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SYMPLECTIC REDUCTION AND A WEIGHTED MULTIPLICITY FORMULA FOR TWISTED SPIN^C-DIRAC OPERATORS*

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Abstract. We extend our earlier work in [TZ1], where an analytic approach to the Guillemin-Sternberg conjecture [GS] was developed, to cases where the Spin^c-complex under consideration is allowed to be further twisted by certain exterior power bundles of the cotangent bundle. The main result is a *weighted* quantization formula in the presence of commuting Hamiltonian actions. The corresponding Morse-type inequalities in holomorphic situations are also established.

0. Introduction. In a previous paper [TZ1], we have developed a direct analytic approach to, as well as certain extensions of, the Guillemin-Sternberg geometric quantization conjecture [GS], which has been proved in various generalities in [DGMW, G, GS, JK, M1, M2, V1, V2]. In this paper, we generalize the results in [TZ1] to cases where the Spin^c-complex under consideration is allowed to be further twisted by certain exterior power bundles of the cotangent bundle. The main result is a *weighted* quantization formula for these twisted Spin^c-complexes in the presence of commuting Hamiltonian actions. We also establish the corresponding Morse-type inequalities in the holomorphic situation.

Let (M, ω) be a closed symplectic manifold admitting a Hamiltonian action of a compact connected Lie group G with Lie algebra $\mathbf{g}$. Let J be an almost complex structure on TM so that $g^{TM}(u,v) = \omega(u, Jv)$ defines a Riemannian metric on TM. After an integration over G if necessary, we can and will assume that G preserves J and g^{TM}.[1]

Let E be a G-equivariant Hermitian vector bundle over M equipped with a G-equivariant Hermitian connection ∇^E.

With these data in hand, for any integer $p \geq 0$, one can construct canonically a formally self-adjoint twisted Spin^c-Dirac operator acting on smooth sections of the twisted Spin^c-vector bundles:

$$D_+^{\wedge^{p,0}(T^*M)\otimes E} : \Omega^{p,\mathrm{even}}(M,E) \to \Omega^{p,\mathrm{odd}}(M,E). \tag{0.1}$$

It gives rise to the finite dimensional virtual vector space

$$Q\left(M, \wedge^{p,0}(T^*M)\otimes E\right) = \ker D_+^{\wedge^{p,0}(T^*M)\otimes E} - \operatorname{coker} D_+^{\wedge^{p,0}(T^*M)\otimes E}. \tag{0.2}$$

Since G preserves everything, one sees easily that $Q(M, \wedge^{p,0}(T^*M)\otimes E)$ is a virtual representation of G. Denote by $Q(M, \wedge^{p,0}(T^*M)\otimes E)^G$ its G-invariant subspace.

Let $\mathbf{g}$ (and thus its dual $\mathbf{g}^*$ also) be equipped with an AdG-invariant metric. Let h_i, $1 \leq i \leq \dim G$, be an orthonormal base of $\mathbf{g}^*$. Let V_i, $1 \leq i \leq \dim G$, be the dual base of $\{h_i\}_{1\leq i\leq \dim G}$.

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[1] In fact, one does not obtain immediately these J and g^{TM}. What one obtains through the direct integration over G is a G-invariant endormorphism $\hat{J}$ with $\hat{J}^2$ negative as well as a G-invariant metric $\hat{g}^{TM}$. One then obtains J, g^{TM} from $\hat{J}$, $\hat{g}^{TM}$ easily.

Let $\mu : M \to \mathbf{g}^*$ be the moment map of the G-action on M. Then it can be written as

$$\mu = \sum_{i=1}^{\dim G} \mu_i h_i, \tag{0.3}$$

with each μ_i a real function on M.

Now for each $V \in \mathbf{g}$, set[2]

$$r_V^E = L_V^E - \nabla_V^E, \tag{0.4}$$

where L_V^E denotes the infinitesimal action of V on E.

DEFINITION 0.1. *We say E is μ-positive if the inequality*

$$\sqrt{-1} \sum_{i=1}^{\dim G} \mu_i(x) r_{V_i}^E(x) > 0 \tag{0.5}$$

holds at every critical point $x \in M \backslash \mu^{-1}(0)$ of $|\mu|^2$, the norm square of the moment map.

As a typical example, the G-equivariant prequantum line bundle L over (M, ω) verifying the Kostant formula ([Ko], cf. [TZ1, (1.13)]), when it exists, is μ-positive. Furthermore, for arbitrary G-equivariant Hermitian vector bundle F over M equipped with a G-equivariant Hermitian connection, there exists $m_0 \in \mathbf{Z}$ such that for all integer $m \geq m_0$, $E = L^m \otimes F$ is μ-positive.

To state the main results of this paper, we now assume that $0 \in \mathbf{g}^*$ is a regular value of μ and, for simplicity, that G acts freely on $\mu^{-1}(0)$. Then one can construct the Marsden-Weinstein reduction (M_G, ω_G), which is a smooth symplectic manifold with $M_G = \mu^{-1}(0)/G$ and the symplectic form ω_G descended from ω. The almost complex structure J also descends to an almost complex structure on TM_G. Furthermore, E descends to a Hermitian vector bundle E_G over M_G with an induced Hermitian connection. Thus one can make the same construction of the twisted Spin^c-Dirac operators as well as the associated virtual vector spaces $Q(M_G, \Lambda^{p,0}(T^*M_G) \otimes E_G)$.

For any integer $k,\ s \geq 0$, let C_s^k be the binomial coefficient given by

$$C_s^k = \frac{s(s-1)\cdots(s-k+1)}{k!}. \tag{0.6}$$

The main result of this paper, which is a generalization of [TZ1, Theorem 4.1] in the Abelian group action case, can be stated as follows.

THEOREM 0.2. *If G is Abelian and E is μ-positive, then the following identity holds for any integer $p \geq 0$,*

$$\dim Q\left(M, \Lambda^{p,0}(T^*M) \otimes E\right)^G = \sum_{k=0}^{p} C_{\dim G}^{p-k} \cdot \dim Q\left(M_G, \Lambda^{k,0}(T^*M_G) \otimes E_G\right). \tag{0.7}$$

When $p = 0$ and E is the prequantum line bundle (when it exists) over (M, ω), (0.7) is the Abelian version of the Guillemin-Sternberg conjecture [GS] proved by Guillemin [G] in a special case and by Meinrenken [M1] and Vergne [V1, 2] in general

[2] When there is no confusion, in this paper we will use the same notation V for the Killing vector field it generates on M.

(see also [DGMW] and [JK]). In some sense one may view (0.7) as a kind of *weighted* quantization formula with the numbers $C_{\dim G}^{p-k}$ as *weighted* coefficients.

Also, as has been pointed out by Siye Wu and the referee, when E is the prequantum line bundle (when it exists) over (M, ω), (0.7) may be viewed as a supersymmetric version of the Guillemin-Sternberg conjecture [GS] in a particular polarization.

We will use the analytic approach developed in [TZ1] to prove Theorem 0.2. However, it should be pointed out that Theorem 0.2 is not a consequence of the result in [TZ1, Theorem 4.1], which itself is a generalization of the Guillemin-Sternberg conjecture [GS]. In particular, the strict inequality in (0.5) can not be relaxed to include the equality as in [TZ1, Theorem 4.2], even when $\mu^{-1}(0) \neq \emptyset$. Furthermore, the Abelian condition on G is essential to both the results as well as their proofs. A notable feature here is that we deal with directly the general case where G may possibly be of higher rank. That is, we do not first prove the result for the $G = S^1$ case and then use the 'reduction in stages' procedure to get the full result.

