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004

K–THEORY FOR TRIANGULATED CATEGORIES III(A): THE THEOREM OF THE HEART*

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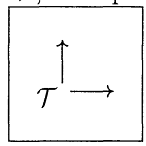
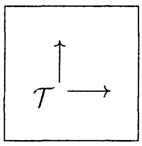
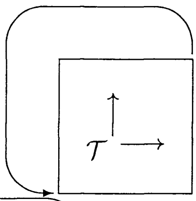
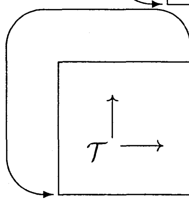
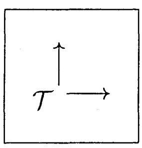
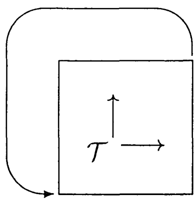
0. Introduction. This is the fourth installment of a series. The main point of the entire series is the following: given a triangulated category $\mathcal{T}$, it is possible to attach to it a K-theory space. Its delooping will be denoted  . Note that, starting with the present article, we no longer wish to consider the construction without the differentials. In the earlier parts of this series, we considered two simplicial sets, namely

 and 

From now on, we wish to consider only  ; all the simplicial sets will be the ones with coherent differentials. We will feel free to omit the differentials in the symbol for the simplicial set. The simplicial sets

 and 

are henceforth to be viewed as identical.

The reason for this is that, after Section II.2, we have nothing more to say about the construction without differentials. In Section II.2, we proved some significant facts about the simplicial set without the differentials. The reader is referred to the introduction of *K-theory for triangulated categories II* for more detail. Anyway, from Section II.3 on, all our simplicial sets come with coherent differentials.

The key theorem of this series of articles is

Strong Theorem I.7.1. *Let $\mathcal{T}$ be a small triangulated category with a non–degenerate t–structure. Let $\mathcal{A}$ be the heart of the t–structure. With the simplicial set*

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$$\boxed{\begin{array}{c}\ \ \uparrow\\ \mathcal{T}\ \longrightarrow\end{array}}$$

defined appropriately, the natural map

$$\boxed{\begin{array}{c}\ \ \uparrow\\ \mathcal{A}\ \longrightarrow\end{array}} \hookrightarrow \boxed{\begin{array}{c}\ \ \uparrow\\ \mathcal{T}\ \longrightarrow\end{array}}$$

induces a homotopy equivalence.

In this theorem, $\boxed{\begin{array}{c}\ \ \uparrow\\ \mathcal{A}\ \longrightarrow\end{array}}$ is homotopy equivalent to Quillen's Q–construction of the abelian category $\mathcal{A}$. The precise definition of $\boxed{\begin{array}{c}\ \ \uparrow\\ \mathcal{T}\ \longrightarrow\end{array}}$ is a somewhat delicate point, discussed in some detail in the introduction to *K–theory for triangulated categories I.* Delicate points aside, a consequence of this theorem is that, given two non–degenerate t–structures on the same triangulated category, the two hearts have isomorphic K–theories. We will prove that many abelian categories have isomorphic K–theories.

The proof of Strong Theorem I.7.1 is the bulk of *K–theory for triangulated categories III.* But in fact, we will be proving more. Let me state for the reader another theorem, which will follow from the same proof. We begin with definitions.

DEFINITION 0.1. *Let $\mathcal{E}$ be an exact category. A sequence*

$$x \longrightarrow y \longrightarrow z$$

in $\mathcal{E}$ is called exact at y *if*

0.1.1. *The map $x \longrightarrow y$ factors as*

$$x \longrightarrow y' \longrightarrow y,$$

with $x \longrightarrow y'$ an admissible epi, and $y' \longrightarrow y$ an admissible mono.

0.1.2. *The map $y \longrightarrow z$ factors as*

$$y \longrightarrow y'' \longrightarrow z,$$

with $y \longrightarrow y''$ an admissible epi, and $y'' \longrightarrow z$ an admissible mono.

0.1.3. *$y' \longrightarrow y \longrightarrow y''$ is an admissible short exact sequence.*

Let $Gr^b(\mathcal{E})$ be the category of bounded, $\mathbb{Z}$–graded objects in $\mathcal{E}$. Let $\Sigma : Gr^b(\mathcal{E}) \longrightarrow Gr^b(\mathcal{E})$ be the shift map. Next, we define a simplicial set.

DEFINITION 0.2. *The bisimplicial set*

$$\boxed{\begin{array}{c}\qquad\ \ \uparrow\\ Gr^b(\mathcal{E})\ \longrightarrow\end{array}}$$

is defined as follows. A (p, q)–simplex is a diagram in $Gr^b(\mathcal{E})$

$$\begin{array}{ccccc} X_{p0} & \longrightarrow & \cdots & \longrightarrow & X_{pq} \\ \uparrow & & & & \uparrow \\ \vdots & & & & \vdots \\ \uparrow & & & & \uparrow \\ X_{00} & \longrightarrow & \cdots & \longrightarrow & X_{0q} \end{array}$$

together with a coherent differential $X_{pq} \longrightarrow \Sigma X_{00}$. The condition is that, for every $0 \leq i \leq i' \leq p$, $0 \leq j \leq j' \leq q$, the sequence

$$\Sigma^{-1} X_{i'j'} \longrightarrow X_{ij} \longrightarrow X_{i'j} \oplus X_{ij'} \longrightarrow X_{i'j'} \longrightarrow \Sigma X_{ij}$$

gives, in each degree, an exact sequence in $\mathcal{E}$.

The proof in this article, which will establish Theorem I.7.1, will also prove the following fact.

Strong Theorem I.4.8. *Let $\mathcal{E}$ be a small exact category. The natural map*

$$\boxed{\begin{array}{l} \ \uparrow \\ \mathcal{E}\ \longrightarrow \end{array}} \hookrightarrow \boxed{\begin{array}{l} \qquad\quad \uparrow \\ Gr^b(\mathcal{E})\ \longrightarrow \end{array}}$$

induces a homotopy equivalence.

Here, $\boxed{\begin{array}{l} \ \uparrow \\ \mathcal{E}\ \longrightarrow \end{array}}$ is homotopy equivalent to Quillen's Q–construction on the exact category $\mathcal{E}$.

Now for a review of the earlier parts of this series. *K–theory for triangulated categories I* contains a proof of the special case of Theorem I.4.8, where $\mathcal{E}$ is an abelian category. *K–theory for triangulated categories II* contains a proof of the special case of Theorem I.7.1, where $\mathcal{T}$ is $D^b(\mathcal{A})$, the bounded derived category of an abelian category $\mathcal{A}$, and the t–structure is the standard one. In a very precise sense, the current article is better. It proves the sharpest and most general results of the series.

This raises the question: what is the point of the earlier articles? Let me try to answer it briefly.

First of all, the three proofs are all different. They look at quite different chains of intermediate simplicial sets. Let us agree that the current theory is unsatisfactory, and that it is to be hoped that there will, some day, be a simpler and more general treatment. Then surely different arguments are of interest. It is unclear which will lead to the better generalisations and simplifications.

The second reason that the earlier articles are of interest, is that they are simpler. The theorems they prove are not optimal; but there is virtue in seeing first a simple proof of a less general statement. The simpler argument is also easier to motivate. Finally, it lends itself more to careful study of alternatives, such as the construction without the differentials. In *K–theory for triangulated categories I* and *II,* we do more than just give the proofs of special cases of the strong theorems stated above. We explain how and why the proofs work.

In the introduction to *K–theory for triangulated categories I,* I divided up the readers of any piece of mathematics into three broad groups, listed in order of probable size:

Group 1: The people who want a rough idea of the contents of the article, and at the very most a sketch of the proofs in an easy special case.

Group 2: The people who want to check the result, because they might consider using it in their own work.

Group 3: The people reading the article because they might work on the problem themselves.

The first two parts of this series, *K–theory for triangulated categories I(A)* and *I(B)*, were intended for a Group 1 audience. The third, *K–theory for triangulated categories II*, is emphatically for the benefit of Group 3. The present part is primarily for Group 2.

In *K–theory for triangulated categories I(A)*, we introduce the definitions and notation (this takes us some 88 pages). Then in *K–theory for triangulated categories I(B)*, we give the simplest proof of the simplest version of our theorem. All the readers of subsequent parts are assumed to be familiar with the notation. So you should have read at least *K–theory for triangulated categories I(A)*, if you proceed beyond this word. In fact, it is highly advisable to have skimmed through the rest of *K–theory for triangulated categories I.* There is a little more notation introduced in the last two sections, but even more relevant, there is a relatively gentle introduction to the way the proofs work, and the type of simplicial sets one constructs.

The first section of this part, Section 1, is again quite soft. There are two types of homotopy that I know, for the simplicial sets that come up in triangulated K–theory. The first type is the trivial homotopies. These are the triangulated analogues of contractions to an initial or a terminal object. The second type of homotopy is the non–trivial homotopies. And one of the key features of this theory is that there is really only one of the non–trivial homotopies.

In Section 1, we make this very precise, showing with explicit examples how to reduce a typical non–trivial homotopy in this theory to a blueprint. This section is really a must for anyone who reads beyond *K–theory for triangulated categories I.* Although not compulsory, it is strongly recommended that the reader also look at Section II.1, the first section of *K–theory for triangulated categories II.* Although *K–theory for triangulated categories II* was written with a Group 3 audience in mind, Section II.1 is only at Group $2\frac{1}{2}$ level. It is quite soft. It discusses, in a general way, the type of simplicial sets and homotopies that come up in the proof, and it also discusses why the various homotopies are well defined. Section 1 of this part, being written for Group 2, focuses on the non–trivial homotopy. It turns out that one can give a very satisfactory treatment of it, and explain why checking that it is well–defined can be reduced to verifying it on a blueprint. Section II.1 is for Group 3, and therefore it tends to focus on potential problems. It turns out that one of the so–called trivial homotopies is less trivial than it seems to be at first sight. This homotopy is the truncation.

Having familiarised himself with the notation, and the type of argument used to show that the homotopies are well–defined, the reader will discover that he has read almost two thirds of this *K–theory for triangulated categories III.* The remaining sections, Sections 2 and 3, contain the proof of the main theorem of the article. The proof we give here is very businesslike. It demonstrates the best theorem I have about the K–theory of triangulated categories, and does so as directly as possible. It is very difficult to say much about the proof, that would be in any way instructive. It is a sequence of maps and homotopies, that get us where we want to be.

This completes the discussion of all the theorems in the article. There are also three conjectures. They may be found in Appendices A, B and C. Appendix A explores the natural map from Waldhausen's K–theory of a Waldhausen category, to the triangulated K–theory of the associated triangulated category. The map need not be a homotopy equivalence. But there is an intermediate space, with a description similar to Waldhausen's, whose homotopy type is conjecturally the same as triangulated K–theory. In Appendix A, I state the conjecture, and show that if true, it implies that for any exact category $\mathcal{E}$, the K–theory of $D^b(\mathcal{E})$ agrees with Quillen's K–theory of $\mathcal{E}$.

Appendix B states a conjecture, generalising Quillen's localisation theorem. The conjecture is straightforward enough to state. In the appendix, I also explain my attempts (so far quite unsuccessful) to generalise Quillen's proof.

Finally, Appendix C gives a vaguely–stated conjecture, generalising Quillen's devissage theorem to triangulated categories.

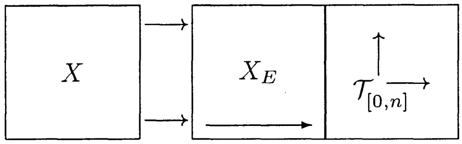
I tried to keep the conjectural appendices very short. Appendix A, on the relation between Waldhausen's K–theory and triangulated K–theory, is the longest. It is the problem I thought about the most. At one point I even thought I had a proof. There is an error in the manuscript I wrote at the time. But even after finding the error, I was under the impression it was fixable. I never checked this carefully, and in the five years that have since passed, I have forgotten all the subtle points. There still exists a 300 page document, *K–theory for triangulated categories IV*, with an outline of what the proof might(?) look like.

I was never under the impression, that I knew how to prove the localisation conjecture of Appendix B. All the appendix offers is the statement of the conjecture, and the statements that would need to be proved, to generalise Quillen's argument from abelian to triangulated categories. Finally, in Appendix C there is not even a clearly formulated conjecture; all there is is an idea.

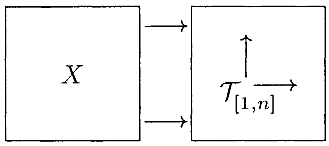
One historical note. The proofs of Theorems I.4.8 and I.7.1 in the weak, special cases date to September, 1988. However, I did not have the proof given here until the following spring, some six months later.

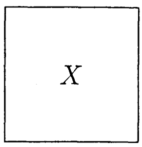
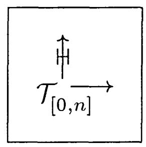
1. Why the Non Trivial Homotopy Is Always Well–Defined. In *K–theory for triangulated categories I*, the reader saw a very simple proof of a weak version of our main theorem. But since we introduced the notation as the proof progressed, there was no possibility of giving a general discussion of our homotopies. It is now time to rectify this shortcoming.

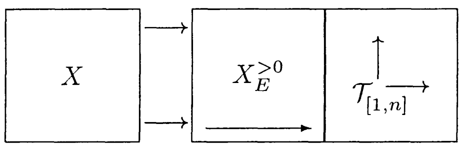
Our homotopies fall into two groups: the trivial and the non–trivial. The trivial homotopies are contractions to an initial or terminal object. Our notation for such homotopies was something like



Sometimes the matter was a little more delicate. For instance, if we consider the simplicial set

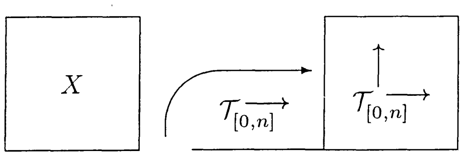


where  lies in the simplicial set  , there is a homotopy



which is very like the contraction to the initial object, except it need not be a contraction. This is because of the non–uniqueness of the differential from the truncation. But those are subtle points for Group 3 readers. I give a very thorough discussion of the non–triviality of the so–called trivial homotopy, in *K–theory for triangulated categories II*. A reader with an interest in this point is referred to Section II.1. The present section occupies itself with the triviality of the non–trivial homotopy. As befits the type of material we present our Group 2 audience, it is possible to give a very satisfactory general discussion, of why the cells in the non–trivial homotopy always do as they should.

We remind the reader of our non–trivial homotopy. It is the homotopy whose shorthand came to something like



In other words, the cells of the homotopy would be

$$
\begin{array}{ccccc}
\begin{array}{|ccc|}\hline X_{m0} \longrightarrow & \cdots & \longrightarrow X_{mn} \\ \uparrow & & \uparrow \\ \vdots & & \vdots \\ \uparrow & & \uparrow \\ X_{00} \longrightarrow & \cdots & \longrightarrow X_{0n} \\ \hline \end{array}
& \longrightarrow &
\underbrace{\overbrace{\begin{array}{ccc} X_{mn}\oplus Y_{00} \longrightarrow & \cdots & \longrightarrow X_{mn}\oplus Y_{0i} \\ \uparrow & & \uparrow \\ \vdots & & \vdots \\ \uparrow & & \uparrow \\ X_{0n}\oplus Y_{00} \longrightarrow & \cdots & \longrightarrow X_{0n}\oplus Y_{0i} \end{array}}}_{(i+1)\ terms}
& \longrightarrow &
\underbrace{\overbrace{\begin{array}{ccc} Y_{mi} \longrightarrow & \cdots & \longrightarrow Y_{mr} \\ \uparrow & & \uparrow \\ \vdots & & \vdots \\ \uparrow & & \uparrow \\ Y_{0i} \longrightarrow & \cdots & \longrightarrow Y_{0r} \end{array}}}_{(n-i+1)\ terms}
\end{array}
$$

In practice, we almost never apply this homotopy as written above. Usually, there are restrictions on the objects allowed, and on the morphisms permitted among them. Following the conventions of Section II.1, we even have a notation which reminds us that the objects and morphisms are restricted, without specifying the restrictions.

The symbol $\boxed{\mathcal{T}_*^{\,?\,?}}$ stands for the simplicial set, in which the exact subcategory (the subscript) is unspecified, and the horizontal and vertical morphisms are also left ambiguous (hence the question marks). In this section, we propose to give general arguments for why the non–trivial homotopy above, and its various close cousins whom we have met in *K–theory for triangulated categories I*, are all well–defined. We need to establish that the purported cells of the homotopy are genuine simplices. For this, it is simpler to assume there are no restrictions on morphisms and objects. In other words, it is easiest to handle the case where

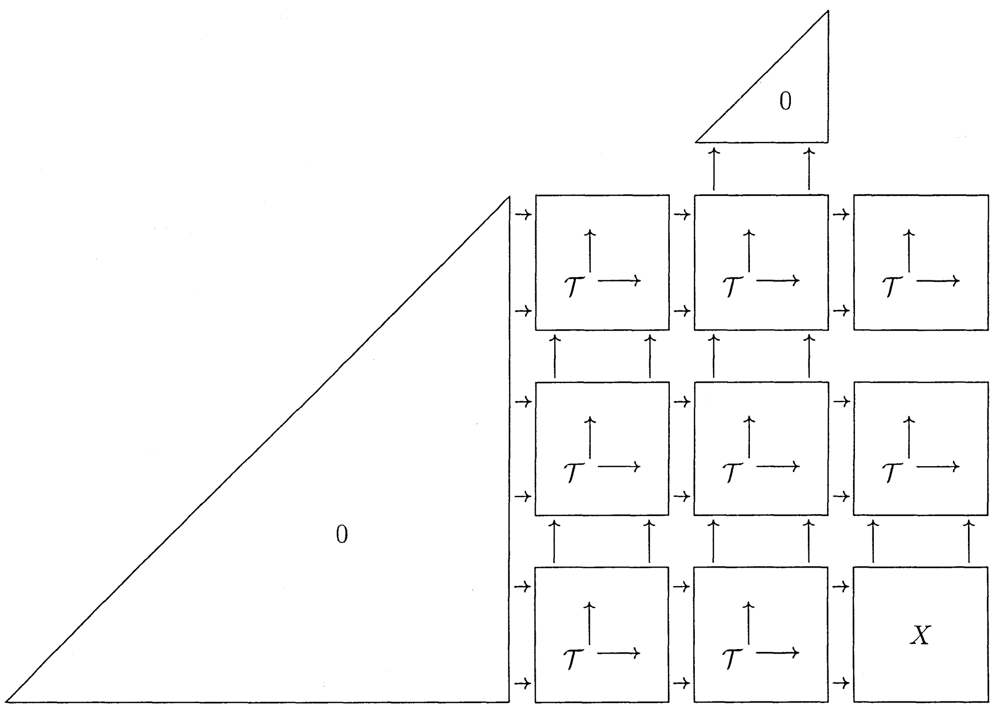
$$
\boxed{\mathcal{T}_*^{\,?\,?}} \qquad = \qquad \boxed{\begin{array}{l} \ \ \uparrow \\ \mathcal{T} \longrightarrow \end{array}}
$$

Thus, we will really not be proving any theorems about $\boxed{\mathcal{T}_*^{\,?\,?}}$. We will be showing that certain homotopies are well–defined in $\boxed{\begin{array}{l} \ \ \uparrow \\ \mathcal{T} \longrightarrow \end{array}}$. Specifically, we will show that the homotopy

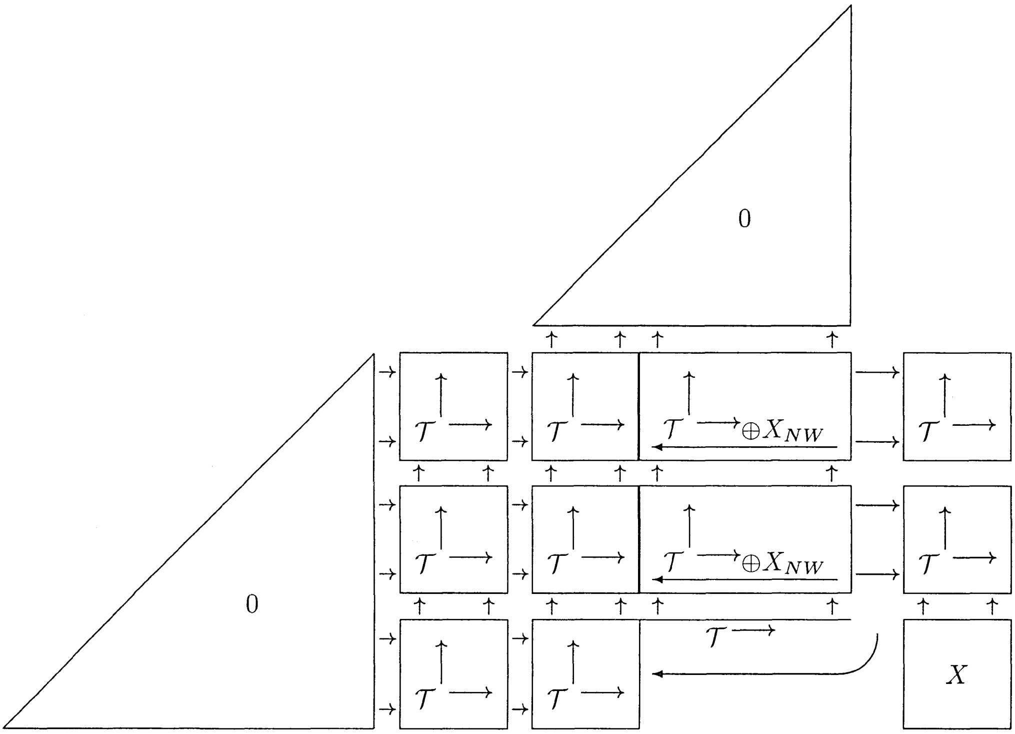
$$
\boxed{X} \qquad \underline{\mathcal{T}\longrightarrow}\ \boxed{\begin{array}{l} \ \ \uparrow \\ \mathcal{T} \longrightarrow \end{array}}
$$

together with its various analogues, are well defined. The particular subcategory of $\mathcal{T}$ that we are dealing with, and the precise nature of the morphisms, play absolutely no role in the main theorem of this section; they are, after all, the source of the subtleties. If we avoid the subtleties like the plague, we can state and prove an honest theorem. After doing that, we will return to showing how this theorem can be applied, even in the subtle situation where the simplicial set is $\boxed{\mathcal{T}_*^{?\,?}}$, that is where the objects and the morphisms are restricted.

