satisfying the condition $\iota : G(\mathbb{Q}_p) \cong GL_n(F_p)$ for $F_p = F \otimes_{\mathbb{Q}} \mathbb{Q}_p$. We fix an isomorphism ι such that it induces for the derived group G° of G

(SL(p)) $$\iota : G^\circ(\mathbb{Q}_p) \cong SL_n(F_p).$$

From time to time, we write $G^\circ(\mathbb{Z}_p)$ for $\iota^{-1}(SL_n(\mathfrak{r}_p))$, where $\mathfrak{r}_p = \mathfrak{r} \otimes_{\mathbb{Z}} \mathbb{Z}_p$. We also suppose, throughout the paper, the strong approximation theorem:

(SA) $$G^\circ(\mathbb{Q}) \textit{ is dense in } G^\circ(\mathbb{A}^{(\infty)}).$$

In Section 6, we will prove the control theorem for the nearly p–ordinary cohomology group and the associated Hecke algebra. The proof is divided into two steps: The first step is the control theory of cohomology groups of arithmetic subgroups of $G^\circ(\mathbb{Q})$ which has been basically taken care of in [H95]. Thus we will resort to a sort of induction process from G° to G in Section 6 in order to prove the control theorem for the nearly p–ordinary sheaf cohomology groups on modular varieties of G.

Let us explain briefly why we think important this problem of controlling Hecke algebras and cohomology groups. Let Φ be a discrete subgroup of $G^\circ(\mathbb{Q}) \cap G^\circ(\mathbb{Z}_p)$

dense in $G^\circ(\mathbb{Z}_p)$. Then for a $(\mathbb{Z}_p, \Phi)$–module L, we study various cohomology groups for Φ, for example, group cohomology $H^q(\Phi, L)$. In particular, we would like to study their p–adic behavior when one shrinks Φ p–adically along a parabolic subgroup P of $GL(n)_{/\mathbb{Z}}$. For example, we take the Borel subgroup B in this preliminary section as P. Shrinking Φ along B means that we take

$$\Phi_1(p^\alpha) = \left\{\gamma \in \Phi \middle| \iota(\gamma) \mod p^\alpha \in N_B(\mathfrak{r}/p^\alpha\mathfrak{r})\right\}$$

for the unipotent radical N_B of B. We study $H^q(\Phi_1(p^\alpha), L)$ and its limit

$$H^q(\Phi_1(p^\infty), L) = \varinjlim_\alpha H^q(\Phi_1(p^\alpha), L) \quad \text{and} \quad \widehat{H}^q(\Phi_1(p^\infty), L) = \varprojlim_\alpha H^q(\Phi_1(p^\alpha), L).$$

Here the limit is taken with respect to the restriction maps and transfer maps.

We also look at the Γ_0–type groups:

$$\Phi_0(p^\alpha) = \left\{\gamma \in \Phi \middle| \iota(\gamma) \mod p^\alpha \in B(\mathfrak{r}/p^\alpha\mathfrak{r})\right\}.$$

Since the maximal split torus T° in $SL(n)$ normalizes N_B and $\Phi_1(p^\alpha)$, we have a natural action of $T^\circ(\mathfrak{r}_p)$ ($\mathfrak{r}_p = r \otimes_{\mathbb{Z}} \mathbb{Z}_p$) on the above cohomology groups. We want to study the $T^\circ(\mathfrak{r}_p)$–module structure of the cohomology groups.

Why is it important to study the limit cohomology groups and the $T^\circ(\mathfrak{r}_p)$–module structure? We answer this question by an example: If L is a rational representation of G°, then we can consider modules $L(A)$ with coefficients in A for various rings, for example $\mathbb{C}$. It is well known after works of Eichler and Shimura on elliptic modular forms, the cohomology groups with coefficients in $L(\mathbb{C})$ is directly related to some specific modular forms (called cohomological modular forms) on G°. Therefore, by cohomological functoriality, on can put a rational or integral structure on the space of modular forms in question. Moreover, for a p–adic ring $\mathcal{O}$, if one can isolate a part of $\widehat{H}^q(\Phi_1(p^\infty), L(\mathcal{O}))$ or $H^q(\Phi_1(p^\infty), L(\mathcal{O}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)$ which is of finite or co-finite type over the completed group ring $\mathcal{O}[[T^\circ(\mathfrak{r}_p)]]$, it gives a p–adic deformation (over an explicitly given $Spec(\mathcal{O}[[T^\circ(\mathfrak{r}_p)]])$) of cohomological modular forms through cohomological functoriality. Here we say a module is of *co-finite* type if its Pontryagin dual is of finite type. In the p–adic situation, the spectrum of p–adic automorphic representations is always continuous. Therefore it is natural to study the limit alongside the individual $H^q(\Phi_1(p^\alpha), L)$ of finite level. Through the action of $T^\circ(\mathfrak{r}_p)$, we may be able to describe the spectrum explicitly. This is a principal reason to study the limit and the action.

There is a general and simple method to isolate such components of finite type over $\mathcal{O}[[T^\circ(\mathfrak{r}_p)]]$. Since $G^\circ \subset G$, we have Hecke operators acting on cohomology groups which preserve integrality. Let $\mathbb{T}$ be one of such operators. If L or its Pontryagin dual is of finite type over $\mathbb{Z}_p$, we have a unique projector $e_{\mathbb{T}}$ acting on $H^q(\Phi_1(p^\alpha), L)$ such that $\mathbb{T}$ is an automorphism on $e_{\mathbb{T}}H^q(\Phi_1(p^\alpha), L)$ and is p–adically nilpotent on $(1 - e_{\mathbb{T}})H^q(\Phi_1(p^\alpha), L)$ (see [H93] 1.11). Here if L is discrete of co-finite type (that is, its Pontryagin dual is of finite type over $\mathbb{Z}_p$), we say that $\mathbb{T}$ is p–adically nilpotent if

$$(1 - e_{\mathbb{T}})H^q(\Phi_1(p^\alpha), L) = \bigcup_{j \geq 1} (1 - e_{\mathbb{T}})H^q(\Phi_1(p^\alpha), L)[\mathbb{T}^j],$$

where the bracket "$[\mathbb{T}^j]$" indicates the kernel of the operator $\mathbb{T}^j$. More generally, if L is an injective limit of discrete $(\mathbb{Z}_p, \Phi)$–modules of co-finite type, we have well defined projector $e_{\mathbb{T}}$. We call a Φ–module L admissible if L is an injective limit of discrete $(\mathbb{Z}_p, \Phi)$–modules of co-finite type. We will see, for $\mathbb{T}$ given by Hecke operator at p (associated to our choice of parabolic subgroup P), the image of $e_{\mathbb{T}}$ has the required property.

Let L^* denote the Pontryagin dual of L. Then we have by Poincaré duality (see [H93] 1.9), if Φ is co-compact,

$$H^q(\Phi_1(p^\alpha), L)^* \cong H^{d-q}(\Phi_1(p^\alpha), L^*) \quad \text{and} \quad H^q(\Phi_1(p^\infty), L)^* \cong \widehat{H}^{d-q}(\Phi_1(p^\infty), L^*),$$

where d is the (real) dimension of the symmetric space of $G^\circ(\mathbb{R})$. Therefore, hereafter, we may assume that L is a p–divisible admissible module.

3. Cohomological automorphic representations for $GL(n)$. In this section, we describe cohomological tempered (modulo center) representations of $GL_n(\mathbb{R})$ and $GL_n(\mathbb{C})$ following Clozel [Cl] Section 3. This is useful in relating cohomology groups to modular forms.

3.1. Rational representations of $GL(n)$. Let T be the standard diagonal torus of $G = GL(n)$. Let $X(T) = \mathrm{Hom}_{alg.gp}(T, \mathbf{G}_m)$. The standard base of the root system with respect to the upper triangular Borel subgroup B is explicitly given by $\{\alpha_i\}$ for $\alpha_i(diag(t_1, t_2, \dots, t_n)) = t_i t_{i+1}^{-1}$ for $i = 1, \dots, n-1$. Then the coroot $\check{\alpha}_i$ of α_i is explicitly given by

$$\check{\alpha}_i(t) = diag(1, \dots, 1, \overset{i}{t}, t^{-1}, 1, \dots, 1).$$

The fundamental dominant weights $\{\omega_i\}$ which form the dual base of $\{\check{\alpha}_i\}$ are given by

$$\omega_i(diag(t_1, t_2, \dots, t_n)) = t_1 \cdot t_2 \cdots t_i \quad (1 \le i \le n-1).$$

We simply write $\omega_n(x) = \det(x)$ for $x \in T$. Thus we may identify $X(T)$ with $\mathbb{Z}^n$ so that $\alpha = (m_1, m_2, \dots, m_n) \in \mathbb{Z}^n$ gives the character

$$\alpha(diag(t_1, t_2, \dots, t_n)) = t_1^{m_1} \cdot t_2^{m_2} \cdots t_n^{m_n}.$$

Then the cone of dominant weights in $X(T)$ is given by

$$C_n = \left\{(m_1, m_2, \dots, m_n) \in \mathbb{Z}^n \middle| m_1 \ge m_2 \ge \dots \ge m_n\right\}.$$

Now we look at the standard parabolic subgroup P of $GL(n)$ corresponding to the partition $n = n_1 + n_2 + \dots + n_r$ of n into r positive integers ($r > 0$). We write $n_i^* = n_1 + n_2 + \dots + n_i$. We may regard T as a maximal split torus of $M = M_P$. Then the center $Z(M)$ of M is a subtorus of T. The set of dominant weights of $M = M_P$ is given by

$$C(M) = C_{n_1} \times \dots \times C_{n_r} \quad \text{in} \quad \mathbb{Z}^n = X(T).$$

For each $\chi \in C(M)$, we have an irreducible representation $\rho_\chi = \mathrm{Ind}_B^P \chi^{-1}$ of M rational over F having χ^{-1} as a highest weight (with respect to the reverse ordering making $-C_n$ positive), which we regard as a representation of P through the projection $pr : P \to M$. By definition, $C(M) \supset C_n$, and $\mathrm{Ind}_P^G \rho_\chi \neq 0$ if and only if $\chi \in C_n$. It is known that $\mathrm{Ind}_P^G \rho_\chi \cong \mathrm{Ind}_B^G \chi^{-1}$ if $\chi \in C_n$, which is absolutely irreducible and is rational over F. For each $\mathbb{Q}$-algebra A, we write $L(\chi; A)$ for the space of $\mathrm{Ind}_B^G \chi^{-1}$ over A.

3.2. Infinity type of cohomological representations of $GL_n(\mathbb{R})$. Here we follow description of cohomological admissible representations of $GL_n(\mathbb{R})$ given in [Cl] Section 3 (see also [H95] Section 8). For a given dominant weight $\chi \in X(T)$, there is at most one irreducible representation $\pi = \pi(\chi)$ of $GL_n(\mathbb{R})$ such that (i) its restriction to $SL_n(\mathbb{R})$ is tempered and

$$\text{(ii)} \qquad H^q(\mathfrak{g}, O_n(\mathbb{R}); H(\pi) \otimes L(\chi; \mathbb{C})) \neq 0,$$

where $H(\pi)$ is the space of smooth vectors of π, and $\mathfrak{g}$ is the complexified Lie algebra of $GL_n(\mathbb{R})$. We call such π cohomological (or χ–cohomological).

In [Cl] Section 3, actually the classification of cohomological π' with non-vanishing

$$H^q(\mathfrak{g}, O_n(\mathbb{R}); H(\pi') \otimes \widetilde{L}(\chi; \mathbb{C})) \neq 0$$

is given for the contragredient $\widetilde{L}(\chi; \mathbb{C})$ of $L(\chi; \mathbb{C})$. Since

$$H^q(\mathfrak{g}, O_n(\mathbb{R}); H(\pi') \otimes \widetilde{L}(\chi; \mathbb{C})) \cong H^q(\mathfrak{g}, O_n(\mathbb{R}); H(\widetilde{\pi}') \otimes L(\chi; \mathbb{C}))$$

(cf. [BW] p.43), the contragredient $\widetilde{\pi}'$ of π' associated to $\widetilde{L}(\chi; \mathbb{C})$ is χ–cohomological. Since π is classified in [Cl] using representation of the Weil group at ∞, what we need to keep in mind is that a representation ρ of the Weil group $W_{\mathbb{C}/\mathbb{R}}$ corresponds to π with non-vanishing $H^q(\mathfrak{g}, O_n(\mathbb{R}); H(\pi) \otimes \widetilde{L}(\chi; \mathbb{C}))$ if and only if ρ^{-1} corresponds to $\pi(\chi)$. Thus what we need to do is to replace $\rho(z)$ in [Cl] by $\rho(z^{-1})$ for $z \in \mathbb{C}^\times \subset W_{\mathbb{C}/\mathbb{R}}$. In this sense, our classification looks a bit different from the one given in [Cl].

Now regard χ as an element of $\mathbb{Z}^n$. Then π is classified by a representation ρ of the Weil group $W_{\mathbb{C}/\mathbb{R}}$ into $GL_n(\mathbb{C})$ (and a character $\chi \in X(T)$). We have that $H^q(\mathfrak{g}, O_n(\mathbb{R}); H(\pi \otimes \varepsilon) \otimes L(\chi; \mathbb{C})) \neq 0$ for a character ε of $\mathbb{R}^\times$ with $\varepsilon^2 = 1$ if and only if for $z \in \mathbb{C}^\times \subset W_{\mathbb{C}/\mathbb{R}}$, writing (see [Cl] p.114-119)

$$\rho(z^{-1}) = \begin{cases} diag(z^{a_1}\overline{z}^{a_2}, z^{a_2}\overline{z}^{a_1}, \dots, z^{a_{2m-1}}\overline{z}^{a_{2m}}, z^{a_{2m}}\overline{z}^{a_{2m-1}}) & \text{if } n = 2m, \\ diag(z^{a_1}\overline{z}^{a_2}, z^{a_2}\overline{z}^{a_1}, \dots, z^{a_{2m-1}}\overline{z}^{a_{2m}}, z^{a_{2m}}\overline{z}^{a_{2m-1}}, (z\overline{z})^{a_{2m+1}}) & \text{if } n = 2m+1, \end{cases}$$

we have

(3.1)

$$a_2 > a_4 > \cdots > a_{2m} > a_{2m-1} > \cdots > a_3 > a_1 \text{ if } n = 2m,$$
$$a_2 > a_4 > \cdots > a_{2m} > a_{2m+1} > a_{2m-1} > \cdots > a_3 > a_1 \text{ if } n = 2m+1,$$

(3.2)

$w = a_{2i-1} + a_{2i}$ $(i = 1, \dots, m)$ and $2a_{2m+1}$ (if $n = 2m+1$) are independent of i,

(3.3)

$$\chi = \begin{cases} (a_2, a_4, \dots, a_{2m}, a_{2m-1}, \dots, a_1) + \mathbf{1} - \delta & \text{if } n = 2m, \\ (a_2, a_4, \dots, a_{2m}, a_{2m+1}, a_{2m-1}, \dots, a_3, , a_1) + \mathbf{1} - \delta & \text{if } n = 2m+1, \end{cases}$$

where $\delta = \omega_1 + \omega_2 + \cdots + \omega_n = (n, n-1, \dots, 1)$ and $\mathbf{1} = (1, 1, \dots, 1)$.

By [Cl] Lemma 3.14, we have

(3.4)

$$H^q(\mathfrak{g}, O_n(\mathbb{R}); H(\pi(\chi)) \otimes L(\chi; \mathbb{C})) \cong \begin{cases} \bigwedge^{q-m^2} \mathbb{C}^{m-1} & \text{if } n = 2m \\ \bigwedge^{q-m(m+1)} \mathbb{C}^m & \text{if } n = 2m+1. \end{cases}$$

Since $H^q(\mathfrak{g}, SO_n(\mathbb{R}); H(\pi(\chi)) \otimes L(\chi;\mathbb{C}))$ really contributes to $H^q(\Phi, L(\chi;\mathbb{C}))$ for a discrete subgroup Φ of the identity connected component $G(\mathbb{R})_+$ of $G(\mathbb{R})$, we deduce from (3.4) combined with [BW] I.5.1 that

$$(3.5)\quad H^q(\mathfrak{g}, SO_n(\mathbb{R}); H(\pi(\chi)) \otimes L(\chi;\mathbb{C})) \cong \begin{cases} \left(\bigwedge^{q-m^2} \mathbb{C}^{m-1}\right) \otimes_{\mathbb{C}} \mathbb{C}[O_n(\mathbb{R})/SO_n(\mathbb{R})] & \text{if } n = 2m \\ \left(\bigwedge^{q-m(m+1)} \mathbb{C}^{m}\right) \otimes_{\mathbb{C}} \mathbb{C}[O_n(\mathbb{R})/SO_n(\mathbb{R})] & \text{if } n = 2m+1. \end{cases}$$

3.3. Infinity type of cohomological representations of $GL_n(\mathbb{C})$. Let $G = Res_{\mathbb{C}/\mathbb{R}}GL(n)$. Then $G(\mathbb{R}) = GL_n(\mathbb{C})$, and every irreducible rational representation of G is isomorphic to $\tau\otimes(\tau'\circ c)$ for a pair (τ,τ') of rational representations of $GL(n)_{/\mathbb{C}}$ for complex conjugation c. Thus $x \in G(\mathbb{R})$ acts via $\tau(x)\otimes\tau'(\overline{x})$ for complex conjugation "$-$". Thus we can view $\tau\otimes(\tau'\circ c)$ as an element χ of $C_n \times C_n$. On the other hand, an admissible irreducible representation π is classified by representations ρ of the Weil group $W_{\mathbb{C}/\mathbb{C}} = \mathbb{C}^\times$. Following [Cl] p.112-3, we have, for admissible irreducible π with tempered restriction to $SL_n(\mathbb{C})$, $H^q(\mathfrak{g}, U_n(\mathbb{R}); H(\pi) \otimes L(\chi;\mathbb{C})) \neq 0$ if and only if for $z \in \mathbb{C}^\times = W_{\mathbb{C}/\mathbb{C}}$

$$(3.6)\quad \rho_\pi(z^{-1}) = diag(z^{p_1}\overline{z}^{q_1}, z^{p_2}\overline{z}^{q_2}, \dots, z^{p_n}\overline{z}^{q_n});$$

$$(3.7)\quad p_i > p_{i+1} \text{ and } q_i < q_{i+1} \text{ for all } i = 0, \dots, n-1;$$

$$(3.8)\quad w = p_i + q_i \text{ is independent of } i;$$

$$(3.9)\quad \chi = (p - \delta + \mathbf{1}, {}^w q - \delta + \mathbf{1}),$$

where the involution "w" indicates the conjugation by the longest element in the Weyl group of T: ${}^w(a_1,\dots,a_n) = (a_n, a_{n-1}, \dots, a_1)$.

Again by [Cl] Lemma 3.14, we have

$$(3.10)\quad H^q(\mathfrak{g}, U_n(\mathbb{R}); H(\pi(\chi)) \otimes L(\chi;\mathbb{C})) \cong \bigwedge^{q-\frac{n(n-1)}{2}} \mathbb{C}^{n-1}.$$

4. Local Hecke algebras. In this section, we reformulate some results in [H95] Section 2 on the structure of local Hecke algebras made of double cosets of Γ_0-type open compact subgroups (with respect to a standard parabolic subgroup) of $GL(n)$. We refer to [H95] for the proof of the results stated in this section.

4.1. Hecke algebras for parabolic subgroups. Let P be a proper standard parabolic subgroup of $GL(n)$ associated with the partition $n = n_1 + n_2 + \cdots + n_r$. Let $\mathcal{V}$ be a discrete valuation ring finite flat over $\mathbb{Z}_\ell$ for a rational prime ℓ. Let ϖ be a prime element of $\mathcal{V}$ and we write $\mathfrak{m} = \mathcal{V}\varpi$ and $k = \mathcal{V}/\mathfrak{m}$. We write v for the valuation with $v(\varpi) = 1$.

Let

$$(4.1)\quad \mathcal{D} = \mathcal{D}_P = \mathcal{D}_P(\mathcal{V}) = \left\{diag(a_1 1_{n_1}, \dots, a_r 1_{n_r}) \in M_n(\mathcal{V}) \middle| \mathcal{V} \supset a_1\mathcal{V} \supset a_2\mathcal{V} \supset \cdots \supset a_r\mathcal{V} \neq 0\right\}.$$

In this section, we often write P for $P(\mathcal{V})$. We then consider $\Delta_\infty = P\mathcal{D}P = P(\mathcal{V})\mathcal{D}P(\mathcal{V}) \subset GL_n(F)$ for the field F of fractions of $\mathcal{V}$. Then it is easy to see

$$(4.2)\quad \Delta_\infty = P\mathcal{D}P \textit{ is a semi-group.}$$

Let $N = N(\mathcal{V})$ be the unipotent radical of P. For a subgroup X of Δ_∞ with $P \triangleright X \supset N$, let $R(X;\Delta_\infty)$ be the space of compactly supported X–bi-invariant functions (with values in $\mathbb{Z}$) compactly supported on Δ_∞. The space $R(X;\Delta_\infty)$ has a natural structure of algebra under the convolution product with respect to the Haar measure μ with $\mu(X) = 1$. We write $XxX \in R(X;\Delta_\infty)$ for the characteristic function of the set given by the double coset XxX. For each $s > 0$, we put $\xi_s = \begin{pmatrix} 1_{n-s} & 0 \\ 0 & \varpi 1_s \end{pmatrix} \in \Delta_\infty$ and let $s_j = n_{r-j+1} + n_{r-j+2} + \cdots + n_r$ for $j = 1, \ldots, r$. Then we write $T_{s_j}(\varpi)$ for $X\xi_{s_j}X$.

PROPOSITION 4.1. *We have the following equality in* $R(X;\Delta_\infty)$*:*

$$X\xi X\zeta X = X\xi\zeta X = X\zeta\xi X = X\zeta X\xi X \quad \text{and} \quad X\xi X \cdot X\zeta X = X\xi\zeta X \tag{4.3}$$

for $\xi, \zeta \in \mathcal{D}$*, and the ring* $R(P;\Delta_\infty)$ *is isomorphic to the polynomial ring* $\mathbb{Z}[T_1, \ldots, T_r]$ *via* $T_{s_j}(\varpi) \mapsto T_j$.

This is proven in [H95] Section 2. A key to the proof is the following explicit coset decomposition of $X\xi X$ of $\xi \in \mathcal{D} = \mathcal{D}_P$: Decompose $X = \bigsqcup_{\eta\in\Xi}(\xi^{-1}X\xi \cap X)\eta$. Then multiplying by $\xi^{-1}X\xi$ from the left, we get

$$\xi^{-1}X\xi X = \bigsqcup_{\eta\in\Xi} \xi^{-1}X\xi\eta \iff X\xi X = \bigsqcup_{\eta\in\Xi} X\xi\eta = \bigsqcup_{\eta\in\xi N_P(\mathcal{V})\xi^{-1}/N_P(\mathcal{V})} X\eta\xi. \tag{4.4}$$

Write $\xi = diag(a_1 1_{n_1}, a_2 1_{n_2}, \ldots, a_r 1_{n_r})$. To describe the group $\xi^{-1}X\xi \cap X$, we write an $n \times n$ matrix A as (A_{ij}) for $n_i \times n_j$ blocks A_{ij}. Then we see

$$\xi^{-1}X\xi \cap X = \left\{ g = (g_{ij}) \in X \middle| g_{ij} \in a_i^{-1}a_j M_{n_i\times n_j}(\mathcal{V}) \text{ for all } j > i \right\}.$$

Thus we may choose the representative set Ξ to be the set of unipotent matrices in $N_P(\mathcal{V})$ such that

$$\Xi \ni \eta \longmapsto (\eta_{ij} \mod a_i^{-1}a_j\mathcal{V})_{j>i} \in \bigoplus_{j>i} M_{n_i\times n_j}(\mathcal{V}/a_i^{-1}a_j\mathcal{V}) \tag{4.5}$$

is an isomorphism.

Let M be the standard Levi-subgroup of P. Then

$$M(A) = \{diag(x_i) | x_i \in GL_{n_i}(A)\}.$$

We write $\pi : P \to M$ for the natural projection, and we put

$$P^\circ = \left\{ x \in P \middle| \pi(x) = diag(x_i) \text{ with } x_i \in SL_{n_i}(A) \text{ for all } i \right\}.$$

Then $P/P^\circ \cong T_M$ via $\det : M \to T_M$ given by $\det(diag(x_i)) = diag(\det(x_i)1_{n_i})$.

COROLLARY 4.2. *We have an algebra isomorphism:*

$$R(P^\circ(\mathcal{V});\Delta_\infty) \cong \mathbb{Z}[T_M(\mathcal{V})][T_1, \ldots, T_r]$$

given by $T_{s_j}(\varpi) \mapsto T_j$ *and* $P^\circ(\mathcal{V})uP^\circ(\mathcal{V}) \mapsto [\det(\pi(u))]$ *for* $u \in P(\mathcal{V})$*, where* $[t]$ *is the group element* t *in the group algebra* $\mathbb{Z}[T_M(\mathcal{V})]$.