Now as in [TZ1, Theorem 0.4 and 4.8], we consider the holomorphic refinement of Theorem 0.2. That is, we assume that (M, ω, J) is Kähler, G acts on M holomorphically and E is a G-equivariant holomorphic Hermitian vector bundle over M with the G-action on E being holomorphic. If for any integers $p,\ q \geq 0$, denote by

$$h^{p,q}(E)^G = \dim H^{0,q}\left(M, \wedge^{p,0}(T^*M) \otimes E\right)^G,$$

$$h^{p,q}(E_G) = \dim H^{0,q}\left(M_G, \wedge^{p,0}(T^*M_G) \otimes E_G\right) \tag{0.8}$$

the corresponding (G-invariant) twisted Hodge numbers, then we can state our refinement of (0.7) as follows.

THEOREM 0.3. *If (M, ω, J) is Kähler, G is Abelian and E is μ-positive, then the following inequality holds for any integers $p,\ q \geq 0$,*

$$h^{p,q}(E)^G - h^{p,q-1}(E)^G + \cdots + (-1)^q h^{p,0}(E)^G$$

$$\leq \sum_{k=0}^{p} C_{\dim G}^{p-k}\left(h^{k,q}(E_G) - h^{k,q-1}(E_G) + \cdots + (-1)^q h^{k,0}(E_G)\right). \tag{0.9}$$

In particular, when $q = 0$, one gets the following inequality for dimensions of spaces of holomorphic sections,

$$h^{p,0}(E)^G \leq \sum_{k=0}^{p} C_{\dim G}^{p-k} \cdot h^{k,0}(E_G). \tag{0.10}$$

Again, Theorem 0.3 is not a consequence of [TZ1, Theorem 4.8].

This paper is organized as follows. In Section 1, we construct the twisted Spin^c-Dirac operators appearing in the context and introduce the corresponding deformations under Hamiltonian actions as in [TZ1]. We also prove a Bochner-type formula for the deformed operators. In Section 2, we extend the methods in [TZ1], which goes back to [BL], to prove Theorems 0.2 and 0.3. The final Section 3 contains some immediate applications as well as further extensions of the above main results. There is also an Appendix in which we provide explicit constructions of certain Spin^c-Dirac operators appearing in Section 3.

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1. Deformations of twisted Spin^c-Dirac operators and a Bochner-type formula. Following [TZ1], we construct in this section the twisted Spin^c-Dirac operators and their deformations to be used in the proof of Theorems 0.2 and 0.3. An important Bochner-type formula for the Laplacians of the deformed operators will be proved.

This section is organized as follows. In a), we construct the above mentioned Dirac operators. In b), following [TZ1], in the situations of Hamiltonian actions we introduce the deformations of the Dirac operators constructed in a). In c), we prove the above mentioned Bochner-type formula for the Laplacians of the deformed operators.

a). Twisted Spin^c-Dirac operators on symplectic manifolds. Let (M,ω) be a closed symplectic manifold. Let J be an almost complex structure on TM such that

$$g^{TM}(v,w) = \omega(v, Jw) \tag{1.1}$$

defines a Riemannian metric on TM. Let $TM_{\mathbf{C}} = TM \otimes \mathbf{C}$ denote the complexification of the tangent bundle TM. Then one has the canonical (orthogonal) splittings

$$TM_{\mathbf{C}} = T^{(1,0)}M \oplus T^{(0,1)}M,$$

$$\wedge^{*,*}(T^*M) = \bigoplus_{i,j=0}^{\dim_{\mathbf{C}} M} \wedge^{i,j}(T^*M), \tag{1.2}$$

where

$$T^{(1,0)}M = \{z \in TM_{\mathbf{C}}; Jz = \sqrt{-1}z\},$$

$$T^{(0,1)}M = \{z \in TM_{\mathbf{C}}; Jz = -\sqrt{-1}z\},$$

$$\wedge^{i,j}(T^*M) = \wedge^i\left(T^{(1,0)*}M\right) \otimes \wedge^j\left(T^{(0,1)*}M\right), \tag{1.3}$$

and $\dim_{\mathbf{C}} M$ is the complex dimension of M.

For any $X \in TM$, which has the decomposition $X = X_1 + X_2 \in T^{(1,0)}M \oplus T^{(0,1)}M$ in the complexification, let $\overline{X}_1^{\,*} \in T^{(0,1)*}M$ (resp. $\overline{X}_2^{\,*} \in T^{(1,0)*}M$) be the metric dual of X_1 (resp. X_2). Set as in [BL, Sect. 5] that

$$c(X) = \sqrt{2}\overline{X}_1^{\,*} \wedge \ -\sqrt{2}i_{X_2}. \tag{1.4}$$

Then $c(X)$ defines the canonical Clifford action of X on $\wedge^{0,*}(T^*M)$. In particular, for any X, $Y \in TM$, one has

$$c(X)c(Y) + c(Y)c(X) = -2g^{TM}(X,Y). \tag{1.5}$$

Let ∇^{TM} be the Levi-Civita connection of g^{TM}. Then ∇^{TM} together with the almost complex structure J induce via projection a canonical Hermitian connection $\nabla^{T^{(1,0)}M}$ on $T^{(1,0)}M$. This in turn induces canonically, for any integer $p \geq 0$, a Hermitian connection $\nabla^{\wedge^{p,0}(T^*M)}$ on $\wedge^{p,0}(T^*M)$. On the other hand, as was shown in [TZ1], ∇^{TM} lifts canonically to a Hermitian connection $\nabla^{\wedge^{0,*}(T^*M)}$ on $\wedge^{0,*}(T^*M)$. Let $\nabla^{\wedge^{p,*}(T^*M)}$ be the Hermitian connection on $\wedge^{p,*}(T^*M)$ obtained from the tensor product of $\nabla^{\wedge^{p,0}(T^*M)}$ and $\nabla^{\wedge^{0,*}(T^*M)}$.

Now let E be a Hermitian vector bundle over M with a Hermitian connection ∇^E. Let $\nabla^{\wedge^{p,*}(T^*M)\otimes E}$ be the tensor product connection of $\nabla^{\wedge^{p,*}(T^*M)}$ and ∇^E on $\wedge^{p,*}(T^*M)\otimes E$.

Denote by $\Omega^{p,*}(M,E)$ the set of smooth sections of $\wedge^{p,*}(T^*M)\otimes E$.

Let $e_1,\ldots,e_{\dim M}$ be an oriented orthonormal base of TM.