THEOREM 1.1. *Consider the simplicial set*

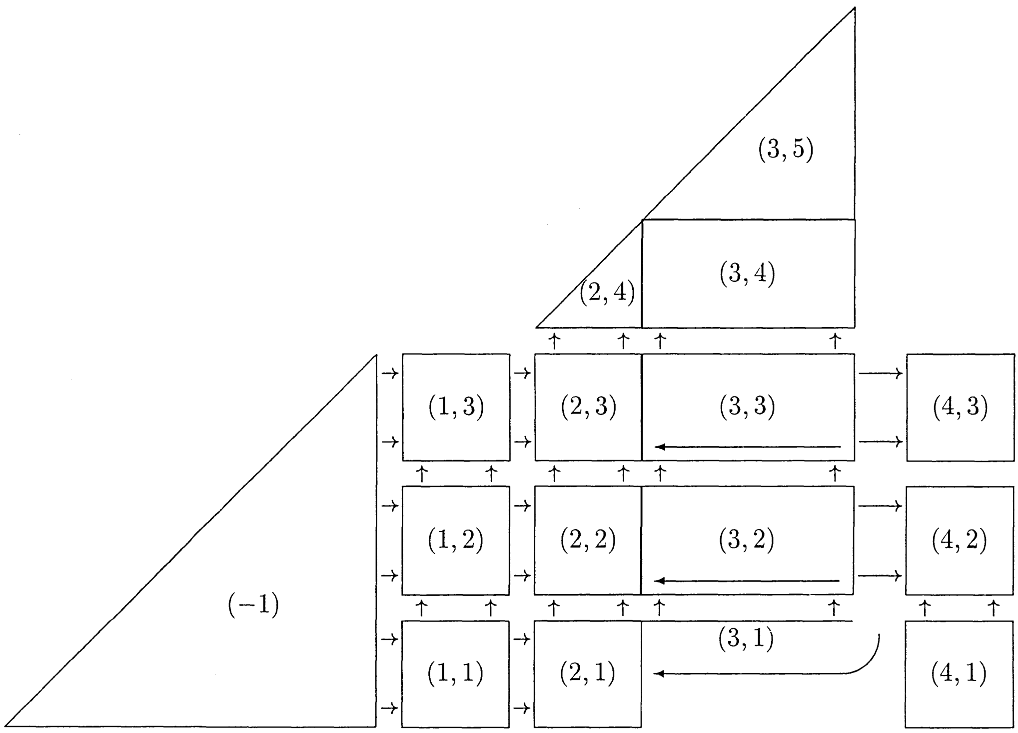


Then on it, the homotopy whose symbol would be



is well defined; its cells are all simplices for the simplicial set.

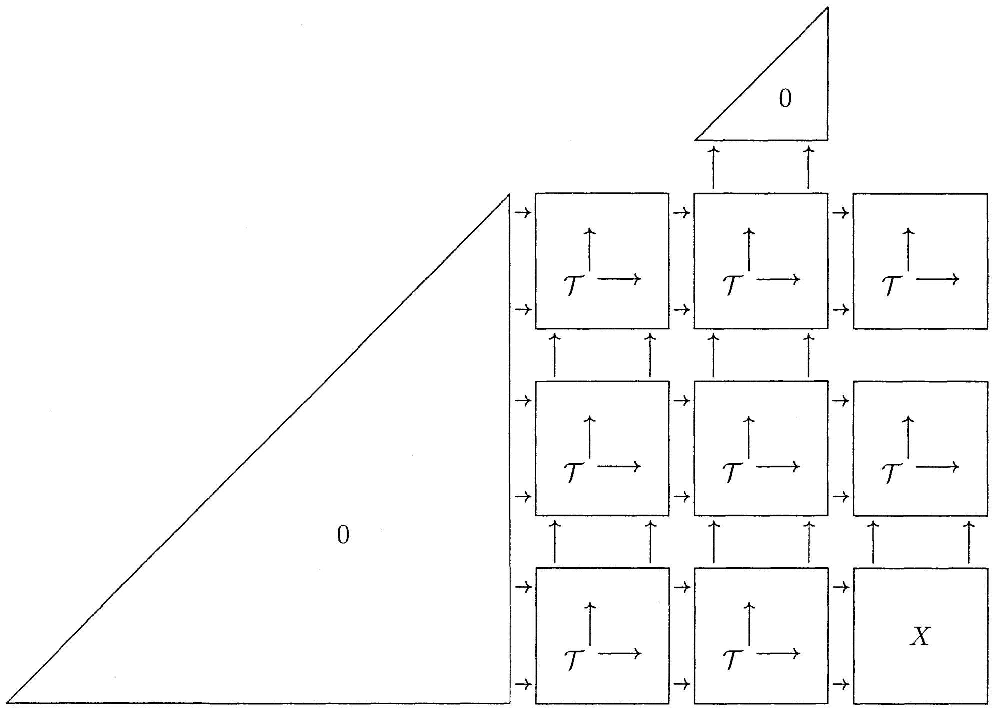
Proof. In this proof, when we say that a square is Mayer–Vietoris, this will mean that, with an obvious choice of a (coherent) differential, the square "folds" to give a semi–triangle. We want to prove all the squares in some large diagram Mayer–Vietoris in this sense. Let us begin by labeling all the regions of this diagram. We rewrite it as



The labels are to allow us to easily refer to any particular region of the homotopy.

What we must show first, is that every square that occurs in this gigantic simplex at the very least folds to give a semitriangle. More precisely, we must show that there is a way to choose the differentials, so that all the squares in the diagram will fold to mapping cones on triangles, or at the very worst direct summands of mapping cones. Furthermore, we will do so in such a way that all the maps, differentials included, will be given maps; they are matrices in maps defining the starting cell of the homotopy.

Any square in the diagram which does not meet Column 3 is a square in



and therefore it is Mayer–Vietoris because by hypothesis, the homotopy started with a simplex. Thus we need only concern ourselves with squares which meet Column 3.

Any square meeting Column 3 must also meet Column (–1) (i.e. the triangle on the right), Column 1, Column 2, Column 3 or Column 4. Columns (–1), 1 and 2 behave identically as far as the following argument goes. We will therefore treat only the cases of Column 1, Column 3 and Column 4.

1.1.1. **Case of Column 1.** *Suppose we are trying to prove that a candidate square in the union of Columns 1 and 3 can be chosen canonically to be a Mayer–Vietoris square. Then there are 3 cases to consider. Either the square is contained in the first row, or it does not meet the first row, or it meets the first row as well as some other row. We discuss these cases separately.*

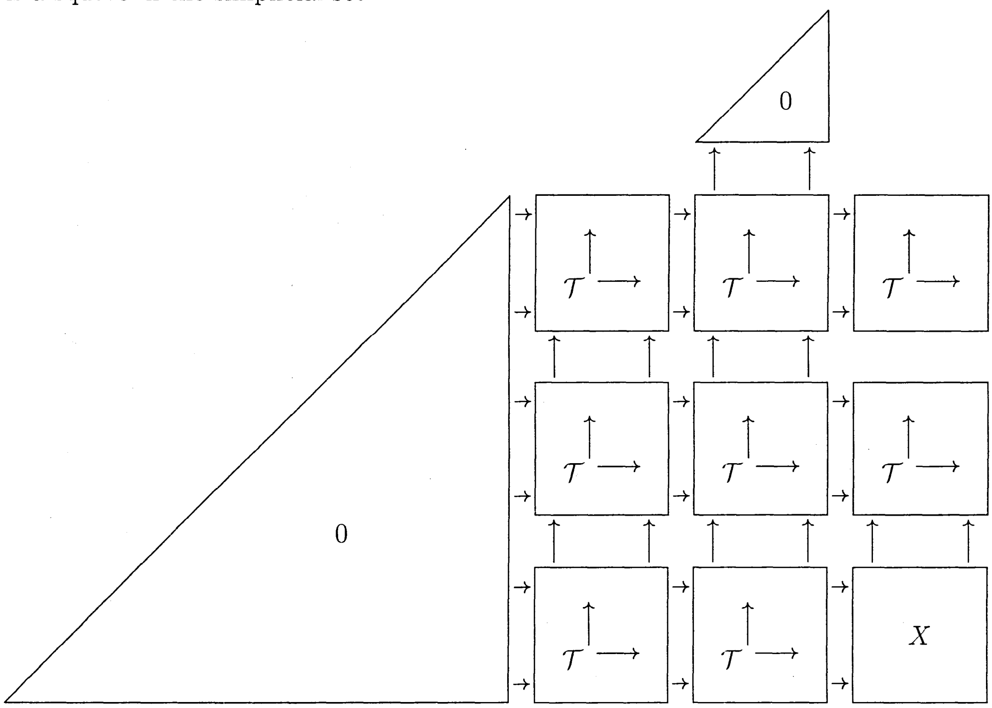
1.1.1.1. Suppose the square is entirely contained in the first row. Then it consists of taking a column in the (1,1) gridbox in the labeled diagram, and pairing it with a column in the (3,1) gridbox. The result is a square

$$\begin{tikzcd} A_{i'j} \arrow[r] & B_{tj'} \oplus X_{i'0} \\ A_{ij} \arrow[r] \arrow[u] & B_{tj'} \oplus X_{i0} \arrow[u] \end{tikzcd}$$

where the A's are a column in the (1,1) box, the X's come from the west column of the (4,1) box and the $B_{tj'}$ is in the north face of the (3,1) box. But

$$\begin{tikzcd} A_{i'j} \arrow[r] & X_{i'0} \\ A_{ij} \arrow[u] \arrow[r] & X_{i0} \arrow[u] \end{tikzcd}$$

is a square in the simplicial set



and is hence automatically Mayer–Vietoris. The square

$$\begin{tikzcd} A_{i'j} \arrow[r] & B_{tj'} \oplus X_{i'0} \\ A_{ij} \arrow[u] \arrow[r] & B_{tj'} \oplus X_{i0} \arrow[u] \end{tikzcd}$$

is isomorphic to the direct sum of

$$\begin{tikzcd} A_{i'j} \arrow[r] & X_{i'0} \\ A_{ij} \arrow[u] \arrow[r] & X_{i0} \arrow[u] \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} 0 \arrow[r] & B_{tj'} \\ 0 \arrow[u] \arrow[r] & B_{tj'} \arrow[u] \end{tikzcd}$$

and is therefore also Mayer–Vietoris.

1.1.1.2. The next possibility is that the square does not meet Row 1 at all. In that case, it must be of the form

$$\begin{tikzcd}
A_{i'j} \arrow[r] & B_{i'j'} \oplus X_{NW} \\
A_{ij} \arrow[u] \arrow[r] & B_{ij'} \oplus X_{NW} \arrow[u]
\end{tikzcd}$$

and can be expressed as a sum of

$$\begin{tikzcd}
A_{i'j} \arrow[r] & B_{i'j'} & & 0 \arrow[r] & X_{NW} \\
A_{ij} \arrow[u] \arrow[r] & B_{ij'} \arrow[u] & \text{and} & 0 \arrow[u] \arrow[r] & X_{NW} \arrow[u]
\end{tikzcd}$$

where, once again, each of the summands is trivially Mayer–Vietoris.

1.1.1.3. The first subtle case, is when both the first row and some other are allowed to occur. In this case, our square takes the form

$$\begin{tikzcd}
C_{i'j} \arrow[r] & D_{i'j'} \oplus X_{t0} \\
A_{ij} \arrow[u] \arrow[r] & B_{tj'} \oplus X_{i0} \arrow[u]
\end{tikzcd}$$

where the A_{ij} comes from the (1,1) box, the $B_{tj'}$ from the north of the (3,1) box, the X_{i0} from the west face of the (4,1) box, the $X_{t0} = X_{NW}$ is the north–west corner of the (4,1) box and the $C_{i'j}$ is in the $(1, x)$ box and $D_{i'j'}$ in the $(3, x)$ box, where $x = 2$ or 3. Although we have already discussed this in *K–theory for triangulated categories I*, we remind the reader of the argument that shows why this square is Mayer–Vietoris. We have a diagram of squares

$$\begin{tikzcd}
C_{i'j} \arrow[r] & D_{i'j'} & \\
A_{tj} \arrow[u] \arrow[r] & B_{tj'} \arrow[u] \arrow[r] & X_{t0} \\
A_{ij} \arrow[u] \arrow[r] & B_{ij'} \arrow[u] \arrow[r] & X_{i0} \arrow[u]
\end{tikzcd}$$

This gives us three semi–triangles

$$(1.1)\qquad A_{ij} \longrightarrow B_{ij'} \oplus A_{tj} \longrightarrow B_{tj'} \longrightarrow \Sigma A_{ij}$$

$$(1.2)\qquad A_{ij} \longrightarrow X_{i0} \oplus A_{tj} \longrightarrow X_{t0} \longrightarrow \Sigma A_{ij}$$

and

$$(1.3)\qquad A_{ij} \longrightarrow B_{ij'} \oplus C_{i'j} \longrightarrow D_{i'j'} \longrightarrow \Sigma A_{ij}$$

But now the square

$$\begin{tikzcd} C_{i'j} \arrow[r] & D_{i'j'} \oplus X_{t0} \\ A_{ij} \arrow[u] \arrow[r] & B_{tj'} \oplus X_{i0} \arrow[u] \end{tikzcd}$$

will be Mayer–Vietoris exactly if the following candidate triangle (1.4) is a semi–triangle

$$(1.4)\qquad A_{ij} \longrightarrow C_{i'j} \oplus B_{tj'} \oplus X_{i0} \longrightarrow D_{i'j'} \oplus X_{t0} \longrightarrow \Sigma A_{ij}$$

and the real point is that (1.4) can be obtained as a direct summand of the mapping cone on the natural map of semitriangles

$$(1.1) \longrightarrow (1.2) \oplus (1.3).$$

What is more, the other direct summand is a contractible triangle. This is left as an exercise to the reader, who can also find a discussion in Remark I.5.3.

One remark should be made now. If it so happens that the semi–triangles

$$A_{ij} \longrightarrow B_{ij'} \oplus A_{tj} \longrightarrow B_{tj'} \longrightarrow \Sigma A_{ij}$$

$$A_{ij} \longrightarrow X_{i0} \oplus A_{tj} \longrightarrow X_{t0} \longrightarrow \Sigma A_{ij}$$

and

$$A_{ij} \longrightarrow B_{ij'} \oplus C_{i'j} \longrightarrow D_{i'j'} \longrightarrow \Sigma A_{ij}$$

are really triangles, or more precisely triangles that come from three short exact sequences

$$0 \longrightarrow A_{ij} \longrightarrow B_{ij'} \oplus A_{tj} \longrightarrow B_{tj'} \longrightarrow 0$$

$$0 \longrightarrow A_{ij} \longrightarrow X_{i0} \oplus A_{tj} \longrightarrow X_{t0} \longrightarrow 0$$

and

$$0 \longrightarrow A_{ij} \longrightarrow B_{ij'} \oplus C_{i'j} \longrightarrow D_{i'j'} \longrightarrow 0$$

of objects in some heart $\mathcal{C}$ of the triangulated category $\mathcal{T}$, and where the differentials are the unique ones possible, then the map of triangles

$$(1.1) \longrightarrow (1.2) \oplus (1.3)$$

is really a map of short exact sequences, and it is easy to show that the mapping cone is indeed a triangle; maps of short exact sequences are good maps of triangles. Thus the direct summand is also a triangle, and it follows that the differential in

$$\begin{tikzcd} C_{i'j} \arrow[r] & D_{i'j'} \oplus X_{t0} \\ A_{ij} \arrow[r] \arrow[u] & B_{tj'} \oplus X_{i0} \arrow[u] \end{tikzcd}$$

is the *unique* map making the above a genuine triangle.

1.1.2. **Case of Column 3.** *The next case to consider, is where we have a square entirely contained in Column 3. Once again, this divides into cases, depending on what rows occur. We distinguish six cases. Case 1.1.2.1 is where only Row 1 occurs. Case 1.1.2.2 is where Row 1 is paired with any of Rows 2, 3 or 4. Case 1.1.2.3 is where Row 1 and Row 5 are paired, Case 1.1.2.4 pairs any of the Rows 2, 3 or 4, Case 1.1.2.5 assumes the square is contained in Row 5, while the last case, Case 1.1.2.6, pairs one of Rows 2, 3 or 4 with Row 5.*

1.1.2.1. Suppose the square we want to show Mayer–Vietoris is embedded in the (3,1) box. Then it takes the form

$$\begin{tikzcd} B_{tj} \oplus X_{i'0} \arrow[r] & B_{tj'} \oplus X_{i'0} \\ B_{tj} \oplus X_{i0} \arrow[r] \arrow[u] & B_{tj'} \oplus X_{i0} \arrow[u] \end{tikzcd}$$

and is therefore the sum of

$$\begin{tikzcd} B_{tj} \arrow[r] & B_{tj'} \\ B_{tj} \arrow[r] \arrow[u] & B_{tj'} \arrow[u] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X_{i'0} \arrow[r] & X_{i'0} \\ X_{i0} \arrow[r] \arrow[u] & X_{i0} \arrow[u] \end{tikzcd}$$

which are both contractible triangles.

1.1.2.2. Suppose that we are dealing with a square, which is in the union of the (3,1) box and the $(3, x)$ box, where $x = 2$, 3 or 4. Then the square is of the form

$$\begin{tikzcd} C_{i'j} \oplus X_{NW} \arrow[r] & C_{i'j'} \oplus X_{NW} \\ B_{tj} \oplus X_{i0} \arrow[r] \arrow[u] & B_{tj'} \oplus X_{i0} \arrow[u] \end{tikzcd}$$

and is therefore the sum of

$$\begin{tikzcd} C_{i'j} \arrow[r] & C_{i'j'} \\ B_{tj} \arrow[r] \arrow[u] & B_{tj'} \arrow[u] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X_{NW} \arrow[r] & X_{NW} \\ X_{i0} \arrow[r] \arrow[u] & X_{i0} \arrow[u] \end{tikzcd}$$

The first of these is part of the simplex s_n, while the second is contractible.

1.1.2.3. If our square is contained in the union of the (3,1) and (3,5) boxes, it is of the form

$$\begin{tikzcd} C_{i'j} \arrow[r] & C_{i'j'} \\ B_{tj} \oplus X_{i0} \arrow[r] \arrow[u] & B_{tj'} \oplus X_{i0} \arrow[u] \end{tikzcd}$$

and is therefore the sum of

$$\begin{tikzcd} C_{i'j} \arrow[r] & C_{i'j'} \\ B_{tj} \arrow[r] \arrow[u] & B_{tj'} \arrow[u] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} 0 \arrow[r] & 0 \\ X_{i0} \arrow[r] \arrow[u] & X_{i0} \arrow[u] \end{tikzcd}$$

Once again, the first is part of the simplex s_n, while the second is contractible.

1.1.2.4. Now suppose that we are dealing with a square inside the third column, and somewhere in rows 2, 3 or 4. Then it has the form

$$\begin{tikzcd} C_{i'j} \oplus X_{NW} \arrow[r] & C_{i'j'} \oplus X_{NW} \\ B_{ij} \oplus X_{NW} \arrow[r] \arrow[u] & B_{ij'} \oplus X_{NW} \arrow[u] \end{tikzcd}$$

and is therefore the sum of

$$\begin{tikzcd} C_{i'j} \arrow[r] & C_{i'j'} \\ B_{ij} \arrow[r] \arrow[u] & B_{ij'} \arrow[u] \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} X_{NW} \arrow[r] & X_{NW} \\ X_{NW} \arrow[r] \arrow[u] & X_{NW} \arrow[u] \end{tikzcd}$$

The first of these is part of the simplex s_n, while the second is contractible.

1.1.2.5. This case assumes that the entire square is in the (3,5) box. In that case, it is just a square in the simplex s_n, quite pure and unadulterated.

1.1.2.6. The last case to consider is where we are dealing with a square inside the third column, containing one of Rows 2, 3 or 4, and Row 5. Then it has the form

$$\begin{tikzcd} C_{i'j} \arrow[r] & C_{i'j'} \\ B_{tj} \oplus X_{NW} \arrow[r] \arrow[u] & B_{tj'} \oplus X_{NW} \arrow[u] \end{tikzcd}$$

and is therefore the sum of

$$\begin{tikzcd} C_{i'j} \arrow[r] & C_{i'j'} \\ B_{tj} \arrow[r] \arrow[u] & B_{tj'} \arrow[u] \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} 0 \arrow[r] & 0 \\ X_{NW} \arrow[r] \arrow[u] & X_{NW} \arrow[u] \end{tikzcd}$$

and, yet again, the first of these is part of the simplex s_n, while the second is contractible.

1.1.3. **Case of Column 4.** *The last series of cases to consider is where the fourth column also occurs. In other words, we are concerned with the possibility that our square is in the union of the third and fourth columns. Again, depending on which rows occur, we deal with cases. Case 1.1.3.1 is where the square is entirely in the first row. Case 1.1.3.2 is where Row 1 and Row 2 occur. The final case, Case 1.1.3.3, is where the square is either entirely in the second row or entirely in the third row (these cases turn out to be identical).*

1.1.3.1. Suppose the square we want to show Mayer–Vietoris is embedded in the union of the (3,1) and (4,1) boxes. Then it takes the form

$$
\begin{tikzcd}
B_{tj} \oplus X_{i'0} \arrow[r] & X_{i'j'} \\
B_{tj} \oplus X_{i0} \arrow[r] \arrow[u] & X_{ij'} \arrow[u]
\end{tikzcd}
$$

and is therefore the sum of

$$
\begin{tikzcd}
X_{i'0} \arrow[r] & X_{i'j'} \\
X_{i0} \arrow[r] \arrow[u] & X_{ij'} \arrow[u]
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
B_{tj} \arrow[r] & 0 \\
B_{tj} \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd}
$$

where the first square is part of the simplex s_n, while the second is contractible.