Suppose that P is the standard Borel subgroup B. Then $T \cong B/N$, and we may regard a characters $\lambda = (\lambda_1, \ldots, \lambda_n) : T(F) \to K^\times$ as a character of B.

Thus $\lambda(diag(a_1,\ldots,a_n)) = \prod_j \lambda_j(a_j)$. Let V be a right $K[B(F)]$–module with λ–eigensubspace $V[\lambda]$. Then we can let $T_s(\varpi) = N\xi_s N = \bigsqcup_i Nx_i$ act on eigenvectors $v \in V[\lambda]$ by $v|T_s(\varpi)(x) = \sum_i vx_i$, and we have

$$v|T_s(\varpi) = \left(|\varpi|_v^{-ts} \prod_{j=t+1}^{n} \lambda_j(\varpi)\right) v, \tag{4.6}$$

where $t = n - s$ and $|x|_v = |\mathcal{V}/\mathfrak{m}|^{-v(x)}$ is the normalized absolute value of $\mathcal{V}$.

4.2. Hecke algebras of local congruence subgroups. We keep the notation introduced in the previous section. We consider the following subgroups:

$$\begin{aligned} I_\alpha = I_{P,\alpha} &= \{g \in GL_n(\mathcal{V}) | g \mod \varpi^\alpha \in P(\mathcal{V}/\varpi^\alpha)\} \\ I_\alpha^\circ = I_{P,\alpha}^\circ &= \{g \in GL_n(\mathcal{V}) | g \mod \varpi^\alpha \in P^\circ(\mathcal{V}/\varpi^\alpha)\}. \end{aligned} \tag{4.7}$$

Let C be an open compact subgroup of $GL_n(\mathcal{V})$ such that

(P) *C contains I_α° and C is contained in I_α.*

Put $\Delta_C = I_\alpha \mathcal{D}_P I_\alpha$. Then we have

PROPOSITION 4.3. *Suppose* (P). *Then we have*

$$\Delta_C \text{ is a semi-group;} \tag{4.8}$$

$$C\xi C \cdot C\zeta C = C\xi\zeta C = C\zeta C \cdot C\xi C \quad \text{for } \xi,\zeta \in \mathcal{D}; \tag{4.9}$$

$$R(C;\Delta_C) \cong \mathbb{Z}[I_\alpha/C][T_1,T_2,\ldots,T_r] \quad \text{by } T_{s_j}(\varpi) \mapsto T_j, \tag{4.10}$$

where $T_{s_j}(\varpi) = C\xi_{s_j}C$.

5. Jacquet modules and nearly p–ordinary part. We recall the definition of Jacquet modules of admissible representations of $GL(n)$ and study its relation to nearly p–ordinary vectors. The result obtained here will be used to show semi-simplicity of the cohomology groups over their Hecke algebras.

5.1. Jacquet modules for $GL(n)$. Let F be a finite extension of $\mathbb{Q}_\ell$ with ℓ–adic integer ring $\mathcal{V}$. Let π be an admissible irreducible representation of $G = GL_n(F)$ over a field K of characteristic 0. We write V for the representation space of π. Since we mainly deal with cohomology group with right $GL_n(F)$–action, here against usual convention, we suppose that V is a *right* $GL_n(F)$–module. For a standard parabolic subgroup P of $GL(n)_{/\mathcal{V}}$ with unipotent radical $N = N_P$, the Jacquet module V_P of V with respect to P is the $N(F)$–co-invariant space of V, that is, $V_P = V/V(P)$ for the subspace $V(P)$ of V generated by

$$\{v\pi(n) - v | n \in N(F) \text{ and } v \in V\}.$$

Since $V(P)$ is stable under P, for the standard Levi subgroup M of P, $M(F)$ naturally acts on V_P. It is known that

1. V_P gives an admissible representation of $M(F)$ if V is admissible (a theorem of Jacquet [BZ] 3.14);
2. The functor $V \mapsto V_P$ is exact ([BZ] 2.35);
3. Let $\xi = diag(1_{n_1}, \varpi 1_{n_2}, \varpi^2 1_{n_3}, \ldots \varpi^{r-1} 1_{n_r}) \in \mathcal{D}_P$. Then $v \in V(P) \iff$ there exists $m > 0$ such that $\int_{\xi^M N(\mathcal{V})\xi^{-M}} v\pi(n)dn = 0$ for all $M \geq m$ for a Haar measure dn of $N(F)$ ([BZ] 2.33);

4. V is super-cuspidal if and only if $V(gN_Pg^{-1}) = V$ for all $g \in G$ and all proper parabolic subgroups P ([BZ] 3.18).

Since $M^\circ(F)T_M(F)$ is of finite index in $M(F)$ for the derived group M° of M, V_P is an admissible representation of $M^\circ(F)$. Since $N(\mathcal{V})$ is a compact group, for any admissible $N(\mathcal{V})$–module X, we have a projector

$$X \to X^{N(\mathcal{V})} = H^0(N(\mathcal{V}), X) \text{ given by } x \mapsto \int_{N(\mathcal{V})} x\pi(n)dn$$

for the Haar measure "dn" normalized so that $\int_{N(\mathcal{V})} dn = 1$. Thus we have a natural exact sequence

$$0 \to V(P)^{N(\mathcal{V})} \to V^{N(\mathcal{V})} \to V_P^{N(\mathcal{V})} \to 0. \tag{5.1}$$

Since each component of the above exact sequence is stable under $N(\mathcal{V})$, it is an exact sequence of $R(N;\Delta_\infty)$–modules.

5.2. Jacquet modules and Hecke operators. We write $\mathbb{T} = \mathbb{T}_P$ for the Hecke operator corresponding to $P\xi P$, where

$$\xi = diag(1_{n_1}, \varpi 1_{n_2}, \varpi^2 1_{n_3}, \dots \varpi^{r-1} 1_{n_r}) \in \mathcal{D}_P.$$

Then we see from (4.4) and (4.5) that

$$v|\mathbb{T}_P^m = \left(\int_{\xi^m N(\mathcal{V})\xi^{-m}} x\pi(n)dn\right)\pi(\xi^m) \quad \text{for } v \in V^{N(\mathcal{V})}. \tag{5.2}$$

Since $N_P(F) = \bigcup_{m\ge1} \xi^m N_P(\mathcal{V})\xi^{-m}$, $\mathbb{T}_P$ is nilpotent on $V(P)^{N(\mathcal{V})}$.

For any open compact subgroup S with $N(\mathcal{V}) \subset S \subset I_{P,1}$, by (4.5), $S\xi S = \bigsqcup_{x\in\Xi} S\xi x$ if $N(\mathcal{V})\xi N(\mathcal{V}) = \bigsqcup_{x\in\Xi} N(\mathcal{V})\xi x$. Thus the action of $\mathbb{T}_P$ on V^S induces the action of the double coset $S\xi S$, and the finite dimensional space V^S is stable under $\mathbb{T}_P$. Then we can decompose $V^S = V^S_{nil} \oplus V^S_{ss}$ so that $\mathbb{T}_P$ is nilpotent on V^S_{nil} and invertible on V^S_{ss}. Since $V^{N(\mathcal{V})} = \bigcup_S V^S$ for S running over all open subgroups of $I_{P,1}$ containing $N(\mathcal{V})$, we have the decomposition:

$$V^{N(\mathcal{V})} = V_P^{nil} \oplus V_P^\circ \tag{5.3}$$

$$\text{such that } V_P^{nil} = \bigcup_S V^S_{nil} = V^{N(\mathcal{V})} \cap V(P) \text{ and } V_P^\circ = \bigcup_S V^S_{ss},$$

where $\mathbb{T}_P$ is invertible on V_P° and nilpotent on V_P^{nil}.

PROPOSITION 5.1. *Let the notation be as above. Let V be an admissible representation of $G(F)$ over a field K of characteristic* 0. *Then $V_P^\circ \cong V_P$ as $R(P(\mathcal{V}), \Delta_\infty)$–modules.*

Proof. By the above construction (5.3), the functor $V \mapsto V_P^\circ$ is an exact functor, and hence, by taking semi-simplification if necessary, we may assume that V_P° and V_P are both semi-simple $R(N;\Delta_\infty)$–modules. By the above splitting, the projection $V \to V_P$ induces an inclusion $V_P^\circ \hookrightarrow V_P$ of $R(N;\Delta_\infty)$–modules. Since an algebraic closure $\overline{K}$ of K is faithfully flat over K, the result for $V \otimes_K \overline{K}$ implies that of V. Thus we may assume that K is an algebraically closed field. Since $\xi \in T_M(F) = Z(M)$, the Hecke operator $\mathbb{T}_P$ is invertible on the admissible $P(F)$-module V_P. Thus we need to show that any $\overline{v} \in V_P$ can be lifted to $v \in V_P^\circ$. Pick $v \in V$ such that $0 \ne \overline{v} = (v$

mod $V(P)) \in V_P$. We may assume that $\overline{v}$ is an eigenvector of $T_M(F)$–action, because we have assumed that V_P is semi-simple $M(F)$–module and K is algebraically closed. By a simple computation, if v is fixed by an open subgroup of $N(\mathcal{V})$, $v\pi(\xi^{-\alpha})$ ($\equiv c\overline{v}$ mod $V(P)$ with $c \neq 0$) is fixed by $N(\mathcal{V})$ for sufficiently large α. Therefore, we may assume that $v \in V^{N(\mathcal{V})}$. Then $v|\mathbb{T}_P^\beta$ for sufficiently large β gets into V_P°. Then $\overline{v}$ is the image of $(v|\mathbb{T}_P^\beta)|\mathbb{T}_P^{-\beta}$ in V_P°, where we have taken the inverse $\mathbb{T}_P^{-\beta}$ inside V_P° on which $\mathbb{T}_P$ is invertible. This shows the desired surjectivity. □

By the above proposition, we get the following isomorphism of $R(P;\Delta_\infty)$–modules

$$V^{N_P(\mathcal{V})} = V_P^\circ \oplus V_P^{nil} \cong V_P^{N_P(\mathcal{V})} \oplus V(P)^{N_P(\mathcal{V})}. \tag{5.4}$$

When $P = B$, then $M = T_M = T$ is commutative, and hence, V_B is finite dimensional. We quote the following theorem of Kazhdan [BZ] 5.21 and 7.12:

PROPOSITION 5.2. *Suppose that either V is absolutely irreducible or is a subquotient of $Ind_B^V\lambda$ for a continuous character $\lambda : T(F) \to K^\times$. Then we have*

$$\dim_K V_B^\circ \le \dim_K V_B \le n! \quad \textit{for } N = N_B.$$

PROPOSITION 5.3. *Suppose that $\pi \subset Ind_P^G\rho$ for an admissible irreducible representation ρ of $M_P(F)$. Let B be the standard Borel subgroup of $GL(n)$. Then the operator $\mathbb{T}_B$ defined above for the Borel subgroup B is nilpotent if ρ is super-cuspidal and $P \neq B$.*

Proof. We suppose the contrary to the assertion: $V_B^\circ \cong V_B$ is non-trivial. Then by [BZ] 3.19, 3.27 and 3.13 (or more precisely, [BW] XI.2), there exists a continuous character $\lambda : T(F) \to K^\times$ of $T(F)$ such that

$$\pi \textit{ can be embedded isomorphically into } Ind_B^G\lambda. \tag{5.5}$$

Thus π cannot be a factor of $Ind_P^G\rho$ for a super-cuspidal ρ of M_P for $P \neq B$. This shows the result. □

Of course, taking the dual of the above statement (5.5), we can realize π as a quotient of $Ind_B^G\lambda'$ for $\lambda' : T(F) \to K^\times$ possibly different from λ.

For a continuous character $\lambda : T(F) \to K^\times$ (with respect to the discrete topology of $K^\times$), writing $\lambda(diag(t_1, \dots, t_n)) = \prod_{j=1}^n \lambda_j(t_j)$, we define a new character $\widetilde{\lambda}$ by $\widetilde{\lambda}(diag(t_1, \dots, t_n)) = \prod_{j=1}^n (\lambda_j(t_j)|t_j|_{\mathfrak{p}_\sigma}^{j-1})$. Then we write $I_B^G(\lambda) = Ind_{B(F)}^{G(F)}(\widetilde{\lambda})$.

For a Haar measure $d\mu_N$ on the unipotent radical $N(F)$ of $B(F)$, we have the modulus character δ_B given by

$$\int_N \phi(x)d\mu_N(x) = \delta_B(b)\int_N \phi(b^{-1}xb)d\mu_N(x).$$

The character is explicitly given by

$$\sqrt{\delta_B}(x) = |\det(x)|_v^{(n-1)/2} \prod_{j=1}^n |t_j|^{1-j}$$

for $x = diag(t_1, \dots, t_n)$. Thus we see $\widetilde{\lambda}(x) = \delta_B(x)^{-1/2}|\det(x)|_v^{(n-1)/2}\lambda(x)$. Since δ_B is the modulus character of the conjugation action of T on the Haar measure μ_N,

$I_B^G(\lambda)$ is the induction in the category of algebraic admissible representations as in [Cl] Definition 1.9 performed for right B–module (not left ones in [Cl]).

PROPOSITION 5.4. *Let W be the Weyl group of $T = T_B$. Let $\lambda : T(F) \to K^\times$ be a continuous character (under the discrete topology on K), and put $V = V(I_B^G(\lambda))$. If λ^w for $w \in W$ are all distinct (where $\lambda^w(t) = \lambda(wtw^{-1})$), then the Jacquet module V_B is a semisimple $T(F)$–module. Moreover $V_B \cong \bigoplus_{w\in W} \widetilde{\lambda^w}$ as $T(F)$–modules; in particular,* $\dim_K V_B = n!$.

Proof. The Jacquet module functor $V \mapsto V_B$ is exact; thus $(X_B)^{ss} = (X^{ss})_B$, where the superscript "ss" indicates the semi-simplification of a representation X. Since $(I_B^G\lambda)^{ss} = |\ |_v^{(n-1)/2} \otimes (\mathrm{Ind}_B^G(\delta_B^{-1/2}\lambda))^{ss} \cong |\ |_v^{(n-1)/2} \otimes (\mathrm{Ind}_B^G(\delta_B^{-1/2}\lambda^w))^{ss} = (I_B^G\lambda^w)^{ss}$ (see [BZ1] 2.9), $\widetilde{\lambda^w}$ is a subquotient of V_B by Frobenius reciprocity ([BZ] 3.13). Thus we have a $T(F)$–linear surjection of $(V_B)^{ss}$ onto $\bigoplus_{w\in W} \widetilde{\lambda^w}$, because $\widetilde{\lambda^w}$ are all distinct. Since $\dim_K V_B \le n!$ and $|W| = n!$, we know that $V_B \cong (V_B)^{ss} = \bigoplus_w \widetilde{\lambda^w}$ and $\dim_K V_B = n!$. □

By (4.6), we get immediately from the above proposition the following assertion:

COROLLARY 5.5. *Let the notation be as in Proposition 5.4. Suppose that λ^w $(w \in W)$ are all distinct. Then for each $\widetilde{\lambda^w}$–eigenvector $v \in V_B \cong V_B^\circ$,*

$$v|T_s(\varpi) = \left(|\varpi|_v^{s(s-1)/2} \prod_{j=n-s+1}^{n} \lambda_j^w(\varpi) \right) v,$$

where $\lambda_j^w(\varpi) = \lambda^w(diag(1,\dots,1,\overset{j}{\varpi},1,\dots,1))$. In particular, V_B° is a semisimple $R(N_B;\Delta_\infty)$–module of dimension $n!$.

6. p–Adic Hecke algebras. In this section, we first define the nearly p–ordinary universal Hecke algebra and prove its finiteness. We shall further prove the control theorem for the universal nearly p–ordinary Hecke algebra (of level p^∞ for G) from the strong multiplicity one theorem for $GL(n)$ and its inner twists.

6.1. Hecke algebra as a double coset algebra. We consider an algebraic group $G_{/\mathbb{Q}} = Res_{F/\mathbb{Q}}G_0$ for a number field F as in the introduction. We write $\mathfrak{l}$ (resp. v) for prime ideals (resp. places) of F and ℓ for primes of $\mathbb{Z}$. We write $G_\ell = G(\mathbb{Q}_\ell)$ and $G_v = G_0(F_v)$. Then $G(\mathbb{A})$ is the restricted product of local groups G_v over all places v of F. Since G_0 is an inner form of $GL(n)_{/F}$, we have

(GL($\mathfrak{l}$)) $\qquad i_\mathfrak{l} : G_\mathfrak{l} \cong GL_n(F_\mathfrak{l})$ *for almost all prime ideals* $\mathfrak{l}$.

Thus there is a finite set of primes Σ for which (GL($\mathfrak{l}$)) fails to hold. We fix an isomorphism $i_\mathfrak{l}$ for every $\mathfrak{l} \notin \Sigma$. We put

$$\mathfrak{r}^{(\Sigma)} = \prod_{\mathfrak{l}\notin\Sigma} \mathfrak{r}_\mathfrak{l} \subset \widehat{\mathfrak{r}} = \prod_\mathfrak{l} \mathfrak{r}_\mathfrak{l}.$$

We fix a rational prime p outside Σ and a standard (proper) parabolic subgroup P of $GL(n)$ associated with the partition $n = n_1 + \cdots + n_r$. Let U be an open compact subgroup of $G(\mathbb{A})$ of the form $U = \prod_\mathfrak{l} U_\mathfrak{l}$ for the projection $U_\mathfrak{l}$ of U in $G_\mathfrak{l}$. We suppose the following conditions on U:

($G_{B,\mathfrak{l}}$) If $U_\mathfrak{l} \ne GL_n(\mathfrak{r}_\mathfrak{l})$ and $\mathfrak{l} \notin \Sigma \cup \{\mathfrak{p}|p\}$, then $I_{B,\alpha} \supset U_\mathfrak{l} \supset I_{B,\alpha}^\circ$,

($G_{P,p}$) $I_{P,\alpha} \supset U_\mathfrak{p} \supset I_{P,\alpha}^\circ$ $(\alpha > 0)$ for all $\mathfrak{p}|p$

for the subgroups $I_{B,1}$ in $GL_n(\mathfrak{r}_\mathfrak{l})$ and $I^\circ_{P,\alpha}$ and $I_{P,\alpha}$ in $GL_n(\mathfrak{r}_\mathfrak{p})$ defined in (4.7).

We write $\mathcal{D}_\mathfrak{l}$ ($\mathfrak{l} \notin \Sigma \cup \{\mathfrak{p}|p\}$) for $\mathcal{D}_B(\mathfrak{r}_\mathfrak{l})$ in (4.1). We put $\mathcal{D}_\mathfrak{p} = \mathcal{D}_P(\mathfrak{r}_\mathfrak{p})$. Then by elementary divisor theory, if $U_\mathfrak{l} = GL_n(\mathfrak{r}_\mathfrak{l})$ and $\mathfrak{l} \notin \Sigma \cup \{\mathfrak{p}|p\}$, then $\Delta_\mathfrak{l} = U_\mathfrak{l}\mathcal{D}_\mathfrak{l}U_\mathfrak{l} = M_n(\mathfrak{r}_\mathfrak{l}) \cap GL_n(F_\mathfrak{l})$. If $U_\mathfrak{l} \neq GL_n(\mathfrak{r}_\mathfrak{l})$ ($\mathfrak{l} \notin \Sigma$), as seen in 4.1,

$$\Delta_\mathfrak{l} = U_\mathfrak{l}\mathcal{D}_\mathfrak{l}U_\mathfrak{l} \quad \text{if } \mathfrak{l} \nmid p, \text{ and } \quad \Delta_\mathfrak{p} = I_{P,\alpha}\mathcal{D}_\mathfrak{p}I_{P,\alpha} \quad \text{if } \mathfrak{p}|p$$

are semi-groups. Anyway $\Delta = \Delta_U = \left(\prod_{\mathfrak{l}\in\Sigma} U_\mathfrak{l}\right) \times \left(\prod_{\mathfrak{l}\notin\Sigma} \Delta_\mathfrak{l}\right)$ is a sub-semi-group in $\mathbf{G}(\mathbb{A}^{(\infty)})$.

We consider the Hecke algebra $R(U;\Delta)$ made of compactly supported (locally constant) U–bi-invariant functions supported on Δ with values in $\mathbb{Z}$. Then by definition, we have

$$\begin{aligned}(6.1)\quad R(U;\Delta_U) &\cong \bigotimes_{\mathfrak{l}\notin\Sigma} R(U_\mathfrak{l};\Delta_\mathfrak{l}) \\ &\cong \left(\bigotimes_{\mathfrak{p}|p} \mathbb{Z}[I_{P,\alpha}/U_\mathfrak{p}][T_{s_1}(\mathfrak{p}),\dots,T_{s_r}(\mathfrak{p})]\right) \otimes \left(\bigotimes_{\mathfrak{l}\notin\Sigma\cup\{\mathfrak{p}|p\}} \mathbb{Z}[T_1(\mathfrak{l}),\dots,T_n(\mathfrak{l})]\right),\end{aligned}$$

for $T_s(\mathfrak{l})$ which is the characteristic function of $U_\mathfrak{l}\xi_sU_\mathfrak{l}$ for $\xi_s = \left(\begin{smallmatrix}1_t & 0\\ 0 & \varpi_\mathfrak{l}1_s\end{smallmatrix}\right) \in \mathcal{D}_\mathfrak{l}$. Here s_j is as in Proposition 4.1, and the above result follows from the proposition combined with [Sh] Theorem 3.20 when $U_\mathfrak{l} = G_0(\mathfrak{r}_\mathfrak{l})$.

We define

$$R = R_P = \left(\bigotimes_{\mathfrak{p}|p} \mathbb{Z}[T_M(\mathfrak{r}_\mathfrak{p})][T_{s_1}(\mathfrak{p}),\dots,T_{s_r}(\mathfrak{p})]\right) \otimes \left(\bigotimes_{\mathfrak{l}\notin\Sigma\cup\{\mathfrak{p}|p\}} \mathbb{Z}[T_1(\mathfrak{l}),\dots,T_n(\mathfrak{l})]\right).$$

Since we have a natural projection $P/P^\circ(\mathfrak{r}_\mathfrak{p}) \cong T_M(\mathfrak{r}_\mathfrak{p}) \to I_{P,\alpha}/U_\mathfrak{p}$ for all U satisfying $(G_{P,p})$, the ring $R(U;\Delta_U)$ is a homomorphic image of R.

6.2. Nearly p–ordinary cohomology groups. Let $G(\mathbb{R})_+$ be the identity connected component of $G(\mathbb{R})$. Let

$$X = X(U) = G(\mathbb{Q})\backslash G(\mathbb{A})/UZ(\mathbb{R})_+C_{\infty+},$$

where $C_{\infty+}$ is a maximal compact subgroup of $G(\mathbb{R})_+$ and $Z(\mathbb{R})_+$ is the center of $G(\mathbb{R})_+$. We now consider the following condition:

$$(GL(\infty)) \qquad G(\mathbb{R}) \cong GL_n(F \otimes_\mathbb{Q} \mathbb{R}).$$

Then if U is sufficiently small, X is a Riemannian manifold. For the moment, we suppose that

(TF) *$X(U)$ is smooth.*

Let L be a left Δ_p^{-1}–module for $\Delta_p = \prod_{\mathfrak{p}|p} \Delta_\mathfrak{p}$ on which the central elements in $\mathfrak{r}_U^\times = \mathfrak{r}^\times \cap (U \cdot G(\mathbb{R})_+)$ act trivially. Then we consider the covering

$$\mathcal{X} = \mathcal{X}(U) = G(\mathbb{Q})\backslash(G(\mathbb{A}) \times L)/UZ(\mathbb{R})_+C_{\infty+},$$

where the action is given by $\gamma(x,\ell)u = (\gamma xu, u_p^{-1}\ell)$. We study the sheaf $\underline{L} = \underline{L}_U$ of locally constant sections of $\mathcal{X}(U)$ over $X(U)$. We have the cohomology groups $H^q(X(U), \underline{L}_U)$ with coefficients in $\underline{L}$. Since U satisfies $(G_{B,\mathfrak{l}})$ and $(G_{P,p})$, we may regard L as a right Δ-module through the projection of Δ_U into Δ_p, that is, for $\delta \in \Delta$ and $\ell \in L$, $\ell\delta = \delta_p^{-1}\ell$.