DEFINITION 1.1. *The twisted Spin^c-Dirac operator $D^{\wedge^{p,0}(T^*M)\otimes E}$ is defined by*

$$D^{\wedge^{p,0}(T^*M)\otimes E}=\sum_{j=1}^{\dim M}c(e_j)\nabla_{e_j}^{\wedge^{p,*}(T^*M)\otimes E}:\Omega^{p,*}(M,E)\to\Omega^{p,*}(M,E). \tag{1.6}$$

Clearly, $D^{\wedge^{p,0}(T^*M)\otimes E}$ is a formally self-adjoint first order elliptic differential operator. Let $D_+^{\wedge^{p,0}(T^*M)\otimes E}$ be the restriction of $D^{\wedge^{p,0}(T^*M)\otimes E}$ on $\Omega^{p,\mathrm{even}}(M,E)$. Set

$$Q\left(M,\wedge^{p,0}(T^*M)\otimes E\right)=\ker D_+^{\wedge^{p,0}(T^*M)\otimes E}-\operatorname{coker}D_+^{\wedge^{p,0}(T^*M)\otimes E}. \tag{1.7}$$

b). Hamiltonian actions and deformations of Dirac operators. Now suppose that (M,ω) admits a Hamiltonian action of a compact connected Lie group G with Lie algebra $\mathbf{g}$. Let $\mu:M\to\mathbf{g}^*$ denote the corresponding moment map. As has been explained, after an integration over G if necessary, we may assume that G preserves J and g^{TM}. We also assume that the G-action on M lifts to a G-action on E preserving the Hermitian metric as well as the Hermitian connection ∇^E on E.

Let $\mathbf{g}$ (and thus $\mathbf{g}^*$ also) be equipped with an AdG-invariant metric. Let $\mathcal{H}=|\mu|^2$ be the norm square of the moment map μ. Then $\mathcal{H}$ is a G-invariant function on M. In particular, its Hamiltonian vector field, denoted by $X^{\mathcal{H}}$, is G-invariant. The following formula for $X^{\mathcal{H}}$ is clear,

$$X^{\mathcal{H}}=-J(d\mathcal{H})^*. \tag{1.8}$$

Let $h_1,\ldots,h_{\dim G}$ be an orthonormal base of $\mathbf{g}^*$. Then μ has the expression

$$\mu=\sum_{i=1}^{\dim G}\mu_i h_i, \tag{1.9}$$

where each μ_i is a real valued function on M. Denote by V_i the Killing vector field on M induced by the dual of h_i. One easily verifies that (cf. [TZ1, Sect. 1b)])

$$J(d\mu_i)^*=-V_i \tag{1.10}$$

and

$$X^{\mathcal{H}}=-2J\sum_{i=1}^{\dim G}\mu_i(d\mu_i)^*=2\sum_{i=1}^{\dim G}\mu_iV_i. \tag{1.11}$$

We are now ready to introduce the crucial deformation following [TZ1, Definition 1.2].

DEFINITION 1.2. *For any $T \in \mathbf{R}$, let $D_T^{\wedge^{p,0}(T^*M)\otimes E}$ be the operator defined by*

$$D_T^{\wedge^{p,0}(T^*M)\otimes E} = D^{\wedge^{p,0}(T^*M)\otimes E} + \frac{\sqrt{-1}T}{2} c\left(X^{\mathcal{H}}\right) : \Omega^{p,*}(M,E) \to \Omega^{p,*}(M,E). \tag{1.12}$$

Clearly, $D_T^{\wedge^{p,0}(T^*M)\otimes E}$ is a formally self-adjoint first order elliptic differential operator. Also, since G preserves everything and $X^{\mathcal{H}}$ is G-invariant, one sees that $D_T^{\wedge^{p,0}(T^*M)\otimes E}$ is G-equivariant. If we denote by $D_{T,+}^{\wedge^{p,0}(T^*M)\otimes E}$ the restriction of $D_T^{\wedge^{p,0}(T^*M)\otimes E}$ on $\Omega^{p,\mathrm{even}}(M,E)$, then

$$Q_T\left(M, \wedge^{p,0}(T^*M)\otimes E\right) = \ker D_{T,+}^{\wedge^{p,0}(T^*M)\otimes E} - \operatorname{coker} D_{T,+}^{\wedge^{p,0}(T^*M)\otimes E} \tag{1.13}$$

is a virtual G-representation. We use as usual a superscript G to denote its G-invariant subspace.

Clearly, the following easy yet important identity holds for any $T \in \mathbf{R}$,

$$\dim Q_T\left(M, \wedge^{p,0}(T^*M)\otimes E\right)^G = \dim Q\left(M, \wedge^{p,0}(T^*M)\otimes E\right)^G. \tag{1.14}$$

c). A Bochner-type formula for the square of $D_T^{\wedge^{p,0}(T^*M)\otimes E}$. For any $V \in \mathbf{g}$, let L_V denote the infinitesimal action induced by V on the corresponding vector bundles. We will in general omit the superscripts of these bundles. Let r_V^E be defined as in (0.4).

For any X, $Y \in TM$, which have the decompositions $X = X_1 + X_2 \in T^{(1,0)}M \oplus T^{(0,1)}M$ and $Y = Y_1 + Y_2 \in T^{(1,0)}M \oplus T^{(0,1)}M$ respectively in the complexification, let $A(X,Y)$ be the endomorphism of $\wedge^{p,0}(T^*M)$ defined by

$$A(X,Y) = \overline{X}_2^* \wedge i_{Y_1}. \tag{1.15}$$

Let $e_1, \cdots, e_{\dim M}$ be an oriented orthonormal base of TM. Then one has the following analogue of [TZ1, Lemma 1.5].

LEMMA 1.3. *The following identity for operators acting on $\Omega^{p,*}(M,E)$ holds,*

$$L_V = \nabla_V + r_V^E - \frac{1}{4}\sum_{j=1}^{\dim M} c(e_j)c\left(\nabla_{e_j}^{TM}V\right) - \frac{1}{2}\mathrm{Tr}\left[\nabla_{\cdot}^{T^{(1,0)}M}V\right] + \sum_{j=1}^{\dim M} A\left(e_j, \nabla_{e_j}^{TM}V\right). \tag{1.16}$$

Proof. By proceeding as in [TZ1, Lemma 1.5], one sees easily that one needs only to calculate $r_V^{\wedge^{p,0}(T^*M)}$.

Recall that V acts on TM by

$$L_V^{TM}X = \nabla_V^{TM}X - \nabla_X^{TM}V, \quad X \in \Gamma(TM), \tag{1.17}$$

from which we have

$$r_V^{TM}(X) = -\sum_{j=1}^{\dim M}\left\langle \nabla_X^{TM}V, e_j\right\rangle e_j = \sum_{j=1}^{\dim M}\left\langle \nabla_{e_j}^{TM}V, X\right\rangle e_j. \tag{1.18}$$

From (1.18) one gets immediately that

$$r_V^{T^*M} = \sum_{j=1}^{\dim M} e_j^* \wedge i_{\nabla_{e_j}^{TM}V}. \tag{1.19}$$

By (1.19), (1.15) and the fact that the almost complex structure J is G-invariant, one deduces easily that for any integer $0 \le p \le \dim_{\mathbf{C}} M$,

$$r_V^{\Lambda^{p,0}(T^*M)} = \sum_{j=1}^{\dim M} A\left(e_j, \nabla_{e_j}^{TM} V\right). \tag{1.20}$$

(1.16) then follows from (1.20) and the arguments in [TZ1, Lemma 1.5]. □

We can now state the following analogue of [TZ1, Theorem 1.6].