1.1.3.2. Suppose the square we want to show Mayer–Vietoris involves Row 1 and Row 2. Then it takes the form

$$
\begin{tikzcd}
C_{i'j} \oplus X_{t0} \arrow[r] & D_{i'j'} \\
A_{tj} \oplus X_{i0} \arrow[r] \arrow[u] & X_{ij'} \arrow[u]
\end{tikzcd}
$$

and this is the second delicate square. The point is that we have a diagram of squares

$$
\begin{tikzcd}
C_{i'j} \arrow[r] & D_{i'0} \arrow[r] & D_{i'j'} \\
A_{tj} \arrow[r] \arrow[u] & X_{t0} \arrow[r] \arrow[u] & X_{tj'} \arrow[u] \\
 & X_{i0} \arrow[r] \arrow[u] & X_{ij'} \arrow[u]
\end{tikzcd}
$$

and the dual of the argument in Case 1.1.1.3 applies. Precisely, we have three semi–triangles

$$
A_{tj} \longrightarrow X_{tj'} \oplus C_{i'j} \longrightarrow D_{i'j'} \longrightarrow \Sigma A_{tj} \tag{1.5}
$$

$$(1.6)\qquad X_{i0} \longrightarrow X_{ij'} \oplus D_{i'0} \longrightarrow D_{i'j'} \longrightarrow \Sigma X_{i0}$$

and

$$(1.7)\qquad X_{t0} \longrightarrow X_{tj'} \oplus D_{i'0} \longrightarrow D_{i'j'} \longrightarrow \Sigma X_{t0}$$

and, exactly dually to Case 1.1.1.3, to prove that

$$\begin{tikzcd} C_{i'j} \oplus X_{t0} \arrow[r] & D_{i'j'} \\ A_{tj} \oplus X_{i0} \arrow[r] \arrow[u] & X_{ij'} \arrow[u] \end{tikzcd}$$

is Mayer–Vietoris, it suffices to establish that there is a semi-triangle

$$(1.8)\qquad A_{tj} \oplus X_{i0} \longrightarrow X_{ij'} \oplus C_{i'j} \oplus X_{t0} \longrightarrow D_{i'j'} \longrightarrow \Sigma\{A_{tj} \oplus X_{i0}\}$$

But (1.8) can be obtained as a direct summand of the mapping cone on the natural map of semi-triangles

$$(1.5) \oplus (1.6) \longrightarrow (1.7).$$

What is more, the other direct summand is a contractible triangle. This is also left as an exercise to the reader, who can once again find a discussion in Remark I.5.3.

As in the proof of Case 1.1.1.3, the reader will see that if we start out with a simplex, where all the objects are in the heart of some t–structure, and all the squares fold to give genuine triangles, then the square produced by the homotopy also folds to give a genuine triangle. In particular, the differential is unique.

1.1.3.3. If Row 1 does not occur, our square must take the form

$$\begin{tikzcd} C_{i'j} \oplus X_{NW} \arrow[r] & D_{i'j'} \\ C_{ij} \oplus X_{NW} \arrow[r] \arrow[u] & D_{ij'} \arrow[u] \end{tikzcd}$$

and is therefore the sum of

$$\begin{tikzcd} C_{i'j} \arrow[r] & D_{i'j'} \\ C_{ij} \arrow[r] \arrow[u] & D_{ij'} \arrow[u] \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} X_{NW} \arrow[r] & 0 \\ X_{NW} \arrow[r] \arrow[u] & 0 \arrow[u] \end{tikzcd}$$

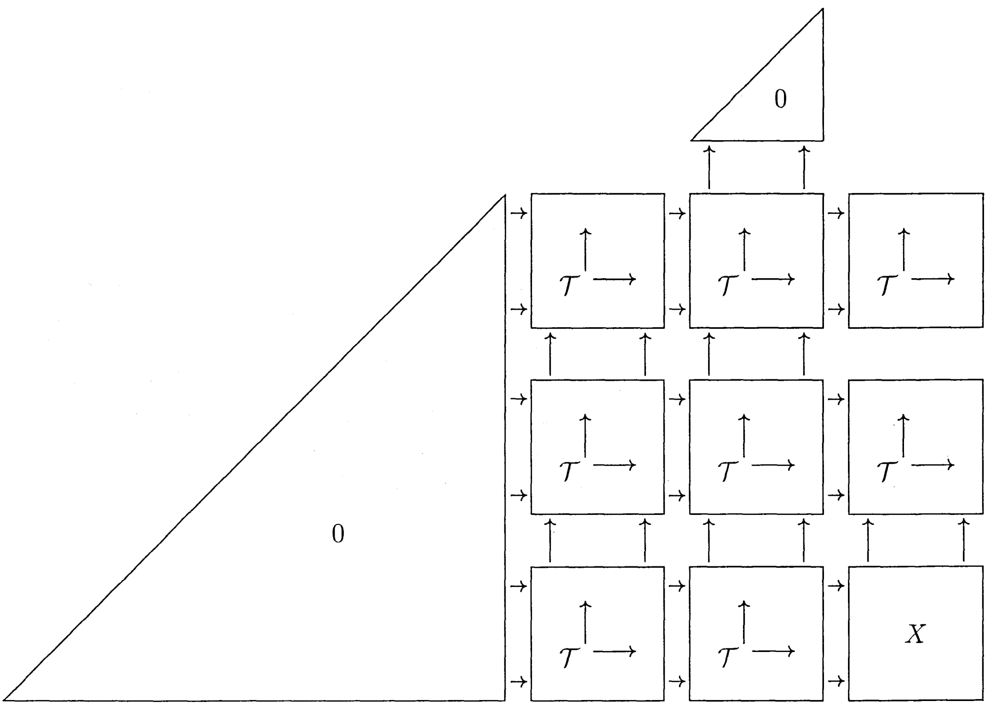
For the last time, the first of these is part of the simplex s_n, while the second is contractible.

This completes the discussion of the cases. We now know that there is a way to choose the maps and differentials, so that at least every square folds to give a semitriangle, and the maps that arise are some universally given matrices in the maps defining the original simplex. They are quite explicitly computable, from the various mapping cones we used.

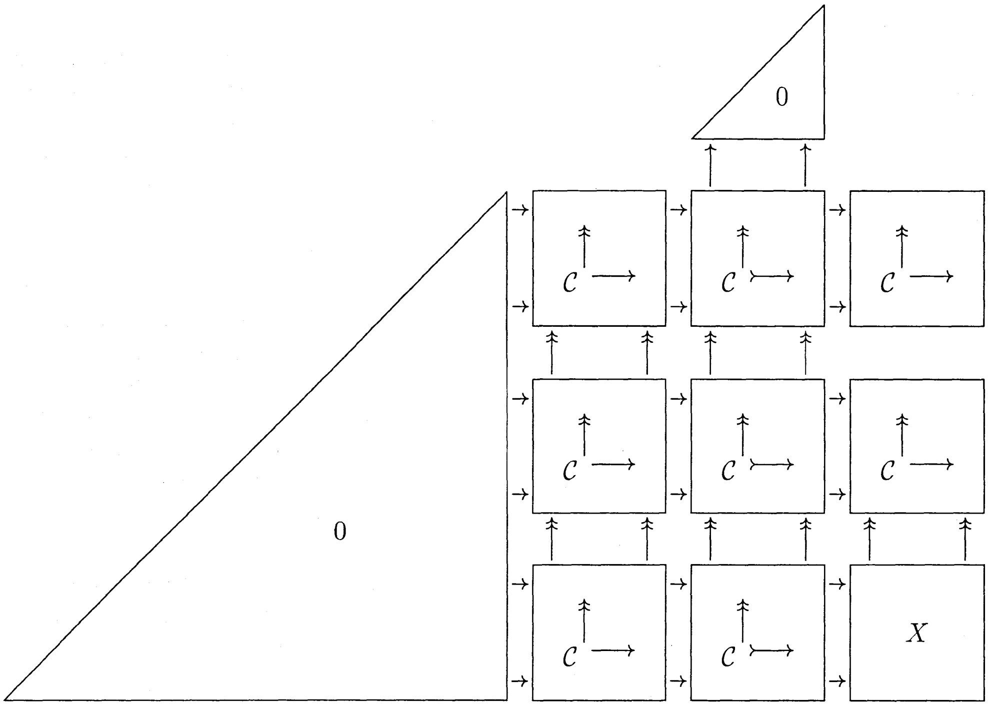
Two things are not immediately clear. The first is why the maps and differentials, that we have just shown how to choose, must be coherent. The second problem is how to show, that if we start with a somewhat restricted diagram, for instance a diagram having a lifting to model categories, the result of the homotopy is another diagram with a lift.

Point 1: The differentials are coherent

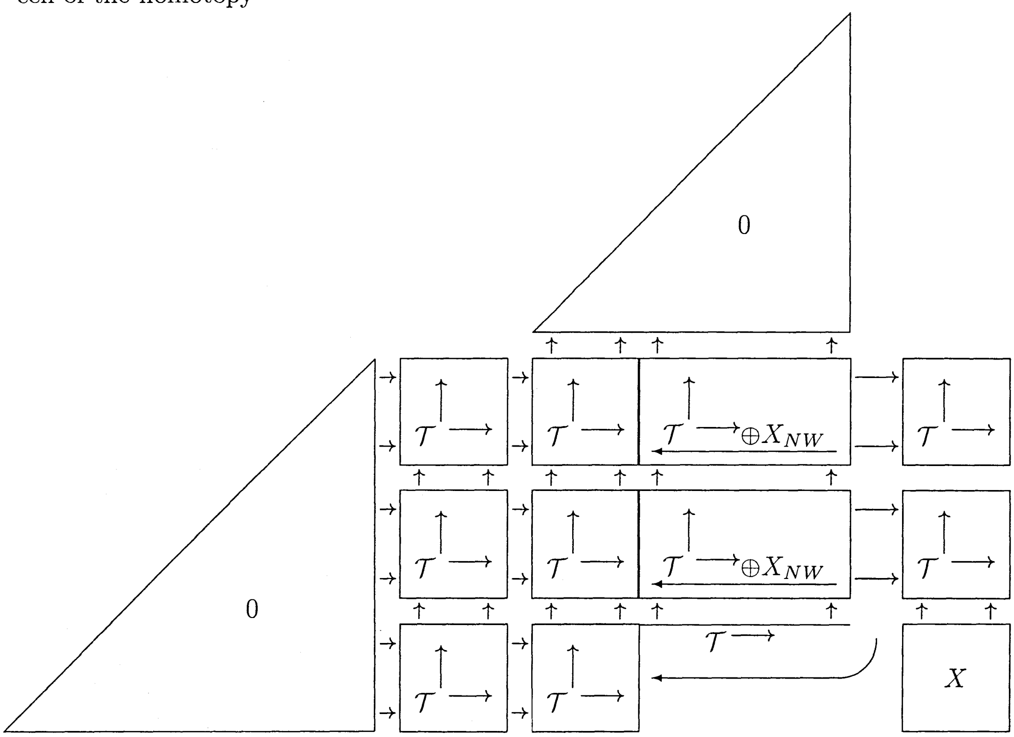
It is, of course, possible to compute this by brute force. But here is a cleaner argument. Suppose that we start with a cell s_n in



where all the objects are in fact in some heart $\mathcal{C}$ of the triagulated category $\mathcal{T}$, for some t-structure. That is, our simplex happens to lie in the smaller simplicial set



Then we have just proved that the morphisms, in the diagram which should give a cell of the homotopy



can be defined, as some universally given matrices in the structure maps of the starting simplex, so that all squares fold to give *triangles.* Note that in the two places where we took mapping cones on triangles in the proof, we observed that in the special case above, they are mapping cones on good maps.

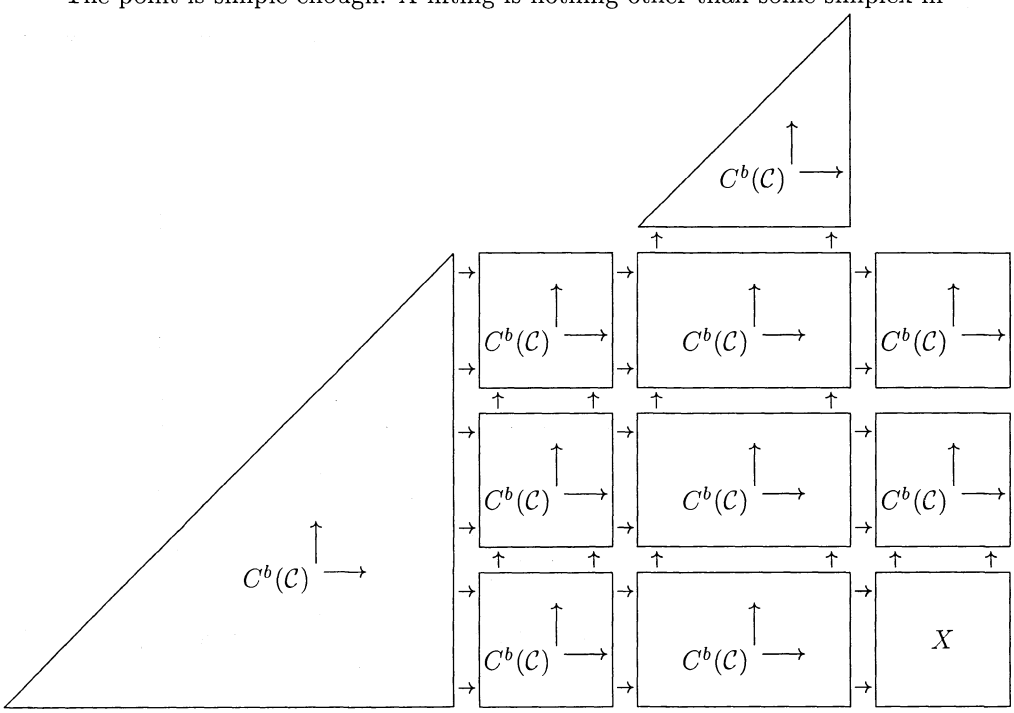
But of course the differentials are unique, and coherent, in this special case. This is because a short exact sequence in a heart of a t–structure corresponds to a unique triangle; the differential is unique. To say that the differentials are coherent, is to say that the restriction of the differential on a large square to a smaller square is equal to the differential on the smaller square. This gives some identity in the universally defined matrices of given maps we have just constructed.

The identity must hold for all simplices all of whose objects are in a heart. This means that the difference between the restriction of a large differential and a small differential is a universally defined matrix of given maps, which vanishes whenever the starting simplex involves only objects of a heart. But in the starting simplex, there was only one map connecting any two objects. The matrix of the difference has components which can only be integer multiples of the given map. If the integer is non–zero, it is easy to construct a simplex of objects in a heart, on which it will not vanish. Hence the difference between a small differential and the restriction of a large differential must be the zero map; the differentials are coherent.

Point 2: There is a Waldhausen lift

If we presume that we started with a simplex that had a lifting to a model category, we wish to show that the universal diagram we have just constructed, in order to define our homotopy, also has such a lifting, in fact to the same model category.

The point is simple enough. A lifting is nothing other than some simplex in

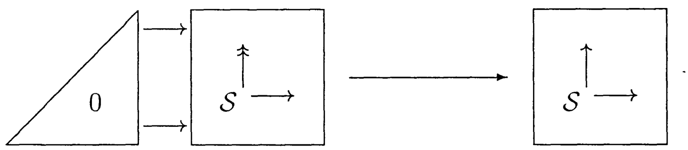


for the abelian category $C^b(\mathcal{C})$, which can always be viewed as the heart of its own derived category. We have shown that, with our universal matrices between direct sums of objects, the homotopy will take a simplex in the heart to a simplex in the heart, with even the differentials being right. Hence the existence of a lifting to the same model category follows.

This completes the proof of Theorem 1.1. $\square$

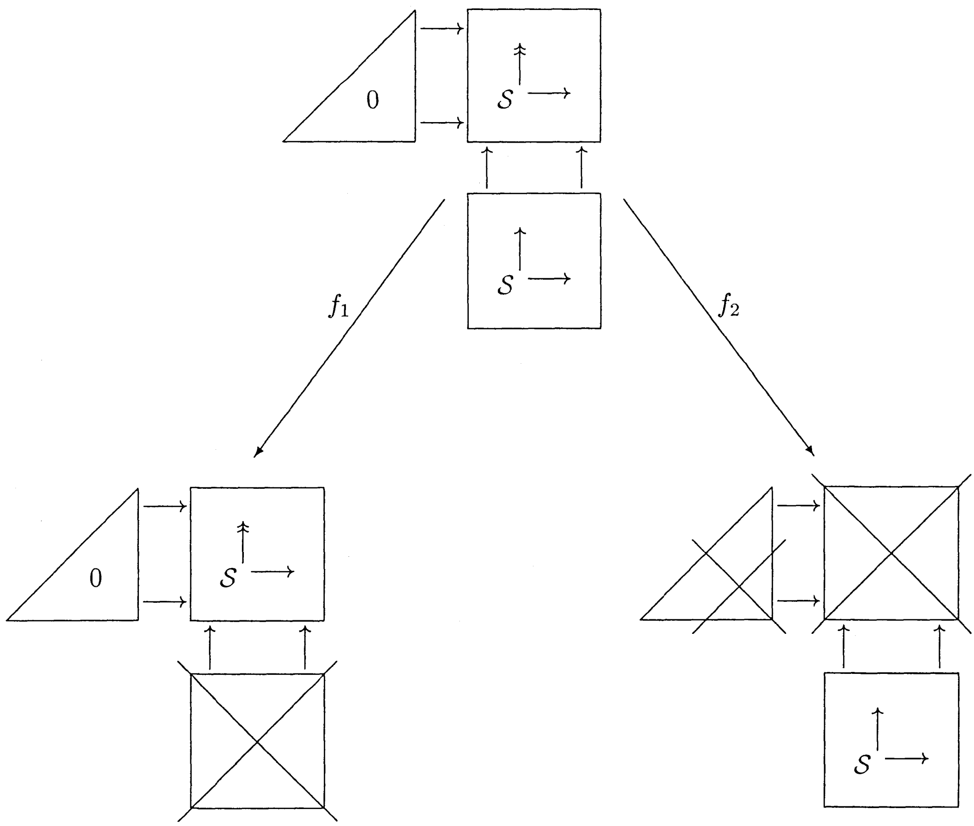
It seems appropriate to give a very simple illustration of the power of Theorem 1.1. Let us now work carefully through the proof of Theorem I.5.1. We remind the reader that in our notation, Theorem I.5.1 stands for Theorem 5.1 in *K–theory for triangulated categories I*. For the reader's benefit, we recall the statement of the theorem.

Theoem I.5.1. *The natural map*

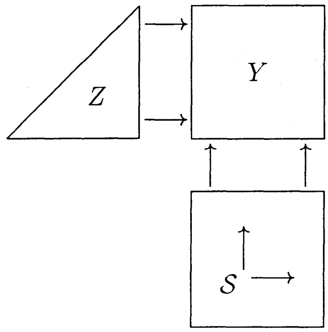


is a homotopy equivalence.

Proof of Theorem I.5.1. We considered the trisimplicial set and two projections

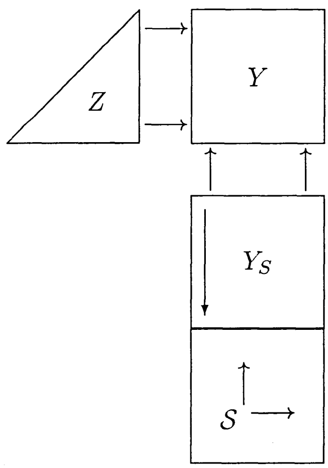


The Segal fiber of the map f_1 is the simplicial set



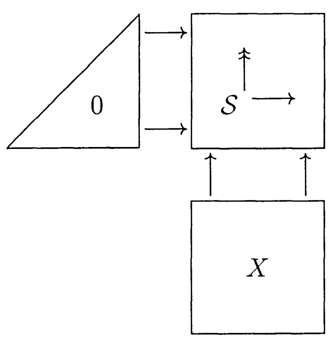
,

which is contracted by the homotopy



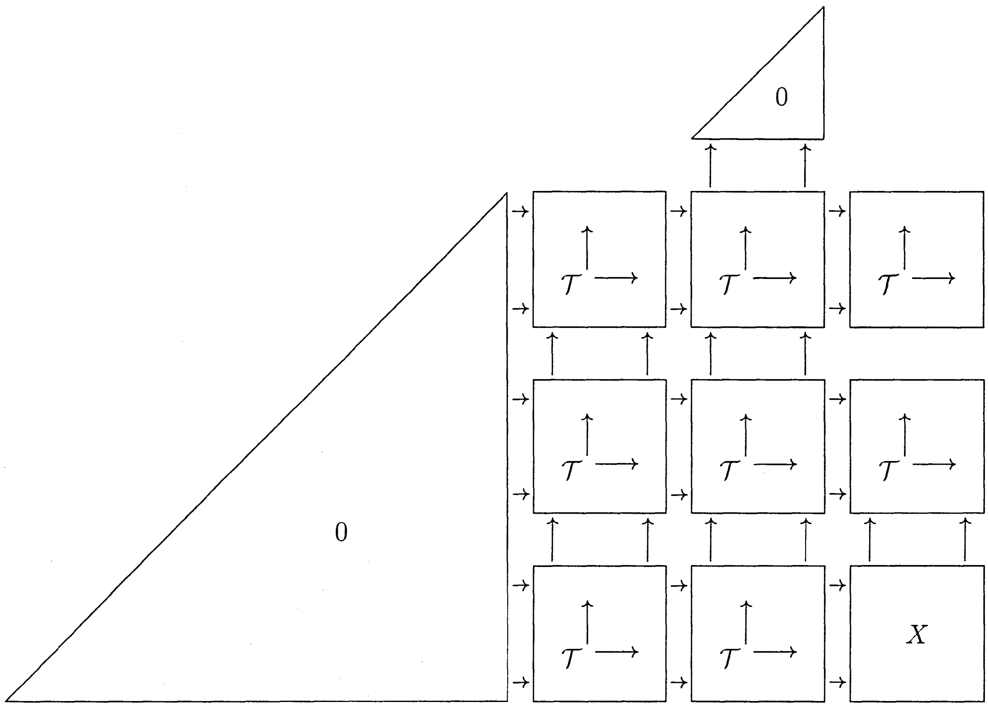
.