For each double coset UxU as an element of $R(U;\Delta)$, we can define the action of $[UxU]$ on $H^q(X(U), \underline{L})$ as follows: We look at the following morphism $[x] : \mathcal{X}(V) \to \mathcal{X}(V^x)$ for $V = xUx^{-1} \cap U$ and $V^x = x^{-1}Vx$ given by $[x](y,\ell) = (yx, x_p^{-1}\ell) = (yx, \ell x_p)$. Then $[x]$ induces a morphism

$$[x] : H^q(X(V), \underline{L}) \to H^q(X(V^x), \underline{L}),$$

and we define

$$[UxU] = Tr_{X(V^x)/X(U)} \circ [x] \circ res_{X(V)/X(U)}. \tag{6.2}$$

Thus under $(G_{P,p})$ and $(G_{B,\mathfrak{l}})$, the (double coset) Hecke algebra R_P acts on the cohomology group $H^q(X(U), \underline{L})$.

Since $G(\mathbb{Q})$ is isomorphic to the multiplicative group of a central simple algebra D over F, choosing a maximal order $\mathfrak{R}$ of D, we can extend G to a group scheme over $\mathbb{Z}$. Hereafter we suppose the following two conditions:

(D1) $G(A) = (\mathfrak{R} \otimes_{\mathbb{Z}} A)^\times$ for a maximal order $\mathfrak{R}$ of a division algebra over F;
(D2) The algebra $\mathfrak{R} \otimes_{\mathfrak{r}} F_{\mathfrak{l}}$ is isomorphic either to $M_n(F_{\mathfrak{l}})$ or a division algebra for all primes $\mathfrak{l}$.

By (D1), it is known that $X(U)$ is compact. The set Σ is made of primes $\mathfrak{l}$ at which $G_{\mathfrak{l}}$ is the multiplicative group of a division algebra central over $F_{\mathfrak{l}}$.

We are going to specify the open compact subgroup U and the Δ^{-1}-module L. As for U, we fix an open subgroup U of $G(\widehat{\mathbb{Z}})$ (containing the center $Z(\mathbb{Z}) = \mathfrak{r}^\times$) satisfying $(G_{B,\mathfrak{l}})$ with $U_p = G(\mathbb{Z}_p)$. Recall that P is a standard parabolic subgroup of $GL(n)$ associated to a partition $n = n_1 + \cdots + n_r$. Let M be the standard Levi-subgroup of P. Thus

$$M(A) = \left\{diag(x_1, \ldots, x_r) \middle| x_i \in GL_{n_i}(A)\right\}.$$

Let M' be a factor of M given by

$$M'(A) = \left\{diag(x_1, \ldots, x_{r-1}) \middle| x_i \in GL_{n_i}(A)\right\},$$

and put

$$M^\circ = \left\{x = diag(x_1, \ldots, x_r) \in M \middle| x_i \in SL(s_i) \text{ for } i = 1, \ldots, r\right\}. \tag{6.3}$$

Let $\pi : P \to M$ be the projection, and write $\mathbf{p}$ for the product of all prime ideals $\mathfrak{p}|p$ in F. Then we define

$$U_{0,P}(\mathbf{p}^\alpha) = \left\{u \in U \middle| u_p \mod \mathbf{p}^\alpha \in P(\mathfrak{r}/\mathbf{p}^\alpha)\right\}$$
$$U_{1,P}(\mathbf{p}^\alpha) = \left\{u \in U_{0,P}(\mathbf{p}^\alpha) \middle| \pi(u_p \mod \mathbf{p}^\alpha) \in M^\circ(\mathfrak{r}/\mathbf{p}^\alpha)\right\}.$$

Let S be an open subgroup of U such that $U_{0,P}(\mathbf{p}^\alpha) \supset S \supset U_{1,P}(\mathbf{p}^\alpha)$ $(\alpha > 0)$. We now specify the Δ_S^{-1}-modules we are going to study. Recall that $N = N_P$ denotes the unipotent radical of P. We put $S_p^\circ = S_p \cap SL_n(\mathfrak{r}_p)$, where S_p is the p–component of S. By abusing the notation, we write $I_{P,\alpha}$ for the product of $I_{P,\alpha}$ with respect to $\mathcal{V} = \mathfrak{r}_\mathfrak{p}$ over all p–adic places $\mathfrak{p}$ of F. Then $Y_S = S_p^\circ/N(\mathfrak{r}_p)$ is an open subspace of the space $Y_\alpha = (I_{P,\alpha} \cap SL_n(\mathfrak{r}_p))/N(\mathfrak{r}_p)$ we studied in [H95]. There are two group actions on the space Y_S. Firstly, since $P(\mathfrak{r}_p) \cap SL_n(\mathfrak{r}_p)$ normalizes $N(\mathfrak{r}_p)$, the quotient $M(\mathfrak{r}_p) \cap SL_n(\mathfrak{r}_p) \cong (P(\mathfrak{r}_p) \cap SL_n(\mathfrak{r}_p))/N(\mathfrak{r}_p)$ naturally acts on Y_S from the right. Secondly, for each choice $\varpi = (\varpi_\mathfrak{p}) \in \mathfrak{r}_p$ with prime element $\varpi_\mathfrak{p} \in \mathfrak{r}_\mathfrak{p}$, we have a natural left action of the semi-group

$$\Delta_{S,p} = S_p \left(\prod_{\mathfrak{p}|p} \mathcal{D}_P(\mathfrak{r}_p) \right) S_p$$

on Y_S as described in [H95] Section 3.

We recall briefly the action of Δ_S on Y_S. Since $\Delta_{S,p} = S_p^\circ T(\mathfrak{r}_p)\mathcal{D}(\mathfrak{r}_p)S_p^\circ$ and S_p° acts on Y_S, we only need to define the action of $T(\mathfrak{r}_p)\mathcal{D}(\mathfrak{r}_p)$ for $\mathcal{D}(\mathfrak{r}_p) = \prod_{\mathfrak{p}|p} \mathcal{D}(\mathfrak{r}_\mathfrak{p})$. First note here that $Y_S \subset (GL(n)/N)(F_p)$ $(F_p = F \otimes_\mathbb{Q} \mathbb{Q}_p)$. Thus there is a natural left action of $SL_n(F_p)$ and right action of $M(F_p) = (P/N)(F_p)$ on $Y = (GL(n)/N)(F_p)$. Then for each $x \in T(\mathfrak{r}_p)\mathcal{D}(\mathfrak{r}_p)$, decompose $S_p x_p S_p = \bigsqcup_{u \in xN(\mathfrak{r}_p)x^{-1}/N(\mathfrak{r}_p)} S_p u x_p$ (see (4.4) and (4.5)). For each class yN in the homogeneous space $Y_S \subset SL_n(F_p)/N(F_p)$ (with $y \in S_p^\circ$), $x_p y = y_u u x_p$ for a unique $u \in xN(\mathfrak{r}_p)x^{-1}/N(\mathfrak{r}_p)$. Then the action of x: $y \mapsto y_u = x_p y x_p^{-1} \mod N(F_p)$ preserves Y_S in $GL_n(F_p)/N(F_p)$. This action induces a left action of $(\Delta_S)^{-1}$ on continuous functions ϕ on Y_S by $x\phi(y) = \phi(x_p^{-1} \circ y)$.

Now we explicitly write down the Δ^{-1}–module L. Let ρ be an absolutely irreducible rational representation of $\overline{M} = Res_{\mathfrak{r}_p/\mathbb{Z}_p} M_{/\mathfrak{r}_p} \subset G_{/\mathbb{Z}_p}$ into $GL(m)$ defined over $\mathcal{O}$ for a valuation ring $\mathcal{O} \subset \overline{\mathbb{Q}}_p$ containing $\sigma(\mathfrak{r})$ for all embeddings $\sigma : F \hookrightarrow \overline{\mathbb{Q}}_p$. Let K be the field of fractions of $\mathcal{O}$. Let $V(\mathcal{O}) \cong \mathcal{O}^m$ be the representation space of the *contragredient* $\widetilde{\rho}$ of ρ, which is again a polynomial representation. For each $\mathcal{O}$–module X, we define an M–module $V(X)$ by $V(\mathcal{O}) \otimes_\mathcal{O} X$. Regard ρ as a representation of $\overline{P} = Res_{\mathfrak{r}_p/\mathbb{Z}_p} P_{/\mathfrak{r}_p} \subset G_{/\mathbb{Z}_p}$ pulling it back to P by the isomorphism $P/N_P \cong M$. Then we suppose that the polynomial representation $L(\rho; K) = \mathrm{Ind}_{\overline{P}}^{G} \widetilde{\rho}$ is non-trivial, which is then absolutely irreducible $G(\mathbb{Q}_p)$–module. Physically we have $L(\rho; K) = \mathrm{Ind}_{P \cap G^\circ}^{G^\circ} \widetilde{\rho}|_{P \cap G^\circ}$:

$$(6.4) \quad L(\rho; K) = \{\phi : (G^\circ/N_P)(\mathbb{Q}_p) \to V(K) | \phi : \textit{polynomial},\ \phi(yx) = \widetilde{\rho}(x^{-1})\phi(y)\}$$

for all $x \in (M \cap SL(n))(F_p)$. We often identify two induced modules $\mathrm{Ind}_{\overline{P}}^{G} \widetilde{\rho}$ and $\mathrm{Ind}_{P \cap G^\circ}^{G^\circ} \widetilde{\rho}|_{P \cap G^\circ}$.

We are going to modify the action of $\Delta_p^{-1} \subset G(\mathbb{Q}_p)$ so that it preserves an $\mathcal{O}$–lattice $L(\rho; \mathcal{O}) \subset L(\rho; K)$. Note that $Y_S \subset (G^\circ/N_P)(\mathbb{Q}_p)$. We then define $L(\rho; \mathcal{O})$ by the subspace of $L(\rho; K)$ made of all functions having values in $V(\mathcal{O})$ on Y_S. If $\phi \in L(\rho; \mathcal{O})$, then for $s \in S_p$, we may write $s = s^\circ t$ with $s^\circ \in S_p^\circ$ and $t \in M(\mathfrak{r}_p)$. Then $\phi(s) = \phi(s^\circ t) = \widetilde{\rho}(t^{-1})\phi(s^\circ) \in V(\mathcal{O})$, and hence $L(\rho; \mathcal{O})$ is an Δ_S^{-1}–module by the pull-back action of the action of Δ_S on Y_S: $b\phi(y) = \phi(b^{-1} \circ y)$ for $\phi \in L(\rho; \mathcal{O})$ and $b \in \Delta_S^{-1}$. This action coincides with the action induced from $L(\rho; K) = \mathrm{Ind}_{\overline{P}}^{G} \widetilde{\rho}$ on S_p° but differs by a scalar factor for general elements in Δ_S^{-1}.

We also consider a far bigger Δ_S^{-1}–module $\mathcal{C}_\rho^S$ made of continuous functions ϕ on Y_S with values in $V(K/\mathcal{O})$ satisfying $\phi(yx) = \widetilde{\rho}(x^{-1})\phi(y)$ for all $x \in M^\circ(\mathfrak{r}_p)$, where $M^\circ = M_P^\circ = SL(n_1) \times \cdots \times SL(n_r) \subset M$. Thus

$$\mathcal{C}_\rho^S = ind_{P^\circ(\mathfrak{r}_p)}^{S_p^\circ} V(K/\mathcal{O}) \quad \text{as } S_p^\circ\text{–module.}$$

The module $\mathcal{C}_\rho^S$ has a natural action of Δ_S^{-1} given by $b\phi(y) = \phi(b_p^{-1} \circ y)$. In particular, we write $\mathcal{C}_\rho$ for $\mathcal{C}_\rho^S$ when $S = U$. Similarly for each topological $\mathcal{O}$–module X, we may consider the following Δ_S^{-1}–modules:

$$\mathcal{C}(Y_S, \rho; X) = \left\{\phi : Y_S \to V(X) : continuous \middle| \phi(yx) = \widetilde{\rho}(x^{-1})\phi(y)\ \forall x \in M^\circ(\mathfrak{r}_p)\right\},$$

where we let $b \in \Delta_S^{-1}$ act on ϕ by $b\phi(y) = \phi(b_p^{-1}y)$.

In [H95], the symbols $L(\rho; A)$ and $\mathcal{C}_\rho^S$ are used to denote induced module from ρ in place of $\widetilde{\rho}$ here. However results obtained there are valid by replacing ρ by $\widetilde{\rho}$; so, hereafter we quote results proven in [H95] without giving any warning about the change of ρ to its contragredient $\widetilde{\rho}$.

We equip $\mathcal{C}(Y_S, \rho; X)$ with a $T_P(\mathfrak{r}_p)$–module structure in the following way: First note that $T_P \cong M_P^\circ \backslash (M_P \cap SL(n))$. Then take $\gamma \in (M_P \cap SL(n))(\mathfrak{r}_p)$. For the class $[\gamma] \in T_P(\mathfrak{r}_p)$ and $\phi \in \mathcal{C}(Y_S, \rho; X)$, we define

$$[\gamma] \cdot \phi(y) = \widetilde{\rho}(\gamma)\phi(y\gamma). \tag{6.5}$$

By our definition of $\mathcal{C}(Y_S, \rho; X)$, for $\gamma^\circ \in M^\circ(\mathfrak{r}_p)$,

$$[\gamma\gamma^\circ] \cdot \phi(y) = \widetilde{\rho}(\gamma\gamma^\circ)\phi(y\gamma\gamma^\circ) = \widetilde{\rho}(\gamma)\widetilde{\rho}(\gamma^\circ)\widetilde{\rho}(\gamma^\circ)^{-1}\phi(y\gamma) = [\gamma] \cdot \phi(y).$$

Thus the action is well defined.

Since Y_S is Zariski dense in the algebraic variety $SL(n)/N$, the induced polynomial representation $L(\rho; K)$, realized on the space of polynomial functions on $SL(n)/N$ with values in $V(K)$, by restriction to Y_S, is sent isomorphically into $\mathcal{C}(Y_S, \rho; K)$. That is, we have an embedding $L(\rho; K) \subset \mathcal{C}(Y_S, \rho; K)$ compatible with the $T_P(\mathfrak{r}_p)$–action. The action of Δ_S^{-1} on Y_S induces a new modified Δ_S^{-1}–action on $L(\rho; K)$, which coincides with the original one on S_p°. For $\varphi = \mathrm{Ind}_P^{GL(n)} \widetilde{\rho}$, noting the fact: $\varphi|_{SL(n)} = \mathrm{Ind}_{P \cap SL(n)}^{SL(n)} \widetilde{\rho}|_{P \cap SL(n)}$ and $\xi \circ y = \xi y \xi^{-1} \mod N(F_p)$ for $\xi \in \mathcal{D}_P(\mathfrak{r}_p)$, we have the new action of ξ on $\phi \in L(\rho; K)$ given by

$$\xi^{-1}\phi(y) = \phi(\xi y \xi^{-1}) = \widetilde{\rho}(\xi)\phi(\xi y) = \omega(\xi)\rho(\xi^{-1})\phi(y),$$

where ω is the central character of $\widetilde{\rho}$, which is equal to $\chi^{-1}|_{T_M}$ if $\widetilde{\rho} = \mathrm{Ind}_B^P \chi^{-1}$.

Let χ be a rational character of $Res_{\mathfrak{r}_p/\mathbb{Z}_p} T_M$. By the determinant map: $P/P^\circ \cong T_M$ as in 1.1, we regard χ a character of $\overline{P}$, and we define $\rho \otimes \chi$ by $\rho \otimes \chi(x)v = \chi(x)\rho(x)v$. We suppose that χ is dominant with respect to ρ (that is, $L(\rho \otimes \chi; K) \neq 0$). Then as already remarked, we can realize $L(\rho \otimes \chi; K)$ in $\mathcal{C}(Y_S, \rho; K)$ uniquely. In particular, the action of T_P on $\mathcal{C}(Y_S, \rho; K)$ preserves $L(\rho \otimes \chi; K)$ and hence induces an action of $T_P(\mathfrak{r}_p)$ on $L(\rho \otimes \chi)$, which is actually given by χ.

We are now going to twist the module $L(\rho; A)$ by a finite order character of T_M. Let ε be a character of $T_M(\mathfrak{r}/\mathbf{p}^\alpha) \to \mathcal{O}^\times$ and $S = U_{0,P}(\mathbf{p}^\alpha)$. We extend ε to

$$T_M(F_p) = T_M(\mathfrak{r}_p) \times \prod_{i=1}^{r} \left(\prod_{\mathfrak{p}|p} \varpi_{\mathfrak{p}}^{\mathbb{Z}} \right)$$

by just putting

$$\varepsilon((1, \ldots, 1, \overset{i}{\varpi_{\mathfrak{p}}}, 1, \ldots, 1)) = 1$$

for all $\mathfrak{p}|p$. For each $x \in \Delta_S$, we write x_{ii} for the i–th $n_i \times n_i$ diagonal block. Then regarding $\det(x) = (\det(x_{ii}))_{0 \le i \le r} \in \mathbf{G}_m^r(F_p)$ as an element of $T_M(F_p)$, we consider ε to be a character $\varepsilon : \Delta_S \to \mathcal{O}^\times$. Thus we may think of $L(\rho \otimes \chi\varepsilon; K)$ which is equal to $L(\rho \otimes \chi; K)$ as a K–vector space but the Δ_S^{-1}–action is twisted as $bv = \varepsilon(b^{-1}) b \circ v$, where "$b \circ v$" indicates the action of $b \in \Delta_S^{-1}$ on $L(\rho \otimes \chi; K)$.

Define a function $\varepsilon_Y : Y_S \to \mathcal{O}^\times$ by $\varepsilon_Y(y) = \varepsilon(\det(y))$ for $\det : P \cap SL(n) \to T_M$ which is well defined because $\det(y)$ for $y \in S_p^\circ \subset \Delta_S$ depends only on the coset $yN(\mathfrak{r}_p)$. Then first realizing $L(\rho \otimes \chi; K)$ in $\mathcal{C}(Y_S, \rho; K)$ and then multiplying functions in $L(\rho \otimes \chi; K)$ by ε_Y, we get the space $L(\rho \otimes \chi\varepsilon; K)$ realized inside $\mathcal{C}(Y_S, \rho; K)$. We then define

$$L(\rho \otimes \chi\varepsilon; \mathcal{O}) = L(\rho \otimes \chi\varepsilon; K) \cap \mathcal{C}(Y_S, \rho; \mathcal{O}),$$

where we take α as small as possible so that ε factors through $T_M(\mathfrak{r}/\mathbf{p}^\alpha)$. Then we write simply

$$L(\rho \otimes \chi\varepsilon) = L(\rho \otimes \chi\varepsilon; \mathcal{O}) \otimes_{\mathcal{O}} K/\mathcal{O}.$$

By definition, $L(\rho \otimes \chi\varepsilon)$ is a p–divisible subspace of $\mathcal{C}_\rho^S$ on which $T_M(\mathfrak{r}_p)$ acts by $\chi\varepsilon$.

We call a continuous homomorphism χ of $T_M(\mathfrak{r}_p)$ into $\overline{\mathbb{Q}}_p^\times$ *arithmetic* if χ coincides with a rational character in $X(Res_{F/\mathbb{Q}} T_M)$ on a small p–adic neighborhood of the identity in $T_M(\mathfrak{r}_p)$. Thus we have shown that $L(\rho \otimes \chi)$ can be realized in a unique way in $\mathcal{C}_\rho^S$ for a suitable α ($S = U_{0,P}(\mathbf{p}^\alpha)$). Then we have well defined cohomology groups:

$$H^q(X(S), \underline{\mathcal{C}}_\rho^S) \quad \text{and} \quad H^q(X(S), \underline{L}(\rho \otimes \chi))$$

on which the double coset algebra $R(S; \Delta_S)$ naturally acts.

We define the p–adic Hecke algebra $\mathbf{h}_{\rho\otimes\chi,q}(S; \mathcal{O})$, for arithmetic χ dominant with respect to ρ, to be the $\mathcal{O}$–subalgebra of $\mathrm{End}_{\mathcal{O}}(H^q(X(S), L(\rho \otimes \chi)))$ generated by the operators in $R(S, \Delta_S)$. Let $\mathbb{T}_j(\mathfrak{p})$ be the operator induced by $T_{s_j}(\varpi_{\mathfrak{p}})$ under the action on $L(\rho \otimes \chi)$ of Δ_S^{-1} described above. Note here that we have modified the original action of $G_{\mathfrak{p}}$ on $L(\rho \otimes \chi; K)$ so that it is minimal among the actions preserving integrality. We write $T_j(\mathfrak{p})$ for the operator induced by $T_{s_j}(\varpi_{\mathfrak{p}})$ using the original action of $G_{\mathfrak{p}}$. The two operators $\mathbb{T}_j(\mathfrak{p})$ and $T_j(\mathfrak{p})$ differ by a constant as specified in (6.6) below.

We can now define the nearly p–ordinary part $\mathbf{h}_{\rho\otimes\chi,q}^{n.ord}(S; \mathcal{O})$ to be the largest direct summand of $\mathbf{h}_{\rho\otimes\chi,q}(S; \mathcal{O})$ on which the image of $\prod_{1 \le j \le r} \prod_{\mathfrak{p}|p} \mathbb{T}_j(\mathfrak{p})$ is a unit.

We write e_P for the projector of $\mathbf{h}_{\rho\otimes\chi,q}(S;\mathcal{O})$ onto $\mathbf{h}^{n.ord}_{\rho\otimes\chi,q}(S;\mathcal{O})$. Then

$$e_P = \lim_{n\to\infty}\left(\prod_{1\le j\le r}\prod_{\mathfrak{p}|p}\mathbb{T}_j(\mathfrak{p})\right)^{n!}$$

is an element in $\mathbf{h}_{\rho\otimes\chi,q}(S;\mathcal{O})$ (see Section 2 and [H93] 1.11). We then define the nearly ordinary cohomology group $H^q_{P-n.ord}(X(S),L(\rho\otimes\chi;A))$ (with respect to P) to be $e_PH^q(X(S),L(\rho\otimes\chi;A))$ as long as we have a well defined e_P acting on it. The notion of near ordinarity depends on the choice of the (conjugacy class of) parabolic subgroup P. Thus for a given S, we may have

$$H^q_{P-n.ord}(X(S),L(\rho\otimes\chi;A)) \neq H^q_{Q-n.ord}(X(S),L(\rho\otimes\chi;A))$$

for two non-conjugate parabolic subgroups P and Q. See Appendix A for a little more intrinsic description of the dependence on P for general reductive groups. Therefore, if confusion is likely, we use the symbol $H^q_{P-n.ord}$ to indicate P–nearly ordinary cohomology groups.

Since the action of Δ_S^{-1} on $L(\rho';X)$ for $\rho'=\rho\otimes\chi'$ is modified, if one uses the original action of $GL(n)$ on $GL(n)/N$ to define the Hecke operator $T_j(\mathfrak{p})=T(\xi_{s_j})$ on the cohomology group, we have the following relation

$$\mathbb{T}_j(\mathfrak{p}) = \omega(\xi_{s_j})T_j(\mathfrak{p}) \quad \text{on } H^q(X(S),\underline{L}(\rho\otimes\chi)), \tag{6.6}$$

where $\xi_s=\begin{pmatrix}1_{n-s} & 0\\ 0 & \varpi 1_s\end{pmatrix}$ in $T(F_\mathfrak{p})$ and ω is the algebraic character in $X(T_M)$ which gives the central character of ρ' on an open neighborhood of the identity of $T_M(\mathfrak{r}_p)$. If $\widetilde{\rho}=\rho_\chi=\mathrm{Ind}_B^P\chi^{-1}$, then $\omega=(\chi\chi')^{-1}$. The operator $T_j(\mathfrak{p})$ is induced by the action of $\xi_{s_j}^{-1}$ under $\widetilde{\rho'}=\mathrm{Ind}_B^G(\chi\chi')^{-1}$ and $\mathbb{T}_j(\mathfrak{p})$ is induced by the action on Y_S of $\xi_{s_j}^{-1}\in\Delta_S^{-1}$. The modified action is given by $\omega(\xi_{s_j}^{-1})^{-1}\rho'(\xi_{s_j}^{-1})=\omega(\xi_{s_j})\rho'(\xi_{s_j}^{-1})$, and hence we get (6.6).