THEOREM 1.4. *The following Bochner-type formula holds,*

$$\left(D_T^{\Lambda^{p,0}(T^*M)\otimes E}\right)^2 = \left(D^{\Lambda^{p,0}(T^*M)\otimes E}\right)^2 - 2\sqrt{-1}T\sum_{i=1}^{\dim G}\mu_i L_{V_i}$$

$$-\frac{\sqrt{-1}T}{2}\operatorname{Tr}\left[\nabla^{T^{(1,0)}M}X^{\mathcal{H}}\right] + 2\sqrt{-1}T\sum_{i=1}^{\dim G}\mu_i r_{V_i}^E$$

$$+\frac{T}{2}\sum_{i=1}^{\dim G}\left(\sqrt{-1}c(JV_i)c(V_i) + |V_i|^2 - 4\sqrt{-1}A(JV_i,V_i)\right)$$

$$+\frac{\sqrt{-1}T}{4}\sum_{j=1}^{\dim M}\left(c(e_j)c\left(\nabla_{e_j}^{TM}X^{\mathcal{H}}\right) + 4A\left(e_j,\nabla_{e_j}^{TM}X^{\mathcal{H}}\right)\right) + \frac{T^2}{4}\left|X^{\mathcal{H}}\right|^2. \tag{1.21}$$

Proof. As in [TZ1, (1.26) and (1.27)], one deduces from (1.12), (1.5) that

$$D_T^2 = D^2 + \frac{\sqrt{-1}T}{2}\sum_{j=1}^{\dim M} c(e_j)c\left(\nabla_{e_j}X^{\mathcal{H}}\right) - \sqrt{-1}T\nabla_{X^{\mathcal{H}}} + \frac{T^2}{4}\left|X^{\mathcal{H}}\right|^2 \tag{1.22}$$

and, by using (1.10), (1.11) and Lemma 1.3, that

$$\nabla_{X^{\mathcal{H}}} = 2\sum_{i=1}^{\dim G}\mu_i L_{V_i} - 2\sum_{i=1}^{\dim G}\mu_i r_{V_i}^E + \frac{1}{4}\sum_{j=1}^{\dim M} c(e_j)c\left(\nabla_{e_j}X^{\mathcal{H}}\right)$$

$$+\frac{1}{2}\operatorname{Tr}\left[\nabla^{T^{(1,0)}M}X^{\mathcal{H}}\right] - \frac{1}{2}\sum_{i=1}^{\dim G} c(JV_i)c(V_i) + \frac{\sqrt{-1}}{2}\sum_{i=1}^{\dim G}|V_i|^2$$

$$-\sum_{j=1}^{\dim M} A\left(e_j,\nabla_{e_j}X^{\mathcal{H}}\right) + 2\sum_{i=1}^{\dim G} A(JV_i,V_i). \tag{1.23}$$

(1.21) follows from (1.22) and (1.23). □

2. Proof of the main theorems. In this section, we apply the methods and techniques in [TZ1, Sects. 2-4], which are closely related to those in [BL], to prove Theorems 0.2 and 0.3. As in [TZ1], the key technical point is a pointwise estimate at each critical point $x \in M\backslash\mu^{-1}(0)$ of $\mathcal{H} = |\mu|^2$.

This section is organized as follows. In a), we prove the key pointwise estimate mentioned above. In b), we prove Theorem 0.2 while Theorem 0.3 will be proved in c).

a). An estimate outside of $\mu^{-1}(0)$. Recall from Definition 0.1 that E is said to be μ-positive if (0.5) holds at every critical point $x \in M\backslash\mu^{-1}(0)$ of $\mathcal{H} = |\mu|^2$.

The main result of this subsection, which is an analogue of [TZ1, Theorem 2.1], can be stated as follows.

THEOREM 2.1. *If G is Abelian and E is μ-positive, then for any open neighborhood U of $\mu^{-1}(0)$, there exist constants $C > 0$, $b > 0$ such that for any $T \geq 1$ and any G-invariant section $s \in \Omega^{p,*}(M, E)$ with* Supp $s \subset M\backslash U$, *one has the following estimate of Sobolev norms,*

$$\left\| D_T^{\wedge^{p,0}(T^*M)\otimes E} s \right\|_0^2 \geq C\left(\|s\|_1^2 + (T-b)\|s\|_0^2\right). \tag{2.1}$$

Proof. By examining the arguments in [TZ1, Sect. 2], one sees that in order to prove Theorem 2.1, one needs only to prove an analogue of [TZ1, Lemma 2.3] in our context.

Thus let $x \in M\backslash\mu^{-1}(0)$ be a critical point of $\mathcal{H}$. Let $e_1, \cdots, e_{\dim M}$ be an orthonormal base of TM near x. Let $(y_1, \cdots, y_{\dim M})$ be the normal coordinate system with respect to $\{e_j\}_{j=1}^{\dim M}$ near x. Clearly, one can choose $e_1, \cdots, e_{\dim M}$ so that $\mathcal{H}$ has the following expression near x,

$$\mathcal{H}(y) = \mathcal{H}(x) + \sum_{j=1}^{\dim M} a_j y_j^2 + O\left(|y|^3\right), \tag{2.2}$$

where the a_j's may possibly be zero.

We can now state our analogue of [TZ1, Lemma 2.3] as follows.

LEMMA 2.2. *If G is Abelian, then the following inequality holds at any critical point $x \in M\backslash\mu^{-1}(0)$ of $\mathcal{H}$,*

$$\begin{aligned}&\frac{\sqrt{-1}}{4}\sum_{j=1}^{\dim M}\left(c(e_j)c\left(\nabla_{e_j}^{TM}X^{\mathcal{H}}\right) + 4A\left(e_j, \nabla_{e_j}^{TM}X^{\mathcal{H}}\right)\right) - \frac{\sqrt{-1}}{2}\mathrm{Tr}\left[\nabla^{T^{(1,0)}M}X^{\mathcal{H}}\right]\\ &+\frac{1}{2}\sum_{i=1}^{\dim G}\left(\sqrt{-1}c(JV_i)c(V_i) + |V_i|^2 - 4\sqrt{-1}A(JV_i, V_i)\right) \geq -\sum_{j=1}^{\dim M}|a_j|.\end{aligned} \tag{2.3}$$

Proof. Since x is a critical point of $\mathcal{H}$, by a result of Kirwan [K, Prop. 3.12], x is a fixed point for the action of the subtorus generated by $\mu(x) \neq 0$. Without loss of generality, we assume that $h_1, \cdots, h_{\dim G}$ has been chosen so that the duals of $h_1, \cdots, h_r$ generate this subtorus denoted by G_1. Let G_2 be the subtorus generated by the duals of $h_{r+1}, \cdots, h_{\dim G}$. Then the original torus has the factorization

$$G = G_1 \cdot G_2 \quad \text{with} \quad G_1 \cap G_2 \ \text{finite}. \tag{2.4}$$

Clearly, one has that

$$\mu_i(x) = 0, \quad r+1 \leq i \leq \dim G. \tag{2.5}$$

Denote by $F_x \subset M$ the connected component containing x of the fixed point set of the G_1-action. Then F_x is a totally geodesic submanifold of M and J preserves the tangent bundle TF_x. Denote $k = \dim F_x$.

Since the G_2-action commutes with the G_1-action, G_2 acts on F_x.