The subtlety of the proof comes from trying to contract the Segal fiber of f_2, that is the simplicial set

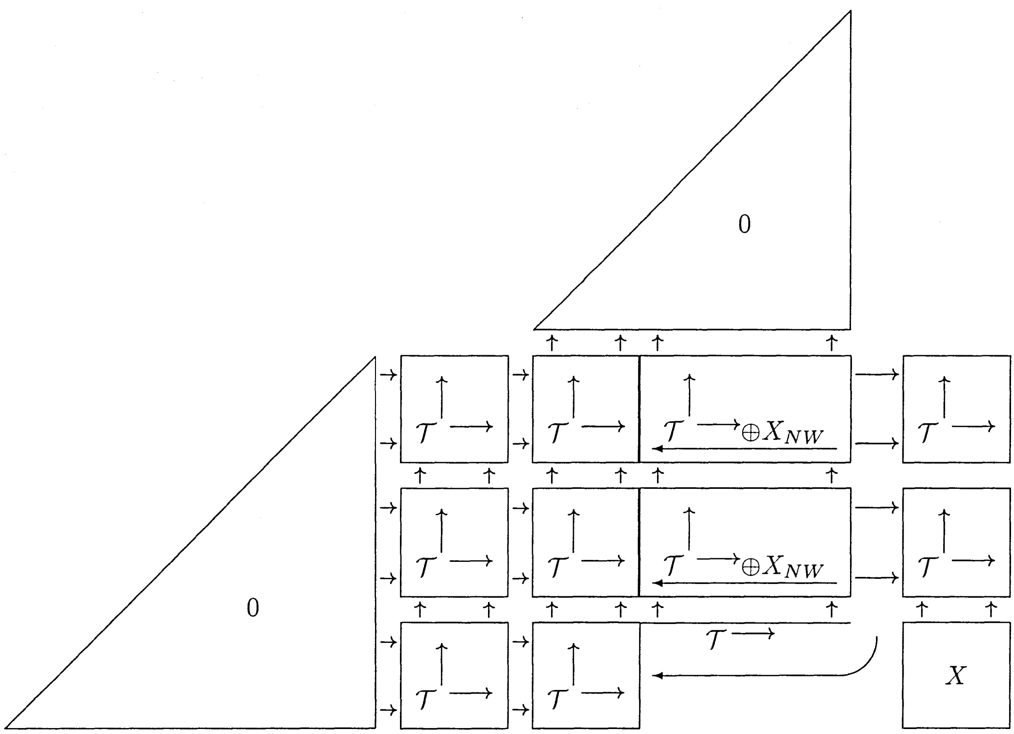


.

Recall the simplicial set

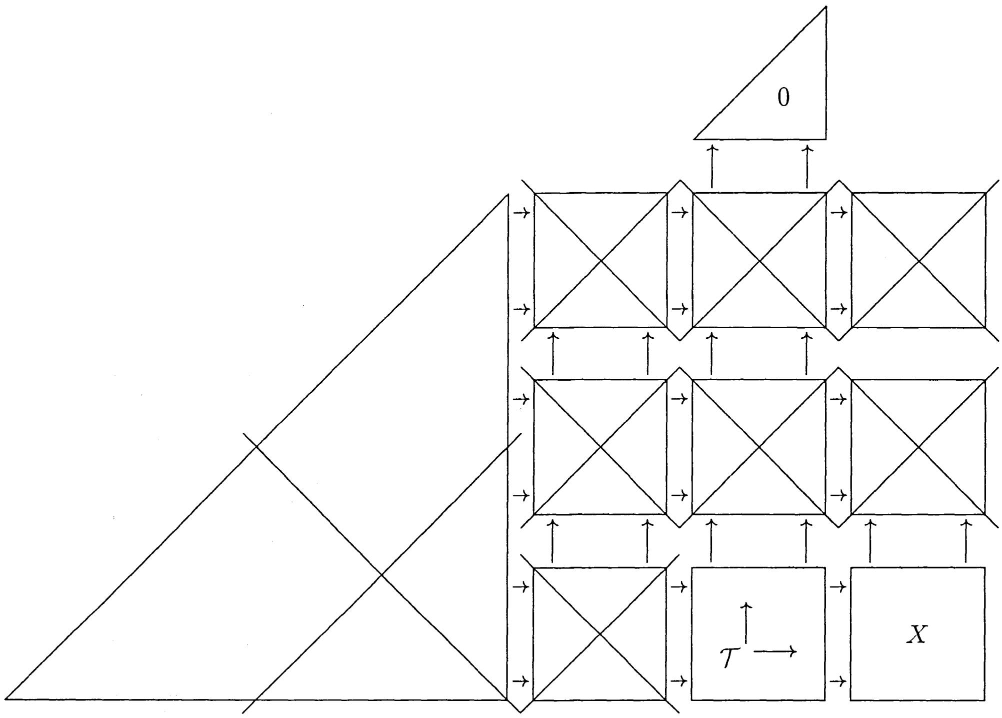


From now on we will refer to the above as the "blueprint set". By Theorem 1.1, this blueprint set admits a homotopy



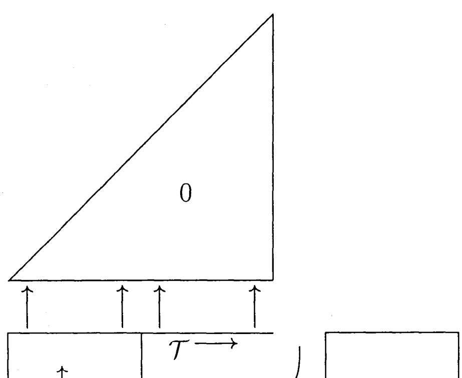
which we will henceforth call the "blueprint homotopy".

Consider now the smaller simplicial set

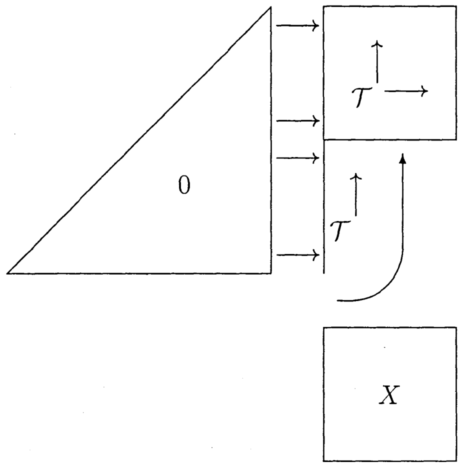


Because this simplicial set is obtained from the blueprint set by deletion, the blueprint homotopy must be defined on it. This follows from the proof of Theorem 1.1. The point is that the proof consisted of checking that certain squares are Mayer–Vietoris. In a diagram where part of the data is deleted, there are fewer squares to check Mayer–Vietoris, and it follows we must have checked them all, back when we proved Theorem 1.1.

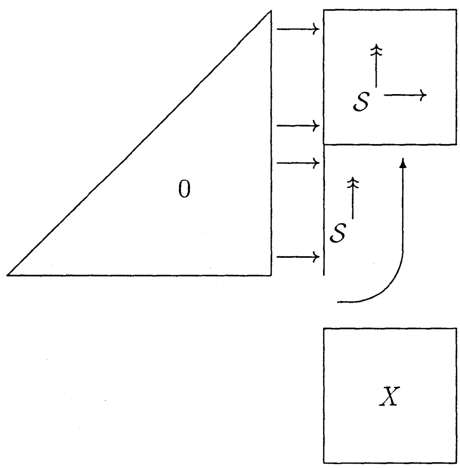
Thus the homotopy, which we write more succinctly as



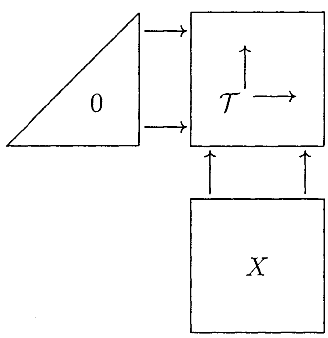
must be well–defined. Dually, we obtain a homotopy



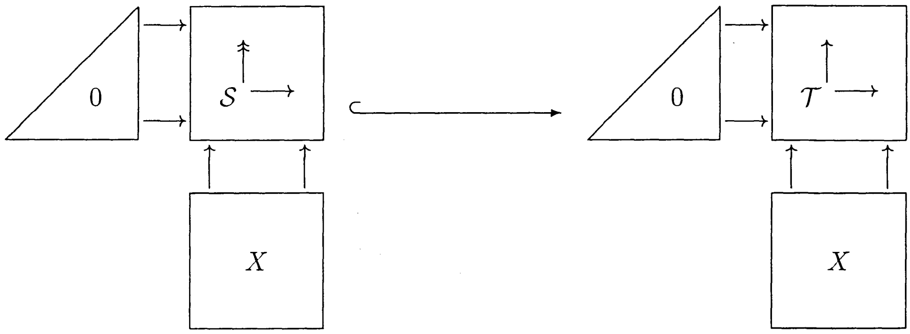
Now, if $\mathcal{S}$ is an exact subcategory of the triangulated category $\mathcal{T}$, we have just shown that the homotopy



comes very close to being well–defined. The image of any homotopy cell in

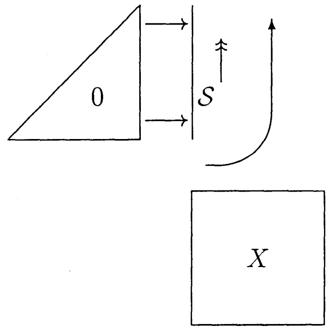


is unmistakeably a simplex. Checking that the homtopy is well defined amounts to verifying only that the homotopy cells never leave



This reduces to showing that the objects stay in $\mathcal{S}$, and the vertical morphisms are epi when they should be. The point of the exercise is that neither reader nor writer need worry, whether the squares are Mayer–Vietoris, or the differentials coherent.

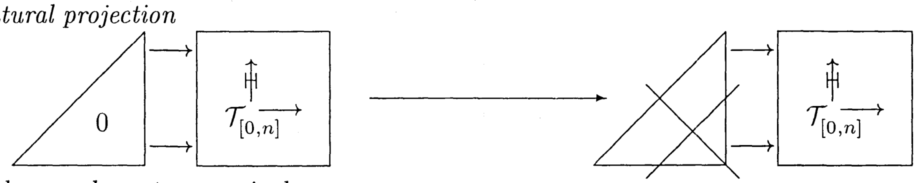
This proves that the identity is homotopic to a map denoted



We refer the reader to the proof of Theorem I.5.1 in *K-theory for triangulated categories I*, for the argument showing that the above is homotopic to the null map. See also the argument starting on page 565 of the present article, under the title "Continuation of the Proof of Theorem 1.2". □

Now that we have given a very simple-minded application of Theorem 1.1, it seems to be time to give a sophisticated one, which we will actually need in this article. Recall that an arrow of type $\xrightarrow{\;\;\text{I}\;\;}$ stands for a morphism in $\mathcal{T}_{[0,n]}$ inducing an isomorphism on H^0.

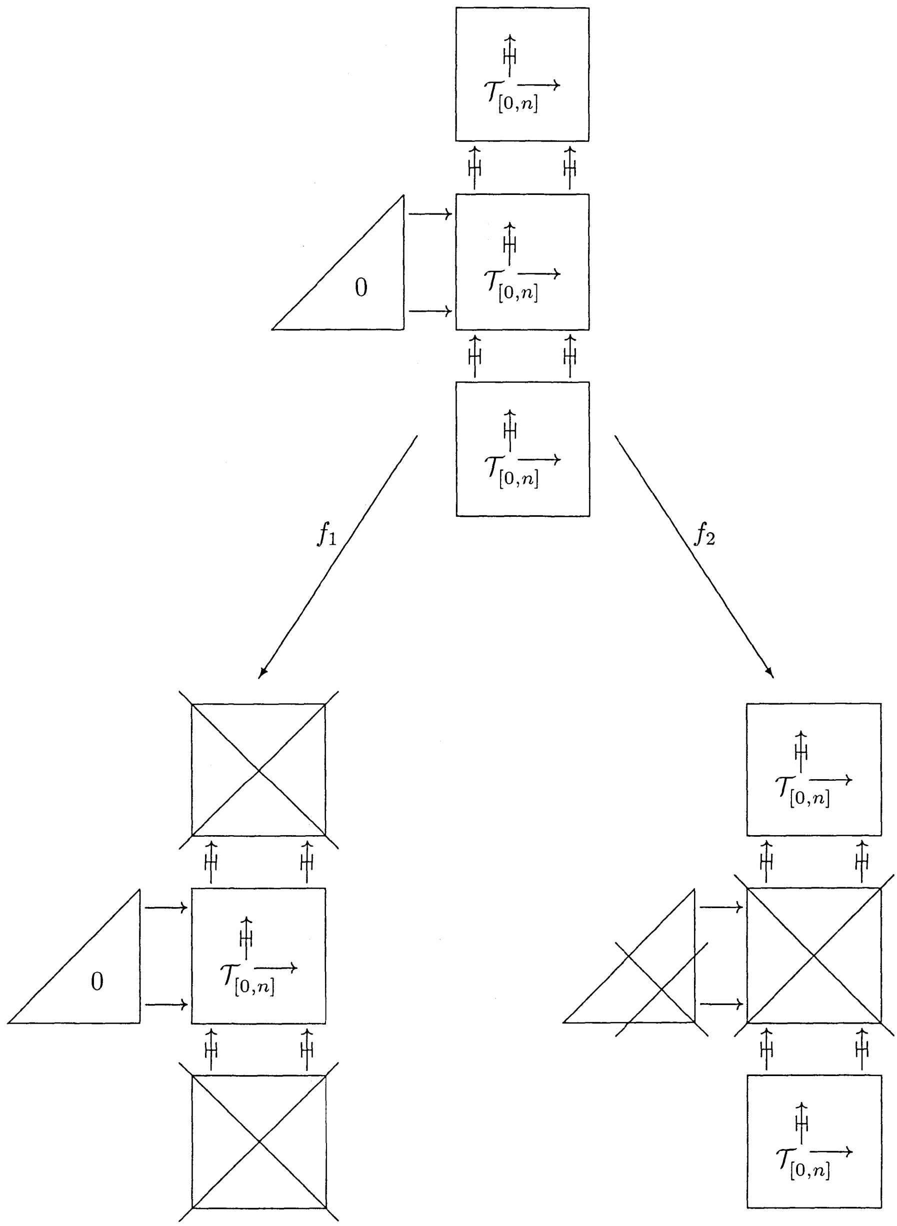
THEOREM 1.2. *Let $\mathcal{T}$ be a triangulated category with a t-structure. Then the natural projection*



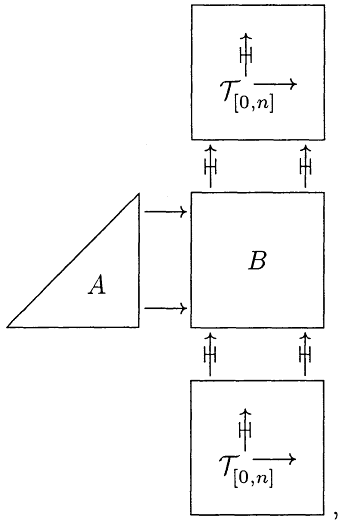
induces a homotopy equivalence.

Proof. We will first give the argument formally. This means, we will not worry about the subtleties, which are caused by having to consider the various modifications introduced in Remark I.3.5, to the basic simplicial set of Construction I.3.3. A Group 3 reader should be able to fill in the details himself. For Group 2 readers, we will discuss in Remark 1.3 why the homotopies in the argument are all well defined, in the particular modification of Construction I.3.3 which I advised the reader to adopt. In other words, Remark 1.3 will show that all the simplices that arise decompose as sums of simplices, each with a lifting to a model category.

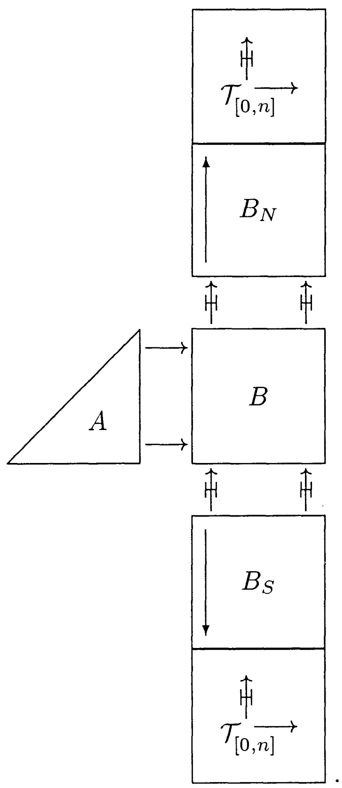
We consider the more complicated diagram of a simplicial set and two projections:



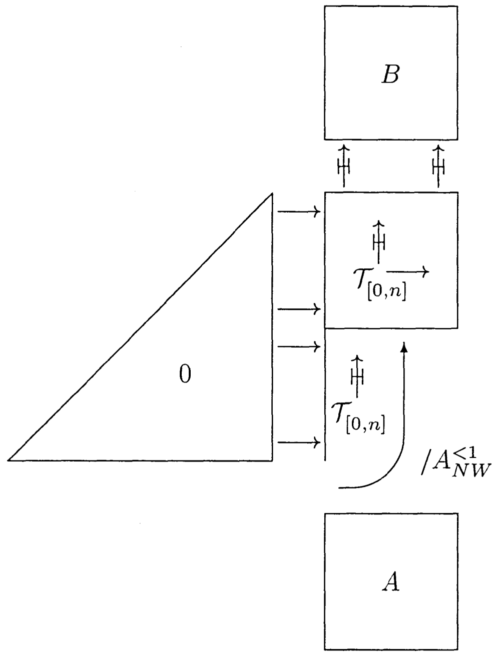
and our theorem will follow, once we establish that f_1 and f_2 are homotopy equivalences. For f_1 this is very easy; the Segal fiber is the simplicial set



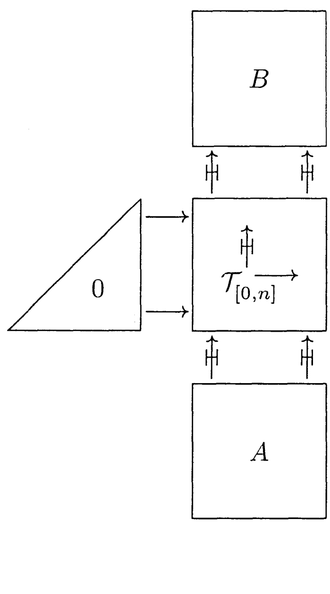
which is contracted by the homotopy



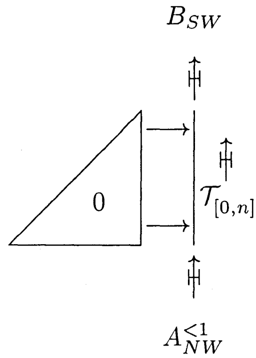
The map f_2 is slightly subtler to prove a homotopy equivalence. The proof amounts to using the homotopy



to factor the identity on the Segal fiber

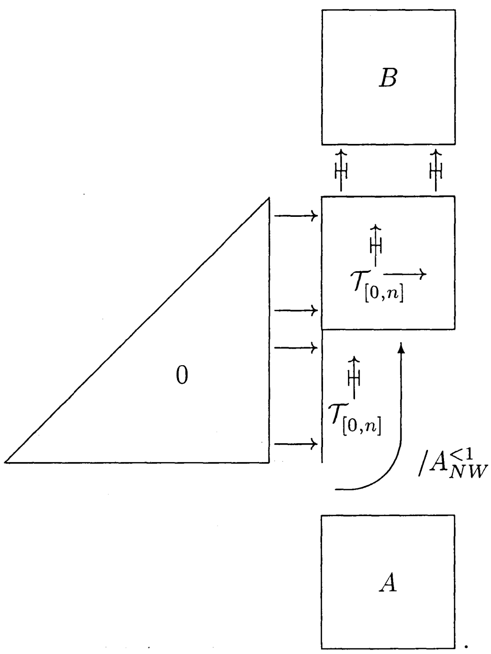


through the contractible simplicial set

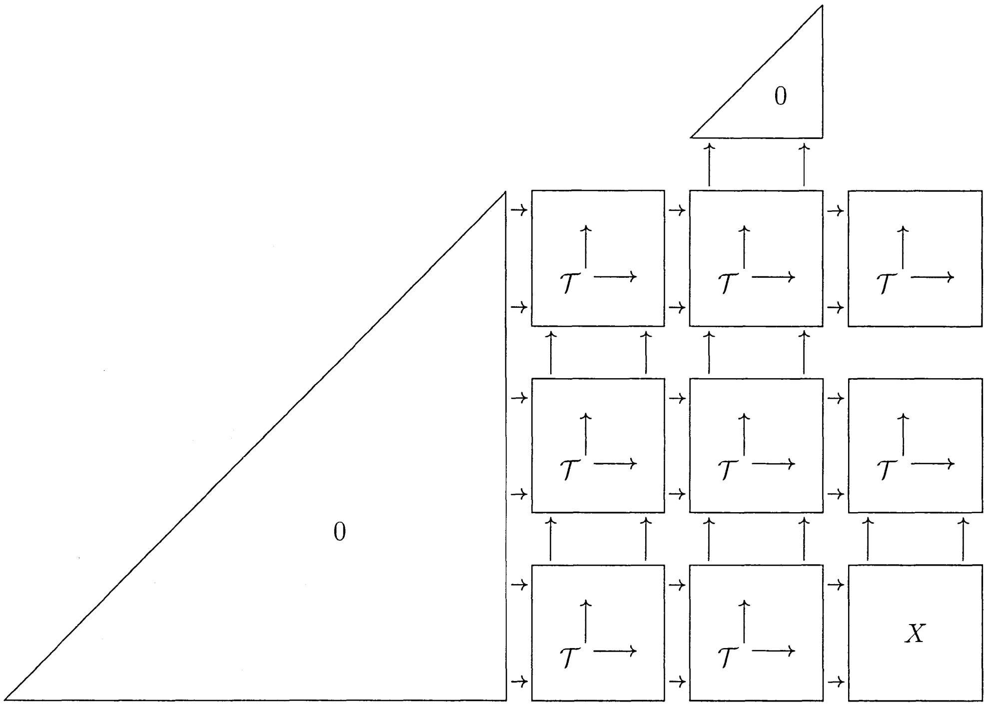


The contractibility of this last simplicial set is by the contraction to the initial object.