Since we have a natural map: $H^q(X(S),\underline{L}(\rho;K))\to H^q(X(S),\underline{L}(\rho))$ with finite cokernel ($L(\rho)=L(\rho;\mathcal{O})\otimes_\mathcal{O}K/\mathcal{O}$), an eigenspace $H^q_{P-n.ord}(X(S),\underline{L}(\rho;K))[\lambda]$ of Hecke operators $T_j(\mathfrak{l})$ (for all $\mathfrak{l}$) with eigenvalue $\lambda(T_j(\mathfrak{l}))$ gives rise to an eigenspace $H^q_{P-n.ord}(X(S),\underline{L}(\rho))[\lambda]$ in the discrete module $H^q(X(S),\underline{L}(\rho))$ with the same eigenvalues. On the other hand, fixing embeddings $i_\infty:\overline{\mathbb{Q}}\hookrightarrow\mathbb{C}$ and $i_p:\overline{\mathbb{Q}}\hookrightarrow\overline{\mathbb{Q}}_p$, we pull back K to $\mathbb{C}$: $K_0=i_\infty(i_p^{-1}(K))$. Then we have

$$H^q(X(S),\underline{L}(\rho;\mathbb{C})) = H^q(X(S),\underline{L}(\rho;K_0))\otimes_{K_0}\mathbb{C},$$

and hence the system of eigenvalues λ is given by an automorphic representation π of $G(\mathbb{A})$. We call such π *nearly p-ordinary* with respect to P. By definition, $e_PH^q(X(S),\underline{L}(\rho))[\lambda]\neq 0$ if and only if $i_p(\lambda(\mathbb{T}_j(\mathfrak{p})))$ is a p–adic unit in K. Thus π is nearly p–ordinary if and only if for a common eigenvector $v\in H^0(S,V(\pi))$ of $T_j(\mathfrak{p})$ (for all $\mathfrak{p}|p$), we have

$$|\lambda(T_j(\mathfrak{p}))|_p = \left|\omega^{-1}(\xi_{s_j})\right|_p \quad \text{for all } j=1,\dots,r \text{ and all } \mathfrak{p}|p. \tag{6.7}$$

In my earlier work [H95] (especially in Theorem 5.1 of the paper), we studied the action of $T_M^\circ(\mathfrak{r}_p)=(T_M\cap SL(n))(\mathfrak{r}_p)$ on $\mathcal{C}_\rho^S$ induced by the above action of $T_P(\mathfrak{r}_p)$

via the isogeny: $T_M^\circ \hookrightarrow M \twoheadrightarrow T_P$. Actually the use of the action of T_P in place of T_M makes the result more transparent, and we obtain, in exactly the same manner as in the proof of [H95] Theorem 5.1 an isomorphism of R_P-modules:

$$\iota_\chi : H^q_{n.ord}(\Gamma^\circ(S_{1,P}(\mathbf{p}^\infty)), L(\rho\otimes\chi)) \cong H^q_{n.ord}(\Gamma^\circ(S_{1,P}(\mathbf{p})), \mathcal{C}^S_\rho) \tag{6.8}$$

satisfying $\iota_\chi(\chi(z)zc) = z\iota_\chi(c)$ for $c \in H^q_{n.ord}(\Gamma^\circ(S_{1,P}(\mathbf{p}^\infty)), L(\rho\otimes\chi))$ under the action of T_P given in (6.5), where $\Gamma^\circ(S) = (S\cdot G^\circ(\mathbb{R}))\cap G^\circ(\mathbb{Q})$. If we assume the vanishing of cohomology degree less than r: $H^q_{n.ord}(\Gamma^\circ(S_{1,P}(\mathbf{p}^\alpha)), L(\rho\otimes\chi; K)) = 0$ for $0 \le q < r$, we have an isogeny (cf. [H95] Theorem 5.2)

$$\iota_\chi : H^r_{n.ord}(\Gamma^\circ(S_{1,P}(\mathbf{p}^\alpha)), L(\rho\otimes\chi)) \to H^r_{n.ord}(\Gamma^\circ(S_{1,P}(\mathbf{p})), \mathcal{C}^S_\rho)[\chi], \tag{6.9}$$

where "$[\chi]$" indicates the χ-eigenspace in the cohomology group under the action of $T_P(\mathfrak{r}_p)$.

6.3. Universal nearly p-ordinary Hecke algebras. Now we fix the following decomposition given by the approximation theorem:

(App)
$$G(\mathbb{A}) = \bigsqcup_{t\in\Xi} G(\mathbb{Q})tU_{0,P}(\mathbf{p}^\alpha)G(\mathbb{R})_+ \quad \text{and} \quad G(\mathbb{A}) = \bigsqcup_{t\in\Xi}\bigsqcup_{s\in\Xi(S)} G(\mathbb{Q})tsSG(\mathbb{R})_+,$$

where $G(\mathbb{R})_+$ is the identity connected component of the real Lie group $G(\mathbb{R})$ and Ξ (resp. $\Xi(S)$) is a finite subset of $G(\mathbb{A}^{(p\infty)})$ (resp. $U_{0,P}(\mathbf{p}^\alpha)_p$). Let $T_M = Z(M)$ and $T_M^\circ = T_M \cap SL(n)$. Since the reduced norm map $\nu : G \to Res_{F/\mathbb{Q}}\mathbf{G}_m$ induces the isomorphism:

$$\nu : G(\mathbb{Q})\backslash G(\mathbb{A})/SG(\mathbb{R})_+ \cong F^\times\backslash F^\times_{\mathbb{A}}/\nu(S)F^\times_{\mathbb{R}+},$$

we may choose the finite sets Ξ and $\Xi(S)$ so that the following four conditions are satisfied:

(i) Ξ is independent of α;
(ii) $t_p = t_\infty = 1$ for $t \in \Xi$;
(iii) $\Xi(S) \subset U_{0,P}(\mathbf{p}^\alpha)_p$ if $S \supset U_{1,P}(\mathbf{p}^\alpha)$;
(iv) $\Xi(S) \subset \Xi(S')$ if $S' \subset S$.

Let $T_M^\circ = T_M \cap SL(n)$. Since $U_{0,P}(\mathbf{p}^\alpha)/U_{1,P}(\mathbf{p}^\alpha) = T_P(\mathfrak{r}/\mathbf{p}^\alpha) \overset{\det}{\cong} T_M^\circ(\mathfrak{r}/\mathbf{p}^\alpha)$ and by strong approximation theorem (SA) in Section 2, we may approximate each element in $T_M^\circ(\mathfrak{r}_p) \subset U_{0,P}(\mathbf{p}^\alpha)$ sufficiently closely by an element in $G(\mathbb{Q}) \cap tU_{0,P}(\mathbf{p}^\alpha)t^{-1}$, ν induces an isomorphism: $\Xi(U_{1,P}(\mathbf{p}^\alpha))$ onto a quotient C of $T_M(\mathfrak{r}/\mathbf{p}^\alpha)/T_M^\circ(\mathfrak{r}/\mathbf{p}^\alpha) \cong (\mathfrak{r}/\mathbf{p}^\alpha)^\times$. The quotient C is the image of $\mathfrak{r}_p^\times$ in $Cl(\nu(S)) = F^\times\backslash F^\times_{\mathbb{A}}/\nu(S)F^\times_{\mathbb{R}+}$. We write $T_M^\circ(\alpha)$ for the subgroup of $T_M(\mathfrak{r}/\mathbf{p}^\alpha)$ containing $T_M^\circ(\mathfrak{r}/\mathbf{p}^\alpha)$ such that $T_M^\circ(\alpha)\backslash T_M^\circ(\mathfrak{r}/\mathbf{p}^\alpha) \cong C$. This group depends on the choice of U. We then define

$$T_M^\circ(\infty) = \varprojlim{}_\alpha T_M^\circ(\alpha).$$

Thus $T_M^\circ(\infty)$ contains $T_M^\circ(\mathfrak{r}_p)$ as a subgroup, and $T_M^\circ(\infty)/T_M^\circ(\mathfrak{r}_p)$ is isomorphic to the closure of $\nu(\mathfrak{r}_U^\times)$ in $\mathfrak{r}_p^\times$, where $\mathfrak{r}_U^\times = \mathfrak{r}^\times \cap (U\cdot G(\mathbb{R}))$.

For each congruence subgroup $\Gamma \subset G(\mathbb{Q})_+$ for $G(\mathbb{Q})_+ = G(\mathbb{Q})\cap G(\mathbb{R})_+$, we define $\overline{\Gamma}$ for its image in $G(\mathbb{Q})/Z(\mathbb{Q})$ for the center Z of G and $\Gamma^\circ = \Gamma \cap G^\circ(\mathbb{Q})$. Then the natural map $\Gamma \twoheadrightarrow \overline{\Gamma}$ induces a homomorphism $\iota : \Gamma^\circ \to \overline{\Gamma}$ with finite kernel and cokernel killed by n. We put $X(\Gamma) = \overline{\Gamma}\backslash G(\mathbb{R})_+/C_{\infty+}Z(\mathbb{R})$. The space $X(\Gamma)$ is compact if D is a division algebra and is a smooth Riemannian manifold if $\overline{\Gamma}$ is torsion-free.

We define $\Gamma_t(S) = G(\mathbb{Q}) \cap tS \cdot G(\mathbb{R})_+ t^{-1}$ in $G(\mathbb{A})$ for $t \in \Xi$,

$$\overline{\Gamma}_t(S) = \Gamma_t(S)/F^\times \cap \Gamma_t(S) \subset PGL_n(F)$$

and $\Gamma_t^\circ(S) = G^\circ(\mathbb{Q}) \cap tS \cdot G(\mathbb{R})_+ t^{-1}$, which is a congruence subgroup of the derived group $G^\circ(\mathbb{Q})$ of $G(\mathbb{Q})$. Since

$$\Gamma_t(U_{0,P}(\mathbf{p}^\alpha))/\Gamma_t(U_{1,P}(\mathbf{p}^\alpha)) \cong T_M^\circ(\alpha),$$

the cohomology group $H^q(\overline{\Gamma}_t(U_{1,P}(\mathbf{p}^\alpha)), L)$ is naturally a $T_M^\circ(\alpha)$–module. By definition $X(S) \cong \bigsqcup_{t\in\Xi} \bigsqcup_{s\in\Xi(S)} X(\Gamma_t(S))$ via $gtsu \mapsto u_\infty$ for $u \in S \cdot G(\mathbb{R})_+$, and

$$H^q(X(S), \underline{L}) \cong \bigoplus_{t\in\Xi} \bigoplus_{s\in\Xi(S)} H^q(\overline{\Gamma}_t(S), L); \tag{6.10}$$

$$H^q(X_{1,P}(\mathbf{p}^\alpha), \underline{L}) = \bigoplus_{t\in\Xi} ind_{T_M^\circ(\alpha)}^{T_M(\mathfrak{r}/\mathbf{p}^\alpha)} H^q(\overline{\Gamma}_t(U_{1,P}(\mathbf{p}^\alpha)), L), \tag{6.11}$$

where we have written $L = L(\rho \otimes \chi)$ for arithmetic χ dominant with respect to ρ and $X_{1,P}(\mathbf{p}^\alpha) = X(U_{1,P}(\mathbf{p}^\alpha))$. Here the involution "$\iota$" in the definition in 1.1 of the induction "$ind_{T_M^\circ(\alpha)}^{T_M(\mathfrak{r}/\mathbf{p}^\alpha)}$" is the identity map. Thus by (6.11), $H^q(X(U_{1,P}(\mathbf{p}^\alpha)), L)$ is a $T_M(\mathfrak{r}/\mathbf{p}^\alpha)$–module. Even if $X(S)$ is not smooth, we can think of the right hand side of (6.10), which also has a natural action of the double coset algebra R_P. Abusing notation, we hereafter write $H^q(X(S), \underline{L})$ for $\bigoplus_{s\in\Xi(S)} H^q(\overline{\Gamma}_t(S), L)$ if (TF) is not satisfied by S.

We consider

$$H^q(X_{1,P}(\mathbf{p}^\infty), \underline{L}) = \varinjlim_\alpha H^q(X_{1,P}(\mathbf{p}^\alpha), \underline{L}). \tag{6.12}$$

Since the restriction map $res : H^q(X_{1,P}(\mathbf{p}^\alpha), \underline{L}) \to H^q(X_{1,P}(\mathbf{p}^\beta), \underline{L})$ for $\beta > \alpha$ is a morphism of R_P–modules, it is compatible with e_P. Thus e_P acts on $H^q(X_{1,P}(\mathbf{p}^\alpha), \underline{L})$ for $\alpha = 1, 2, \ldots, \infty$, and we have, writing $H^q_{n.ord}(X_{1,P}(\mathbf{p}^\alpha), \underline{L})$ for $e_P H^q(X_{1,P}(\mathbf{p}^\alpha), \underline{L})$,

$$\begin{aligned} H^q_{n.ord}(X_{1,P}(\mathbf{p}^\infty), \underline{L}) &= \varinjlim_\alpha H^q_{n.ord}(X_{1,P}(\mathbf{p}^\alpha), \underline{L}) \\ &= \bigoplus_{t\in\Xi} ind_{T_M^\circ(\infty)}^{T_M(\mathfrak{r}_p)} H^q_{n.ord}(\Gamma_t(U_{1,P}(\mathbf{p}^\infty)), \underline{L}), \end{aligned} \tag{6.13}$$

where $T_M^\circ(\infty) = \varprojlim_\alpha T_M^\circ(\alpha)$ and

$$H^q(\overline{\Gamma}_t(U_{1,P}(\mathbf{p}^\infty)), \underline{L}) = \varinjlim_\alpha H^q(\overline{\Gamma}_t(U_{1,P}(\mathbf{p}^\alpha)), \underline{L}).$$

We define the p–adic Hecke algebra $\mathbf{h}^{n.ord}_{\rho\otimes\chi,q}(U_{1,P}(\mathbf{p}^\infty); \mathcal{O})$ of level $\mathbf{p}^\infty$ by the subalgebra of $\mathrm{End}_{\mathbf{\Lambda}}(H^q_{n.ord}(X_{1,P}(\mathbf{p}^\infty), \underline{L}(\rho \otimes \chi)))$ generated over $\mathbf{\Lambda} = \mathcal{O}[[T_M(\mathfrak{r}_p)]]$ by operators in R_P. By definition, $\mathbf{h}^{n.ord}_{\rho\otimes\chi,q}(U_{1,P}(\mathbf{p}^\alpha); \mathcal{O})$ is an $\mathcal{O}[[T_M(\mathfrak{r}_p)]]$–algebra.

Although in [H95] Section 5, results are formulated using $I_{P,\alpha}$ and Y_α, the result proven there is valid without modification replacing $I_{P,\alpha}$ and Y_α by S_p and Y_S. All arguments, as easily checked, go through without modification. In particular e_P is well defined on $H^q(\Gamma_t(S), \underline{\mathcal{C}}_\rho)$, and by (6.8), we get a canonical isomorphism

(6.14)
$$\iota_\chi : H^q_{n.ord}(\Gamma^\circ_t(U_{1,P}(\mathbf{p}^\infty)), L(\rho \otimes \chi)) \cong H^q_{n.ord}(\Gamma^\circ_t(U_{0,P}(\mathbf{p})), \mathcal{C}_\rho)$$

such that $\iota_\chi(\chi(z)zc) = z\iota_\chi(c)$ for $c \in H^q(\Gamma^\circ_t(U_{1,P}(\mathbf{p}^\infty)), L(\rho \otimes \chi))$ and $z \in T^\circ_M(\mathfrak{r}_p)$.

Let $\mathfrak{r}^\times_+ = \mathfrak{r}^\times \cap Z(\mathbb{A}^{(\infty)}) \cdot G(\mathbb{R})_+$. The quotient

$$H = \mathfrak{r}^\times_+\Gamma_t(U_{1,P}(\mathbf{p}^\alpha))/\mathfrak{r}^\times_+\Gamma^\circ_t(U_{1,P}(\mathbf{p}^\alpha))$$

is independent of $\alpha \gg 0$ and t and is a subgroup of a finite group $\mathfrak{r}^\times_+/(\mathfrak{r}^\times_+)^n$. The finite group H is canonically isomorphic to $\overline{\Gamma}_t(U_{1,P}(\mathbf{p}^\alpha))/\overline{\Gamma}^\circ_t(U_{1,P}(\mathbf{p}^\alpha))$. Similarly, the kernel $H' = \mathrm{Ker}(\Gamma^\circ_t(U_{1,P}(\mathbf{p}^\alpha)) \to \overline{\Gamma}_t(U_{1,P}(\mathbf{p}^\alpha)))$ is finite, of exponent n and independent of $\alpha \gg 0$ and t.

Recall that $L(\rho; \mathcal{O})$ is a representation of S_p (not just that of S°_p). We assume that

1. the central elements in $\mathfrak{r}^\times_U = (\mathfrak{r}^\times_+ \cap U \cdot G(\mathbb{R})_+)$ act trivially on $L(\rho; \mathcal{O})$;
2. $\chi(\mathfrak{r}^\times_U) = 1$.

Thus we may regard $L(\rho; \mathcal{O})$ as a $\overline{\Gamma}_t(U)$–module. We assume that $p \nmid n$. Then by the Hochschild-Serre spectral sequence applied to the exact sequence:

$$1 \to H' \to \Gamma^\circ_t(U_{1,P}(\mathbf{p}^\alpha)) \to \overline{\Gamma}^\circ_t(U_{1,P}(\mathbf{p}^\alpha)) \to 1,$$

we have

(6.15)
$$\iota_\chi : H^q_{n.ord}(\overline{\Gamma}^\circ_t(U_{1,P}(\mathbf{p}^\infty)), L(\rho \otimes \chi)) \cong H^q_{n.ord}(\Gamma^\circ_t(U_{1,P}(\mathbf{p}^\infty)), L(\rho \otimes \chi)),$$

because $H^q(H', X) = 0$ if $q > 0$ and X is a $\mathbb{Z}_p$–module. Thus

$$H^q_{n.ord}(\overline{\Gamma}^\circ_t(U_{1,P}(\mathbf{p}^\infty)), L(\rho \otimes \chi))$$

is independent of χ by (6.14).

Again by Hochschild-Serre spectral sequence applied to the exact sequence:

$$1 \to \overline{\Gamma}^\circ_t(U_{1,P}(\mathbf{p}^\alpha)) \to \overline{\Gamma}_t(U_{1,P}(\mathbf{p}^\alpha)) \to H \to 1,$$

we have

(6.16)
$$\iota_\chi : H^q_{n.ord}(\overline{\Gamma}_t(U_{1,P}(\mathbf{p}^\infty)), L(\rho \otimes \chi)) \cong H^0(H, H^q_{n.ord}(\Gamma^\circ_t(U_{1,P}(\mathbf{p}^\infty)), L(\rho \otimes \chi))).$$

Thus $H^q_{n.ord}(\overline{\Gamma}_t(U_{1,P}(\mathbf{p}^\infty)), L(\rho \otimes \chi))$ is again independent of χ.

The space $\mathcal{C}_\rho$ is naturally a $T_P(\mathfrak{r}_p)$–module as described above. This induces the action of T_P on the right hand side of (6.14). We have a natural isomorphism induced by determinant 1.1: $T_P \cong T^\circ_M$. Thus we can view $\mathcal{C}_\rho$ as a $T^\circ_M(\mathfrak{r}_p)$–module. The unit group $\nu(\mathfrak{r}^\times_U)$ acts trivially on $H^q_{n.ord}(\overline{\Gamma}_t(U_{1,P}(\mathbf{p}^\infty)), L(\rho \otimes \chi))$. Since

$$T^\circ_M(\infty) = \overline{T^\circ_M(\mathfrak{r}_p)(\nu(\mathfrak{r}^\times_U))}$$

in $T_M(\mathfrak{r}_p)$, we can extend the action of $T^\circ_M(\mathfrak{r}_p)$ to $T^\circ_M(\infty)$ making $\nu(\mathfrak{r}_U^\times)$ act on it trivially. Then we have a canonical isomorphism

(6.17)
$$\iota_\chi : H^q_{n.ord}(X_{1,P}(\mathbf{p}^\infty), L(\rho\otimes\chi)) \cong \bigoplus_{t\in\Xi} ind^{T_M(\mathfrak{r}_p)}_{T^\circ_M(\infty)} H^q_{n.ord}(\overline{\Gamma}_t(U_{1,P}(\mathbf{p}^\infty)), L(\rho)),$$

satisfying $\iota_\chi(\chi(z)zc) = z\iota_\chi(c)$ for $z \in T_M(\mathfrak{r}_p)$. This shows

THEOREM 6.1. *Suppose* (D1-2) , $(SL(p))$ *and that* $p \nmid n$. *Then we have an algebra isomorphism*

$$\iota_\chi : \mathbf{h}^{n.ord}_{\rho\otimes\chi,q}(U_{1,P}(\mathbf{p}^\infty);\mathcal{O}) \cong \mathbf{h}^{n.ord}_{\rho,q}(U_{1,P}(\mathbf{p}^\infty);\mathcal{O})$$

taking $T_j(\mathfrak{l})$ *to* $T_j(\mathfrak{l})$ $(1 \le j \le n)$ *for all* $\mathfrak{l} \notin \Sigma \cap \{\mathfrak{p}|p\}$ *and* $\mathbb{T}_j(\mathfrak{p})$ *to* $\mathbb{T}_j(\mathfrak{p})$ $(1 \le j \le r)$. *In particular, if* $P = B$, *the Hecke algebra* $\mathbf{h}^{n.ord}_{\chi,q}(U_{1,P}(\mathbf{p}^\infty);\mathcal{O})$ *is independent of* χ.

Proof. By our construction, the isomorphism in (6.17) is equivariant under Hecke operators listed in the theorem (or more precisely, it is an isomorphism of R_P–modules). Since the Hecke algebra $\mathbf{h}^{n.ord}_{\rho\otimes\chi,q}(U_{1,P}(\mathbf{p}^\infty);\mathcal{O})$ acts faithfully on the cohomology group $H^q_{n.ord}(X_{1,P}(\mathbf{p}^\infty), L(\rho\otimes\chi))$ and generated by these Hecke operators, we get the identity of the algebras by (6.17). □

Let r_1 (resp. r_2) for the number of real (resp. complex) places of F. We write $\mathbf{h}^{n.ord}_q(U_{1,P}(\mathbf{p}^\infty);\mathcal{O})$ for $\mathbf{h}^{n.ord}_{\chi,q}(U_{1,P}(\mathbf{p}^\infty);\mathcal{O})$ if $P = B$. Hereafter we assume $(GL(\infty))$ and that q is equal to the following number r given by

(BD)
$$r = \begin{cases} r_1m^2 + r_2m(n-1) & \text{if } n = 2m, \\ r_1m(m+1) + r_2mn & \text{if } n = 2m+1. \end{cases}$$

The above number r gives the bottom degree of the cuspidal cohomology group of $GL(n)$ under $(GL(\infty))$. Here the word "bottom degree" means that

$$H^j_{cusp}(X(S), L(\rho;\mathbb{C})) = 0 \ \text{ if } j < r, \text{ and}$$
$$H^j_{cusp}(X(S), L(\rho;\mathbb{C})) = 0 \ \text{ for all } j \ \text{ if } \ H^r_{cusp}(X(S), L(\rho;\mathbb{C})) = 0.$$

Then the above explicit value of the bottom degree follows from (3.5) and (3.10).