To summarize, one has

LEMMA 2.3. *a). If $1 \leq i \leq r$, then $V_i|_{F_x} = 0$ and $\mu_i|_{F_x}$ is constant; b). if $r+1 \leq i \leq \dim G$, then $\mu_i(x) = 0$ and $(d\mu_i)^*|_{F_x} \in \Gamma(TF_x)$.*

Proof. Since G_2 acts on F_x, for any $r+1 \leq i \leq \dim G$, $V_i|_{F_x} \in \Gamma(TF_x)$. Thus by (1.10), $(d\mu_i)^* = JV_i \in \Gamma(TF_x)$. The other parts of the lemma are clear. □

Without loss of generality, one may choose $e_1, \cdots, e_{\dim M}$ near x so that $e_1, \cdots, e_k$ is an orthonormal base of TF_x. Let $\{y_j\}_{1 \leq j \leq \dim M}$ be the corresponding normal coordinates near x. Then from Lemma 2.3 one deduces that, near x, this orthonormal base can be further arranged so that

$$\sum_{i=1}^{r} |\mu_i(y)|^2 = |\mu(x)|^2 + \sum_{j=k+1}^{\dim M} a_j y_j^2 + O\left(|y|^3\right) \tag{2.6}$$

and

$$\sum_{i=r+1}^{\dim G} |\mu_i(y)|^2 = \sum_{j=1}^{k} a_j y_j^2 + O\left(|y|^3\right). \tag{2.7}$$

(2.6) and (2.7) together provide a splitting of $\mathcal{H}$ near x according to the splitting (2.4).

From (1.10) and (2.6) one deduces, at x, that

$$\sum_{i=1}^{r} \nabla_{e_j}^{TM}(\mu_i V_i) = \begin{cases} -a_j J e_j, & \text{for } k+1 \leq j \leq \dim M, \\ 0, & \text{for } 1 \leq j \leq k. \end{cases} \tag{2.8}$$

From (2.8), one deduces, at x, that

$$\frac{\sqrt{-1}}{4} \sum_{j=1}^{\dim M} \sum_{i=1}^{r} \left(c(e_j) c\left(2\nabla_{e_j}^{TM}(\mu_i V_i)\right) + 4A\left(e_j, 2\nabla_{e_j}^{TM}(\mu_i V_i)\right)\right)$$

$$-\frac{\sqrt{-1}}{2} \sum_{i=1}^{r} \operatorname{Tr}\left[2\nabla^{T^{(1,0)}M}(\mu_i V_i)\right]$$

$$= -\sum_{j=k+1}^{\dim M} a_j \left(\frac{\sqrt{-1}}{2} c(e_j) c(Je_j) + 2\sqrt{-1} A(e_j, Je_j) + \frac{1}{2}\right) \geq -\sum_{j=k+1}^{\dim M} |a_j|, \tag{2.9}$$

where the last inequality follows from the obvious inequalities that

$$|c(e_j) c(Je_j)| \leq 1 \tag{2.10}$$

and

$$\left|2\sqrt{-1}A(e_j, Je_j) + \frac{1}{2}\right| = \left|\frac{1}{2} - 2\overline{e_j^{0,1}}^* \wedge i_{e_j^{1,0}}\right| \leq \frac{1}{2}, \tag{2.11}$$

with $e_j^{1,0} \in T^{(1,0)}M$ (resp. $e_j^{0,1} \in T^{(0,1)}M$) the (1,0) (resp. (0,1)) component of the complexification of e_j.

On the other hand, by Lemma 2.3, for each $r+1 \leq i \leq \dim G$, μ_i can be written, near x, as

$$\mu_i(y) = \sum_{j=1}^{k} b_{ij} y_j + O\left(|y|^2\right). \tag{2.12}$$

By (2.7) and (2.12), one deduces that

$$\sum_{i=r+1}^{\dim G} \sum_{j=1}^{k} b_{ij}^2 = \sum_{j=1}^{k} a_j, \tag{2.13}$$

which, together with (1.10), imply

$$\sum_{i=r+1}^{\dim G} |V_i(x)|^2 = \sum_{j=1}^{k} a_j. \tag{2.14}$$

From Lemma 2.3, (2.14), (1.10) and (2.10), one deduces, at x, that

$$\frac{\sqrt{-1}}{2} \sum_{j=1}^{\dim M} \sum_{i=r+1}^{\dim G} \left(c(e_j) c\left(\nabla_{e_j}^{TM}(\mu_i V_i)\right) + 4A\left(e_j, \nabla_{e_j}^{TM}(\mu_i V_i)\right)\right)$$

$$+ \sum_{i=r+1}^{\dim G} \left(-\sqrt{-1}\operatorname{Tr}\left[\nabla^{T^{(1,0)}M}(\mu_i V_i)\right] + \frac{\sqrt{-1}c(JV_i)c(V_i) + |V_i|^2}{2}\right)$$

$$-2\sqrt{-1} \sum_{i=r+1}^{\dim G} A(JV_i, V_i) = \sum_{i=r+1}^{\dim G} \sqrt{-1}c(JV_i)c(V_i) \geq -\sum_{j=1}^{k} |a_j|. \tag{2.15}$$

From (1.11), (2.9), (2.15) and Lemma 2.3, one gets (2.3). The proof of Lemma 2.2 is completed. □

Since E is μ-positive, using Theorem 1.4, (0.5) and Lemma 2.2, one can proceed in the same way as in [TZ1, Sect. 2], with almost no changes, to prove Theorem 2.1. That is, we first prove pointwise estimates analogous to [TZ1, Prop. 2.2] around each point outside of $\mu^{-1}(0)$, in using Lemma 2.2 when dealing with critical points of $\mathcal{H}$, and then glue them together to get the global estimate (2.1). The essential point in this last gluing step is again as in [TZ1] that each L_{V_i}, $1 \leq i \leq \dim G$, vanishes when acting on G-invariant sections. We leave the details to the interested reader. □

REMARK 2.4. A notable difference between Lemma 2.2 and [TZ1, Lemma 2.3] is that even with some of the a_j's being negative, one does not have in general a strict inequality in (2.3). This means that the μ-positivity of E is necessary for Theorem 2.1 (compare with [TZ1, Remark 2.4]).

b). Proof of Theorem 0.2. If F is a G-equivariant Hermitian vector bundle over M, we denote by F_G its induced bundle on M_G (cf. [TZ1, Sect. 4a)]).

As in [TZ1], Theorem 2.1 allows us to localize our problem to sufficiently small neighborhoods of $\mu^{-1}(0)$. While near $\mu^{-1}(0)$, we can directly apply the analysis and results in [TZ1, Sects. 3 and 4a)], which are closely related to [BL, Sects. 8, 9], to the G-invariant restriction of $D_T^{\wedge^{p,0}(T^*M)\otimes E}$.

Combining the above arguments, one deduces using (1.14) the following analogue of [TZ1, (4.3)] (of course for different twisted bundles and conditions),

$$\dim Q\left(M, \wedge^{p,0}(T^*M)\otimes E\right)^G = \dim Q\left(M_G, \left(\wedge^{p,0}(T^*M)\otimes E\right)_G\right). \tag{2.16}$$

Now since G is Abelian, the normal bundle to $\mu^{-1}(0)$ is equivariantly trivial. From this fact one deduces directly the *weighted* splitting

$$\left(\wedge^{p,0}(T^*M)\right)_G = \bigoplus_{k=0}^{p} C_{\dim G}^{p-k}\cdot \wedge^{k,0}\left(T^*M_G\right), \tag{2.17}$$

where the numbers $C_{\dim G}^{p-k}$ have been defined in (0.6).