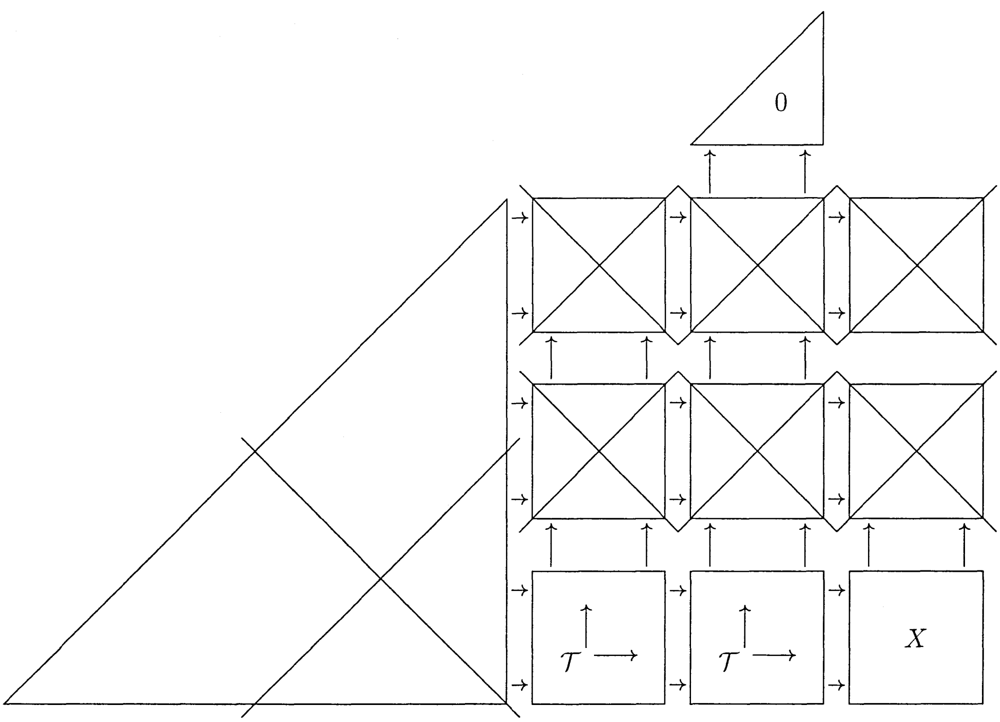
But in this section we are in the business of worrying why the non–trivial homotopies are well defined. In particular, we should concern ourselves with the homotopy



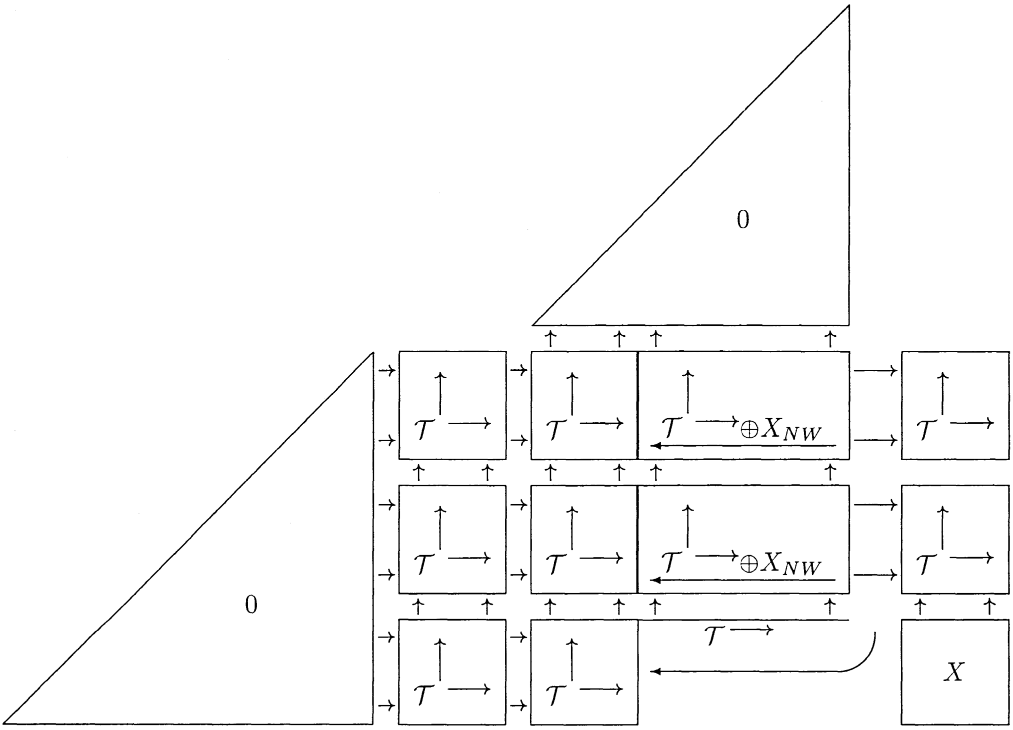
To this end, we recall our simplicial set



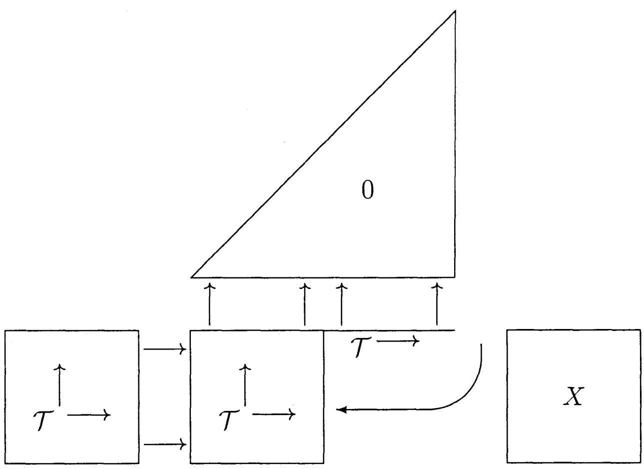
By deleting some of the data, we obtain the smaller simplicial set



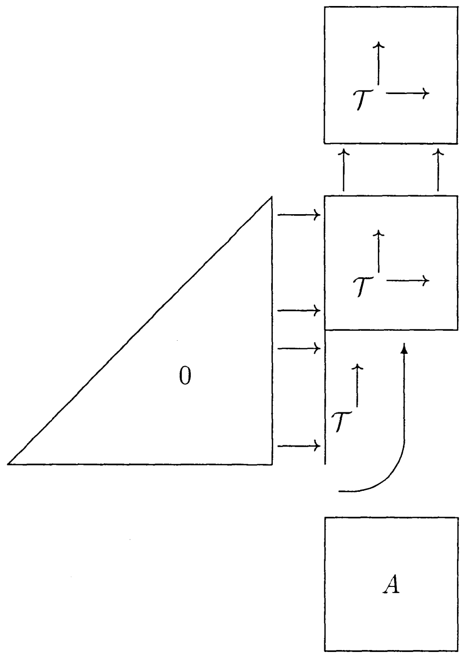
Theorem 1.1 guarantees that there is a well–defined homotopy



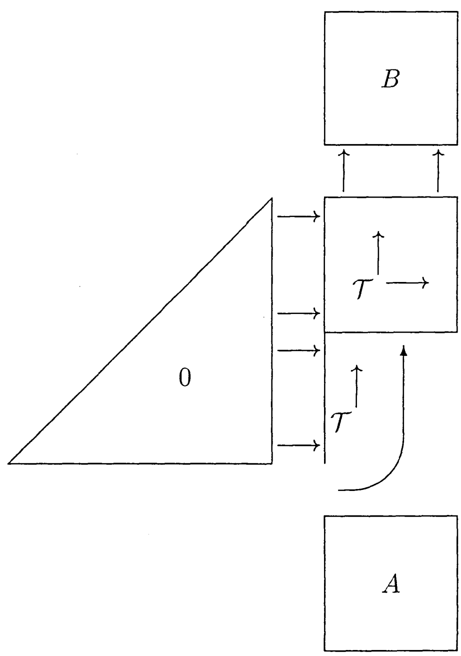
and from the proof, it follows the homotopy remains well-defined when we delete some of the structure. In particular, on our smaller simplicial set we obtain a homotopy which we will denote



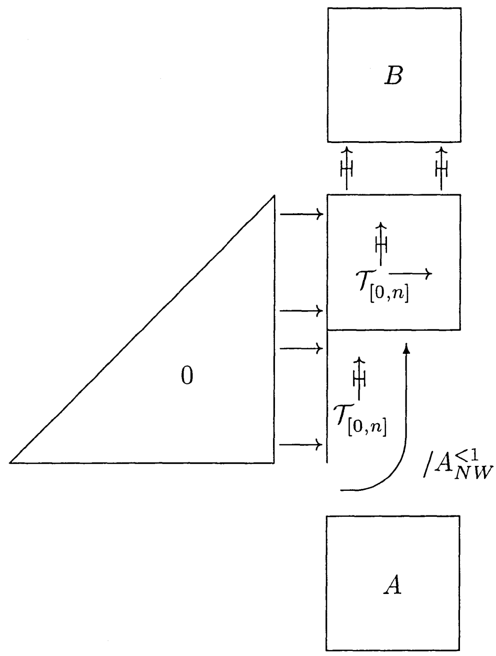
This homotopy is dual to



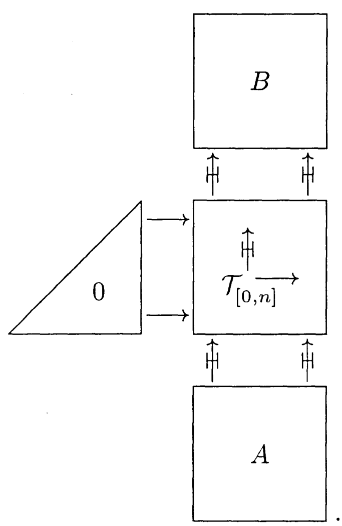
In the homotopy, the contents of the top box are fixed. So we may also view it as a homotopy on the simplicial set, where the top is constrained to be fixed; that is, the homotopy whose symbol would be



So far this argument has been completely painless, appealing directly to the proof of Theorem 1.1. To obtain the homotopy



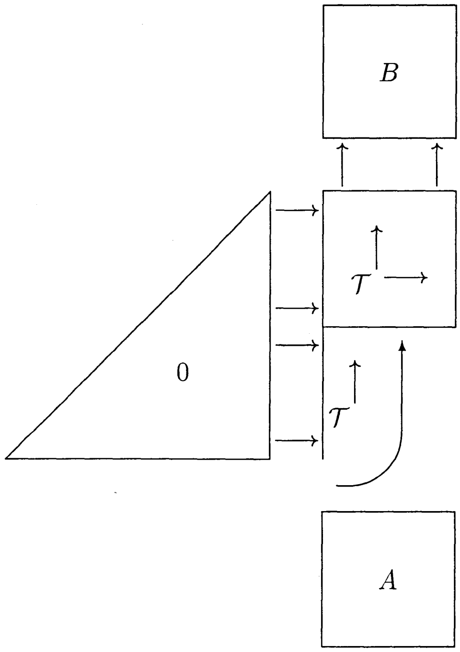
we have to worry a little about the t–structure truncation. Suppose we start with a simplex s_n in the simplicial set



It is given as a diagram

$$
s_n = \left(
\begin{tikzcd}
 & & & B_{r0} \arrow[r] & \cdots \arrow[r] & B_{rq} \\
 & & & \vdots \arrow[u, tail] & & \vdots \arrow[u, tail] \\
 & & & B_{00} \arrow[r] \arrow[u, tail] & \cdots \arrow[r] & B_{0q} \arrow[u, tail] \\
 & & 0 \arrow[r] & S_{n0} \arrow[r] \arrow[u, tail] & \cdots \arrow[r] & S_{nq} \arrow[u, tail] \\
 & \ddots & \vdots \arrow[u] & \vdots \arrow[u, tail] & & \vdots \arrow[u, tail] \\
0 \arrow[r] & \cdots \arrow[r] & R_{0n} \arrow[r] \arrow[u] & S_{00} \arrow[r] \arrow[u, tail] & \cdots \arrow[r] & S_{0q} \arrow[u, tail] \\
 & & & A_{p0} \arrow[r] \arrow[u, tail] & \cdots \arrow[r] & A_{pq} \arrow[u, tail] \\
 & & & \vdots \arrow[u, tail] & & \vdots \arrow[u, tail] \\
 & & & A_{00} \arrow[r] \arrow[u, tail] & \cdots \arrow[r] & A_{0q} \arrow[u, tail]
\end{tikzcd}
\right)
$$

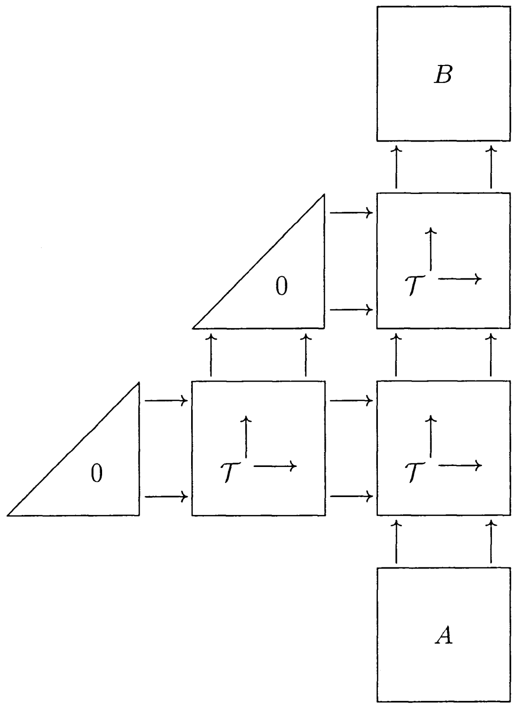
Then the homotopy



takes s_n to $n+1$ distinct $(n+1)$–simplices, the i^{th} of which is a diagram

$$
\begin{tikzcd}
 & & & & & & B_{r0} \arrow[r] & \cdots \arrow[r] & B_{rq} \\
 & & & & & & \vdots \arrow[u, tail] & & \vdots \arrow[u, tail] \\
 & & & & & & B_{00} \arrow[u, tail] \arrow[r] & \cdots \arrow[r] & B_{0q} \arrow[u, tail] \\
 & & & & & 0 \arrow[r] & S_{n0} \arrow[u, tail] \arrow[r] & \cdots \arrow[r] & S_{nq} \arrow[u, tail] \\
 & & & & & \vdots \arrow[u] & \vdots \arrow[u, tail] & & \vdots \arrow[u, tail] \\
 & & & 0 \arrow[r] & \cdots \arrow[r] & R_{in} \arrow[u] \arrow[r] & S_{i0} \arrow[u, tail] \arrow[r] & \cdots \arrow[r] & S_{iq} \arrow[u, tail] \\
 & & 0 \arrow[r] & R_{ii}\oplus A_{NW} \arrow[u] \arrow[r] & \cdots \arrow[r] & R_{in}\oplus A_{NW} \arrow[u] \arrow[r] & S_{i0}\oplus A_{p0} \arrow[u] \arrow[r] & \cdots \arrow[r] & S_{i0}\oplus A_{pq} \arrow[u] \\
 & & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] & \vdots \arrow[u, tail] & & \vdots \arrow[u, tail] \\
0 \arrow[r] & \cdots \arrow[r] & R_{0i} \arrow[u] \arrow[r] & R_{0i}\oplus A_{NW} \arrow[u] \arrow[r] & \cdots \arrow[r] & R_{0n}\oplus A_{NW} \arrow[u] \arrow[r] & S_{00}\oplus A_{p0} \arrow[u, tail] \arrow[r] & \cdots \arrow[r] & S_{00}\oplus A_{pq} \arrow[u, tail] \\
 & & & & & & A_{p0} \arrow[u] \arrow[r] & \cdots \arrow[r] & A_{pq} \arrow[u] \\
 & & & & & & \vdots \arrow[u, tail] & & \vdots \arrow[u, tail] \\
 & & & & & & A_{00} \arrow[u, tail] \arrow[r] & \cdots \arrow[r] & A_{0q} \arrow[u, tail]
\end{tikzcd}
$$

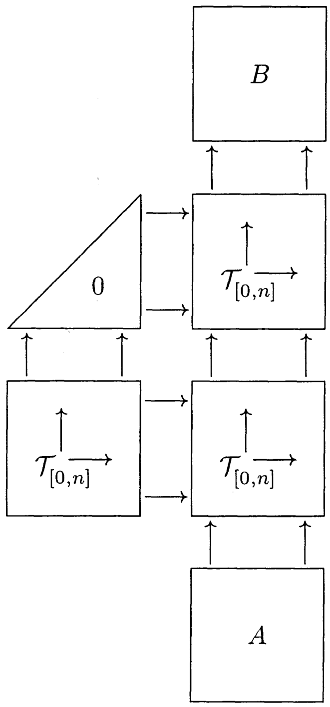
and Theorem 1.1 assures us that this is a simplex in the simplicial set



By deletion,

$$
\begin{tikzcd}[column sep=small]
 & & & B_{r0} \arrow[r] & \cdots \arrow[r] & B_{rq} \\
 & & & \vdots \arrow[u, tail] & & \vdots \arrow[u, tail] \\
 & & & B_{00} \arrow[r] \arrow[u, tail] & \cdots \arrow[r] & B_{0q} \arrow[u, tail] \\
 & & 0 \arrow[r] & S_{n0} \arrow[r] \arrow[u, tail] & \cdots \arrow[r] & S_{nq} \arrow[u, tail] \\
 & \cdot^{\cdot^{\cdot}} & \vdots \arrow[u] & \vdots \arrow[u, tail] & & \vdots \arrow[u, tail] \\
0 \arrow[r] & \cdots \arrow[r] & R_{in} \arrow[r] \arrow[u] & S_{i0} \arrow[r] \arrow[u, tail] & \cdots \arrow[r] & S_{iq} \arrow[u, tail] \\
R_{ii} \oplus A_{NW} \arrow[r] \arrow[u] & \cdots \arrow[r] & R_{in} \oplus A_{NW} \arrow[r] \arrow[u] & S_{i0} \oplus A_{p0} \arrow[r] \arrow[u] & \cdots \arrow[r] & S_{i0} \oplus A_{pq} \arrow[u] \\
\vdots \arrow[u] & & \vdots \arrow[u] & \vdots \arrow[u, tail] & & \vdots \arrow[u, tail] \\
R_{0i} \oplus A_{NW} \arrow[r] \arrow[u] & \cdots \arrow[r] & R_{0n} \oplus A_{NW} \arrow[r] \arrow[u] & S_{00} \oplus A_{p0} \arrow[r] \arrow[u, tail] & \cdots \arrow[r] & S_{00} \oplus A_{pq} \arrow[u, tail] \\
 & & & A_{p0} \arrow[r] \arrow[u] & \cdots \arrow[r] & A_{pq} \arrow[u] \\
 & & & \vdots \arrow[u, tail] & & \vdots \arrow[u, tail] \\
 & & & A_{00} \arrow[r] \arrow[u, tail] & \cdots \arrow[r] & A_{0q} \arrow[u, tail]
\end{tikzcd}
$$

can certainly be viewed as a simplex in the simplicial set



and the relevant point is that, for the special simplices above, there is a mono of $A^{<1}_{NW}$ into the second row of this simplicial set, whose composite to the third row is zero. Precisely, the injection $A^{<1}_{NW} \hookrightarrow A_{NW}$ gives a compatible family of morphisms $A^{<1}_{NW} \hookrightarrow R_{ij} \oplus A_{NW}$, whose composites with the projection $R_{ij} \oplus A_{NW} \longrightarrow R_{ij}$ is zero. This gives a map of $A^{<1}_{NW}$ into the part of the diagram below surrounded by a dashbox

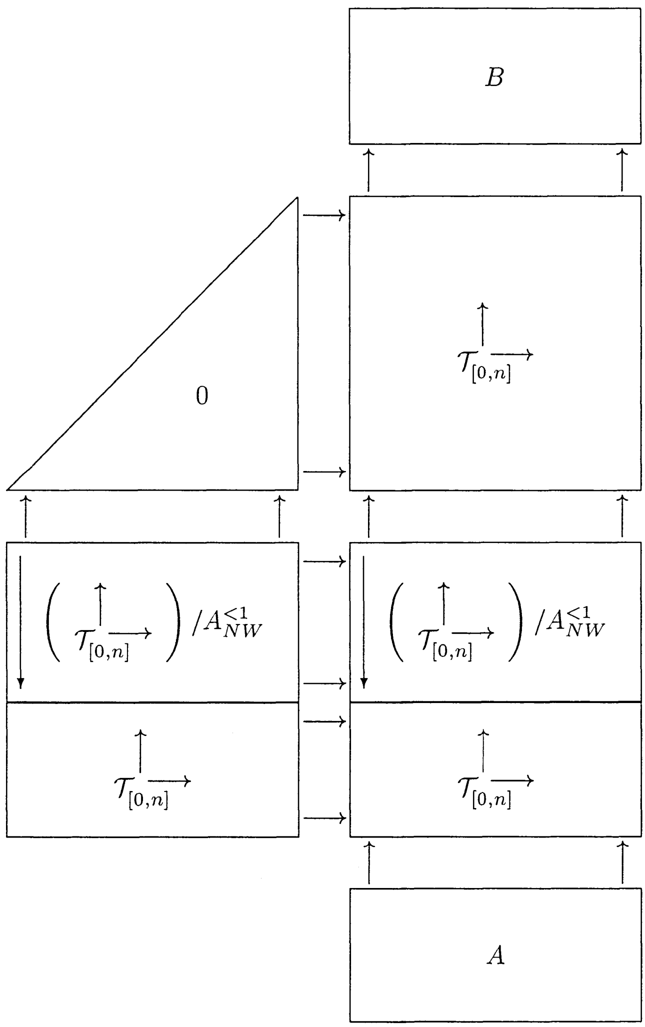
$$
\begin{tikzcd}
 & & & B_{r0} \arrow[r] & \cdots \arrow[r] & B_{rq} \\
 & & & \vdots \arrow[u, tail] & & \vdots \arrow[u, tail] \\
 & & & B_{00} \arrow[u, tail] \arrow[r] & \cdots \arrow[r] & B_{0q} \arrow[u, tail] \\
 & & 0 \arrow[r] & S_{n0} \arrow[u, tail] \arrow[r] & \cdots \arrow[r] & S_{nq} \arrow[u, tail] \\
 & \iddots & \vdots \arrow[u] & \vdots \arrow[u, tail] & & \vdots \arrow[u, tail] \\
0 \arrow[r] & \cdots \arrow[r] & R_{in} \arrow[u] \arrow[r] & S_{i0} \arrow[u, tail] \arrow[r] & \cdots \arrow[r] & S_{iq} \arrow[u, tail] \\
R_{ii} \oplus A_{NW} \arrow[u] \arrow[r] & \cdots \arrow[r] & R_{in} \oplus A_{NW} \arrow[u] \arrow[r] & S_{i0} \oplus A_{p0} \arrow[u] \arrow[r] & \cdots \arrow[r] & S_{i0} \oplus A_{pq} \arrow[u] \\
\vdots \arrow[u] & & \vdots \arrow[u] & \vdots \arrow[u, tail] & & \vdots \arrow[u, tail] \\
R_{0i} \oplus A_{NW} \arrow[u] \arrow[r] & \cdots \arrow[r] & R_{0n} \oplus A_{NW} \arrow[u] \arrow[r] & S_{00} \oplus A_{p0} \arrow[u, tail] \arrow[r] & \cdots \arrow[r] & S_{00} \oplus A_{pq} \arrow[u, tail] \\
 & & & A_{p0} \arrow[u] \arrow[r] & \cdots \arrow[r] & A_{pq} \arrow[u] \\
 & & & \vdots \arrow[u, tail] & & \vdots \arrow[u, tail] \\
 & & & A_{00} \arrow[u, tail] \arrow[r] & \cdots \arrow[r] & A_{0q} \arrow[u, tail]
\end{tikzcd}
$$

and what is especially good about this special case, is that the maps from $A^{<1}_{NW}$ into the region within the dashbox are all monos; that is, all the objects in the diagram are in $\mathcal{T}^{\geq 0}$, and the map from the object $A^{<1}_{NW}$ into any of the objects inside the dashbow is mono in $\mathcal{T}^{\geq 0}$. Furthermore, by the time we compose the map from $A^{<1}_{NW}$ with any map leaving the dashbox, the composite vanishes.