Under this choice of q, we write $\mathbf{h}^{n.ord}_\rho$ for $\mathbf{h}^{n.ord}_{\rho,r}$. When $P = B$, we simply write $\mathbf{h}^{n.ord}$ for $\mathbf{h}^{n.ord}_\chi$ which is independent of $\chi \in X(Res_{F/\mathbb{Q}}T)$. We write $\mathcal{W} = \mathcal{W}_{P,\rho}$ for

$$\bigoplus_{t\in\Xi} ind^{T_M(\mathfrak{r}_p)}_{T^\circ_M(\mathfrak{r}_p)} H^r_{n.ord}(\overline{\Gamma}_t(U_{0,P}(\mathbf{p})), \mathcal{C}_\rho).$$

Let $W = W_P$ be the Pontryagin dual module of $\mathcal{W}_{P,\rho}$. We put $W(K) = W \otimes_{\mathcal{O}} K$. Then using (6.16), we can deduce from (6.9) the following result:

THEOREM 6.2. *Suppose* (D1-2), $(GL(\infty))$, $(SL(p))$ *and that* $p \nmid n$. *Let* χ *be an arithmetic character of* T_M *dominant with respect to* ρ *such that* $\chi\chi_0^{-1}$ *factors through* $T_M(\mathfrak{r}/\mathbf{p}^\alpha)$ *for* $\chi_0 \in X(T_M)$. *Then for the ideal* $\mathcal{P} = Ker(\chi)$ *in* $\mathcal{O}[[T_M(\mathfrak{r}_p)]]$, *we have*

$$W(K)/\mathcal{P}W(K) \cong H^r_{n.ord}(X(U_{0,P}(\mathbf{p}^\alpha)), L(\rho\otimes\chi;K)).$$

Proof. We have (from [H95] Theorems 5.1 and 5.2) a Hecke equivariant isogeny

$$H^r_{n.ord}(\Gamma^\circ_t(U_{0,P}(\mathbf{p}^\alpha)), L(\rho\otimes\chi)) \to H^r_{n.ord}(\Gamma^\circ_t(U_{1,P}(\mathbf{p}^\infty)), L(\rho))[\chi].$$

We have $[\chi]$ in place of $[\omega_\chi]$ in the theorems in [H95], because present normalization of the action of T_M uses the T_P–action given in (6.5), which is different from the one used in [H95]. Since we have an exact sequence:

$$1 \to H' \to \Gamma^\circ_t(\mathbf{p}^\alpha) \to \overline{\Gamma}_t(\mathbf{p}^\alpha) \to H \to 1$$

with finite group H and H' independent of exponent $\alpha \gg 0$ and of t, we get an isogeny of $T^\circ_M(\mathfrak{r}_p)$–modules:

$$H^r_{n.ord}(\overline{\Gamma}_t(U_{0,P}(\mathbf{p}^\alpha)), L(\rho\otimes\chi)) \to H^r_{n.ord}(\overline{\Gamma}_t(U_{1,P}(\mathbf{p}^\infty)), L(\rho))[\chi].$$

By Shapiro's lemma applied to the right-hand-side of (6.17), we have

$$\mathcal{W}_{P,\rho}[\chi] \cong \bigoplus_{t\in\Xi} H^r_{n.ord}(\overline{\Gamma}_t(U_{1,P}(\mathbf{p}^\infty)), L(\rho))[\chi].$$

This gives rise to the Hecke equivariant isogeny:

$$H^r_{n.ord}(X_{0,P}(\mathbf{p}^\alpha), \underline{L}(\rho\otimes\chi)) \cong \bigoplus_{t\in\Xi} H^r_{n.ord}(\overline{\Gamma}_t(U_{0,P}(\mathbf{p}^\alpha)), L(\rho\otimes\chi)) \to \mathcal{W}_{P,\rho}[\chi].$$

The desired isomorphism is induced from the Pontryagin dual map of the above isogeny, after tensoring K. □

By the above proposition, it is easy to see the Pontryagin dual module $W_{P,\rho}$ of $\mathcal{W}_{P,\rho}$ is of finite type over $\mathcal{O}[[\Gamma_{T_M}]]$ for the maximal torsion-free subgroup Γ_{T_M} of $T_M(\mathfrak{r}_p)/\overline{\mathfrak{r}^\times}$, which is canonically isomorphic to the maximal torsion-free subgroup of $T_M(\mathfrak{r}_p)/\overline{\nu(\mathfrak{r}^\times_U)}$. We simply write $\mathbf{h}$ for $\mathbf{h}^{n.ord}_\rho(U_{1,P}(\mathbf{p}^\infty);\mathcal{O})$. Then the Hecke algebras $\mathbf{h}$ and $\mathbf{h}(K) = \mathbf{h}\otimes_\mathcal{O} K$ act faithfully on W and $W(K)$, respectively. In particular, $\mathbf{h}$ is finite over $\mathcal{O}[[\Gamma_{T_M}]]$. Moreover if $n > 2$ or F has at least one complex place (that is, $r_2 > 0$), we can find by (3.3) and (3.9) a dominant character $\chi \in X(Res_{F/\mathbb{Q}}T)$ such that there is no cuspidal cohomological automorphic representation whose infinity type is χ (see the following Section 7 for more details). Then for $P = B$, $W(K)/\mathcal{P}W(K) = 0$ ($\mathcal{P} = \mathrm{Ker}(\chi)$), and hence $W(K)$ is a torsion $\mathcal{O}[[\Gamma_T]]$–module for the maximal torsion-free subgroup Γ_T of $T(\mathfrak{r}_p)$. Note that $\Lambda = \mathcal{O}[[\Gamma_T]]$ is isomorphic to the power series ring $\mathcal{O}[[T_1,\dots,T_d]]$ for $d = r[F:\mathbb{Q}] + 1 + \delta - r_1 - r_2$, where $\delta = \delta_p$ is the defect of the Leopoldt conjecture given by $\dim_\mathbb{Q}(\mathfrak{r}^\times \otimes_\mathbb{Z} \mathbb{Q}) - \dim_{\mathbb{Q}_p}(\overline{\mathfrak{r}^\times} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$ for the p–adic closure $\overline{\mathfrak{r}^\times}$ in $\mathfrak{r}_p$. Then the Hecke algebra has dimension less than $d+1$ if $n > 2$ or $r_2 > 0$. We thus have

COROLLARY 6.3. *The Hecke algebra $\mathbf{h}^{n.ord}_\rho(U_{1,P}(\mathbf{p}^\infty);\mathcal{O})$ (of bottom degree) is finite over $\mathcal{O}[[\Gamma_{T_M}]]$. In particular,*

$$\begin{aligned}\dim(\mathbf{h}^{n.ord}_\rho(U_{1,P}(\mathbf{p}^\infty);\mathcal{O})) &\le \dim(\mathcal{O}[[\Gamma_{T_M}]])\\ &= \mathrm{rank}(Res_{F/\mathbb{Q}}T_M) + 2 + \delta_p - r_1 - r_2 = (r-1)[F:\mathbb{Q}] + 2 + \delta_p + r_2,\end{aligned}$$

where r_1 (resp. r_2) is the number of real (resp. complex) places of F and P is associated to the partition of n into r parts. If $n > 2$ or F has at least one complex place, then the above inequality is strict for $P = B$.

We now assume

(JL) The global Jacquet-Langlands correspondence compatible with the local correspondence holds for G and $Res_{F/\mathbb{Q}}GL(n)$.

The local Jacquet-Langlands correspondence is known by [DKV] and [R]. The existence of global correspondence is known under a certain ramification condition (see [AC]) Theorem B and [Cl1] Theorem 3.3). In particular, (JL) holds under the conditions (D1-2) (as long as $\Sigma \neq \emptyset$). Under this assumption, the Hecke module $H^i(\overline{\Gamma}, L(\rho;\mathbb{C}))$ can be embedded into $H^i_{sq}(\overline{\Gamma}', L(\rho;\mathbb{C}))$ for a suitable congruence subgroup Γ' of $GL_n(F)$ ([Cl] Section 3.5), where the latter cohomology group is the square integrable cohomology groups. Then from the strong multiplicity one theorem (e.g. [Cl] Theorem 1.1 and [Cl1] Theorem 3.3) valid for $Res_{F/\mathbb{Q}}GL(n)$, we know the strong multiplicity one theorem for G.

As an application of the strong multiplicity one, we have

PROPOSITION 6.4. *Suppose that $U = \widehat{\mathfrak{R}}^\times$ for a maximal order $\mathfrak{R}$ of D satisfying (D1-2), $(SL(p))$ and $(GL(\infty))$, where $\widehat{\mathfrak{R}} = \mathfrak{R}\otimes_{\mathbb{Z}}\widehat{\mathbb{Z}}$. Suppose that the strong multiplicity one theorem holds for cuspidal automorphic representations of $G(\mathbb{A})$. Then we have*

1. *The commutative K-algebra $\mathbf{h}^{n.ord}_\chi(U_{0,B}(\mathbf{p}^\alpha);K)$ is semi-simple;*
2. *The cohomology group*

$$H^r_{n.ord}(X(U_{0,B}(\mathbf{p}^\alpha)), L(\chi;K))$$

is free of finite rank over the Hecke algebra $\mathbf{h}^{n.ord}_\chi(U_{0,B}(\mathbf{p}^\alpha);K)$.

Proof. Since K is a finite extension of $\mathbb{Q}_p$, we can embed K into $\mathbb{C}$ (algebraically). We fix such an embedding. We have a natural action of $C = C_\infty/C_{\infty+} \cong \{\pm 1\}^{r_1}$ on $H^r(X(S), L(\rho\otimes\chi;\mathbb{C}))$ which commutes with Hecke operators. We fix a character $\varepsilon : C \to \{\pm 1\}$ and consider the ε-eigenspace: $H^r(X(S), L(\rho\otimes\chi;\mathbb{C}))[\varepsilon]$. Then this space is isomorphic to a space $\mathcal{S}$ of cusp forms on $G(\mathbb{A})$ invariant under the group S with a fixed infinity type (see (3.5) and (3.10) and [H95] Section 8 for the description of $\mathcal{S}$). We will see later in Corollary 8.3 from p-near ordinarity that if π is an automorphic representation of $G(\mathbb{A})$ intervening in $H^r_{n.ord}(X(U_{0,B}(\mathbf{p})), L(\rho\otimes\chi;K))[\varepsilon]$, the local component $\pi_\mathfrak{p}$ of π at $\mathfrak{p}|p$ can be embedded into $I^{G_\mathfrak{p}}_{B(F_\mathfrak{p})}\lambda$ for a character $\lambda : T(F_\mathfrak{p}) \to K^\times$ with distinct λ^w for all elements w in the Weyl group W of T. By Corollary 5.5, we now know that $V(\pi_\mathfrak{p})^\circ$ is a semi-simple $R(N_B(\mathfrak{r}_\mathfrak{p}), \Delta_\infty)$-module, and each character of $R(N_B(\mathfrak{r}_\mathfrak{p}), \Delta_\infty)$ appears on $V(\pi_\mathfrak{p})^\circ$ at most multiplicity 1. Actually we will see that $e_B V(\pi_\mathfrak{p})^\circ$ is one dimensional (Corollary 8.3). For $\mathfrak{l}\in\Sigma$, since $U_{0,B}(\mathbf{p}^\alpha)_\mathfrak{l}$ is the maximal compact subgroup of $D^\times_\mathfrak{l}$ for a division algebra $D_{\mathfrak{l}/F_\mathfrak{l}}$, the $\mathfrak{l}$-component $\pi_\mathfrak{l}$ of π is a one dimensional representation. If $\mathfrak{l}\notin\Sigma\cup\{\mathfrak{p}|p\}$, $U_{0,B}(\mathbf{p}^\alpha)_\mathfrak{l}$ is a maximal compact subgroup of $GL_n(F_\mathfrak{l})$ and hence $\pi_\mathfrak{l}$ is spherical, and $H^0(U_{0,B}(\mathbf{p}^\alpha)_\mathfrak{l}, V(\pi_\mathfrak{l}))$ is one dimensional. Thus by the strong multiplicity one theorem, $\mathcal{S}$ is a semi-simple module over R_B and each character of R_B has multiplicity at most one. In other words, $H^r_{n.ord}(X(U_{0,B}(\mathbf{p}^\alpha)), L(\rho\otimes\chi;\mathbb{C}))[\varepsilon]$ is free of rank 1 over $\mathbf{h}^{n.ord}_{\rho\otimes\chi}(U_{0,P}(\mathbf{p}^\alpha);K)\otimes_K\mathbb{C}$ (fixing an embedding $K\hookrightarrow\mathbb{C}$), which is semi-simple. This shows the desired result by a descent from $\mathbb{C}$ to K, since $\mathbb{C}$ is a faithfully flat K-module. □

Actually under $(GL(\infty))$ and (D1-2) with $\Sigma\neq\emptyset$, the strong multiplicity one theorem is known (see [AC] Theorem B and [Cl1] Theorem 3.3); so, the above proposition holds.

For simplicity, we write $\mathbf{h}$ for $\mathbf{h}^{n.ord}_{\mathrm{id},r}(U_{1,B}(\mathbf{p}^\infty);\mathcal{O})$. Let $\mathcal{P}$ be the kernel of the algebra homomorphism $\chi : \mathcal{O}[[T(\mathfrak{r}_p)]] \to \mathcal{O})$ induced by χ. We study the localization

$\mathbf{h}_{\mathcal{P}}$. By definition, $\mathbf{h}$ acts faithfully on $\bigoplus_\varepsilon H^r_{n.ord}(X_{1,B}(\mathbf{p}^\infty), K/\mathcal{O})[\varepsilon]$. Thus for an ideal $\mathfrak{a}_\varepsilon$ of $\mathbf{h}$, $\mathbf{h}/\mathfrak{a}_\varepsilon$ acts faithfully on $H^r_{n.ord}(X_{1,B}(\mathbf{p}^\infty), K/\mathcal{O})[\varepsilon]$ for a given $\varepsilon : C \to \{\pm 1\}$. Write H_ε for the Pontryagin dual module of $H^r_{n.ord}(X_{1,B}(\mathbf{p}^\infty), K/\mathcal{O})[\varepsilon]$. Note that $\mathcal{P}\mathbf{h}_{\mathcal{P}} \supset \mathfrak{a}_{\varepsilon,\mathcal{P}}$ for all ε by Proposition 6.4. Since

$$\mathbf{h}_{\mathcal{P}}/\mathcal{P}\mathbf{h}_{\mathcal{P}} \cong \frac{(\mathbf{h}/\mathfrak{a}_\varepsilon\mathbf{h})_{\mathcal{P}}}{\mathcal{P}(\mathbf{h}/\mathfrak{a}_\varepsilon\mathbf{h})_P} \twoheadrightarrow (H_\varepsilon/\mathcal{P}H_\varepsilon)\otimes K \cong H^r_{n.ord}(X_{0,B}(\mathbf{p}^\alpha), L(\chi;K))[\varepsilon]$$

by Theorem 6.2 and Proposition 6.4, choosing $\overline{x} \in H^r_{n.ord}(X_{0,B}(\mathbf{p}^\alpha), L(\chi;K))$ which is the image of $1 \in \mathbf{h}/\mathfrak{a}_\varepsilon$, we can define an $\mathbf{h}$–linear map $\mathbf{h} \to H_\varepsilon$ by $T \mapsto Tx$ for x in H_ε such that $x \mod \mathcal{P} = \overline{x}$. Then by Theorem 6.2, the induced map $\mathbf{h}_{\mathcal{P}} \to H_{\varepsilon,\mathcal{P}}$ is surjective (by Nakayama's lemma). Since the action is faithful, we have $(\mathbf{h}/\mathfrak{a}_\varepsilon)_{\mathcal{P}} \cong W(K)[\varepsilon]_{\mathcal{P}}$, and we get

THEOREM 6.5. *Suppose that $U = \widehat{\mathfrak{R}}^\times$ for a maximal order $\mathfrak{R}$ of D satisfying* (D1-2) $(GL(\infty))$ *and* $(SL(p))$. *For each dominant arithmetic character χ of $T(\mathfrak{r}_p)$, writing $\mathcal{P}$ for* $\mathrm{Ker}(\chi : \mathcal{O}[[T(\mathfrak{r}_p)]] \to \mathcal{O})$, *we have a natural algebra isomorphism:*

$$\left(\mathbf{h}^{n.ord}(U_{1,B}(\mathbf{p}^\infty);\mathcal{O})/\mathcal{P}\mathbf{h}^{n.ord}(U_{1,B}(\mathbf{p}^\infty);\mathcal{O})\right)\otimes_{\mathcal{O}} K \cong \mathbf{h}^{n.ord}_\chi(U_{0,B}(\mathbf{p}^\alpha);K),$$

which takes $T_j(\mathfrak{l})$ to $T_j(\mathfrak{l})$ for all $\mathfrak{l}$ outside Σ.

7. A conjecture on Krull dimension of $\mathbf{h}^{n.ord}$. In this section, we restate the conjecture in the introduction and give supporting arguments in terms of infinity types and functoriality.

7.1. Statement of the conjecture. We have seen in Corollary 6.3 that $\dim \mathbf{h}^{n.ord}$ is finite for $\mathbf{h}^{n.ord} = \mathbf{h}^{n.ord}(U_{1,B}(\mathbf{p}^\infty);\mathcal{O})$. The following conjecture giving the upper bound of the dimension is a slightly stronger version of the conjecture in the introduction:

CONJECTURE 7.1. *Let $\mathbf{h}^{n.ord} = \mathbf{h}^{n.ord}(U_{1,B}(\mathbf{p}^\infty);\mathcal{O})$ for $\rho = \mathrm{id}$. Then we have*

$$\dim(\mathbf{h}^{n.ord}) \le r(n,F)+1$$

where

$$r(n,F) = \begin{cases} m[F:\mathbb{Q}]+1, & \text{if } n = 2m, \\ m[F:\mathbb{Q}]+r_2+1, & \text{if } n = 2m+1. \end{cases}$$

The conjecture is equivalent to the Leopoldt conjecture if $n = 1$. As for $n = 2$, the conjecture is known ([H94a] and [H95]), if we assume the following two conditions:

1. The Leopoldt conjecture holds for F and p;
2. F has at most one complex place: $r_2 \le 1$.

As we will explain more, later in this subsection, the conjecture holds under (D1-2), $(SL(p))$ and $(GL(\infty))$ when $3 \le n \le 4$ and $F = \mathbb{Q}$, because we know from Corollary 6.3 that $\dim(\mathbf{h}^{n.ord}) \le n = r(n;\mathbb{Q})+1$ if $3 \le n \le 4$.

We shall give a heuristic argument for the conjecture in terms of the infinity type of cohomological automorphic representation. We will see some other supporting evidences for this conjecture in the following subsection and also study some implication of the validity of the conjecture.

Let π be a cuspidal cohomological automorphic representation of $G(\mathbb{A})$. For each embedding $\sigma : F \hookrightarrow \mathbb{C}$, we write π_σ for the local component at the infinite place $[\sigma]$ induced by the embedding. Thus $\pi_\sigma = \pi_{c\sigma}$ for complex conjugation c. In Section 3, we have associated to π_σ, its infinity type $\chi_\sigma \in X(T)$ for real embedding σ and a pair $(\chi_\sigma, \chi_{c\sigma}) \in X(T)^2$ for complex σ. Actually χ_σ is an element of positive weights C_n.

We fix an open compact subgroup U of $G(\mathbb{A}^{(\infty)})$ so that $U \supset G(\mathbb{Z}_p)$. Let $\mathcal{X}(U)$ be the set of dominant arithmetic characters $\chi : T(\mathfrak{r}_p) \to K^\times$ of cohomological automorphic representations occurring in $H^r(X(U_{1,B}(\mathbf{p}^\alpha)), \underline{L}(\chi; K))$ for some $\alpha > 0$, where r is the bottom degree defined in (BD). We write $\mathcal{X}_\infty(U)$ for the set of infinity types χ_∞ for $\chi \in \mathcal{X}(U)$. Choosing a complete representative subset $\infty \subset I$ for infinite places of F, we may identify the set $\mathcal{X}_\infty(U)$ with a subset of $X(T)[I] = X(Res_{F/\mathbb{Q}}T)$ by the result of Section 3, where I is the set of embedding of F into $\mathbb{C}$ and $X(T)[I]$ is the set of formal linear combinations of elements in I with coefficients in the abelian group $X(T)$. In other words, decomposing $\infty = \infty(\mathbb{R}) \bigsqcup \infty(\mathbb{C})$ for real embeddings $\infty(\mathbb{R})$ and complex embeddings $\infty(\mathbb{C})$, we have associated to each π the sum

$$\chi_\infty(\pi) = \sum_{\sigma \in \infty(\mathbb{R})} \chi_\sigma \sigma + \sum_{\tau \in \infty(\mathbb{C})} (\chi_\tau \tau + \chi_{c\tau} c\tau).$$

Let W be the Weyl group of T in $GL(n)$ and w be the longest element of W. Then $t \mapsto wt^{-1}w$ is an automorphism of T preserving the positivity with respect to B. For each character $\chi : T(\mathfrak{r}_p) \to \mathcal{O}^\times$, we write $\chi^w(t) = \chi(wtw^{-1})$. For the determinant composed with the norm character, we write $\nu : T(F) \xrightarrow{\det} F^\times \xrightarrow{norm} \mathbb{Q}^\times$. Then by the description in Section 3, we get

(7.1)
$$\mathcal{X}_\infty(U) \subset \mathcal{X} = C_n[I] \bigcap \{\chi \in X(T)[I] | \chi^w + \chi^c = [\chi]\nu \text{ with } [\chi] \in \mathbb{Z}\},$$

where complex conjugation c acts on the values of χ so that $\chi(t)^c = c(\chi(t))$, identifying $X(T) = \mathbb{Z}^n$, ν corresponds to $\sum_\sigma \mathbf{1}\sigma$ with $\mathbf{1} = (1, 1, \ldots, 1)$, and $\chi^c = \sum_{\sigma \in I} \chi_\sigma c\sigma$. Although in Section 3, the property characterizing χ is stated using $\widetilde{\chi} = \chi + \delta - \nu$ for $\delta = \sum_\sigma (n, n-1, \ldots, 1)\sigma$, the description of $\mathcal{X}$ does not change since $\delta^w + \delta^c = (n+1)\nu$.

The set $\mathcal{X}_\infty(U)$ may not coincide with $\mathcal{X}$. For example, if $n = 1$ and F contains no CM fields, $\mathcal{X}_\infty(U) = \mathbb{Z}\nu$. However $\mathcal{X}$ has rank $r_2 + 1$ when $n = 1$. It is interesting to study when $\mathcal{X} \neq \bigcup_U \mathcal{X}_\infty(U)$ happens for general $n > 1$.

Let Γ_T be the maximal p–profinite subgroup of $T(\mathfrak{r}_p)/\overline{\mathfrak{r}^\times}$. Then $\Lambda = \mathcal{O}[[\Gamma_T]]$ is isomorphic to the power series ring of d variables for $d = [F : \mathbb{Q}]n + 1 + \delta_p - r_1 - r_2$. Thus $Spec(\Lambda)(\overline{\mathbb{Q}}_p)$ is the product of d copies of the open unit disk in $\overline{\mathbb{Q}}_p$. Suppose that F is either totally real or a CM field. By Langlands functoriality, for sufficiently large α, we expect (as we will see later) to be able to create a non-trivial nearly p–ordinary π of level p^α with $\chi = \chi(\pi)$ for any given $\chi \in \mathcal{X}$. If this is the case, by Theorem 6.5, the support on $Spec(\mathcal{O}[[T(\mathfrak{r}_p)]] \otimes_\mathbb{Z} \mathbb{Q})$ of $\mathbf{h}^{n.ord} \otimes_\mathbb{Z} \mathbb{Q}$ contains the closure X of $\mathcal{X}$ in $Spec(\Lambda \otimes_\mathbb{Z} \mathbb{Q})$. Since the dimension of the closure X is equal to the $\mathbb{Z}_p$–rank of the p–adic closure of $\mathcal{X}$, which is the rank of the $\mathbb{Z}$–module generated by $\mathcal{X}$. Thus if either $n > 2$ or $r_2 > 0$, the linear span of $\mathcal{X}$ is smaller than $X(T)[I]$, and hence we can find $\chi \in C_n[I]$ which is never an infinity type of a cohomological automorphic representation. This fact is used to show that the Hecke algebra is a torsion Λ–module in Corollary 6.3 under the condition that $n > 2$ or $r_2 > 0$. In particular, if $3 \leq n \leq 4$

and $F = \mathbb{Q}$, it is enough to conclude $\dim \mathbf{h}^{n.ord} \leq 3$; so, the conjecture holds in this case. Anyway it is easy to see that the rank of $\mathcal{X}$ is equal to the upper bound of the dimension in the conjecture, and this gives a reason for the conjecture.

REMARK 7.1. *It is a part of a much general conjecture of Langlands that for each cuspidal cohomological automorphic representation π of $G(\mathbb{A})$, there exists a pure motive M defined over F with coefficients in a number field E such that $L(s,\pi) = L(s, M_{/F})$. Since the infinity type of π determines the gamma factor of $L(s,\pi)$, we can describe the Hodge type of M in terms of $\chi_\infty(\pi)$. The outcome is as follows: Write $\chi(\pi)_\sigma + \delta - \mathbf{1} = (m_{\sigma,1}, \dots, m_{\sigma,n})$. Then the Hodge type of M at the infinite place σ is given by*

$$(7.2) \qquad (m_{\sigma,1}, m_{c\sigma,n}), (m_{\sigma,2}, m_{c\sigma,n-1}), \dots, (m_{\sigma,j}, m_{c\sigma,n-j+1}), \dots.$$

Since M is supposed to be pure, the Hodge numbers (p_i, q_i) of M has to satisfy $p_i + q_i = w$ for the weight w of M. Thus the description of the infinity type of cohomological representations given above is a consequence of the purity of M, admitting Langlands' conjecture.

Fix a complex embedding $i_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Now suppose that F is either a CM field or a totally real field. Thus we have well defined complex conjugation ρ acting on F such that $c \circ i_\infty \circ \sigma = i_\infty \circ \sigma \circ \rho$ for all $\sigma \in I$. If F is totally real, ρ is just the identity map of F. Then for $T^\circ = SL(n) \cap T$, we consider the automorphism $\iota : T^\circ(\mathfrak{r}_p) \to T^\circ(\mathfrak{r}_p)$ given by $\iota(t) = w\rho(t)^{-1}w^{-1}$ for the longest element w of the Weyl group of T. Let Γ° be the torsion-free part of $T^\circ(\mathfrak{r}_p)$ and we write $\Gamma_j^\circ = (\Gamma^\circ)^{p^j}$. The involution $\iota : T^\circ(\mathfrak{r}_p) \to T^\circ(\mathfrak{r}_p)$ induces an involution ι of $\Lambda_j = \mathcal{O}[[\Gamma_j^\circ]]$ into itself. Then Conjecture 7.1 follows from the Leopoldt conjecture for F and p combined with the following vanishing property of the nearly p–ordinary cohomology groups:

CONJECTURE 7.2. *Suppose that F is either totally real or a CM field. Then for any given open compact subgroup $U \subset G(\mathbb{A}^{(\infty)})$ with $U \supset G(\mathbb{Z}_p)$, there exists $j \geq 0$ such that $H^r_{n.ord}(\overline{\Gamma}(U_{1,B}(\mathbf{p}^\infty)), K/\mathcal{O})$ is annihilated by the ideal $\Lambda_j(\iota - 1)\Lambda_j$, where r is the bottom degree for G.*

For any dominant arithmetic character $\chi : T^\circ(\mathfrak{r}_p) \to \mathcal{O}^\times$ we can find $j > 0$ such that $\chi|_{\Gamma_j^\circ}$ is induced by a dominant rational character χ_0 in $X(T^\circ)[I]$. By the definition of ι, the character χ_0 is in $\mathcal{X}$ if and only if the $\mathcal{O}$–algebra homomorphism $\chi : \Lambda_j \to \mathcal{O}$ induced by χ factors through $\Lambda_j/\Lambda_j(\iota-1)\Lambda_j$. Thus if Conjecture 7.2 holds, then the Pontryagin dual module W° of $H^r_{n.ord}(\overline{\Gamma}(U_{1,B}(\mathbf{p}^\infty)), K/\mathcal{O})$ is supported by $Spec(\Lambda_j/\Lambda_j(\iota - 1)\Lambda_j)$, which has relative dimension $r(n,F) - (r_2 + 1)$ over $\mathcal{O}$. It is then clear from the induction process (6.17) that the Pontryagin dual module W of $H^r_{n.ord}(X_{1,B}(\mathbf{p}^\infty)), K/\mathcal{O})$ is supported by the spectrum of

$$(\Lambda_j/\Lambda_j(\iota - 1)\Lambda_j)[[Cl_F(p^\infty)]],$$

whose relative dimension over $\mathcal{O}$ is $r(n,F)+\delta_p$ for the defect of the Leopoldt conjecture δ_p of F.