Theorem 0.2 then follows from (2.16) and (2.17). □

REMARK 2.5. In view of Remark 2.4, the μ-positivity condition (0.5) can not be weakened in general to include the equality as in [TZ1, Theorem 4.2], even when $\mu^{-1}(0)$ is nonempty.

REMARK 2.6. Though its proof is of the same method, Theorem 0.2 can not be deduced from results in [TZ1] without imposing further conditions. The point is that if one wants to apply directly the results in [TZ1] to our situation, one needs the condition that at every critical point $x \in M\backslash\mu^{-1}(0)$ of $\mathcal{H}$,

$$\sqrt{-1}\sum_{i=1}^{\dim G}\mu_i r_{V_i}^{\wedge^{p,0}(T^*M)\otimes E} \geq 0, \tag{2.18}$$

which clearly does not imply (0.5).

c). Proof of Theorem 0.3. We now assume that (M,ω,J) is Kähler and G acts on M holomorphically. Furthermore, we assume that E is a G-equivariant holomorphic Hermitian vector bundle over M on which G acts holomorphically and that ∇^E is the unique holomorphic Hermitian connection.

The key observation is, similarly as in [TZ1, Remark 1.4], that for any $T \in \mathbf{R}$ we have in this situation

$$D_T^{\wedge^{p,0}(T^*M)\otimes E} = \sqrt{2}\left(e^{-T\mathcal{H}/2}\overline{\partial}^{\wedge^{p,0}(T^*M)\otimes E}e^{T\mathcal{H}/2} + e^{T\mathcal{H}/2}\left(\overline{\partial}^{\wedge^{p,0}(T^*M)\otimes E}\right)^* e^{-T\mathcal{H}/2}\right). \tag{2.19}$$

Furthermore, one has clearly an analogue of [TZ1, (3.54)]. Thus by the same reason as in [TZ1, Sect. 4d)], all the arguments before this subsection preserve the $\mathbf{Z}$-grading nature of the twisted Dolbeault complex on M with coefficient bundle $\wedge^{p,0}(T^*M)\otimes E$, and this leads to the following holomorphic refinement of (2.16) which holds for any integer $q \geq 0$,

$$h^{p,q}(E)^G - h^{p,q-1}(E)^G + \cdots + (-1)^q h^{p,0}(E)^G \leq h^{0,q}\left(\left(\wedge^{p,0}(T^*M)\otimes E\right)_G\right)$$

$$-h^{0,q-1}\left(\left(\wedge^{p,0}(T^*M)\otimes E\right)_G\right)+\cdots+(-1)^q h^{0,0}\left(\left(\wedge^{p,0}(T^*M)\otimes E\right)_G\right). \tag{2.20}$$

On the other hand, one verifies directly that the splitting (2.17) is holomorphic in this situation.

(0.9) is then a consequence of (2.17) and (2.20). While (0.10) follows clearly from (0.9).

The proof of Theorem 0.3 is completed. □

3. Applications and further extensions. In this section, we discuss some immediate applications and possible extensions of Theorems 0.2, 0.3 as well as the methods and techniques involved in their proofs.

This section is organized as follows. In a), we apply Theorem 0.2 to get a vanishing multiplicity result for twisted de Rham-Hodge operators and a weighted multiplicity formula for twisted Signature operators. In b), we prove a negative analogue of Theorem 0.2, that is, we prove a weighted multiplicity formula in the case that '>' is replaced by '<' in (0.5). We also show that the strict inequalities are necessary. In c), we discuss briefly the case where $0\in\mathbf{g}^*$ is not a regular value of the moment map μ. Finally, we discuss in d) the applications to the typical example where E is the prequantum line bundle over (M,ω).

a). Applications to twisted de Rham-Hodge and Signature operators. Set

$$Q_{\mathrm{dR}}(M,E)=\bigoplus_{p=\mathrm{even}} Q\left(M,\wedge^{p,0}(T^*M)\otimes E\right)-\bigoplus_{p=\mathrm{odd}} Q\left(M,\wedge^{p,0}(T^*M)\otimes E\right) \tag{3.1}$$

and

$$Q_{\mathrm{Sig}}(M,E)=\bigoplus_{p=0}^{\dim_{\mathbf{C}} M} Q\left(M,\wedge^{p,0}(T^*M)\otimes E\right). \tag{3.2}$$

One verifies easily that $Q_{\mathrm{dR}}(M,E)$ (resp. $Q_{\mathrm{Sig}}(M,E)$) is exactly the virtual vector space associated to the twisted (by E) de Rham-Hodge (resp. Signature) operator on M. The following result gives the corresponding multiplicity formulas for these operators.

THEOREM 3.1. *Under the same assumptions as in Theorem 0.2, the following identities hold,*

$$\dim Q_{\mathrm{dR}}(M,E)^G=0, \tag{3.3}$$

$$\dim Q_{\mathrm{Sig}}(M,E)^G=2^{\dim G}\dim Q_{\mathrm{Sig}}(M_G,E_G). \tag{3.4}$$

Proof. Theorem 3.1 follows easily from Theorem 0.2 and the definitions (3.1), (3.2) with some elementary computation. □

REMARK 3.2. The assumption that $0\in\mathbf{g}^*$ is a regular value of μ is essential, particularly for the vanishing property (3.3). This will be discussed further in c) and d).

b). Weighted multiplicity formula for μ-negative bundles. A G-equivariant Hermitian vector bundle E over (M,ω) with G-invariant Hermitian connection

∇^E is said to be μ-negative if at every critical point $x \in M\backslash\mu^{-1}(0)$ of $\mathcal{H} = |\mu|^2$, one has

$$\sqrt{-1}\sum_{i=1}^{\dim G} \mu_i(x) r^E_{V_i}(x) < 0, \tag{3.5}$$

instead of (0.5).

For any μ-negative bundle E, one can introduce the same deformation of twisted Spin^c-Dirac operators as in Definition 1.2, but take $T \to -\infty$, instead of $+\infty$, to prove the following result.

THEOREM 3.3. *If G is Abelian and E is μ-negative, then the following identity holds for any integer $p \geq 0$,*

$$\dim Q\left(M, \Lambda^{p,0}(T^*M) \otimes E\right)^G$$

$$= (-1)^{\dim G} \sum_{k=0}^{p} C^{p-k}_{\dim G} \cdot \dim Q\left(M_G, \Lambda^{k,0}(T^*M_G) \otimes E_G\right). \tag{3.6}$$

Proof. One can proceed similarly as in Section 2 to prove Theorem 3.2. The key point to note is that now the analogue of Lemma 2.2 should take the following form at any critical point $x \in M\backslash\mu^{-1}(0)$ of $\mathcal{H} = |\mu|^2$,

$$\frac{\sqrt{-1}}{4}\sum_{j=1}^{\dim M}\left(c(e_j)c\left(\nabla^{TM}_{e_j}X^{\mathcal{H}}\right) + 4A\left(e_j, \nabla^{TM}_{e_j}X^{\mathcal{H}}\right)\right) - \frac{\sqrt{-1}}{2}\mathrm{Tr}\left[\nabla^{T^{(1,0)}M}X^{\mathcal{H}}\right]$$

$$+\frac{1}{2}\sum_{i=1}^{\dim G}\left(\sqrt{-1}c(JV_i)c(V_i) + |V_i|^2 - 4\sqrt{-1}A(JV_i, V_i)\right) \leq \sum_{j=1}^{\dim M}|a_j|. \tag{3.7}$$

We leave the details to the interested reader (Compare also with [TZ1, Remark 4.5]). □

As immediate applications, one gets analogues of Theorem 3.1 for μ-negative bundles. One also gets Morse-type inequalities as refinements of (3.6) in holomorphic situations.