The unsubtle truncation homotopy allows us then to define a homotopy whose cells are

$$
\begin{tikzcd}
 & & & B_{r0} \arrow[r] & \cdots \arrow[r] & B_{rq} \\
 & & & \vdots \arrow[u] & & \vdots \arrow[u] \\
 & & & B_{00} \arrow[u] \arrow[r] & \cdots \arrow[r] & B_{0q} \arrow[u] \\
 & & 0 \arrow[r] & S_{n0} \arrow[u] \arrow[r] & \cdots \arrow[r] & S_{nq} \arrow[u] \\
 & \iddots & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] \\
0 \arrow[r] & \cdots \arrow[r] & R_{in} \arrow[u] \arrow[r] & S_{i0} \arrow[u] \arrow[r] & \cdots \arrow[r] & S_{iq} \arrow[u] \\
\frac{R_{ii}\oplus A_{NW}}{A^{<1}_{NW}} \arrow[u] \arrow[r] & \cdots \arrow[r] & \frac{R_{in}\oplus A_{NW}}{A^{<1}_{NW}} \arrow[u] \arrow[r] & \frac{S_{i0}\oplus A_{p0}}{A^{<1}_{NW}} \arrow[u] \arrow[r] & \cdots \arrow[r] & \frac{S_{i0}\oplus A_{pq}}{A^{<1}_{NW}} \arrow[u] \\
\vdots \arrow[u] & & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] \\
\frac{R_{hi}\oplus A_{NW}}{A^{<1}_{NW}} \arrow[u] \arrow[r] & \cdots \arrow[r] & \frac{R_{hn}\oplus A_{NW}}{A^{<1}_{NW}} \arrow[u] \arrow[r] & \frac{S_{h0}\oplus A_{p0}}{A^{<1}_{NW}} \arrow[u] \arrow[r] & \cdots \arrow[r] & \frac{S_{h0}\oplus A_{pq}}{A^{<1}_{NW}} \arrow[u] \\
R_{hi}\oplus A_{NW} \arrow[u] \arrow[r] & \cdots \arrow[r] & R_{hn}\oplus A_{NW} \arrow[u] \arrow[r] & S_{h0}\oplus A_{p0} \arrow[u] \arrow[r] & \cdots \arrow[r] & S_{h0}\oplus A_{pq} \arrow[u] \\
\vdots \arrow[u] & & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] \\
R_{0i}\oplus A_{NW} \arrow[u] \arrow[r] & \cdots \arrow[r] & R_{0n}\oplus A_{NW} \arrow[u] \arrow[r] & S_{00}\oplus A_{p0} \arrow[u] \arrow[r] & \cdots \arrow[r] & S_{00}\oplus A_{pq} \arrow[u] \\
 & & & A_{p0} \arrow[u] \arrow[r] & \cdots \arrow[r] & A_{pq} \arrow[u] \\
 & & & \vdots \arrow[u] & & \vdots \arrow[u] \\
 & & & A_{00} \arrow[u] \arrow[r] & \cdots \arrow[r] & A_{0q} \arrow[u]
\end{tikzcd}
$$

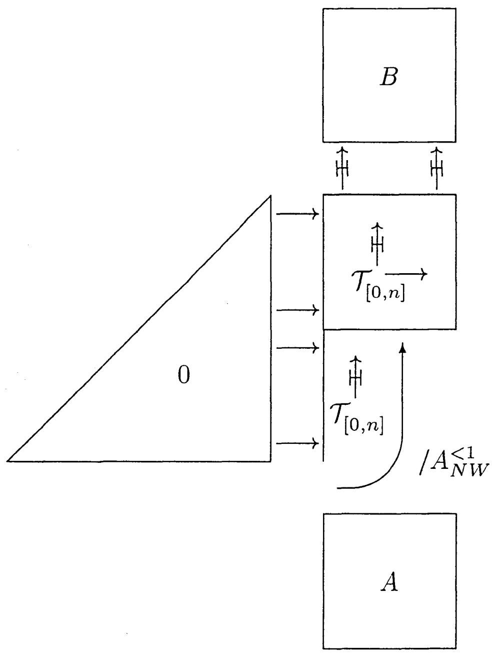
We remind the reader that, in our shorthand, this homotopy would be denoted something like



The last cell of this homotopy has, for one of its faces, the following diagram

$$\begin{tikzcd}
& & & B_{r0} \arrow[r] & \cdots \arrow[r] & B_{rq} \\
& & & \vdots \arrow[u] & & \vdots \arrow[u] \\
& & & B_{00} \arrow[u] \arrow[r] & \cdots \arrow[r] & B_{0q} \arrow[u] \\
& & 0 \arrow[r] & S_{n0} \arrow[u] \arrow[r] & \cdots \arrow[r] & S_{nq} \arrow[u] \\
& \iddots & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] \\
0 \arrow[r] & \cdots \arrow[r] & R_{in} \arrow[u] \arrow[r] & S_{i0} \arrow[u] \arrow[r] & \cdots \arrow[r] & S_{iq} \arrow[u] \\
\frac{R_{ii}\oplus A_{NW}}{A^{<1}_{NW}} \arrow[u] \arrow[r] & \cdots \arrow[r] & \frac{R_{in}\oplus A_{NW}}{A^{<1}_{NW}} \arrow[u] \arrow[r] & \frac{S_{i0}\oplus A_{p0}}{A^{<1}_{NW}} \arrow[u] \arrow[r] & \cdots \arrow[r] & \frac{S_{i0}\oplus A_{pq}}{A^{<1}_{NW}} \arrow[u] \\
\vdots \arrow[u] & & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] \\
\frac{R_{0i}\oplus A_{NW}}{A^{<1}_{NW}} \arrow[u] \arrow[r] & \cdots \arrow[r] & \frac{R_{0n}\oplus A_{NW}}{A^{<1}_{NW}} \arrow[u] \arrow[r] & \frac{S_{00}\oplus A_{p0}}{A^{<1}_{NW}} \arrow[u] \arrow[r] & \cdots \arrow[r] & \frac{S_{00}\oplus A_{pq}}{A^{<1}_{NW}} \arrow[u] \\
& & & A_{p0} \arrow[u] \arrow[r] & \cdots \arrow[r] & A_{pq} \arrow[u] \\
& & & \vdots \arrow[u] & & \vdots \arrow[u] \\
& & & A_{00} \arrow[u] \arrow[r] & \cdots \arrow[r] & A_{0q} \arrow[u]
\end{tikzcd}$$

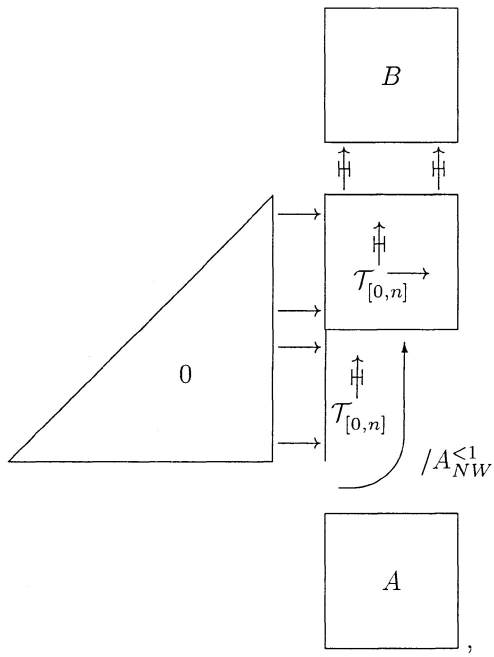
which, lo and behold, is almost exactly what we need. It is practically exactly a cell in the homotopy whose shorthand is



Precisely, in the diagram

$$
\begin{tikzcd}
& & & & & & B_{r0} \arrow[r] & \cdots \arrow[r] & B_{rq} \\
& & & & & & \vdots \arrow[u] & & \vdots \arrow[u] \\
& & & & & & B_{00} \arrow[u] \arrow[r] & \cdots \arrow[r] & B_{0q} \arrow[u] \\
& & & & & 0 \arrow[r] & S_{n0} \arrow[u] \arrow[r] & \cdots \arrow[r] & S_{nq} \arrow[u] \\
& & & & & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] \\
& & & 0 \arrow[r] & \cdots \arrow[r] & R_{in} \arrow[u] \arrow[r] & S_{i0} \arrow[u] \arrow[r] & \cdots \arrow[r] & S_{iq} \arrow[u] \\
& & 0 \arrow[r] & \frac{R_{ii}\oplus A_{NW}}{A^{<1}_{NW}} \arrow[u] \arrow[r] & \cdots \arrow[r] & \frac{R_{in}\oplus A_{NW}}{A^{<1}_{NW}} \arrow[u] \arrow[r] & \frac{S_{i0}\oplus A_{p0}}{A^{<1}_{NW}} \arrow[u] \arrow[r] & \cdots \arrow[r] & \frac{S_{i0}\oplus A_{pq}}{A^{<1}_{NW}} \arrow[u] \\
& & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] \\
0 \arrow[r] & \cdots \arrow[r] & R_{0i} \arrow[u] \arrow[r] & \frac{R_{0i}\oplus A_{NW}}{A^{<1}_{NW}} \arrow[u] \arrow[r] & \cdots \arrow[r] & \frac{R_{0n}\oplus A_{NW}}{A^{<1}_{NW}} \arrow[u] \arrow[r] & \frac{S_{00}\oplus A_{p0}}{A^{<1}_{NW}} \arrow[u] \arrow[r] & \cdots \arrow[r] & \frac{S_{00}\oplus A_{pq}}{A^{<1}_{NW}} \arrow[u] \\
& & & & & & A_{p0} \arrow[u] \arrow[r] & \cdots \arrow[r] & A_{pq} \arrow[u] \\
& & & & & & \vdots \arrow[u] & & \vdots \arrow[u] \\
& & & & & & A_{00} \arrow[u] \arrow[r] & \cdots \arrow[r] & A_{0q} \arrow[u]
\end{tikzcd}
$$

which is a typical cell in the homotopy



we have shown that everything inside the dashbox is OK. All the squares are naturally Mayer–Vietoris. But any square in the diagram is contained in the union of the dashbox above and the dashbox below.

$$
\begin{tikzcd}
& & & & & & B_{r0} \arrow[r] & \cdots \arrow[r] & B_{rq} \\
& & & & & & \vdots \arrow[u, tail] & & \vdots \arrow[u, tail] \\
& & & & & & B_{00} \arrow[u, tail] \arrow[r] & \cdots \arrow[r] & B_{0q} \arrow[u, tail] \\
& & & & & 0 \arrow[r] & S_{n0} \arrow[u, tail] \arrow[r] & \cdots \arrow[r] & S_{nq} \arrow[u, tail] \\
& & & & \ddots & \vdots \arrow[u] & \vdots \arrow[u, tail] & & \vdots \arrow[u, tail] \\
& & & 0 \arrow[r] & \cdots \arrow[r] & R_{in} \arrow[u] \arrow[r] & S_{i0} \arrow[u, tail] \arrow[r] & \cdots \arrow[r] & S_{iq} \arrow[u, tail] \\
& & 0 \arrow[r] & \frac{R_{ii}\oplus A_{NW}}{A^{<1}_{NW}} \arrow[u] \arrow[r] & \cdots \arrow[r] & \frac{R_{in}\oplus A_{NW}}{A^{<1}_{NW}} \arrow[u] \arrow[r] & \frac{S_{i0}\oplus A_{p0}}{A^{<1}_{NW}} \arrow[u, tail] \arrow[r] & \cdots \arrow[r] & \frac{S_{i0}\oplus A_{pq}}{A^{<1}_{NW}} \arrow[u, tail] \\
& \ddots & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] & \vdots \arrow[u, tail] & & \vdots \arrow[u, tail] \\
0 \arrow[r] & \cdots \arrow[r] & R_{0i} \arrow[u] \arrow[r] & \frac{R_{0i}\oplus A_{NW}}{A^{<1}_{NW}} \arrow[u] \arrow[r] & \cdots \arrow[r] & \frac{R_{0n}\oplus A_{NW}}{A^{<1}_{NW}} \arrow[u] \arrow[r] & \frac{S_{00}\oplus A_{p0}}{A^{<1}_{NW}} \arrow[u, tail] \arrow[r] & \cdots \arrow[r] & \frac{S_{00}\oplus A_{pq}}{A^{<1}_{NW}} \arrow[u, tail] \\
& & & & & & A_{p0} \arrow[u, tail] \arrow[r] & \cdots \arrow[r] & A_{pq} \arrow[u, tail] \\
& & & & & & \vdots \arrow[u, tail] & & \vdots \arrow[u, tail] \\
& & & & & & A_{00} \arrow[u, tail] \arrow[r] & \cdots \arrow[r] & A_{0q} \arrow[u, tail]
\end{tikzcd}
$$

It suffices therefore to show that

$$
\begin{tikzcd}
 & & 0 \arrow[r] & \frac{R_{ii}\oplus A_{NW}}{A_{NW}^{<1}} \arrow[r] & \cdots \arrow[r] & \frac{R_{in}\oplus A_{NW}}{A_{NW}^{<1}} \arrow[r] & \frac{S_{i0}\oplus A_{p0}}{A_{NW}^{<1}} \arrow[r] & \cdots \arrow[r] & \frac{S_{i0}\oplus A_{pq}}{A_{NW}^{<1}} \\
 & \ddots & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] \\
0 \arrow[r] & \cdots \arrow[r] & R_{0i} \arrow[r] \arrow[u] & \frac{R_{0i}\oplus A_{NW}}{A_{NW}^{<1}} \arrow[r] \arrow[u] & \cdots \arrow[r] & \frac{R_{0n}\oplus A_{NW}}{A_{NW}^{<1}} \arrow[r] \arrow[u] & \frac{S_{00}\oplus A_{p0}}{A_{NW}^{<1}} \arrow[r] \arrow[u] & \cdots \arrow[r] & \frac{S_{00}\oplus A_{pq}}{A_{NW}^{<1}} \arrow[u]
\end{tikzcd}
$$

is a simplex (i.e. all squares are naturally Mayer–Vietoris). But

$$
\begin{tikzcd}
 & & 0 \arrow[r] & R_{ii}\oplus A_{NW} \arrow[r] & \cdots \arrow[r] & R_{in}\oplus A_{NW} \arrow[r] & S_{i0}\oplus A_{p0} \arrow[r] & \cdots \arrow[r] & S_{i0}\oplus A_{pq} \\
 & \ddots & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] \\
0 \arrow[r] & \cdots \arrow[r] & R_{0i} \arrow[r] \arrow[u] & R_{0i}\oplus A_{NW} \arrow[r] \arrow[u] & \cdots \arrow[r] & R_{0n}\oplus A_{NW} \arrow[r] \arrow[u] & S_{00}\oplus A_{p0} \arrow[r] \arrow[u] & \cdots \arrow[r] & S_{00}\oplus A_{pq} \arrow[u]
\end{tikzcd}
$$

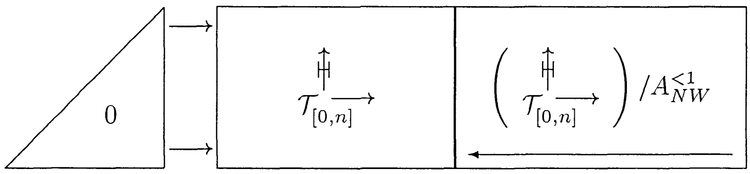
certainly is, being part of the well–defined homotopy cell

$$\begin{tikzcd}
 & & & & & & B_{r0} \arrow[r] & \cdots \arrow[r] & B_{rq} \\
 & & & & & & \vdots \arrow[u] & & \vdots \arrow[u] \\
 & & & & & & B_{00} \arrow[u] \arrow[r] & \cdots \arrow[r] & B_{0q} \arrow[u] \\
 & & & & & 0 \arrow[r] & S_{n0} \arrow[u] \arrow[r] & \cdots \arrow[r] & S_{nq} \arrow[u] \\
 & & & & & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] \\
 & & & 0 \arrow[r] & \cdots \arrow[r] & R_{in} \arrow[u] \arrow[r] & S_{i0} \arrow[u] \arrow[r] & \cdots \arrow[r] & S_{iq} \arrow[u] \\
 & & 0 \arrow[r] & R_{ii}\oplus A_{NW} \arrow[u] \arrow[r] & \cdots \arrow[r] & R_{in}\oplus A_{NW} \arrow[u] \arrow[r] & S_{i0}\oplus A_{p0} \arrow[u] \arrow[r] & \cdots \arrow[r] & S_{i0}\oplus A_{pq} \arrow[u] \\
 & & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] \\
0 \arrow[r] & \cdots \arrow[r] & R_{0i} \arrow[u] \arrow[r] & R_{0i}\oplus A_{NW} \arrow[u] \arrow[r] & \cdots \arrow[r] & R_{0n}\oplus A_{NW} \arrow[u] \arrow[r] & S_{00}\oplus A_{p0} \arrow[u] \arrow[r] & \cdots \arrow[r] & S_{00}\oplus A_{pq} \arrow[u] \\
 & & & & & & A_{p0} \arrow[u] \arrow[r] & \cdots \arrow[r] & A_{pq} \arrow[u] \\
 & & & & & & \vdots \arrow[u] & & \vdots \arrow[u] \\
 & & & & & & A_{00} \arrow[u] \arrow[r] & \cdots \arrow[r] & A_{0q} \arrow[u]
\end{tikzcd}$$

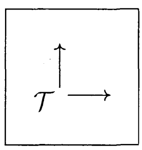
and once again the cell

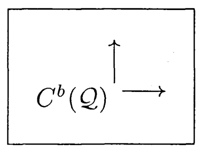
$$\begin{tikzcd}[column sep=small]
 & & 0 \arrow[r] & \frac{R_{ii}\oplus A_{NW}}{A^{<1}_{NW}} \arrow[r] & \cdots \arrow[r] & \frac{R_{in}\oplus A_{NW}}{A^{<1}_{NW}} \arrow[r] & \frac{S_{i0}\oplus A_{p0}}{A^{<1}_{NW}} \arrow[r] & \cdots \arrow[r] & \frac{S_{i0}\oplus A_{pq}}{A^{<1}_{NW}} \\
 & & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] \\
0 \arrow[r] & \cdots \arrow[r] & R_{0i} \arrow[r] \arrow[u] & \frac{R_{0i}\oplus A_{NW}}{A^{<1}_{NW}} \arrow[r] \arrow[u] & \cdots \arrow[r] & \frac{R_{0n}\oplus A_{NW}}{A^{<1}_{NW}} \arrow[r] \arrow[u] & \frac{S_{00}\oplus A_{p0}}{A^{<1}_{NW}} \arrow[r] \arrow[u] & \cdots \arrow[r] & \frac{S_{00}\oplus A_{pq}}{A^{<1}_{NW}} \arrow[u]
\end{tikzcd}$$

is obtained from the above by a trivial truncation homotopy. It is the homotopy which, in our shorthand, has the symbol



REMARK 1.3. Now we have to discuss why the above homotopy takes cells with a lifting to some model category $C^b(\mathcal{Q})$, to bigger cells which also lift. The problem is related to the fact that it is not entirely clear, why t–structure truncations should take a diagram with a lifting to another diagram with a lifting. In other words, because the t–structure is not assumed to be in any way related to the model category $C^b(\mathcal{Q})$ to which the simplex lifts, why should one expect the lifting to be preserved by truncation?

Consider now a simplex in 

with a lifting to 

. That is, we have a diagram of bicartesian squares in $C^b(\mathcal{Q})$:

$$\begin{tikzcd}
B_{m0} \arrow[r] & \cdots \arrow[r] & B_{mn} \\
\vdots \arrow[u] & & \vdots \arrow[u] \\
B_{00} \arrow[r] \arrow[u] & \cdots \arrow[r] & B_{0n} \arrow[u]
\end{tikzcd}$$

Of course, this diagram is entirely determined by pullback from the diagram

$$\begin{tikzcd} B_{m0} \arrow[r] & \cdots \arrow[r] & B_{mn} \\ & & \vdots \arrow[u] \\ & & B_{0n} \arrow[u] \end{tikzcd}$$

What we want to do in this Remark, is reinterpret this obvious fact in terms of sheaves on some space.

Consider the partially ordered set $\mathcal{B}$, given below:

$$\begin{tikzcd} b_{m0} \arrow[r] & \cdots \arrow[r] & b_{mn} \\ & & \vdots \arrow[u] \\ & & b_{0n} \arrow[u] \end{tikzcd}$$

That is, the elements of the set $\mathcal{B}$ are

$$b_{ij} \qquad \text{where} \qquad \begin{cases} i = m & \text{and} \quad 0 \leq j \leq n \\ & \textbf{or} \\ 0 \leq i \leq m & \text{and} \qquad j = n \end{cases}$$

The partial ordering is given by setting $b_{ij} < b_{i'j'}$ if there is an arrow in the diagram from b_{ij} to $b_{i'j'}$.

Now we make $\mathcal{B}$ into a topological space. The open subsets of $\mathcal{B}$ are the subsets closed under majorisation. Precisely, $U \subset \mathcal{B}$ is open if, whenever $x \in U$ and $x < y$, then $y \in U$.