When F is neither a CM field nor totally real, the situation is rather murky, and even conjecturally, there is no clear-cut description of the annihilator of the cohomology group $H^r_{n.ord}(\overline{\Gamma}(U_{1,B}(\mathbf{p}^\infty)), K/\mathcal{O})$ or the Hecke algebra $\mathbf{h}^{n.ord}(\mathbf{p}^\infty; \mathcal{O})$ except for CM components. The annihilator is computed for CM components of the Hecke algebra in [H94b] Section 5 when $r_2 = 1$ and D is a quaternion algebra positive definite over all real places of F. In this case, the annihilator is directly related to the units of the quadratic extension of F (containing a CM field). We hope to come back this question in a forthcoming paper.

7.2. Compatibility with automorphic functoriality. In the previous subsection, we described how the conjectural dimension of the universal nearly p–ordinary Hecke algebra is deduced by the linear rank of the module of infinity types of cuspidal cohomological automorphic representations. Here we show that the conjectural dimension formula is compatible with some of the Langlands functorialities: tensor products, base-change and automorphic induction.

Since we have a (conjectural) motivic interpretation of cohomological automorphic representations, it would be easier to deal with the set of Hodge weight $(m_{\sigma,1},\ldots,m_{\sigma,n})$ of motives rather than $\mathcal{X}$. We therefore define for the standard diagonal torus $T = T_n \subset GL(n)$, under the notation of Remark 7.1,

$$\mathcal{H}_n = \mathcal{H}_{n/F} = \left\{\chi \in X(T_n)[I] \middle| \chi^c + \chi^w = [\chi]\nu\right\}.$$

Then we associate to each rank n pure motive M a Hodge weight

$$\chi(M) = \sum_{\sigma}(m_{\sigma,1},\ldots,m_{\sigma,n})\sigma \in \mathcal{H}_n$$

if the Hodge type of M is given by (7.2). Then $\mathcal{X}$ gives rise to a spanning cone of $\mathcal{H}_n$ via $\chi \mapsto \chi + \delta - \nu$. We call $\chi = \sum_\sigma (m_{\sigma,1},\ldots,m_{\sigma,n})\sigma \in \mathcal{H}_n$ *regular* if $m_{\sigma,i} \neq m_{\sigma,j}$ for all $i \neq j$ and σ. We call the motive *regular* if $\chi(M)$ is regular. Let $\mathcal{H}_n^+$ be the set of regular elements in $\mathcal{H}_n$. Thus our reason for the conjecture is that regular elements in $\mathcal{H}_n$ span the full module $\mathcal{H}_n$ over $\mathbb{Z}$, and its rank should give the dimension of $\mathbf{h}^{n.ord}$ over K.

Let F/F' be a finite field extension. We write $I_F = \{\sigma : F \hookrightarrow \overline{\mathbb{Q}}\}$ to distinguish I_F and $I_{F'}$. Starting from rank n regular pure motive $M_{/F}$, we consider the restriction of scalar $Res_{F/F'}M$, then

$$\chi(Res_{F/F'}(M)) = \sum_{\sigma\in I_{F'}}\left(\bigoplus_{\tau:Res(\tau)=\sigma}\chi_\tau(M)\right)\sigma.$$

This induces a linear map $Res_{F/F'} : \mathcal{H}_{n/F} \to \mathcal{H}_{n[F:F']/F'}$, and the map is obviously injective. Then comparing the rank of the source and the target, we get the following result

PROPOSITION 7.3. *For a finite extension F/F', we have $r(n;F) \leq r(n[F:F'],F')$, and the equality holds if and only if one of the following conditions is satisfied:*

1. *n is even;*
2. *F is totally imaginary;*
3. *$n[F:F']$ is odd, and there is at most one real place of F over each real place of F'.*

Proof. Let $\mathcal{H} = \{\chi \in \mathcal{H}_n | [\chi] = 0\}$. Then $r(n,F) = \operatorname{rank}\mathcal{H} + 1$. When n is even, the assertion is obvious; so, we may assume that n is odd. We write $r_1(\sigma)$ (resp. $r_2(\sigma)$) for the number of real (resp. complex) places of F over a place σ of F'. Then the contribution of each σ to $\operatorname{rank}\mathcal{H}$ is given by

$$mr_1(\sigma) + (2m+1)r_2(\sigma) = m[F:F'] + r_2(\sigma).$$

Thus $r(n,F) = m[F:\mathbb{Q}] + r_2(F) + 1$. Similarly we can compute

$$r(n[F:F'],F') = \begin{cases} m[F:\mathbb{Q}] + \ell[F':\mathbb{Q}] + 1 & \text{if } [F:F'] = 2\ell, \\ m[F:\mathbb{Q}] + \ell[F':\mathbb{Q}] + r_2(F') + 1 & \text{if } [F:F'] = 2\ell+1. \end{cases}$$

From this, we conclude $r(n;F) \leq r(n[F:F'],F')$ and the equality holds if and only if F satisfies (2) or (3). □

There is some hope that assuming the Leopoldt conjecture for the base field, we might be able to prove Conjecture 7.1 over the given field. Anyway this is the case where $n \leq 2$ and $r_2 \leq 1$. As we will see in 8.1, as long as $\chi(Res_{F/F'}(M))$ remains regular, automorphic induction for F/F' preserves near p–ordinarity if all the prime factors of p in F' split in F. The Hodge type $\chi(Res_{F/F'}(M))$ of the restriction of scalar remains regular for $\chi(M_{/F})$ in a full dimensional cone inside $\mathcal{H}_{n/F}$ if F is either a CM field or a totally real field. If F is neither a CM field nor a totally real field, preserving regularity requirement imposes a strong restriction. Anyway, by the above proposition, if automorphic induction exists for $GL(n)$, we can reduce the general (totally real) case of the conjecture to the special case where the Leopoldt conjecture is valid for F':

COROLLARY 7.4. *Let L be a number field, and we choose a real abelian extension $F'/\mathbb{Q}$ so that every prime factor of p of F' splits in $F = LF'$. Then the Leopoldt conjecture for the number field L follows from Conjecture 7.1 for F' and p under the following conditions:*

1. *L is totally real;*
2. *Automorphic induction exists for cuspidal cohomological automorphic representations from $GL(2)_{/F}$ to $GL(2[F:F'])_{/F'}$.*

Proof. It is easy to see that the Leopoldt conjecture for F and p implies that of L and p; so, we may assume that $L = F$. Let G be the algebraic group $Res_{F'/\mathbb{Q}}(D')^\times$ for a division algebra $D'_{/L}$ satisfying (D1-2) with $\dim_{F'} D' = 4[L:F']^2$. We write $H = Res_{L/\mathbb{Q}} D^\times$ for a quarternion algebra $D_{/L}$ satisfying the following conditions:

(a) D is unramified at every finite place;

(b) D is either totally definite or $D \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R}) \times \mathbb{H} \times \cdots \times \mathbb{H}$.

Then we look at the universal nearly p–ordinary Hecke algebra $\mathbf{h} = \mathbf{h}^{n.ord}(N\mathbf{p}^\infty; \underline{\mathcal{O})}$ for H defined in [H89]. Let Γ be the maximum torsion-free subgroup of $T_H(\mathbb{Z}_p)/\overline{\mathfrak{r}^\times}$ for the p–adic closure $\overline{\mathfrak{r}^\times}$ in $\mathfrak{r}_p^\times$. Then $\mathrm{rank}_{\mathbb{Z}_p}\Gamma = [L:\mathbb{Q}] + 1 + \delta$ for the defect of the Leopoldt conjecture δ for (L,p).

Let $\mathbb{P}$ be a minimal prime ideal of $\mathbf{h}$ and put $\mathbb{I} = \mathbf{h}/\mathbb{P}$. We call an $\mathcal{O}$–algebra homomorphism $P : \mathbb{I} \to \mathcal{O}$ arithmetic if P induces a dominant arithmetic character $\chi_P : T_H(\mathfrak{r}_p) \to \mathcal{O}^\times$. The projection $\mathbf{h} \to \mathbb{I}$ induces, for each arithmetic $\mathcal{O}$–algebra homomorphism $P : \mathbb{I} \to \mathcal{O}$, an $\mathcal{O}$–algebra homomorphism $\mathbf{h}/\operatorname{Ker}(\chi_P)\mathbf{h} \to \mathcal{O}$, which in turn gives rise to the Hecke eigenvalue system of an automorphic representation $\pi(P)$ of $H(\mathbb{A})$. For a place $\mathfrak{l}$ of F prime to p, it is known that if $\pi(P)_\mathfrak{l}$ is special (resp. super-cuspidal) for one point P, then $\pi(P)_\mathfrak{l}$ is special (resp. super-cuspidal) for all P, because the corresponding Galois representation restricted to the inertia group $I_\mathfrak{l}$ is rigid if $\mathfrak{l} \nmid p$. For $\mathfrak{p}|p$, $\pi(P)_\mathfrak{p}$ is always principal except possibly when $\chi_P = 0$, for which it could be special. For a given cohomological automorphic representation π of $H(\mathbb{A})$ of cohomological weight 0, if $\pi_\mathfrak{p}$ for all $\mathfrak{p}|p$ is special, then π is automatically p–ordinary. Computing limit multiplicity by the trace formula, we can find π such that π is of cohomological weight 0 and $\pi_\mathfrak{l}$ is special for a given finite set $\Sigma \cup \{\mathfrak{p}|p\}$ of primes. Then we can find $\mathbb{I}$ as above so that $\pi(P) = \pi$ for a prime ideal P of $\mathbb{I}$.

Now we choose Σ so that

1. every $\mathfrak{l} \in \Sigma$ is totally split in L/F';
2. If $\mathfrak{l}$ is a prime ideal of F' with non-split $D'_\mathfrak{l}$, Σ contains all places of F over $\mathfrak{l}$.

Then the automorphic representation $\pi(P)$ has automorphic induction $\Pi(P)$ in cuspidal (cohomological) automorphic representations of G, because $\Pi(P)_\mathfrak{l}$ for $\mathfrak{l} \in \Sigma$ is a

Steinberg representation and hence square integrable ([B] Section 5).

Let $\Lambda_j = \mathcal{O}[[\Gamma^{p^j}]]$. We choose a regular weight $\chi \in \mathcal{X}$ (which is a square in $X(T_H)$) for a p–split torus T_H of H so that $\chi' = Res_{F/F'}\chi$ is still regular. Then $\mathbf{h} \otimes_{\Lambda_j,\chi} K$ is the Hecke algebra of finite level. We find an open compact subgroup $U \subset G(\mathbb{A}^{(\infty)})$ and have an algebra homomorphism $\Pi_j : \mathbf{h}^{n,ord}_{\chi'}(U_{1,B}(\mathbf{p}^{\alpha(j)}), K) \to \mathbb{I} \otimes_{\Lambda_j,\chi} K$ for suitable level $\alpha(j)$ depending on j such that $P \circ \Pi_j$ corresponds to $\Pi(P)$ for P factoring through $\mathbb{I} \otimes_{\Lambda_j,\chi} K$. The group U is independent of j if we choose it sufficiently small. Let $\mathbf{h}' = \mathbf{h}^{n.ord}(U_{1,B}(\mathbf{p}^\infty); \mathcal{O})$. Since we have a natural $\mathcal{O}$–algebra homomorphism $\mathbf{h}' \to \mathbf{h}^{n,ord}_{\chi'}(U_{1,B}(\mathbf{p}^{\alpha(j)}), K)$ induced by the isomorphism in Theorem 6.2, we may pull back Π_j to a unique algebra homomorphism $\Pi_j : \mathbf{h}' \to \mathbb{I} \otimes_{\Lambda_j,\chi} K$ (still denoted by Π_j). We have a Galois representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/L) \to GL_2(\mathbb{I})$ such that $Tr(\rho(Frob_{\mathfrak{l}})) = T_1(\mathfrak{l})$ for almost all primes $\mathfrak{l}$ of L. Thus Π_j actually has values in $\mathbb{I} \otimes_{\Lambda_j,\chi} \mathcal{O}$, because the image is generated by the coefficients of the characteristic polynomial of $ind_F^{F'} \rho(Frob_{\mathfrak{l}})$. Taking the projective limit of Π_j with respect to j, we get $\Pi : \mathbf{h}' \to \mathbb{I}$. Since $\mathbb{I}$ is generated by Hecke operators $T_1(\mathfrak{l})$ for almost all primes $\mathfrak{l}$, and image of the map Π is generated by $Tr(ind_L^{F'} \rho(Frob_{\mathfrak{l}}))$. From this, it is clear that $\mathbb{I}$ is an $\mathbf{h}'$–module of finite type. It is known by [H89] that $\dim \mathbb{I} = [L : \mathbb{Q}] + 2 + \delta$. This combined with Conjecture 7.1 shows

$$[L : \mathbb{Q}] + 2 + \delta = \dim \mathbb{I} \leq \dim \mathbf{h} = [F' : \mathbb{Q}][L : F'] + 2.$$

Thus $\delta = 0$ and the Leopoldt conjecture for L follows. □

REMARK 7.2. *If Conjecture 7.1 is valid for the split $GL(n)_{/F'}$ (instead of G associated to division algebra D'), we can apply the same argument as in the proof of the corollary to $GL(1)_{/L}$ for a CM extension L/F' and $GL([L : F'])_{/F'}$ in place of $H_{/L}$ and $G_{/F'}$. The result is the same, that is, the Leopoldt conjecture for (L, p) follows from Conjecture 7.1 for $GL([L : F'])_{/F'}$. The difficulty of applying the conjecture for G associated to a division algebra is that the automorphic induction image of an arithmetic Hecke character of L has to be super-cuspidal at ramified places of D', which is impossible if the place split in L/F' (which is not always but often the case). Anyway the Leopoldt conjecture for a CM field L is equivalent to that for its maximal totally real field; so, this case is basically covered by Corollary 7.4. Anyway, this remark shows that the function $(n, F) \mapsto r(n, F)$ is the minimal assignment of dimensions so that Proposition 7.3 holds for the given $r(1, F)$ predicted by the Leopoldt conjecture.*

We look into the base-change functoriality from $Res_{F'/\mathbb{Q}}GL(n)$ to $Res_{F/\mathbb{Q}}GL(n)$ for a finite extension F/F'. Thus the functoriality is induced by $\pi(M_{/F'}) \mapsto \pi(M_{/F})$ for pure regular motive $M_{/F'}$ of rank n. The corresponding linear map of the infinity types of π (or Hodge types of M) is the the inflation map $Inf_{F/F'} : \mathcal{H}_{n/F'} \to \mathcal{H}_{n/F}$ given by $Inf_{F/F'}(\chi)_\sigma = \chi_{\sigma|_{F'}}$. This is a representation theoretic dual of $Res_{F/F'}$. Since $\mathcal{H}_{n/F}$ is naturally a Galois module, we see easily that $\mathrm{Im}(Inf_{F/F'}) = H^0(\mathrm{Gal}(\overline{\mathbb{Q}}/F'), \mathcal{H}_{n/F'})$, and $Inf_{F'/F}$ is obviously injective. We see that $r(2, F) - r(1, F) = r_1 + r_2$. Thus we have

PROPOSITION 7.5. *We have $r(n, F) \geq r(n, F')$ for a finite extension F/F'. If F is a CM field with maximal real subfield F', $r(2, F) - r(1, F) = r(2, F') - r(1, F')$. Thus under the Leopoldt conjecture for F' and p, Conjecture 7.1 for $(F, n = 2, p)$ implies that of $(F', n = 2, p)$.*

We consider the linear map $\boxplus : \mathcal{H}_{n/F} \oplus \mathcal{H}_{\ell/F} \to \mathcal{H}_{n\ell/F}$ given by $(m_i) \oplus (n_j) \mapsto$

$(m_i - n_j)_{0\le i\le n, 0\le j\le \ell}$ fixing an order of the indices (i,j). This is associated to

$$\chi(M) \oplus \chi(N) \mapsto \chi(M \otimes \check{N})$$

for the dual motive $\check{N}$ of N. It is easy to see that $\mathrm{Ker}(\boxplus) = (\sum_\sigma \mathbb{Z}(\mathbf{1}_n, \mathbf{1}_\ell)\sigma) \cap (\mathcal{H}_n \oplus \mathcal{H}_\ell)$, where $\mathbf{1}_n = \overbrace{(1,1,\dots,1)}^{n}$. Take $x = \sum_\sigma x_\sigma(\mathbf{1}_n, \mathbf{1}_\ell)\sigma \in \mathrm{Ker}(\boxplus)$. Then we have $x^c + x^w = \sum_\sigma (x_\sigma + x_{c\sigma})(\mathbf{1}_n, \mathbf{1}_\ell)\sigma = [x]\nu$. This shows $2x_\sigma = [x]$ for real place σ and $x_\sigma + x_{c\sigma} = [x]$ for complex places σ. Therefore we have $\mathrm{rank}_{\mathbb{Z}} \mathrm{Ker}(\boxplus) = r_2 + 1$, and

PROPOSITION 7.6. *Suppose that $n \ge \ell \ge 1$. We have*

$$\mathrm{rank}_{\mathbb{Z}} \mathrm{Im}(\boxplus) = r(n;F) + r(\ell : F) - r_2 - 1 \le r(\ell n; F),$$

and the equality holds only when either $n = \ell = 2$ and $r_2 = 0$ or $\ell = 1$.

Iterating the above process, we can think of $\chi_1 \boxplus \cdots \boxplus \chi_k \in \mathcal{H}_{n_1 \dots n_k/F}$ for $(\chi_1, \dots \chi_k) \in \mathcal{H}_{N_1/F} \oplus \cdots \oplus \mathcal{H}_{n_k/F}$. This map corresponds to the tensor product of k–motives: $(M_1, \dots, M_k) \mapsto M_1 \otimes M_2 \otimes \cdots \otimes M_k$, and by induction on k, we get:

$$\mathrm{rank}_{\mathbb{Z}}(\mathrm{Im}((\chi_1, \dots \chi_k) \mapsto \chi_1 \boxplus \cdots \boxplus \chi_k)) \le r(n_1 \times n_2 \times \cdots \times n_k; F).$$

Thus our conjecture is compatible with tensor functoriality.

8. Ordinarity condition via Newton polygon. In this section, we study what admissible representations of $GL_n(F_\mathfrak{p})$ give rise to the p–component of a P–nearly ordinary automorphic representation. We fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. Then we fix two embeddings $i_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $i_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. For each field embedding $\sigma : F \hookrightarrow \overline{\mathbb{Q}}$, we write ∞_σ for the infinite place associated to $i_\infty \circ \sigma : F \to \mathbb{C}$ and $\mathfrak{p}_\sigma$ for the p–adic place associated to $i_p \circ \sigma$.

8.1. Newton polygon and Hodge polygon. We fix one embedding $\sigma : F \hookrightarrow \overline{\mathbb{Q}}$ and an irreducible cohomological automorphic representation π of $G(\mathbb{A})$ of infinity type χ. We recall the (modified) induced module $I_B^G(\lambda) = Ind_{B(F_{\mathfrak{p}_\sigma})}^{G_{\mathfrak{p}_\sigma}} \widetilde{\lambda}$ defined in Section 5 for a continuous character $\lambda : T(F_{\mathfrak{p}_\sigma}) \to \overline{\mathbb{Q}}^\times$ (with respect to the discrete topology on $\overline{\mathbb{Q}}$): Writing $\lambda(diag(t_1, \dots, t_n)) = \prod_{j=1}^n \lambda_j(t_j)$, the modified character $\widetilde{\lambda}$ is given by

$$\widetilde{\lambda}(diag(t_1, \dots, t_n)) = \prod_{j=1}^n \lambda_j(t_j)|t_j|_{\mathfrak{p}_\sigma}^{j-1},$$

where we normalize the $\mathfrak{p}_\sigma$–adic absolute value so that $|\varpi|_{\mathfrak{p}_\sigma} = |\mathfrak{r}_{\mathfrak{p}_\sigma}/\varpi\mathfrak{r}_{\mathfrak{p}_\sigma}|^{-1}$ for a prime element $\varpi \in \mathfrak{r}_{\mathfrak{p}_\sigma}$. We assume that

1. $\mathfrak{p}_\sigma \notin \Sigma$ and $\pi_{\mathfrak{p}_\sigma} \hookrightarrow I_B^G(\lambda)$ for a continuous character $\lambda : T(F_{\mathfrak{p}_\sigma}) \to \overline{\mathbb{Q}}^\times$;
2. π_∞ is cohomological and associated to a dominant character $\chi \in \mathcal{X}$ as in 3.2 and 3.3.

When π is nearly p–ordinary with respect to B, we know that the condition (1) is satisfied by (5.5). We write $[\mathfrak{p}_\sigma]$ for the set of embedding $\tau : F \hookrightarrow \overline{\mathbb{Q}}$ such that $\mathfrak{p}_\tau = \mathfrak{p}_\sigma$.

We define the (reciprocal) Hecke polynomial of π by

$$H_{\mathfrak{p}_\sigma}(T) = \prod_{i=1}^n (1 - \lambda_i(\varpi)T) = \sum_{s=0}^n (-1)^s |\varpi|_{\mathfrak{p}_\sigma}^{-s(s-1)/2} a_s T^s. \tag{8.1}$$

By [BZ1] 2.9, if $\pi_{\mathfrak{p}_\sigma} \hookrightarrow I_B^G(\lambda')$ for another character λ', then $\lambda' = \lambda^w$ for an element $w \in W$; so, $H_{\mathfrak{p}_\sigma}(T)$ is well defined independent of the choice of λ. When $\pi_{\mathfrak{p}_\sigma}$ is unramified (and hence $\pi_{\mathfrak{p}_\sigma} \cong I_B^G(\lambda)$), this polynomial is the Hecke polynomial of the Hecke operators $T_j(\mathfrak{p})$ (without modification, see (6.6)) as in [Sh] Theorem 3.21. However, when $\pi_{\mathfrak{p}_\sigma}$ is ramified, the above polynomial differs from the one constructed in [Sh] by these Hecke operators.

Let $V = V(\pi_{\mathfrak{p}_\sigma})$, and write V_B° for the semi-simple part for $\mathbb{T}_B = \prod_{j=1}^n T_j(\varpi)$ as in Section 5. Then as seen in Section 5, V_B° is canonically isomorphic to the Jacquet module V_B, which is a semi-simple T-module. Write $V_B^\circ[\widetilde{\lambda}] \subset V_B^\circ$ for the space corresponding to the $\widetilde{\lambda}$-eigensubspace $V_B[\widetilde{\lambda}] \subset V_B$. Then $V_B^\circ[\widetilde{\lambda}]$ is an eigenspace of $T_s(\varpi)$, and its eigenvalue is given by $|\varpi|_{\mathfrak{p}_\sigma}^{s(s-1)/2} \prod_{j=1}^s \lambda_{n-s+j}(\varpi)$ (see Corollary 5.5).