REMARK 3.4. We use a simple example to illustrate that even with $\mu^{-1}(0) \neq \emptyset$, one can not weaken the μ-positive (resp. μ-negative) assumption in (0.5) (resp. (3.5)) by an equality to get a weighted multiplicity formula similar to what happens in [TZ1, Theorem 4.2]: Taking M to be $\mathbf{P}^1$ with its standard S^1-action and $E = M \times \mathbf{C}$, one verifies directly that $\dim Q_{\mathrm{dR}}(M, \mathbf{C})^G = \dim Q_{\mathrm{dR}}(M, \mathbf{C}) = 2 \neq 0$.

REMARK 3.5. It is also clear that the Abelian condition on G is essential for our argument. In fact, an example due to Vergne (cf. [JK, pp.686]) shows that the Abelian condition on G is necessary for Theorem 3.3 in the case where $p = 0$ and E is the dual of the prequantum line bundle over (M, ω) (when the latter exists).

c). The case where $0 \in \mathbf{g}^*$ is a singular value of μ. We now make a brief discussion on the possible generalizations of our main results to the case where $0 \in \mathbf{g}^*$ is a singular value of μ.

When $p=0$ and E is the prequantum line bundle over (M,ω) (when it exists), the quantization formula for singular reduction has been established by Meinrenken-Sjamaar [MS] (see also [TZ3] for an analytic treatment as well as extensions in certain situations). One notable feature in this case is the phenomenon that 'the singular value 0 is removable' in the perturbative singular quantization formula [MS, Theorems 2.5, 2.9]. However, as we will see, the situation for the case where p is nonzero is rather different.

In the simplest case where G is the circle, a fairly general singular localization formula, which can indeed be applied to our situation, has been proved in [TZ2, Theorem 6.7][3]. To be more precise, let V be the Killing vector field on M generated by the unit base of $\mathbf{g}$ and let $F_0=\mu^{-1}(0)\cap\{x\in M; V(x)=0\}$ be the subset of the fixed points of the G-action contained in $\mu^{-1}(0)$. Then one has the following result, which can be proved by a combination of the arguments in Section 2 with those in the proof of [TZ2, Theorem 6.7] (Compare also with [Br]).

THEOREM 3.6. *If $G=S^1$, E is μ-positive and $0\in\mathbf{g}^*$ is a singular value of μ, then*

$$\dim Q\left(M,\Lambda^{p,0}(T^*M)\otimes E\right)^G$$

$$=\sum_{k=0}^{p} C_{\dim G}^{p-k}\cdot\dim Q\left(M_{G,0^-},\Lambda^{k,0}(T^*M_{G,0^-})\otimes E_{G,0^-}\right)+\operatorname{ind} D_{F_0,+}^{\Lambda^{p,0}(T^*M)\otimes E}(V)$$

$$=\sum_{k=0}^{p} C_{\dim G}^{p-k}\cdot\dim Q\left(M_{G,0^+},\Lambda^{k,0}(T^*M_{G,0^+})\otimes E_{G,0^+}\right)+\operatorname{ind} D_{F_0,+}^{\Lambda^{p,0}(T^*M)\otimes E}(-V), \tag{3.8}$$

where $M_{G,0^\pm}$ are the symplectic reductions $\mu^{-1}(\pm\varepsilon)$ with $\varepsilon>0$ sufficiently small and $E_{G,0^\pm}$ the induced bundles from E, while $D_{F_0,+}^{\Lambda^{p,0}(T^*M)\otimes E}(\pm V)$ will be defined in Appendix.

COROLLARY 3.7. *Under the same conditions as in Theorem 3.6, one has*

$$\dim Q_{\mathrm{dR}}(M,E)^G=\operatorname{ind} D_{F_0,+}^{\Lambda^{\mathrm{even},0}(T^*M)\otimes E}(V)-\operatorname{ind} D_{F_0,+}^{\Lambda^{\mathrm{odd},0}(T^*M)\otimes E}(V)$$

$$=\operatorname{ind} D_{F_0,+}^{\Lambda^{\mathrm{even},0}(T^*M)\otimes E}(-V)-\operatorname{ind} D_{F_0,+}^{\Lambda^{\mathrm{odd},0}(T^*M)\otimes E}(-V). \tag{3.9}$$

REMARK 3.8. In the next subsection, we will show that even when E is the prequantum line bundle over (M,ω), the contributions of F_0 to the right hand sides of (3.8), (3.9) may well be nonzero. This explains the essential difference between the $p=0$ and $p\neq 0$ cases as mentioned above.

REMARK 3.9. If G is of higher rank, one can apply Theorems 3.6, 3.7 inductively to get localization formulas for $\dim Q(M,\Lambda^{p,0}(T^*M)\otimes E)^G$ and $\dim Q_{\mathrm{dR}}(M,E)^G$, respectively.

d). The case where E is the prequantum line bundle over (M,ω). In this section, we assume E is the prequantum line bundle L over (M,ω), of which we

[3]See also [Br] for a direct analytic treatment as well as extensions to the holomorphic case.

assume the existence. Then L is a G-equivariant Hermitian vector bundle over M with G-invariant Hermitian connection ∇^L such that

$$\frac{\sqrt{-1}}{2\pi}\left(\nabla^L\right)^2 = \omega. \tag{3.10}$$

Furthermore, we assume that μ verifies the Kostant formula [Ko] (cf. [TZ1, (1.13)])

$$L_V s = \nabla^L_V s - 2\pi\sqrt{-1}\langle\mu, V\rangle s, \;\; s \in \Gamma(L), \;\; V \in \mathbf{g}. \tag{3.11}$$

The following result is clear.

PROPOSITION 3.10. *L is μ-positive.*

One of the novelties for the prequantum line bundle is that there is a standard shifting trick to reduce the computation of dimensions of nontrivial components of the G-representation $Q(M, \wedge^{p,0}(T^*M) \otimes L)$ to those of trivial components of G-representations of the form $Q(M, \wedge^{p,0}(T^*M) \otimes E)$ with E to be 'μ-positive'.

To be more precise, for any $\xi_i \in \mathbf{Z}$, $1 \leq i \leq \dim G$, set

$$\xi = \xi_1 h_1 + \cdots \xi_{\dim G} h_{\dim G} \in \mathbf{g}^*. \tag{3.12}$$

Let $\mathbf{C}_\xi$ be the G-equivariant complex line bundle $M \times \mathbf{C}$ over M with G-invariant Hermitian connection $\nabla^{\mathbf{C}_\xi}$, on which G acts by

$$L_V s = \nabla^{\mathbf{C}_\xi}_V s + 2\pi\sqrt{-1}\langle\xi, V\rangle s, \;\; s \in \Gamma(\mathbf{C}_\xi), \;\; V \in \mathbf{g}. \tag{3.13}$$

The existence of $\mathbf{C}_\xi$ is clear.