What is a sheaf over $\mathcal{B}$ with values in the abelian category $\mathcal{Q}$? It is nothing more nor less than a diagram

$$\begin{tikzcd} B_{m0} \arrow[r] & \cdots \arrow[r] & B_{mn} \\ \vdots \arrow[u] & & \vdots \arrow[u] \\ B_{00} \arrow[u] \arrow[r] & \cdots \arrow[r] & B_{0n} \arrow[u] \end{tikzcd}$$

where each $B_{ij} \in \mathcal{Q}$, and every square is a pullback square. After all, the open sets in $\mathcal{B}$ are all of the form $\mathcal{B}_{ij}$, for some pair of integers $0 \leq i \leq m$, and $0 \leq j \leq n$. $\mathcal{B}_{ij}$ just consists of the elements

$$
\begin{tikzcd}
b_{mj} \arrow[r] & \cdots \arrow[r] & b_{mn} \\
 & & \vdots \arrow[u] \\
 & & b_{in} \arrow[u]
\end{tikzcd}
$$

Clearly, $\mathcal{B}_{ij} = \mathcal{B}_{mj} \cup \mathcal{B}_{in}$, and therefore to give an arbitrary sheaf $\mathcal{S}$ on $\mathcal{B}$, it suffices to give its values on the open sets $\mathcal{B}_{mj}$ for $0 \leq j \leq n$, and $\mathcal{B}_{in}$ for $0 \leq i \leq m$. Let B_{ij} stand for $\Gamma(\mathcal{B}_{ij}, \mathcal{S})$. The diagram

$$
\begin{tikzcd}
B_{m0} \arrow[r] & \cdots \arrow[r] & B_{mn} \\
 & & \vdots \arrow[u] \\
 & & B_{0n} \arrow[u]
\end{tikzcd}
$$

determines the values of the sheaf $\mathcal{S}$ on the open sets $\mathcal{B}_{mj}$ and $\mathcal{B}_{in}$, which form a basis for the topology. The value on $\mathcal{B}_{ij}$ is determined by the pullback diagram

$$
\begin{tikzcd}
B_{mj} \arrow[r] & B_{mn} \\
B_{ij} \arrow[u] \arrow[r] & B_{in} \arrow[u]
\end{tikzcd}
$$

and hence a sheaf on this topological space really is nothing other than a diagram

$$
\begin{tikzcd}
B_{m0} \arrow[r] & \cdots \arrow[r] & B_{mn} \\
\vdots \arrow[u] & & \vdots \arrow[u] \\
B_{00} \arrow[u] \arrow[r] & \cdots \arrow[r] & B_{0n} \arrow[u]
\end{tikzcd}
$$

where every square is a pullback square.

The topological space $\mathcal{B}$ and all of its open subsets are finite sets. In particular, they are paracompact, and the Čech cohomology agrees with the derived functor cohomology. Given a sheaf $\mathcal{S}$ on $\mathcal{B}$, that is a diagram

$$\begin{tikzcd}
B_{m0} \arrow[r] & \cdots \arrow[r] & B_{mn} \\
\vdots \arrow[u] & & \vdots \arrow[u] \\
B_{00} \arrow[u] \arrow[r] & \cdots \arrow[r] & B_{0n} \arrow[u]
\end{tikzcd}$$

one can reasonably ask to compute the cohomology of this sheaf. Given an open set $\mathcal{B}_{ij} \subset \mathcal{B}$, what are the groups $H^k(\mathcal{B}_{ij}, \mathcal{S})$?

It is immediate that every cover of $\mathcal{B}_{ij}$ contains the cover $\mathcal{B}_{ij} = \mathcal{B}_{mj} \cup \mathcal{B}_{in}$ as a refinement. The Čech cohomology, which agrees with the derived functor cohomology, can therefore be computed on this cover. We deduce that the cohomology $H^k(\mathcal{B}_{ij}, \mathcal{S})$ is just the cohomology of the complex

$$B_{in} \oplus B_{mj} \longrightarrow B_{mn}.$$

Therefore, to say that the diagram

$$\begin{tikzcd}
B_{m0} \arrow[r] & \cdots \arrow[r] & B_{mn} \\
\vdots \arrow[u] & & \vdots \arrow[u] \\
B_{00} \arrow[u] \arrow[r] & \cdots \arrow[r] & B_{0n} \arrow[u]
\end{tikzcd}$$

consists of *bicartesian* squares, is nothing more nor less than to require that the sheaf $\mathcal{S}$ have vanishing higher cohomology on every open subset $\mathcal{B}_{ij} \subset \mathcal{B}$. Let us call a sheaf $\mathcal{S}$ *acyclic* if its cohomology vanishes on every $\mathcal{B}_{ij} \subset \mathcal{B}$.

Let $D^b(\mathcal{B})$ be the bounded derived category of the category of all sheaves on $\mathcal{B}$ with values in $\mathcal{Q}$. The objects of $D^b(\mathcal{B})$ are complexes of sheaves $\mathcal{S}$ on $\mathcal{B}$. Every such complex is quasi–isomorphic to a complex of acyclic sheaves. Take for instance a flabby resolution. To begin with, we know we can always replace a complex by a (possibly infinite) flabby resolution. But in fact, the resolution by acyclics may be taken finite, because every one of the finitely many open sets in $\mathcal{B}$ has finite homological dimension.

Thus $D^b(\mathcal{B})$ is the derived category of a category we denote $\boxed{\begin{array}{cc} & \uparrow \\ C^b(\mathcal{Q}) & \longrightarrow \end{array}}$,

whose objects are diagrams

$$\begin{array}{ccccc}
B_{m0} & \longrightarrow & \cdots & \longrightarrow & B_{mn} \\
\uparrow & & & & \uparrow \\
\vdots & & & & \vdots \\
\uparrow & & & & \uparrow \\
B_{00} & \longrightarrow & \cdots & \longrightarrow & B_{0n}
\end{array}$$

of bicartesian squares in $C^b(\mathcal{Q})$. Now the real point of the Remark is that $\mathcal{B}$ is a stratified space, with every point being a stratum. Thus, given an arbitrary perversity function on the strata, there corresponds a t–structure on $D^b(\mathcal{B})$. In particular, there corresponds a t–structure truncation. And the point is that the truncated simplices used in the proof of Theorem 1.2 are all obtainable from simplices in $\boxed{C^b(\mathcal{Q})\ {\uparrow}\!\!\longrightarrow}$ by a t–structure truncation for some perverse t–structure on the category $D^b(\mathcal{B})$.

Note. It is slightly criminal to denote our category by $\boxed{C^b(\mathcal{Q})\ {\uparrow}\!\!\longrightarrow}$, because we tend to think of $\boxed{C^b(\mathcal{Q})\ {\uparrow}\!\!\longrightarrow}$ as a simplicial set, containing diagrams

$$\begin{array}{ccccc}
B_{m0} & \longrightarrow & \cdots & \longrightarrow & B_{mn} \\
\uparrow & & & & \uparrow \\
\vdots & & & & \vdots \\
\uparrow & & & & \uparrow \\
B_{00} & \longrightarrow & \cdots & \longrightarrow & B_{0n}
\end{array}$$

where m and n are allowed to vary. Our category has objects in which m and n are fixed. It would perhaps be better notation to denote the category $\left(\boxed{C^b(\mathcal{Q})\ {\uparrow}\!\!\longrightarrow}\right)_{mn}$. However, the author happens to find this notation so æsthetically revolting, that he preferred the slightly misleeding notation.

Now I have told the reader in general terms what we will do. It is time to turn very specific. We need to show how to construct the cell

$$\begin{tikzcd}
 & & & & & & B_{r0} \arrow[r] & \cdots \arrow[r] & B_{rq} \\
 & & & & & & \vdots \arrow[u] & & \vdots \arrow[u] \\
 & & & & & & \boxed{B_{00}} \arrow[r] \arrow[u] & \cdots \arrow[r] & B_{0q} \arrow[u] \\
 & & & & & 0 \arrow[r] & S_{n0} \arrow[r] \arrow[u] & \cdots \arrow[r] & S_{nq} \arrow[u] \\
 & & & & & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] \\
 & & & 0 \arrow[r] & \cdots \arrow[r] & R_{in} \arrow[r] \arrow[u] & S_{i0} \arrow[r] \arrow[u] & \cdots \arrow[r] & S_{iq} \arrow[u] \\
 & & 0 \arrow[r] & \frac{R_{ii}\oplus A_{NW}}{A^{<1}_{NW}} \arrow[r] \arrow[u] & \cdots \arrow[r] & \frac{R_{in}\oplus A_{NW}}{A^{<1}_{NW}} \arrow[r] \arrow[u] & \frac{S_{i0}\oplus A_{p0}}{A^{<1}_{NW}} \arrow[r] \arrow[u] & \cdots \arrow[r] & \frac{S_{i0}\oplus A_{pq}}{A^{<1}_{NW}} \arrow[u] \\
 & & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] \\
0 \arrow[r] & \cdots \arrow[r] & R_{0i} \arrow[r] \arrow[u] & \frac{R_{0i}\oplus A_{NW}}{A^{<1}_{NW}} \arrow[r] \arrow[u] & \cdots \arrow[r] & \frac{R_{0n}\oplus A_{NW}}{A^{<1}_{NW}} \arrow[r] \arrow[u] & \frac{S_{00}\oplus A_{p0}}{A^{<1}_{NW}} \arrow[r] \arrow[u] & \cdots \arrow[r] & \frac{S_{00}\oplus A_{pq}}{A^{<1}_{NW}} \arrow[u] \\
 & & & & & & A_{p0} \arrow[r] \arrow[u] & \cdots \arrow[r] & A_{pq} \arrow[u] \\
 & & & & & & \vdots \arrow[u] & & \vdots \arrow[u] \\
 & & & & & & A_{00} \arrow[r] \arrow[u] & \cdots \arrow[r] & A_{0q} \arrow[u]
\end{tikzcd}$$

out of the cell

$$
\begin{tikzcd}
 & & & & & & B_{r0} \arrow[r] & \cdots \arrow[r] & B_{rq} \\
 & & & & & & \vdots \arrow[u] & & \vdots \arrow[u] \\
 & & & & & & B_{00} \arrow[u] \arrow[r] & \cdots \arrow[r] & B_{0q} \arrow[u] \\
 & & & & & 0 \arrow[r] & S_{n0} \arrow[u] \arrow[r] & \cdots \arrow[r] & S_{nq} \arrow[u] \\
 & & & & & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] \\
 & & & 0 \arrow[r] & \cdots \arrow[r] & R_{in} \arrow[u] \arrow[r] & S_{i0} \arrow[u] \arrow[r] & \cdots \arrow[r] & S_{iq} \arrow[u] \\
 & & 0 \arrow[r] & R_{ii}\oplus A_{NW} \arrow[u] \arrow[r] & \cdots \arrow[r] & R_{in}\oplus A_{NW} \arrow[u] \arrow[r] & S_{i0}\oplus A_{p0} \arrow[u] \arrow[r] & \cdots \arrow[r] & S_{i0}\oplus A_{pq} \arrow[u] \\
 & & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] \\
0 \arrow[r] & \cdots \arrow[r] & R_{0i} \arrow[u] \arrow[r] & R_{0i}\oplus A_{NW} \arrow[u] \arrow[r] & \cdots \arrow[r] & R_{0n}\oplus A_{NW} \arrow[u] \arrow[r] & S_{00}\oplus A_{p0} \arrow[u] \arrow[r] & \cdots \arrow[r] & S_{00}\oplus A_{pq} \arrow[u] \\
 & & & & & & A_{p0} \arrow[u] \arrow[r] & \cdots \arrow[r] & A_{pq} \arrow[u] \\
 & & & & & & \vdots \arrow[u] & & \vdots \arrow[u] \\
 & & & & & & A_{00} \arrow[u] \arrow[r] & \cdots \arrow[r] & A_{0q} \arrow[u]
\end{tikzcd}
$$

using nothing other than perverse t–structure truncations on suitably chosen finite topological spaces. To begin with, observe that to give a lifting of

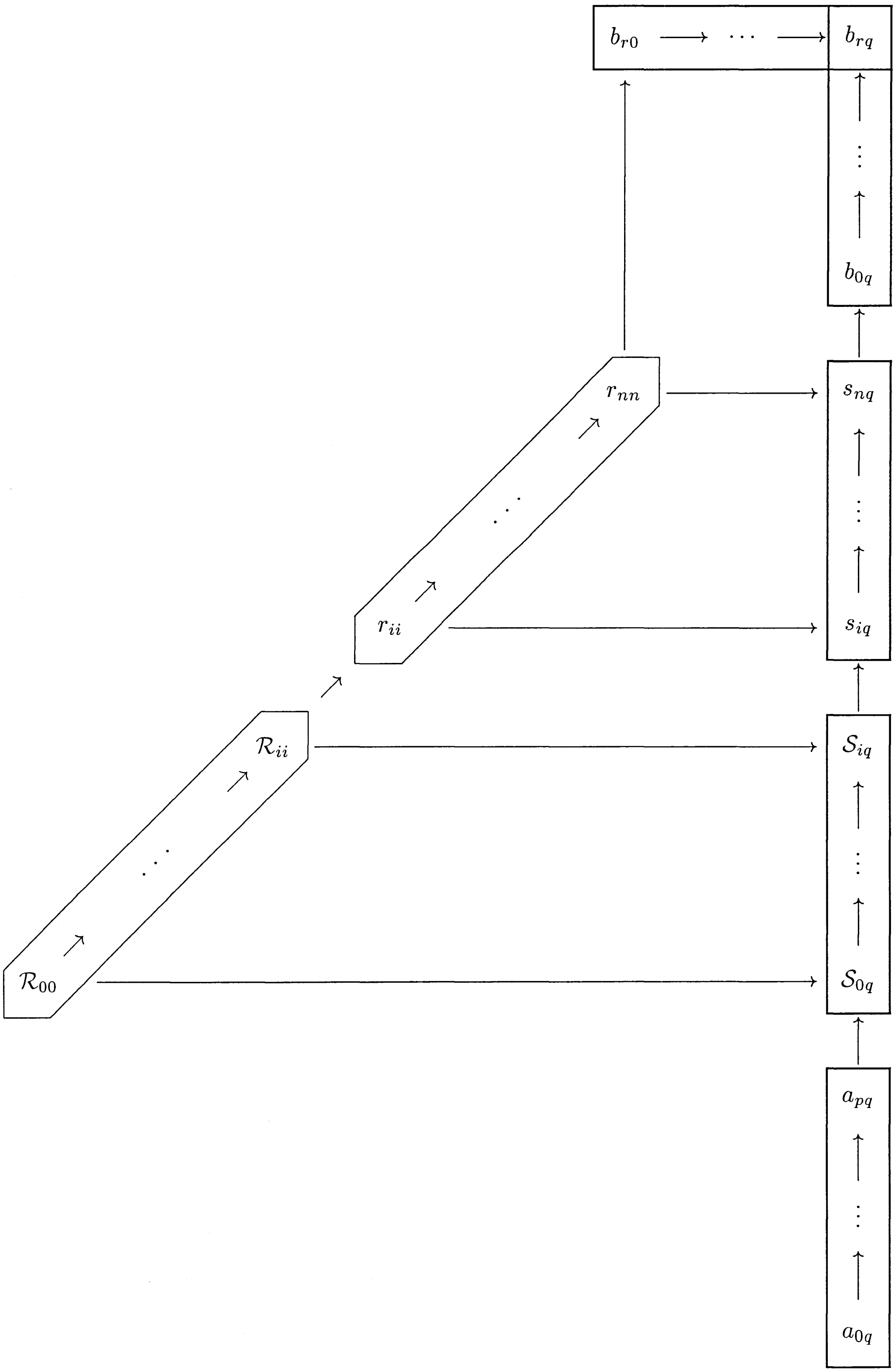
$$
\begin{tikzcd}[column sep=small]
 & & & & & & B_{r0} \arrow[r] & \cdots \arrow[r] & B_{rq} \\
 & & & & & & \vdots \arrow[u] & & \vdots \arrow[u] \\
 & & & & & & B_{00} \arrow[r] \arrow[u] & \cdots \arrow[r] & B_{0q} \arrow[u] \\
 & & & & & 0 \arrow[r] & S_{n0} \arrow[r] \arrow[u] & \cdots \arrow[r] & S_{nq} \arrow[u] \\
 & & & & \cdot^{\cdot^{\cdot}} & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] \\
 & & & 0 \arrow[r] & \cdots \arrow[r] & R_{in} \arrow[r] \arrow[u] & S_{i0} \arrow[r] \arrow[u] & \cdots \arrow[r] & S_{iq} \arrow[u] \\
 & & 0 \arrow[r] & R_{ii}\oplus A_{NW} \arrow[r] \arrow[u] & \cdots \arrow[r] & R_{in}\oplus A_{NW} \arrow[r] \arrow[u] & S_{i0}\oplus A_{p0} \arrow[r] \arrow[u] & \cdots \arrow[r] & S_{i0}\oplus A_{pq} \arrow[u] \\
 & \cdot^{\cdot^{\cdot}} & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] \\
0 \arrow[r] & \cdots \arrow[r] & R_{0i} \arrow[r] \arrow[u] & R_{0i}\oplus A_{NW} \arrow[r] \arrow[u] & \cdots \arrow[r] & R_{0n}\oplus A_{NW} \arrow[r] \arrow[u] & S_{00}\oplus A_{p0} \arrow[r] \arrow[u] & \cdots \arrow[r] & S_{00}\oplus A_{pq} \arrow[u] \\
 & & & & & & A_{p0} \arrow[r] \arrow[u] & \cdots \arrow[r] & A_{pq} \arrow[u] \\
 & & & & & & \vdots \arrow[u] & & \vdots \arrow[u] \\
 & & & & & & A_{00} \arrow[r] \arrow[u] & \cdots \arrow[r] & A_{0q} \arrow[u]
\end{tikzcd}
$$

is the same as to give a diagram $C^b(\mathcal{Q})$

$$\begin{tikzcd}
 & & & & & & \tilde B_{r0} \arrow[r] & \cdots \arrow[r] & \tilde B_{rq} \\
 & & & & & & \vdots \arrow[u, "\wr"'] & & \vdots \arrow[u, "\wr"'] \\
 & & & & & & \tilde B_{00} \arrow[r] \arrow[u, "\wr"'] & \cdots \arrow[r] & \tilde B_{0q} \arrow[u, "\wr"'] \\
 & & & & & \tilde R_{nn} \arrow[r] & \tilde S_{n0} \arrow[r] \arrow[u, "\wr"'] & \cdots \arrow[r] & \tilde S_{nq} \arrow[u, "\wr"'] \\
 & & & & & \vdots \arrow[u] & \vdots \arrow[u, "\wr"'] & & \vdots \arrow[u, "\wr"'] \\
 & & & \tilde R_{ii} \arrow[r] & \cdots \arrow[r] & \tilde R_{in} \arrow[r] \arrow[u] & \tilde S_{i0} \arrow[r] \arrow[u, "\wr"'] & \cdots \arrow[r] & \tilde S_{iq} \arrow[u, "\wr"'] \\
 & & \tilde R_{ii} \arrow[r] & \tilde R_{ii}\oplus\tilde A_{NW} \arrow[r] \arrow[u] & \cdots \arrow[r] & \tilde R_{in}\oplus\tilde A_{NW} \arrow[r] \arrow[u] & \tilde S_{i0}\oplus\tilde A_{p0} \arrow[r] \arrow[u] & \cdots \arrow[r] & \tilde S_{i0}\oplus\tilde A_{pq} \arrow[u] \\
 & & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] & \vdots \arrow[u, "\wr"'] & & \vdots \arrow[u, "\wr"'] \\
\tilde R_{00} \arrow[r] & \cdots \arrow[r] & \tilde R_{0i} \arrow[r] \arrow[u] & \tilde R_{0i}\oplus\tilde A_{NW} \arrow[r] \arrow[u] & \cdots \arrow[r] & \tilde R_{0n}\oplus\tilde A_{NW} \arrow[r] \arrow[u] & \tilde S_{00}\oplus\tilde A_{p0} \arrow[r] \arrow[u, "\wr"'] & \cdots \arrow[r] & \tilde S_{00}\oplus\tilde A_{pq} \arrow[u, "\wr"'] \\
 & & & & & & \tilde A_{p0} \arrow[r] \arrow[u] & \cdots \arrow[r] & \tilde A_{pq} \arrow[u] \\
 & & & & & & \vdots \arrow[u, "\wr"'] & & \vdots \arrow[u, "\wr"'] \\
 & & & & & & \tilde A_{00} \arrow[r] \arrow[u, "\wr"'] & \cdots \arrow[r] & \tilde A_{0q} \arrow[u, "\wr"']
\end{tikzcd}$$

where the $\tilde R_{ii}$ happen to be isomorphic to zero in $D^b(\mathcal{Q})$. To be more precise, they come with an isomorphism to zero, and all other objects in the lifted diagram come with isomorphisms to the objects they lift, and the isomorphisms commute with all the maps that are part of the definition of a simplex.