Let P be a standard parabolic subgroup associated with a partition $n = n_1 + n_2 + \cdots + n_r$ of n into r–parts. Define a tuple $S = (0 = s_0 \le s_1 \le s_2 \le \cdots \le s_r)$ of $r+1$ integers by $n_{r-i} = s_{i+1} - s_i$ for all $i = 0, \cdots, r$. We take an embedding $\sigma : F \hookrightarrow \overline{\mathbb{Q}}$ and write $e = e_\sigma$ for the ramification index of $F_{\mathfrak{p}_\sigma}/\mathbb{Q}_p$. Then for the cohomological weight $\chi = \sum_\tau (m_{\tau,1}, m_{\tau,2}, \ldots, m_{\tau,n})\tau \in \mathcal{X} \subset X(T)[I]$ of π (thus $m_{\tau,1} \ge m_{\tau,2} \ge \cdots \ge m_{\tau,n}$), we define the Hodge P–polygon $\Delta_P = \Delta_{P,\sigma}$ by the convex hull of the following vertices:

$$(8.2)\qquad \left\{ (s_{\sigma,i}, e_\sigma^{-1} \sum_{\tau \in [\mathfrak{p}_\sigma]} \sum_{j=n-s_i+1}^{n} (m_{\tau,j} + n - j)) \big| i = 0, 1, \ldots, r \right\}.$$

In particular, the slope $\mu_{\sigma,i} = \mu_i$ of the i-th edge of the polygon $\Delta_{P,\sigma}$ is given by

$$(8.3)\qquad \mu_{\sigma,i} = \frac{\sum_{j=n-s_i+1}^{n-s_{i-1}} \sum_{\tau \in [\mathfrak{p}_\sigma]} (m_{\tau,j} + n - j)}{e_\sigma n_{r-i+1}}.$$

On the other hand, we define the Newton P–polygon $\Delta_{P,\sigma}^N$ of π by the convex hull of the vertices over S (in the horizontal x–axis) of the Newton polygon of $H_{\mathfrak{p}_\sigma}(T)$, which is above or on the Newton polygon of $H_{\mathfrak{p}_\sigma}(T)$ and coincides with the Newton polygon only when $P = B$.

8.2. Newton polygon is above Hodge polygon. We keep the notation introduced in the previous subsection. Let $S_\sigma = (s_0 = s_{\sigma,0}, \ldots, s_r = s_{\sigma,r})$ be the maximal tuple of integers in increasing order such that

$$|i_p(a_s)|_p = \left| i_p \left(\prod_{\tau \in [\mathfrak{p}_\sigma]} \chi_\tau(\xi_s) \right) \right|_p$$

for the p–adic absolute value $|\ |_p$ of $\overline{\mathbb{Q}}_p$, where $\xi_s = diag(1, \ldots, 1, \overbrace{\varpi, \ldots, \varpi}^{s}) \in GL_n(F_{\mathfrak{p}_\sigma})$. In particular, we have $s_0 = 0$ and $s_r = n$ because $a_0 = 1$.

From the data S_σ, we would like to determine the parabolic subgroup P which is minimal among parabolic subgroups Q so that π is Q–ordinary. Let $n = n'_1 + n'_2 + \cdots + n'_{r'}$ be the partition associated to Q. Define a tuple $S_Q = (s'_i)_{i=0,\ldots,r'}$ by $n'_{r'-i} = s'_{i+1} - s'_i$ for all $i = 0, \ldots, r'-1$ with $s'_0 = 0$. Since a_s is the sum of eigenvalues of $T_s(\varpi)$ on $V_B^\circ \cong V_B$ (by Corollary 5.5), we conclude from (6.6) and (6.7) that

$$(8.4)\qquad \pi \text{ is } Q\text{–ordinary} \iff S_Q \subset S_\sigma.$$

Here we remark that ω in (6.6) and (6.7) is given by χ^{-1} by our definition of the algebraic induction (6.4). Then we have

THEOREM 8.1. *Let the assumption and the notation be as above. Then we have*

1. *If the cohomological automorphic representation π of infinity type $\chi \in \mathcal{X}$ is nearly Q-ordinary at $\mathfrak{p}_\sigma | p$ for a standard parabolic subgroup Q, then the Newton Q-polygon of $H_{\mathfrak{p}_\sigma}(T)$ coincides with the Hodge Q-polygon $\Delta_{Q,\sigma}$ defined above;*
2. *If $\pi_{\mathfrak{p}_\sigma} \cong I_B^G(\lambda)$, the parabolic subgroup P determined by S_σ is the smallest standard parabolic subgroup for which π is nearly P-ordinary at $\mathfrak{p}_\sigma$;*
3. *For every irreducible cohomological automorphic representation π with $\pi_{\mathfrak{p}_\sigma}$ embedded into $I_B^G(\lambda)$ and having infinity type χ, the Newton polygon of the Hecke polynomial $H_{\mathfrak{p}_\sigma}(T)$ is on or above the Hodge B-polygon $\Delta_{B,\sigma}$.*

Proof. We have already proven the assertion (1) and (2). By the definition of the idempotent e_B, we know that

$$|i_p(a_s)|_p \leq \left| i_p \left(\prod_{\tau \in [\mathfrak{p}_\sigma]} \chi_\tau(\xi_s) \right) \right|_p .$$

This shows that Δ_B is under or on the Newton polygon of $H_{\mathfrak{p}_\sigma}(T)$. □

REMARK 8.1. *A general conjecture (due to Langlands) is that for each cohomological cuspidal automorphic representation π, there should exist a pure motive $M_{/F}$ of rank n such that $L(s,\pi) = L(s,M)$. Since we can write down the Γ-factor of $L(s,\pi)$ explicitly by the infinity type of π, we can predict the Hodge type of $M \times_{F,i_\infty \sigma} \mathbb{C}$, which would be given by*

$$((m_{\sigma,1}, m_{\sigma,n}), (m_{\sigma,2}, m_{\sigma,n-1}), \ldots, (m_{\sigma,j}, m_{\sigma,n-j+1}), \ldots). \tag{8.5}$$

Thus if p totally splits in F, Δ_B is exactly the Hodge polygon of M at the infinite place ∞_σ. If π is unramified at $\mathfrak{p}_\sigma$, we expect that M is crystalline at $\mathfrak{p}_\sigma$. Then the crystalline realization of M at $\mathfrak{p}_\sigma$ has the crystalline Frobenius map Φ. Although Φ is not a linear map, its power $\phi = \Phi^{[\mathfrak{r}/\mathfrak{p}_\sigma:\mathbb{F}_p]}$ has a well defined characteristic polynomial $H_{cris,\sigma}(T)$ of degree n. A standard conjecture is that the crystalline characteristic polynomial $H_{cris,\sigma}(T)$ should coincide with the characteristic polynomial $H_{\mathfrak{p}_\sigma}(T)$ of the Frobenius at $\mathfrak{p}_\sigma$ of an $\mathfrak{l}$-adic étale realization of M ($\mathfrak{l} \nmid p$). Thus by a well known result of Fontaine and Mazur, if p completely splits in F, the Newton polygon of $H_{\mathfrak{p}_\sigma}(T) = H_{cris,\sigma}(T)$ is on or above the Hodge polygon Δ_B. This gives a philosophical explanation of the above theorem 8.1. The above theorem is hence a bit stronger than this geometric fact, because (i) Δ_B is really above the Hodge polygon if p does not completely split in F, and (ii) the theorem also gives an information even when π ramifies at $\mathfrak{p}_\sigma$ (which should corresponds to a non-crystalline motive).

Since near p-ordinarity of a cuspidal cohomological automorphic representation (with respect to the Borel subgroup) implies that $\pi_{\mathfrak{p}_\sigma}$ for all $\mathfrak{p}_\sigma$ is a subrepresentation of an induced representation, as a direct consequence of the above theorem, we see

COROLLARY 8.2. *For cuspidal cohomological automorphic representations, we write tensor product functoriality as $\pi(M) \boxtimes \pi(M') = \pi(M \otimes M')$ from $GL(m) \times GL(\ell)$ to $GL(n\ell)$, automorphic induction functoriality as $\pi(M) \mapsto \pi(Res_{F/F'} M)$ from $Res_{F/\mathbb{Q}} GL(n)$ to $Res_{F'/\mathbb{Q}} GL([F:F']n)$ and base-change functoriality as $\pi(M_{/F'}) \mapsto \pi(M_{/F})$ for a finite extension F/F'. Then, as long as the functorial image remains cohomological (that is, the infinity type of the image is regular), automorphic induction* *preserves near p-ordinarity (with respect to B) provided that all prime factors of p in F' split in F, and tensor product and base-change preserve near p-ordinarity (with respect to B) unconditionally.*

The following corollary guarantees semi-simplicity of the nearly p–ordinary Hecke algebra:

COROLLARY 8.3. *Let N be the unipotent radical of B. Suppose that π is nearly ordinary at $\mathfrak{p}_\sigma$ with respect to B. Then $e_B H^0(N(\mathfrak{r}_{\mathfrak{p}_\sigma}), V(\pi_{\mathfrak{p}_\sigma}))$ is a one dimensional space on which $B(F_{\mathfrak{p}_\sigma})$ acts by a character λ such that $|\lambda|_p = \left|\prod_{\tau\in[\mathfrak{p}_\sigma]}\chi_\tau\nu_\tau^{-1}\delta_\tau\right|_p$, where $\delta = \sum_\sigma (n, n-1, \dots, 1)\sigma$ and $\nu = \sum_\sigma \mathbf{1}_n\sigma$.*

Proof. By near B–ordinarity, we have from the argument in Section 5 that $\pi_{\mathfrak{p}_\sigma}$ is a subquotient of $I_B^G(\lambda)$ for a character $\lambda : T(F_{\mathfrak{p}_\sigma}) \to \overline{\mathbb{Q}}^\times$. By near B–ordinarity, we know from the theorem that we can choose λ so that $|\lambda|_p = \left|\prod_{\tau\in[\mathfrak{p}_\sigma]}\chi'_\tau\right|_p$ for $\chi' = \chi\nu^{-1}\delta$. Writing $\chi'_\tau = (m_{\tau,1}, \dots, m_{\tau,n})$, we have $m_{\tau,1} > m_{\tau,2} > \cdots > m_{\tau,n}$ for all $\tau \in [\mathfrak{p}_\sigma]$. Thus if $w \neq 1$,

$$\prod_{s=1}^{n} |\lambda^w(\xi_s)|_p < \left|\prod_{\tau\in[\mathfrak{p}_\sigma]}\prod_{s=1}^{n}\chi'_\tau(\xi_s)\right|_p .$$

In particular, λ^w for $w \in W$ are all distinct. Then by Proposition 5.4, we know that $e_B V_B \subset \bigoplus_{w\in W}\widetilde{\lambda^w}$ as $B(F_{\mathfrak{p}_\sigma})$–modules for $V = V(\pi_{\mathfrak{p}_\sigma})$, and as we have already seen in Corollary 5.5 and (4.6), for $\widetilde{\lambda^w}$–eigenvector $v \in V_B$, we have

$$v|\mathbb{T}_s(\varpi) = \left(\prod_{\tau\in[\mathfrak{p}_\sigma]}\chi_\tau(\xi_s)\right)^{-1}|\varpi|_{\mathfrak{p}_\sigma}^{-ts}\widetilde{\lambda^w}(\xi_s)v = \left(|\lambda(\xi_s)|_p^{-1}\lambda^w(\xi_s)\right)v.$$

Thus from $|\lambda^{-1}\lambda^w(\xi_s)|_p < 1$ for at least one s if $w \neq 1$, we conclude that e_B kills the $\widetilde{\lambda^w}$–eigenspace if $w \neq 1$. Therefore $e_B V_B$ is at most 1–dimensional and actually is equal to the $\widetilde{\lambda}$–eigenspace under the above isomorphism. Since $V_B^\circ \cong V_B$ as $R(B, \Delta_\infty)$–modules (Proposition 5.1), this shows the desired assertion. □

Appendix A. Semi-simplicity of Hecke algebras for reductive groups. We can generalize from cuspidal cohomology of $GL(n)$ to interior cohomology of general split reductive groups the argument which proves semi-simplicity of the nearly ordinary Hecke algebra of p–power level, which we describe here.

Let $G_{0/F}$ be a connected reductive group over a number field F, and we put $G = Res_{F/\mathbb{Q}}G_0$. If G_0 is split over $F_p = F \otimes_{\mathbb{Q}} \mathbb{Q}_p$, we shall prove semi-simplicity of the commutative Hecke algebra acting on the nearly ordinary cohomology group $H^q_{n.ord}(X(U), L) \subset H^q_!(X(U), L)$ for a modular variety $X(U)$ associated to an arbitrary p–power level open compact subgroup U of $G(\mathbb{A}^{(\infty)})$. Here the locally constant sheaf L on $X(U)$ is associated to a rational representation of G, and $H^q_!$ indicates the image of the compactly supported cohomology group in the cohomology group without the support condition.

There could be two ways of proving the semi-simplicity. The first one is a bit far-fetched: We interpolate p–adically the cohomology group $H^q_{n.ord}(X(U), L)$ varying rational representations L, getting a space V, which is a module of finite type over the Iwasawa algebra $\Lambda = \mathcal{O}[[T(\mathbb{Z}_p)]]$ of the torus $T = Res_{F/\mathbb{Q}}T_0$ for an F_p–split torus

$T_0 \subset G_0$. Then one proceed to prove that $V_\chi = V \otimes_{\Lambda,\chi} \mathcal{O}$ for each dominant weight χ is isomorphic to $H^q_{n.ord}(X(U), L)$ for the rational representation of highest weight χ^{-1} and the maximal compact subgroup $U \subset G(\mathbb{A}^\infty)$. If we find densely populated χ (in $Spec(\Lambda)(\mathcal{O})$) with the above specialization property, the Hecke algebra $\mathbf{h}$ of V (over Λ) specializes along χ to the semi-simple one of level 1, and hence $\mathbf{h}$ must be reduced. Then by specializing $\mathbf{h}$ to p–power level Hecke algebra, we get the desired semi-simplicity for almost all such specializations. This method has been exploited for $GSp(2g)$ in [TU] and probably works well for reductive groups G yielding Shimura varieties. However this method is ineffective to prove semi-simplicity for all χ and all p–power level. Also this method is probably not feasible for general G whose modular variety does not have complex structure, because in such cases, $\mathbf{h}$ is a torsion Λ–module.

We should emphasize here semi-simplicity (or unramifiedness) at arithmetic primes of the universal nearly ordinary Hecke algebra h is important in constructing p–adic L–functions on the spectrum $Spec(h)$ and relating its values with complex L–values of automorphic L–functions.

In earlier works of the author, the semi-simplicity of such Hecke algebras for $GL(2)_{/F}$ is proven using the theory of old and new forms. A key point of this method is to prove one-dimensionality of nearly ordinary vectors in each irreducible (cohomological) automorphic representation, carefully analyzing old vectors. The purpose of this paper is to prove directly the semi-simplicity for all arithmetic characters $\chi : T(\mathbb{Z}_p) \to \overline{\mathbb{Q}}_p^\times$ and all reductive G split at p. We prove via the theory of Jacquet modules of local automorphic representations ([BZ] and [BZ1]) that the nearly ordinary vector is unique up to constant multiple if the representation is irreducible. This is a generalization of the argument in Section 5 to general reductive groups. Since the automorphic representation occurring in the cohomology group is unramified outside p, the one-dimensionality gives rise to the semi-simplicity. Our final result is Corollary A.4.

A.1. Jacquet modules for reductive groups. Let G be a split connected reductive group over a finite extension F of $\mathbb{Q}_p$. Let π be an admissible representation of $G(F)$ on a vector space V over a field K of characteristic different from p. We suppose that G acts on V from the right. Fix a Borel subgroup B with split torus $T = B/N$ for the unipotent radical N. A parabolic subgroup $P \supset B$ is called standard. We fix a standard parabolic subgroup P with unipotent radical N_P.

Since the characteristic of K is different from p, we have a Haar measure of $N_P(F)$. We then define

$$V(P) = V(P,\pi) = \{v - v\pi(n) \in V(\pi) \big| v \in V(\pi),\ n \in N_P(F)\},$$

and put $V_P = V_P(\pi) = V/V(P)$, which is called the Jacquet module. We take a sufficiently large open compact subgroup $U_w \subset N(F)$ for each $w = v - v\pi(n) \in V(P)$ so that $n \in U_w$. Then we see that $\int_U v\pi(u)du = 0$ for every open subgroup U of $N(F)$ containing U_w. Write $U_0 = N(\mathcal{V})$ for the p–adic integer ring $\mathcal{V} \subset F$, and choose the Haar measure du so that $\int_{U_0} du = 1$. We also choose an increasing sequence of open compact subgroups U_i indexed by $0 \le i \in \mathbb{Z}$ such that $\bigcup_i U_i = N(F)$. Then the map

$$v \mapsto \lim_{i\to\infty} (U_i : U_0)^{-1} \int_{U_i} v\pi(u)du$$

gives rise to a section of $V \twoheadrightarrow V_P$. Thus $V \cong V_P \oplus V(P)$ canonically, and the association $V \mapsto V_P$ is an exact functor from the category of admissible representations of $G(F)$ into the category of admissible $T(F)$–modules.

PROPOSITION A.1. *Suppose K has characteristic 0 and that π is absolutely irreducible. Let W be the Weyl group of T in G. Then we have $\dim_K V_B \leq |W|$ for each Borel subgroup B.*

Proof. By extending scalar, we may assume that K is algebraically closed. If $V_B = 0$, there is nothing to prove. Thus we suppose that $V_B \neq 0$. Then we can find a character $\lambda : T(F) \to K^\times$ which gives the representation of $T = B/N_B$ on a simple factor of V_B. Then π is a factor of $\mathrm{Ind}_{B(F)}^{G(F)} \lambda$ by Frobenius reciprocity [BZ1] 2.3. By [BZ1] 2.8, the length of the composition series of V_B as $T(F)$–modules is then bounded by $|W|$. Since T is abelian, this shows that $\dim_K V_B \leq |W|$. □

For each character $\lambda : T(F) \to K^\times$ and $w \in W$, we write λ^w for another character of T given by $\lambda^w(t) = \lambda(wtw^{-1})$.

COROLLARY A.2. *Suppose that $V_B[\widetilde{\lambda}] \neq 0$, where $\widetilde{\lambda} = \delta_B^{1/2}\lambda$ for the right module character δ_B on B. Then $\pi \subset \mathrm{Ind}_{B(F)}^{G(F)} \widetilde{\lambda}$. If $\lambda^w(t)$ for $w \in W$ are all distinct, $V_B \subset \bigoplus_{w \in W} \widetilde{\lambda^w}$ as $T(F)$–modules.*

The proof is exactly the same as that of Proposition 5.4; so, we leave it to the reader.

A.2. Double coset algebras. Let

$$D = \{x \in T(F) | xN_B(\mathcal{V})x^{-1} \supset N_B(\mathcal{V})\}$$

be the expanding sub-semigroup of $T(F)$. When $G = GL(n)$, this semi-group is exactly equal to the one $\mathcal{D}_B$ defined in (4.1). Let Σ be the set of simple positive roots of T with respect to B. Then for each subset $\Theta \subset \Sigma$, we have a parabolic subgroup $P_\Theta = BW_\Theta B$ for the subgroup W_Θ of the Weyl group W generated by reflections associated to $\Sigma - \Theta$. Traditionally the above standard parabolic subgroup is denoted by $P_{\Sigma-\Theta}$ (e.g. [J] II.1.8), but we use the symbol P_Θ for that. Write L_Θ for the Levi subgroup of P_Θ and write Z_Θ for the identity connected component of the center of L_Θ. We consider the orthogonal complement

$$X_*(\Theta) = \{\xi \in X_*(T) | \theta(\xi) = 0 \ \forall \theta \in \Sigma - \Theta\}$$

in $X_*(T) = \mathrm{Hom}_{alg-gp}(\mathbf{G}_m, T)$. Then $X_*(\Theta) = X_*(Z_\Theta)$ and $\mathrm{rank}_{\mathbb{Z}} X_*(\Theta) = |\Theta| + \mathrm{rank}\, Z$ for the center $Z \subset G$. For each $\alpha \in \Sigma$, we therefore find a unique generator $\xi_\alpha \in X_*(\{\alpha\})$ modulo $X_*(Z)$ so that $\xi_\alpha(\varpi) \in D$ for a prime element $\varpi \in \mathcal{V}$. Then D is generated by $\xi_\alpha(\varpi)$ for $\alpha \in \Sigma$ and $T(\mathcal{V})Z(F)$. We write D_0 for the sub-semigroup of D generated by $\{\xi_\alpha(\varpi) | \alpha \in \Sigma\}$ and $Z(F)$. We write simply $P_\alpha = P_{\{\alpha\}}$ and $L_\alpha = L_{\{\alpha\}}$.

We consider $\Delta = \Delta_B = B(\mathcal{V})DB(\mathcal{V})$, which is a sub-semigroup of $G(F)$. For simplicity, we write B for $B(\mathcal{V})$ and N for $N_B(\mathcal{V})$. Since D commutes with T, for $\xi \in D$,

$$B = \bigsqcup_{u \in N/\xi^{-1}N\xi} (\xi^{-1}B\xi \cap B)u \quad \text{and} \quad N = \bigsqcup_{u \in N/\xi^{-1}N\xi} (\xi^{-1}N\xi \cap N)u.$$

This shows

$$\text{(A.1)}\quad N\xi N = \bigsqcup_{u\in N/\xi^{-1}N\xi} N\xi u = \bigsqcup_{u\in \xi N\xi^{-1}/N} Nu\xi \text{ and}$$
$$B\xi B = \bigsqcup_{u\in N/\xi^{-1}N\xi} B\xi u = \bigsqcup_{u\in \xi N\xi^{-1}/N} Bu\xi.$$

In particular, $N\xi N/N \cong B\xi B/B$.

We now consider the double coset algebra $R = R(N,\Delta)$ spanned over $\mathbb{Z}$ by NxN for $x \in D$. We let R act on $v \in V^N = H^0(N(\mathcal{V}), V)$ by

$$\text{(A.2)}\qquad v|[N\xi N] = \sum_{u\in N/\xi^{-1}N\xi} v\pi(\xi u) = \int_{\xi N\xi^{-1}} v\pi(\xi)du.$$

Let $\{\xi_\alpha(\varpi)\}_{\alpha\in\Sigma}$ be the generators of D modulo center. Then for $\xi = \prod_\alpha \xi_\alpha$, we have

$$N(F) = \cup_{j=0}^{\infty}\xi^j N\xi^{-j}.$$

Thus writing $T(\xi) = N\xi N$ as an operator on V, we see easily from (A.1) that $T(\xi^j) = T(\xi)^j$ and $T(\xi)$ is nilpotent on $V(B)^N$ by (A.2).

Let ${}^tN$ be the opposite unipotent subgroup of N. We put $U_1(r) = {}^tN(r)T(r)N$, where $X(r)$ is the kernel of the reduction map: $X(\mathcal{V}) \to X(\mathcal{V}/\varpi^r\mathcal{V})$ for an algebraic group $X_{/\mathcal{V}}$. Then $U_1(r)$ is a decreasing sequence of open compact subgroups of $G(F)$ with $\cap_r U_1(r) = N$ and $U_1(r)\xi U_1(r) = \bigsqcup_{u\in\xi N\xi^{-1}/N} U_1(r)u\xi$ for $\xi \in D$. Thus $V^N = \bigcup_r H^0(U_1(r), V)$, and the finite dimensional space $H^0(U_1(r), V)$ is stable under $T(\xi)$. Thus we can decompose $T(\xi) = s + n$ for commuting sum of a unique nilpotent operator n and a unique semi-simple operator s first on each $H^0(U_1(r), V)$ and then over the union $V^N = \bigcup_r H^0(U_1(r), V)$. Thus we find $V^N = V(B)^N \oplus s(V^N)$ and the projection $V \twoheadrightarrow V_B$ is injective on $s(V^N)$.

Since $v\pi(\xi^{-j})$ is N–invariant for a sufficiently large j, the projection $V^N \to V_B$ is surjective. Thus $s(V^N) \cong V_B$, we get a R–linear isomorphism

$$V^N \cong V(B)^N \oplus V_B.$$

Let $\lambda : T(F) \to K^\times$ be a character intervening in the $T(F)$–module V_B. Since T is abelian, we can always find a subspace $V_B[\lambda]$ on which $T(F)$ acts via the character λ. We write Ad_N for the adjoint representation of $Res_{F/\mathbb{Q}_p}(T)$ on the Lie algebra of $Res_{F/\mathbb{Q}}(N)$. Then $\det(Ad_N(x)) = \rho^2(x)$ for the half sum of positive roots ρ with respect to $Res_{F/\mathbb{Q}_p}(B)$. Since N acts trivially on V_B, we have

$$\text{(A.3)}\quad v|[NxN] = [N : x^{-1}Nx]\lambda(x)v = |\det(Ad_N(x))|_p\lambda(x)v = |\rho^2(x)|_p\lambda(x)v,$$

where "$|\ |_p$" is the standard p–adic absolute value such that $|p|_p^{-1} = p$ and Ad is the adjoint representation of $Res_{F/\mathbb{Q}_p}(T)$ on the Lie algebra of $Res_{F/\mathbb{Q}_p}(N)$.