Set

$$L_\xi = L \otimes \mathbf{C}_\xi. \tag{3.14}$$

Then L_ξ is canonically a G-equivariant Hermitian vector bundle over M with the tensor product connection ∇^{L_ξ}. Furthermore, G acts on L_ξ through the formula

$$L_V s = \nabla^{L_\xi}_V s - 2\pi\sqrt{-1}\langle\mu - \xi, V\rangle s, \;\; s \in \Gamma(L_\xi), \;\; V \in \mathbf{g}. \tag{3.15}$$

Since G is Abelian, $\mu_\xi := \mu - \xi$ may also be regarded as a moment map for the Hamiltonian action of G on (M, ω). One then has the following extension of Proposition 3.10, which can be verified directly.

PROPOSITION 3.11. *L_ξ is μ_ξ-positive.*

Let $Q(M, \wedge^{p,0}(T^*M) \otimes L)_\xi$ denote the ξ-eigenspace of the G-representation $Q(M, \wedge^{p,0}(T^*M) \otimes L)$. That is, L_V acts on $Q(M, \wedge^{p,0}(T^*M) \otimes L)_\xi$ by multiplication by $2\pi\sqrt{-1}\langle\xi, V\rangle$. Then one verifies directly that

$$\dim Q\left(M, \wedge^{p,0}(T^*M) \otimes L\right)_\xi = \dim Q\left(M, \wedge^{p,0}(T^*M) \otimes L_\xi\right)^G. \tag{3.16}$$

This is the shifting trick mentioned above.

Now when ξ is a regular value of μ, in view of Proposition 3.11 one can apply Theorem 0.2 to get a weighted multiplicity formula calculating $\dim Q(M, \wedge^{p,0}(T^*M) \otimes L_\xi)^G$ and thus the ξ-multiplicity of $Q(M, \wedge^{p,0}(T^*M) \otimes L)$ in terms of quantities on the μ_ξ-symplectic reduction $M_{G,\xi} = \mu^{-1}(\xi)/G$. When ξ is not a regular value of μ, one may first apply Theorem 3.6 and then an induction procedure to calculate $\dim Q(M, \wedge^{p,0}(T^*M) \otimes L)_\xi$ via (3.16).

By summing over all these ξ's, one has clearly that

$$\dim Q\left(M, \wedge^{p,0}(T^*M)\otimes L\right) = \sum_{\xi} \dim Q\left(M, \wedge^{p,0}(T^*M)\otimes L\right)_{\xi}. \tag{3.17}$$

In particular, in the case where $G = S^1$, if for any such ξ denote by $F_\xi = \mu^{-1}(\xi)\cap F$ where F is the fixed point set of the G-action on M, then one can combine the above reasoning with Corollary 3.7 to get

$$\dim Q_{\mathrm{dR}}(M, L) = \sum_{\xi}\left(\operatorname{ind} D_{F_\xi,+}^{\wedge^{\mathrm{even},0}(T^*M)\otimes L_\xi}(V) - \operatorname{ind} D_{F_\xi,+}^{\wedge^{\mathrm{odd},0}(T^*M)\otimes L_\xi}(V)\right). \tag{3.18}$$

REMARK 3.12. Note that if ξ is not contained in the image of the μ, then it is automatically a regular value of μ. In this case, $M_{G,\xi} = \emptyset$. By Theorem 0.2, one sees immediately that in the summations in (3.17) and (3.18), the ξ actually runs through the integral lattice points contained in the image of μ.

Now if the Euler characteristic $\chi(M)$ of M is nonzero, then from (3.18) and the Atiyah-Singer index theorem [AS], which gives that

$$\dim Q_{\mathrm{dR}}(M, L) = \chi(M), \tag{3.19}$$

one deduces by (3.17) that at least one of the terms $\dim Q(M, \wedge^{\mathrm{even},0}(T^*M)\otimes L)_\xi$ and $\dim Q(M, \wedge^{\mathrm{odd},0}(T^*M)\otimes L)_\xi$ should be nonzero. In view of (3.16), (3.18), Theorem 3.6 and Corollary 3.7, this provides a concrete example mentioned in Remark 3.8.

Appendix. The construction of the Dirac operators appearing in Theorem 3.6. Let W be a S^1-equivariant Hermitian vector bundle equipped with a S^1-equivariant Hermitian connection. The purpose of this appendix is to make explicit constructions of the Spin^c-Dirac operators $D^W_{F_0,+}(\pm V)$ appearing in Theorem 3.6 (Compare with the Appendix in [TZ2]).

Let N be the normal bundle to F_0, then N inherits naturally an almost complex structure J_N, a Hermitian metric g^N as well as a Hermitian connection ∇^N.

Since V is a generator of the S^1-action, $\sqrt{-1}L_V$ acts on N as a covariantly constant invertible self-adjoint operator commuting with J_N. Let N_+, N_- be the positive and negative eigenbundles of $\sqrt{-1}L_V|_N$ respectively. Then J_N preserves $N_\pm$, and one has the canonical splittings

$$N_\pm \otimes \mathbf{C} = N_\pm^{(1,0)} \oplus N_\pm^{(0,1)}. \tag{A.1}$$

Let $\mathrm{Sym}(N_+^{(1,0)})$ (resp. $\mathrm{Sym}(N_-^{(0,1)})$) be the total symmetric power of $N_+^{(1,0)}$ (resp. $N_-^{(0,1)}$). Then $\mathrm{Sym}(N_-^{(0,1)})\otimes \mathrm{Sym}(N_+^{(1,0)})\otimes \det(N_+^{(1,0)})\otimes W|_{F_0}$ is an infinite dimensional vector bundle over F_0, on which $\sqrt{-1}L_V$ acts as a covariantly constant self-adjoint operator. Furthermore, its zero eigenbundle, denoted by $(\mathrm{Sym}(N_-^{(0,1)})\otimes \mathrm{Sym}(N_+^{(1,0)})\otimes \det(N_+^{(1,0)})\otimes W|_{F_0})^{S^1}$, is of finite dimension.

DEFINITION A.1. *The operator $D^W_{F_0,+}(V)$ is defined as the (twisted) Spin^c-Dirac operator on F_0,*

$$D^W_{F_0,+}(V) : \Omega^{0,\mathrm{even}}\left(F_0, \left(\mathrm{Sym}\left(N_-^{(0,1)}\right)\otimes \mathrm{Sym}\left(N_+^{(1,0)}\right)\otimes \det\left(N_+^{(1,0)}\right)\otimes W|_{F_0}\right)^{S^1}\right)$$

$$\to \Omega^{0,\mathrm{odd}}\left(F_0, \left(\mathrm{Sym}\left(N_-^{(0,1)}\right)\otimes \mathrm{Sym}\left(N_+^{(1,0)}\right)\otimes \det\left(N_+^{(1,0)}\right)\otimes W|_{F_0}\right)^{S^1}\right). \tag{A.2}$$

If we change V to $-V$, we get the similar definition of $D^W_{F_0,+}(-V)$.

By setting $W = \wedge^{p,0}(T^*M) \otimes E$, one gets the Dirac operators appearing in Theorem 3.6.

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