A.3. Rational representations of G. We now suppose that F is a number field and G_0 is a reductive group defined over F split at $F_\mathfrak{p}$ for all primes $\mathfrak{p}|p$. We then consider $G = Res_{F/\mathbb{Q}}G_0$. We suppose that G_0 is actually defined over $O_p = O_F \otimes_\mathbb{Z} \mathbb{Z}_p$ as a smooth (relatively connected) split group scheme over O_p, where O_F

is the integer ring of F. Then G is a generic fibre of $Res_{O_p/\mathbb{Z}_p}G_{0/O_p}$. We sometimes write $G(\mathbb{Z}_p)$ and $G_0(O_\mathfrak{p})$ for $\mathbb{Z}_p$-points (resp. $O_\mathfrak{p}$-points) of these schemes defined over p-adic integers. We fix a split Borel subgroup $B_0 \subset G_{0/O_p}$ with unipotent radical N_0, and write $B_\mathfrak{p} = B_0(F_\mathfrak{p})$ whose unipotent radical $N_\mathfrak{p}$ is given by $N_0(F_\mathfrak{p})$. We define $B = Res_{O_p/\mathbb{Z}_p}B_0$, $N = Res_{O_p/\mathbb{Z}_p}N_0$ and $T = B/N = Res_{O_p/\mathbb{Z}_p}T_0$ for $T_0 = B_0/N_0$. Let $G(\widehat{\mathbb{Z}}) \subset G(\mathbb{A}^\infty)$ denote a maximal compact subgroup (by abusing notation) as maximal as possible (this means that we assume a local component of $G(\widehat{\mathbb{Z}})$ to be hyperspecial if one need to assume it in order to assure that the spherical representation has only one fixed vector under the maximal compact; see [T] 3.8 and [Ca] III). We assume that the p-component of $G(\widehat{\mathbb{Z}})$ is given by $G(\mathbb{Z}_p)$.

We write $D_\mathfrak{p}$ (resp. $D_{0,\mathfrak{p}}$) for D (resp. D_0) with respect to $B_\mathfrak{p}$ and the split torus $T_\mathfrak{p} = B_\mathfrak{p}/N_\mathfrak{p}$. Let $\mathbf{p}$ be the Jacobson radical of $O_p = O_F \otimes_\mathbb{Z} \mathbb{Z}_p$ for the integer ring O_F of F. Thus $\mathbf{p} = \prod_{\mathfrak{p}|p} \mathfrak{p}O_p$. Define

(A.4) $$U_0(r) = \left\{u \in G(\mathbb{Z}_p) \middle| u \mod \mathbf{p}^r \in B_0(O/\mathbf{p}^r O)\right\}$$
$$U_1(r) = \left\{u \in G(\mathbb{Z}_p) \middle| u \mod \mathbf{p}^r \in N_0(O/\mathbf{p}^r O)\right\}.$$

We fix a subgroup S such that $U_0(r) \supset S \supset Z(\mathbb{Z}_p)U_1(r)$ for $r > 0$. Then

(A.5) $$\Delta = \Delta_p = U_0(r)\Big(\prod_{\mathfrak{p}|p} D_{0,\mathfrak{p}}\Big)U_0(r) = U_0(r)\Big(\prod_{\mathfrak{p}|p} D_\mathfrak{p}\Big)U_0(r) \textit{ is a semi-group.}$$

We consider for a topological module A, the space of continuous functions: $\mathcal{C}(A) = \mathcal{C}_S(A) = \{\phi : S/Z(\mathbb{Z}_p)N(\mathbb{Z}_p) \to A\}$. We would like to make $\mathcal{C}(A)$ a left Δ_p^{-1}-module for the opposite semi-group Δ_p^{-1}. For that, we first define a left action of Δ_p on $Y_S = S/Z(\mathbb{Z}_p)N(\mathbb{Z}_p)$. Since S acts on $Y_S = S/Z(\mathbb{Z}_p)N(\mathbb{Z}_p)$ from left and $U_0(r) = ST(\mathbb{Z}_p)$, we only need to define a left action of $D_0 = \prod_{\mathfrak{p}|p} D_{0,\mathfrak{p}}$. Pick $y \in S$ and consider $yN(\mathbb{Z}_p)$. Then for $d \in D_0$, $dyN(\mathbb{Z}_p)d^{-1} = dyd^{-1}dN(\mathbb{Z}_p)d^{-1} \subset dyd^{-1}N(\mathbb{Q}_p)$ and $dyd^{-1}N(\mathbb{Q}_p)$ is well defined in $G(\mathbb{Q}_p)/Z(\mathbb{Z}_p)N(\mathbb{Q}_p)$. Writing ${}^tN$ for the opposite unipotent subgroup of G, we have the Iwahori decomposition $U_0(r) = U'T(\mathbb{Z}_p)N(\mathbb{Z}_p)$ and $S = U'T'N(\mathbb{Z}_p)$ for open subgroups $U' \subset {}^tN(\mathbb{Z}_p)$ and $T' \subset T(\mathbb{Z}_p)$. Then we see $dU'd^{-1} \subset U'$ and $dN(\mathbb{Z}_p)d^{-1} \supset N(\mathbb{Z}_p)$ by the definition of D. This shows that the coset $dyd^{-1}N(F) \cap S$ is well defined single coset of $N(\mathbb{Z}_p)$, which we designate to be the image of the action of $d \in D_0$. This action extends to that of the semi-group Δ by an obvious way. We now let Δ_p^{-1} act on $\mathcal{C}_S(A)$ by $d\phi(y) = \phi(d^{-1}y)$. In this way, $\mathcal{C}_S(A)$ becomes a Δ_p^{-1}-module.

We now fix a finite extension K over $\mathbb{Q}_p$ which contains all conjugates of F in $\overline{\mathbb{Q}}_p$. Let $\mathcal{O}$ be the p-adic integer ring of K. We now assume that A is either an $\mathcal{O}$-module of finite or co-finite type or a vector space over K. We put the p-adic topology on module of finite type and vector spaces over K and the discrete topology on module of co-finite type. We consider the quotient scheme G/N defined over K, and its structure sheaf $\mathcal{O}_{G/N}$. We consider the algebraic induction module:

(A.6) $$L(\chi; K) = \left\{(\phi : G/N \to K) \in H^0(G/ZN, \mathcal{O}_{G/N}) \middle| \phi(yt) = \chi(t)\phi(y)\ \forall t \in T\right\},$$

where $\chi \in X(T) = \mathrm{Hom}_{alg-gp}(T, \mathbf{G}_m)$. We let G act on $L(\chi; K)$ by $g\phi(y) = \phi(g^{-1}y)$. Then $L(\chi; K) = \mathrm{Ind}_B^G \chi^{-1}$, which is the induction in the category of scheme theoretic representations (that is, polynomial representations). We write this representation as $\rho_\chi : G \to GL(L(\chi; K))$.

Suppose that χ is trivial on the center for the moment. We restrict functions in $L(\chi; K)$ to $Y_S = S/Z(\mathbb{Z}_p)N(\mathbb{Z}_p)$ and get an embedding $L(\chi; K) \hookrightarrow \mathcal{C}_S(K)$. The image is stable under the action of Δ_p^{-1} and for $\xi \in D^{-1}$ and $u \in N(\mathbb{Z}_p)$,

$$(u\xi)\phi(y) = \phi(\xi^{-1}u^{-1}y) = \chi(\xi)\rho((u\xi))\phi(y) \text{ for } \phi \in L(\chi; K).$$

When we call $L(\chi; K)$ a Δ_p^{-1}–module, we take the action induced by $\mathcal{C}_S(A)$. When we call $L(\chi; K)$ a G–module, we take the action given by ρ_χ. The two actions differ by scalar. We then define $L_S(\chi; \mathcal{O}) = \mathcal{C}_S(\mathcal{O}) \cap L(\chi; K)$, which is a Δ_p^{-1}–module (but not a $G(\mathbb{Q}_p)$–module). By definition, the action of Δ_p^{-1} factors through $\Delta_p^{-1}/Z(\Delta_p^{-1})$ for the center $Z(\Delta_p^{-1})$ of Δ_p^{-1}. We then define $L_S(\chi; K/\mathcal{O}) = L_S(\chi; \mathcal{O}) \otimes_{\mathcal{O}} K/\mathcal{O}$. For each character $\varepsilon : T(O/\mathbf{p}^r) \to \mathcal{O}^\times$, we may regard ε as a character of $U_0(r)$ because $U_0(r)/U_1(r) \cong T_0(O/\mathbf{p}^r)$. Then we define a Δ_p^{-1}–module by

$$\text{(A.7)} \quad L(\chi\varepsilon; \mathcal{O}) = \varepsilon L(\chi; \mathcal{O}) \subset \mathcal{C}_{U_0(r)}(\mathcal{O}) \text{ and } L(\chi\varepsilon; K/\mathcal{O}) = L(\chi\varepsilon; \mathcal{O}) \otimes_{\mathcal{O}} K/\mathcal{O} \subset \mathcal{C}_{U_0(r)}(K/\mathcal{O}).$$

Let $\chi : T(\mathbb{Z}_p) \to \mathcal{O}^\times$ be a general character which may not be trivial on Z but induces an algebraic character χ_0 on an open neighborhood of the identity. We can always find an algebraic character $\psi \in X(G) = \mathrm{Hom}_{alg-gp}(G, \mathbf{G}_m)$ such that $\psi|_Z = \chi_0^h$ for some positive integer h. We take a character $\psi_0 : G(\mathbb{Q}_p) \to \mathcal{O}^\times$ such that $\psi_0^h = \psi$ and $\psi_0 = \chi$ on an open neighborhood of the identity in $Z(\mathbb{Z}_p)$. Then we have a unique subspace in $\mathcal{C}_S(K) \otimes \psi_0$ isomorphic to $L(\chi; K)$ as S_p–modules, which we again denote by $L(\chi; K)$. We fix such a ψ_0 and define $\mathcal{C}_S(\psi, A)$ by $\mathcal{C}(A) \otimes \psi_0$. The choice of ψ_0 does not matter for our purpose. We then define $L(\chi; \mathcal{O}) = L(\chi; K) \cap \mathcal{C}_S(\psi; \mathcal{O})$ and $L(\chi; K/\mathcal{O}) = L(\chi; \mathcal{O}) \otimes K/\mathcal{O}$. These are well defined Δ_p^{-1}–modules. The action of Δ_p^{-1} may depend on the choice of ψ_0, but the difference is only a unit-scalar multiple (in $\mathcal{O}$).

A.4. Nearly p–ordinary representations. Let U be an open subgroup of $G(\widehat{\mathbb{Z}})$. We consider the associated modular variety:

$$X(U) = G(\mathbb{Q})\backslash G(\mathbb{A})/UC_{\infty+}Z(\mathbb{A}),$$

where $C_{\infty+}$ is the identity connected component of the maximal compact subgroup of the Lie group $G(\mathbb{R})$ and $Z \subset G$ is the center of G. We fix a character $\chi : T(\mathbb{Z}_p) \to \mathcal{O}^\times$, which is algebraic on a neighborhood of the identity. We assume that $\varepsilon = \chi\chi_0^{-1}$ factors through $T_0(O/\mathbf{p}^r)$ for $\chi_0 \in X(T)$. We call χ dominant if $L(\chi; K) \neq 0$. Then we fix a p–adic Hecke character $\widehat{\psi} : Z(\mathbb{A})/Z(\mathbb{Q})Z(\mathbb{R})_+ \to \mathcal{O}^\times$ such that $\widehat{\psi}$ coincides with χ on $T(\mathbb{Z}_p) \cap Z(\mathbb{Q}_p)$. This condition tells us that $\widehat{\psi}$ is a p–adic avatar of a complex Hecke character $\psi : Z(\mathbb{A})/Z(\mathbb{Q}) \to \mathbb{C}^\times$. This means that $i_\infty^{-1}(\psi(x)) = i_p^{-1}(\widehat{\psi}(x))$ for all $x \in Z(\mathbb{A})$ with $x_p = x_\infty = 1$ and the ∞–type of ψ is given by $\chi_0|_Z$. We then define a right action, if $U_p \subset U_0(r)$ of $uz \in UC_{\infty+}Z(\mathbb{A})$ ($u \in UC_{\infty+}$ and $z \in Z(\mathbb{A})$) on $L(\chi; A)$ by $\phi|uz = \widehat{\psi}(z)\varepsilon(u_p)\rho_\chi(u_p^{-1})\phi$. The action is well defined since $\chi = \varepsilon\chi_0 = \widehat{\psi}$ on $Z(\mathbb{Q}_p) \cap U$. We write this right $Z(\mathbb{A})U$–module by $L(\chi, \widehat{\psi}; A)$.

We define the covering space $\mathcal{X}(U)$ of $X(U)$ by

$$\text{(A.8)} \qquad \mathcal{X}(U) = G(\mathbb{Q})\backslash(G(\mathbb{A}) \times L(\chi, \widehat{\psi}; A))/UC_{\infty+}Z(\mathbb{A}),$$

where $\gamma(x, \phi)u = (\gamma xu, \phi|u)$ for $\gamma \in G(\mathbb{Q})$ and $u \in UC_{\infty+}Z(\mathbb{A})$. We use the same symbol $L(\chi, \widehat{\psi}; A)$ for the sheaf of locally constant sections of $\mathcal{X}(U)$ over $X(U)$.

We consider the limit, shrinking U,

$$\mathcal{L}(A) = \mathcal{L}^q(\chi, \widehat{\psi}; A) = \varinjlim_U H^q_!(X(U), L(\chi, \widehat{\psi}; A)). \tag{A.9}$$

On the space $\mathcal{L}(K)$ the group $G(\mathbb{A}^\infty)$ acts from the right via a smooth representation, which is completely reducible. The complete reducibility follows from the fact that the interior cohomology $H^q_!$ is embedded in the square-integrable cohomology and hence the representation over $\mathbb{C}$ is unitary ([Cl] 3.17-18). Here we regard $L(\chi, \widehat{\psi}; K)$ as a $G(\mathbb{Q}_p)$–module via the representation ρ_χ. Thus in particular, we have an action on $H^0(U, \mathcal{L}^q(\chi, \widehat{\psi}; K)) = \mathcal{L}^q(\chi, \widehat{\psi}; K)^U$ of the double coset algebra $R_U = R(U^{(p)} \times B(\mathbb{Z}_p), G(\mathbb{A}^{p\infty}) \times \Delta_p)$ of double cosets UxU with $x \in G(\mathbb{A}^{p\infty}) \times \Delta_p$, where $U = U_p \times U^{(p)}$ and we have assumed that $U_p \subset U_0(1)$.

As already described, we have a unique co-character $\xi_\alpha : \mathbf{G}_m \to T$ for each $\alpha \in \Sigma$ such that $\xi_\alpha(\varpi)$ ($\alpha \in \Sigma$ and parameters ϖ of $F_\mathfrak{p}$) gives a minimum set of generators of $D_\mathfrak{p}$ modulo $Z(F_p)$. We consider the double coset $T(\alpha) = U\xi_\alpha(p)U$. The double coset acts on $\mathcal{L}(\chi, \widehat{\psi}; A)$ for $A = \mathcal{O}$ and $K/\mathcal{O}$ via the Δ_p^{-1}–module structure on $L(\chi, \widehat{\psi}; A)$. The corresponding operator will be written as $\mathbb{T}(\alpha)$. The operator $\mathbb{T}(\alpha)$ is determined (up to unit multiples in $\mathcal{O}$) independently of the choice of ξ_α, because $Z(\Delta_p^{-1})$ acts on $L(\chi, \widehat{\psi}; A)$ by a character ψ_0 with values in $\mathcal{O}^\times$, and $\xi_\alpha(\varpi)$ is unique modulo center. We then make a product $\mathbb{T} = \prod_{\alpha\in\Sigma} \mathbb{T}(\alpha)$.

Similarly, the double coset $T(\alpha)$ acts on $\mathcal{L}^q(\chi, \widehat{\psi}; K)^{N(\mathbb{Z}_p)}$ through the action of $G(\mathbb{A}^\infty)$. The corresponding operator will be written by the same symbol $T(\alpha)$. Then we put $T = \prod_{\alpha\in\Sigma} T(\alpha)$. Since the action of $G(\mathbb{A}^\infty)$ is defined using the G–action ρ_χ on $L(\chi; K)$. Then the two operators $\mathbb{T}(\alpha)$ and $T(\alpha)$ are related on the image of $\mathcal{L}^q(\chi, \widehat{\psi}; \mathcal{O})^{N(\mathbb{Z}_p)}$ by

$$\mathbb{T}(\alpha) = \chi(\xi_\alpha(p))^{-1} T(\alpha) \quad \text{up to } p\text{–adic units if } \chi|_Z \neq 1. \tag{A.10}$$

The limit $e = \lim_{n\to\infty} \mathbb{T}^{n!}$ exists on $H^q(X(U), L(\chi, \widehat{\psi}; A))$ for $A = \mathcal{O}$, K and $K/\mathcal{O}$ (see Section 2). Thus the limit e exists on $\mathcal{L}^q(\chi, \widehat{\psi}; A)^{N(\mathbb{Z}_p)}$ for $A = \mathcal{O}$, $K/\mathcal{O}$ and K. It is easy to see, if $U_p \supset N(\mathbb{Z}_p)$,

$$H^0(U, e\mathcal{L}^q(\chi, \widehat{\psi}; K)^{N(\mathbb{Z}_p)}) = e\left(H^q(X(U), L(\chi, \widehat{\psi}; K))\right). \tag{A.11}$$

We write $\mathcal{L}^q_{n.ord}(\chi, \widehat{\psi}; A)$ for $e\mathcal{L}^q(\chi, \widehat{\psi}; A)^{N(\mathbb{Z}_p)}$. An irreducible representation π of $G(\mathbb{A}^\infty)$, which is a subquotient of $\mathcal{L}^q(\chi, \widehat{\psi}; K)$, is called *nearly ordinary* of p–type χ if $e\left(V(\pi)^{N(\mathbb{Z}_p)}\right) \neq 0$ for the representation space $V(\pi)$ of π. For a subset $\Theta \subset \Sigma$, we can think of $\mathbb{T}(\Theta) = \prod_{\alpha\in\Theta} \mathbb{T}(\alpha)$ and $e_\Theta = \lim_{n\to\infty} \mathbb{T}_\Theta^{n!}$. If $e_\Theta V(\pi) \neq 0$, we call π *nearly* Θ*–ordinary* or *nearly* P_Θ*–ordinary.*

A.5. Semi-simplicity of interior cohomology groups. Recall that 2ρ is the sum of positive root of T with respect to B. If π is a local component of a nearly p–ordinary representation of p–type χ, then for its p–component π_p (acting on V), its Jacquet module $V_B \neq 0$, and hence, by Corollary A.2 and (A.3), we find a character $\lambda : T(\mathbb{Q}_p) \to K^\times$ such that $V_B[\widetilde{\lambda}] \neq 0$ and $|\rho^{-2}\widetilde{\lambda}(x)|_p = |\chi_0(x)|_p$ (because $\left||\rho(x)|_p\right|_p = |\rho(x)^{-1}|_p$), where "$|\ |_p$" is the p–adic absolute value on K normalized so that $|p| = \frac{1}{p}$.

By definition, the right modulus function δ_B is given by

$$\int_{N(\mathbb{Q}_p)} \phi(u)du = \delta_B(b) \int_{N(\mathbb{Q}_p)} \phi(b^{-1}xb)du.$$

This shows that

$$\delta_B = |\rho^2|_p^{-1}, \tag{A.12}$$

where $\rho^2 = \det \circ Ad_N$ is the sum of positive roots, and ρ is a sum of fundamental weights with respect to B. This shows

$$|\lambda|_p = |\chi_0\rho|_p. \tag{A.13}$$

Note that χ_0 is non-negative with respect to B because χ_0 is dominant. Since $\chi_0 \geq 0$, $\chi_0\rho > 0$, that is, $\chi_0\rho$ is in the interior of the Weyl chamber of B. This shows that if $w \neq 1$,

$$|\lambda^w(d)|_p < |\lambda(d)|_p \quad \text{for all } d \in D, \tag{A.14}$$

because W acts simply transitively on Weyl chambers and each element in the interior of the chamber of λ has the maximum p-adic absolute value on D in its conjugates under W. In particular, we get

THEOREM A.3. *Let π be an irreducible nearly ordinary representation of p-type χ. Then there exists a character $\lambda : T(\mathbb{Q}_p) \to K^\times$ such that $\widetilde{\lambda} \hookrightarrow V_B(\pi_p) \hookrightarrow \oplus_{w\in W}\widetilde{\lambda^w}$ and $|\lambda|_p = |\rho\chi|_p$, where ρ is the sum of fundamental weight with respect to B and $|\ |_p$ is the absolute value on K. Moreover $eH^0(N(\mathbb{Z}_p), V(\pi_p))$ is one dimensional, on which $T(\xi) = U\xi U$ for $\xi \in D$ acts by scalar $|\rho(\xi)|_p\lambda(\xi)$.*

Now suppose that $U = U_p \times G(\widehat{\mathbb{Z}}^p)$ with $U_1(r) \subset U_p \subset U_0(r)$ for $r > 0$. For prime ideals $\mathfrak{l} \nmid p$ of O_F, we consider $T_\mathfrak{l}(\alpha) = U\xi_\alpha(\varpi_\mathfrak{l})U$ for the prime element $\varpi_\mathfrak{l}$ in $O_\mathfrak{l}$. Then we define the Hecke algebra $h_q^{n.ord}(\chi, \widehat{\psi}; K)$ by the K-subalgebra of $\mathrm{End}_K(eH_!^q(X(U), L(\chi, \widehat{\psi}; K)))$ generated by the operators $T(\alpha)$ for all $\alpha \in \Sigma$ and Hecke operators associated to double cosets $U\xi U$ with $\xi_p = 1$. Since

$$eH_!^q(X(U), L(\chi, \widehat{\psi}; K)) = H^0(U, e\mathcal{L}^q(\chi, \widehat{\psi}; K)^{N(\mathbb{Z}_p)}),$$

we get the following semi-simplicity of the Hecke algebra from the fact that the spherical irreducible representation of $G_0(F_\mathfrak{l})$ has a unique vector fixed by the maximal compact subgroup:

COROLLARY A.4. *Let the notation and the assumption be as above. Then we have*

1. *The Hecke module $eH_!^q(X(U), L(\chi, \widehat{\psi}; K))$ has a base made of common eigenvectors of $h_q^{n.ord}(\chi, \widehat{\psi}; K)$ if K is algebraically closed;*
2. *The Hecke algebra $h_q^{n.ord}(\chi, \widehat{\psi}; K)$ is semi-simple.*

Appendix B. Correction to [H95]. Since we quoted often results proven in [H95], we here list some of serious misprints and corrections to the result in the paper. One serious mistake is the sign of the character χ in Section 7 (basically, we need to change χ by χ^{-1} in Lemma 7.2 and its proof). Also in Lemma 7.2, ω_i $(i = 1, \ldots, n)$ have to be dominant characters with respect to $(G_1, {}^tB)$ for the lower triangular Borel subgroup ${}^tB$ instead of the upper trianguler one. In the list below, P.5 L.5b indicates the fifth line from the bottom of the page 5.

Page, line	Statement in [H95]	Correction
P. 453, L. 5 P. 454, L. 6b	$H^q_{P-n.ord}$	$H^r_{P-n.ord}$ for $r = r(\chi)$
P. 454, L. 6b	$H^r_{P-n.ord}(\Phi_{0,P}(p), \mathcal{C}_{P,\rho}[\omega_\chi])$	$H^r_{P-n.ord}(\Phi_{0,P}(p), \mathcal{C}_{P,\rho})[\omega_\chi]$
P. 463, L. 11 P. 466, L. 9b	$\chi(\pi(\xi))^{-1}\rho_\chi(\xi^{-1})$	$\chi(\pi(\xi))\rho_\chi(\xi^{-1})$
P. 466 L. 3b	$\lvert\chi(\pi(d))\rvert_p \geq \lvert\eta(\pi(d))\rvert_p$	$\lvert\chi(\pi(d))\rvert_p \leq \lvert\eta(\pi(d))\rvert_p$
P. 467 L. 1b	$\omega_i(diag(t_1, \ldots, t_n)) = \prod_{1\leq j\leq i} t_j$	$\omega_i(diag(t_1, \ldots, t_n)) = \prod_{1\leq j\leq i} t_j^{-1}$
P. 468 L. 2	$\chi(diag(t_1, \ldots, t_n)) = \prod_{1\leq i\leq n-1} t_j^{j_i}$	$\chi(diag(t_1, \ldots, t_n)) = \prod_{1\leq i\leq n-1} t_j^{-j_i}$
P. 468 (**)	$(\omega w u)$	$(\xi w u)$
P. 468 (**)	$\chi(\pi(\xi))$	$\chi(\pi(\xi))^{-1}$
P. 468 L. 11	$\chi(\pi(d))$	$\chi(\pi(d))^{-1}$
P. 468 L. 16	$\lvert\lambda(\xi_s)\rvert_p < \lvert\chi(\pi(\xi))\rvert_p$	$\lvert\lambda(\xi_s)\rvert_p < \lvert\chi(\pi(\xi))^{-1}\rvert_p$

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