This diagram can be viewed as a sheaf on the topological space $\mathcal{X}$, which we will schematically denote



The way this diagram should be read is the following. The partially ordered set has points b_{rj} where $0 \leq j \leq q$, b_{jq} with $0 \leq j \leq r$, r_{jj} with $i \leq j \leq r$, $\mathcal{R}_{jj}$ with $0 \leq j \leq i$, s_{jq} with $i \leq j \leq r$, $\mathcal{S}_{jq}$ with $0 \leq j \leq i$ and a_{jq} with $0 \leq \gamma \leq p$. The labels of the points are meant to be suggestive; each point corresponds to the object having the same name but with Roman capitals, in the diagram of the simplex. The only exception is $\mathcal{S}_{jq}$, which corresponds to $S_{j0} \oplus A_{pq}$.

$$\begin{tikzcd}
 & & & & & & \tilde{B}_{r0} \arrow[r] & \cdots \arrow[r] & \tilde{B}_{rq} \\
 & & & & & & \vdots \arrow[u] & & \vdots \arrow[u] \\
 & & & & & & \tilde{B}_{00} \arrow[u] \arrow[r] & \cdots \arrow[r] & \tilde{B}_{0q} \arrow[u] \\
 & & & & & \tilde{R}_{nn} \arrow[r] & \tilde{S}_{n0} \arrow[u] \arrow[r] & \cdots \arrow[r] & \tilde{S}_{nq} \arrow[u] \\
 & & & & & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] \\
 & & & \tilde{R}_{ii} \arrow[r] & \cdots \arrow[r] & \tilde{R}_{in} \arrow[u] \arrow[r] & \tilde{S}_{i0} \arrow[u] \arrow[r] & \cdots \arrow[r] & \tilde{S}_{iq} \arrow[u] \\
 & & \tilde{R}_{ii} \arrow[r] & \tilde{R}_{ii}\oplus\tilde{A}_{NW} \arrow[u] \arrow[r] & \cdots \arrow[r] & \tilde{R}_{in}\oplus\tilde{A}_{NW} \arrow[u] \arrow[r] & \tilde{S}_{i0}\oplus\tilde{A}_{p0} \arrow[u] \arrow[r] & \cdots \arrow[r] & \tilde{S}_{i0}\oplus\tilde{A}_{pq} \arrow[u] \\
 & & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] \\
\tilde{R}_{00} \arrow[r] & \cdots \arrow[r] & \tilde{R}_{0i} \arrow[u] \arrow[r] & \tilde{R}_{0i}\oplus\tilde{A}_{NW} \arrow[u] \arrow[r] & \cdots \arrow[r] & \tilde{R}_{0n}\oplus\tilde{A}_{NW} \arrow[u] \arrow[r] & \tilde{S}_{00}\oplus\tilde{A}_{p0} \arrow[u] \arrow[r] & \cdots \arrow[r] & \tilde{S}_{00}\oplus\tilde{A}_{pq} \arrow[u] \\
 & & & & & & \tilde{A}_{p0} \arrow[u] \arrow[r] & \cdots \arrow[r] & \tilde{A}_{pq} \arrow[u] \\
 & & & & & & \vdots \arrow[u] & & \vdots \arrow[u] \\
 & & & & & & \tilde{A}_{00} \arrow[u] \arrow[r] & \cdots \arrow[r] & \tilde{A}_{0q} \arrow[u]
\end{tikzcd}$$

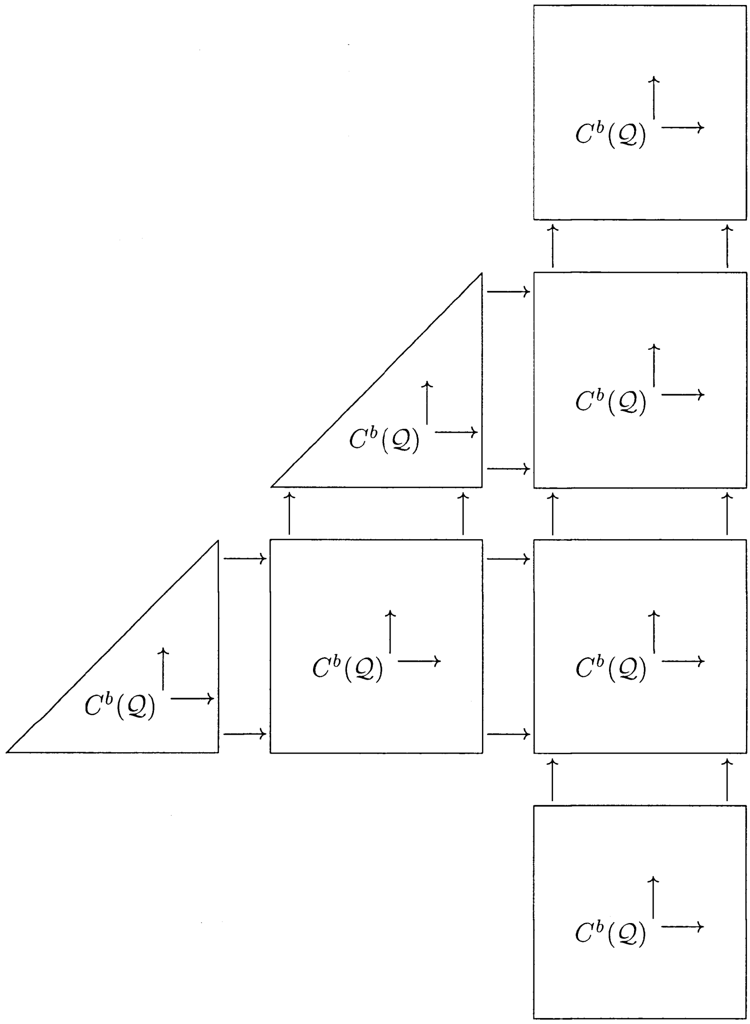
The partial ordering is that b's are as in the example discussed above, the r's are smaller than the b's, the $\mathcal{R}$'s are smaller than the r's, the s's are smaller than the b_{iq}'s, the $\mathcal{S}$'s are smaller than the s's, and the a's are smaller than the $\mathcal{S}$'s. Furthermore, for every j, $r_{jj} < s_{jq}$ and $\mathcal{R}_{jj} < \mathcal{S}_{jq}$. Perhaps a simpler way to state this, is to say that the points of the partially ordered set are all the objects of the diagram on the right and top fringe, together with the diagonal entries of the triangles on the left (these are declared to be honorary members of the top fringe). A point in this partially ordered set is less than another, if there is an arrow joining the lesser to the greater.

Once again, one can turn this partially ordered set into a topological space, by declaring sets closed under majorisation to be open. I leave it to the reader to check that an acyclic sheaf on this space is nothing more nor less than a diagram

$$
\begin{tikzcd}
 & & & & & & \tilde{B}_{r0} \arrow[r] & \cdots \arrow[r] & \tilde{B}_{rq} \\
 & & & & & & \vdots \arrow[u] & & \vdots \arrow[u] \\
 & & & & & & \tilde{B}_{00} \arrow[u] \arrow[r] & \cdots \arrow[r] & \tilde{B}_{0q} \arrow[u] \\
 & & & & & \tilde{R}_{nn} \arrow[r] & \tilde{S}_{n0} \arrow[u] \arrow[r] & \cdots \arrow[r] & \tilde{S}_{nq} \arrow[u] \\
 & & & & & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] \\
 & & & \tilde{R}_{ii} \arrow[r] & \cdots \arrow[r] & \tilde{R}_{in} \arrow[u] \arrow[r] & \tilde{S}_{i0} \arrow[u] \arrow[r] & \cdots \arrow[r] & \tilde{S}_{iq} \arrow[u] \\
 & & \tilde{R}_{ii} \arrow[r] & \tilde{R}_{ii}\oplus\tilde{A}_{NW} \arrow[u] \arrow[r] & \cdots \arrow[r] & \tilde{R}_{in}\oplus\tilde{A}_{NW} \arrow[u] \arrow[r] & \tilde{S}_{i0}\oplus\tilde{A}_{p0} \arrow[u] \arrow[r] & \cdots \arrow[r] & \tilde{S}_{i0}\oplus\tilde{A}_{pq} \arrow[u] \\
 & & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] \\
\tilde{R}_{00} \arrow[r] & \cdots \arrow[r] & \tilde{R}_{0i} \arrow[u] \arrow[r] & \tilde{R}_{0i}\oplus\tilde{A}_{NW} \arrow[u] \arrow[r] & \cdots \arrow[r] & \tilde{R}_{0n}\oplus\tilde{A}_{NW} \arrow[u] \arrow[r] & \tilde{S}_{00}\oplus\tilde{A}_{p0} \arrow[u] \arrow[r] & \cdots \arrow[r] & \tilde{S}_{00}\oplus\tilde{A}_{pq} \arrow[u] \\
 & & & & & & \tilde{A}_{p0} \arrow[u] \arrow[r] & \cdots \arrow[r] & \tilde{A}_{pq} \arrow[u] \\
 & & & & & & \vdots \arrow[u] & & \vdots \arrow[u] \\
 & & & & & & \tilde{A}_{00} \arrow[u] \arrow[r] & \cdots \arrow[r] & \tilde{A}_{0q} \arrow[u]
\end{tikzcd}
$$

of bicartesian squares of objects, in the abelian category $\mathcal{Q}$. The derived category

$D^b(\mathcal{X})$ has for a model a category which we will denote



consisting of diagrams of bicartesian squares in $C^b(\mathcal{Q})$, as above.

To define the t–structure we want, we need first a slightly larger topological space. We enlarge the partially ordered set $\mathcal{X}$ by one point, obtaining a space we will denote $\mathcal{Y}$. Let us call the extra point, i.e. the unique point in $\mathcal{Y} - \mathcal{X}$, by the name p. We declare p to be less than

$$\mathcal{S}_{iq} < s_{iq} < \cdots < s_{nq} < b_{0q} < \cdots < b_{rq}$$

and no other points. The only points less than p are the a's and the $\mathcal{R}$'s.

An acyclic sheaf on $\mathcal{Y}$ with values in $\mathcal{Q}$ is a three dimensional diagram of bicartesian squares. To represent it on a planar piece of paper, I will write two planar sections of it. It consists of a diagram

$$\begin{tikzcd}
 & & & \tilde{B}^1_{r0} \arrow[r] & \cdots \arrow[r] & \tilde{B}^1_{rq} \\
 & & & \vdots \arrow[u] & & \vdots \arrow[u] \\
 & & & \tilde{B}^1_{00} \arrow[u] \arrow[r] & \cdots \arrow[r] & \tilde{B}^1_{0q} \arrow[u] \\
 & & \tilde{R}^1_{nn} \arrow[r] & \tilde{S}^1_{n0} \arrow[u] \arrow[r] & \cdots \arrow[r] & \tilde{S}^1_{nq} \arrow[u] \\
 & \cdot^{\cdot^{\cdot}} & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] \\
\tilde{R}^1_{ii} \arrow[r] & \cdots \arrow[r] & \tilde{R}^1_{in} \arrow[u] \arrow[r] & \tilde{S}^1_{i0} \arrow[u] \arrow[r] & \cdots \arrow[r] & \tilde{S}^1_{iq} \arrow[u] \\
\{\tilde{R}_{ii} \oplus \tilde{A}_{NW}\}^1 \arrow[u] \arrow[r] & \cdots \arrow[r] & \{\tilde{R}_{in} \oplus \tilde{A}_{NW}\}^1 \arrow[u] \arrow[r] & \{\tilde{S}_{i0} \oplus \tilde{A}_{p0}\}^1 \arrow[u] \arrow[r] & \cdots \arrow[r] & \{\tilde{S}_{i0} \oplus \tilde{A}_{pq}\}^1 \arrow[u] \\
\vdots \arrow[u] & & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] \\
\{\tilde{R}_{0i} \oplus \tilde{A}_{NW}\}^1 \arrow[u] \arrow[r] & \cdots \arrow[r] & \{\tilde{R}_{0n} \oplus \tilde{A}_{NW}\}^1 \arrow[u] \arrow[r] & \{\tilde{S}_{00} \oplus \tilde{A}_{p0}\}^1 \arrow[u] \arrow[r] & \cdots \arrow[r] & \{\tilde{S}_{00} \oplus \tilde{A}_{pq}\}^1 \arrow[u]
\end{tikzcd}$$

together with another diagram

$$\begin{tikzcd}
 & & \tilde{R}^0_{ii} \arrow[r] & \{\tilde{R}_{ii} \oplus \tilde{A}_{NW}\}^0 \arrow[r] & \cdots \arrow[r] & \{\tilde{R}_{in} \oplus \tilde{A}_{NW}\}^0 \arrow[r] & \{\tilde{S}_{i0} \oplus \tilde{A}^0_{p0}\}^0 \arrow[r] & \cdots \arrow[r] & \{\tilde{S}_{i0} \oplus \tilde{A}^0_{pq}\}^0 \\
 & \cdot^{\cdot^{\cdot}} & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] \\
\tilde{R}^0_{00} \arrow[r] & \cdots \arrow[r] & \tilde{R}^0_{0i} \arrow[u] \arrow[r] & \{\tilde{R}_{0i} \oplus \tilde{A}_{NW}\}^0 \arrow[u] \arrow[r] & \cdots \arrow[r] & \{\tilde{R}_{0n} \oplus \tilde{A}_{NW}\}^0 \arrow[u] \arrow[r] & \{\tilde{S}_{00} \oplus \tilde{A}^0_{p0}\}^0 \arrow[u] \arrow[r] & \cdots \arrow[r] & \{\tilde{S}_{00} \oplus \tilde{A}^0_{pq}\}^0 \arrow[u] \\
 & & & & & & \tilde{A}^0_{p0} \arrow[u] \arrow[r] & \cdots \arrow[r] & \tilde{A}^0_{pq} \arrow[u] \\
 & & & & & & \vdots \arrow[u] & & \vdots \arrow[u] \\
 & & & & & & \tilde{A}^0_{00} \arrow[u] \arrow[r] & \cdots \arrow[r] & \tilde{A}^0_{0q} \arrow[u]
\end{tikzcd}$$

together with maps from any object with a superscript 0 to the object of the same label with a superscript 1, so that every square, even in the third direction, is bicartesian. Concretely, the open sets containing $p \in \mathcal{Y} - \mathcal{X}$ correspond to the objects with superscript 0, while the open sets not containing p have superscript 1.

The perversity we want to consider on $\mathcal{Y}$ is actually relatively simple. Let Z be the closure of p in $\mathcal{Y}$, and let $U = \mathcal{Y} - Z$. On Z we take the trivial t–structure, where $D^b(Z)^{\geq 0} = D^b(Z)$. On U we take the standard t–structure, where $D^b(U)^{\geq 0}$ is the category of complexes of sheaves whose homology is supported in positive degrees. These t–structures glue to give a t–structure on all of $D^b(\mathcal{Y})$. The actual simplex we started with, in the proof of Theorem 1.2, had a lifting

$$\begin{tikzcd}
 & & & & & & \tilde B_{r0} \arrow[r] & \cdots \arrow[r] & \tilde B_{rq} \\
 & & & & & & \vdots \arrow[u] & & \vdots \arrow[u] \\
 & & & & & & \tilde B_{00} \arrow[u] \arrow[r] & \cdots \arrow[r] & \tilde B_{0q} \arrow[u] \\
 & & & & & \tilde R_{nn} \arrow[r] & \tilde S_{n0} \arrow[u] \arrow[r] & \cdots \arrow[r] & \tilde S_{nq} \arrow[u] \\
 & & & & \ddots & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] \\
 & & & \tilde R_{ii} \arrow[r] & \cdots \arrow[r] & \tilde R_{in} \arrow[u] \arrow[r] & \tilde S_{i0} \arrow[u] \arrow[r] & \cdots \arrow[r] & \tilde S_{iq} \arrow[u] \\
 & & \tilde R_{ii} \arrow[r] & \tilde R_{ii} \oplus \tilde A_{NW} \arrow[u] \arrow[r] & \cdots \arrow[r] & \tilde R_{in} \oplus \tilde A_{NW} \arrow[u] \arrow[r] & \tilde S_{i0} \oplus \tilde A_{p0} \arrow[u] \arrow[r] & \cdots \arrow[r] & \tilde S_{i0} \oplus \tilde A_{pq} \arrow[u] \\
 & \ddots & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] \\
\tilde R_{00} \arrow[r] & \cdots \arrow[r] & \tilde R_{0i} \arrow[u] \arrow[r] & \tilde R_{0i} \oplus \tilde A_{NW} \arrow[u] \arrow[r] & \cdots \arrow[r] & \tilde R_{0n} \oplus \tilde A_{NW} \arrow[u] \arrow[r] & \tilde S_{00} \oplus \tilde A_{p0} \arrow[u] \arrow[r] & \cdots \arrow[r] & \tilde S_{00} \oplus \tilde A_{pq} \arrow[u] \\
 & & & & & & \tilde A_{p0} \arrow[u] \arrow[r] & \cdots \arrow[r] & \tilde A_{pq} \arrow[u] \\
 & & & & & & \vdots \arrow[u] & & \vdots \arrow[u] \\
 & & & & & & \tilde A_{00} \arrow[u] \arrow[r] & \cdots \arrow[r] & \tilde A_{0q} \arrow[u]
\end{tikzcd}$$

and is thus a complex $\mathcal{S}$ of sheaves on $\mathcal{X}$. The inclusion $\mathcal{X} \hookrightarrow \mathcal{Y}$ has a continuous retraction $\pi : \mathcal{Y} \longrightarrow \mathcal{X}$, sending p to $\mathcal{S}_{iq}$. Then $\pi^*\mathcal{S}$ is a complex of sheaves on $\mathcal{Y}$, which lies in $D^b(\mathcal{Y})^{\geq 0}$ for our choice of perversity. The object α in $D^b(\mathcal{Y})_{[0,0]}$, which is given by the pair of diagrams

$$
\begin{array}{ccccccccccc}
 & & & & & & 0^1 & \longrightarrow & \cdots & \longrightarrow & 0^1 \\
 & & & & & & \uparrow & & & & \uparrow \\
 & & & & & & \vdots & & & & \vdots \\
 & & & & & & \uparrow & & & & \uparrow \\
 & & & & & & 0^1 & \longrightarrow & \cdots & \longrightarrow & 0^1 \\
 & & & & & & \uparrow & & & & \uparrow \\
 & & & & 0^1 & \longrightarrow & 0^1 & \longrightarrow & \cdots & \longrightarrow & 0^1 \\
 & & & & \uparrow & & \uparrow & & & & \uparrow \\
 & & \iddots & & \vdots & & \vdots & & & & \vdots \\
 & & & & \uparrow & & \uparrow & & & & \uparrow \\
0^1 & \longrightarrow & \cdots & \longrightarrow & 0^1 & \longrightarrow & 0^1 & \longrightarrow & \cdots & \longrightarrow & 0^1 \\
\uparrow & & & & \uparrow & & \uparrow & & & & \uparrow \\
\{\tilde{A}^{<1}_{NW}\}^1 & \longrightarrow & \cdots & \longrightarrow & \{\tilde{A}^{<1}_{NW}\}^1 & \longrightarrow & \{\tilde{A}^{<1}_{NW}\}^1 & \longrightarrow & \cdots & \longrightarrow & \{\tilde{A}^{<1}_{NW}\}^1 \\
\uparrow & & & & \uparrow & & \uparrow & & & & \uparrow \\
\vdots & & & & \vdots & & \vdots & & & & \vdots \\
\uparrow & & & & \uparrow & & \uparrow & & & & \uparrow \\
\{\tilde{A}^{<1}_{NW}\}^1 & \longrightarrow & \cdots & \longrightarrow & \{\tilde{A}^{<1}_{NW}\}^1 & \longrightarrow & \{\tilde{A}^{<1}_{NW}\}^1 & \longrightarrow & \cdots & \longrightarrow & \{\tilde{A}^{<1}_{NW}\}^1
\end{array}
$$

together with

$$
\begin{array}{ccccccccccccccccc}
 & & & & 0^0 & \to & 0^0 & \longrightarrow & \cdots & \longrightarrow & 0^0 & \to & 0^0 & \longrightarrow & \cdots & \longrightarrow & 0^0 \\
 & & & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & & & \uparrow \\
 & & & \cdot^{\cdot^{\cdot}} & \vdots & & \vdots & & & & \vdots & & \vdots & & & & \vdots \\
 & & & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & & & \uparrow \\
0^0 & \longrightarrow & \cdots & \longrightarrow & 0^0 & \to & 0^0 & \longrightarrow & \cdots & \longrightarrow & 0^0 & \to & 0^0 & \longrightarrow & \cdots & \longrightarrow & 0^0 \\
 & & & & & & & & & & & & \uparrow & & & & \uparrow \\
 & & & & & & & & & & & & 0^0 & \longrightarrow & \cdots & \longrightarrow & 0^0 \\
 & & & & & & & & & & & & \uparrow & & & & \uparrow \\
 & & & & & & & & & & & & \vdots & & & & \vdots \\
 & & & & & & & & & & & & \uparrow & & & & \uparrow \\
 & & & & & & & & & & & & 0^0 & \longrightarrow & \cdots & \longrightarrow & 0^0
\end{array}
$$

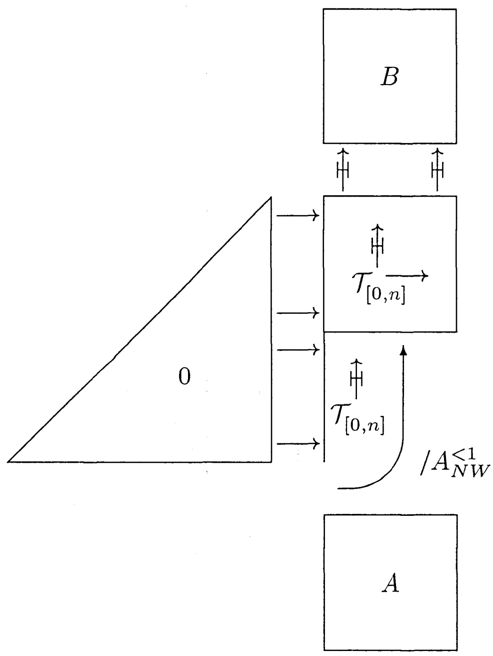
injects into $\pi^*\mathcal{S}$. The quotient $\pi^*\mathcal{S}/\alpha$ is quasi-isomorphic to a complex of acyclic sheaves on $\mathcal{Y}$, and in it we easiiy recognize a subset, giving a lifting of the diagram

$$
\begin{tikzcd}
 & & & & & & B_{r0} \arrow[r] & \cdots \arrow[r] & B_{rq} \\
 & & & & & & \vdots \arrow[u] & & \vdots \arrow[u] \\
 & & & & & & B_{00} \arrow[u] \arrow[r] & \cdots \arrow[r] & B_{0q} \arrow[u] \\
 & & & & & 0 \arrow[r] & S_{n0} \arrow[u] \arrow[r] & \cdots \arrow[r] & S_{nq} \arrow[u] \\
 & & & & \cdot^{\cdot^{\cdot}} & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] \\
 & & & 0 \arrow[r] & \cdots \arrow[r] & R_{in} \arrow[u] \arrow[r] & S_{i0} \arrow[u] \arrow[r] & \cdots \arrow[r] & S_{iq} \arrow[u] \\
 & & 0 \arrow[r] & \frac{R_{ii}\oplus A_{NW}}{A^{<1}_{NW}} \arrow[u] \arrow[r] & \cdots \arrow[r] & \frac{R_{in}\oplus A_{NW}}{A^{<1}_{NW}} \arrow[u] \arrow[r] & \frac{S_{i0}\oplus A_{p0}}{A^{<1}_{NW}} \arrow[u] \arrow[r] & \cdots \arrow[r] & \frac{S_{i0}\oplus A_{pq}}{A^{<1}_{NW}} \arrow[u] \\
 & \cdot^{\cdot^{\cdot}} & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] \\
0 \arrow[r] & \cdots \arrow[r] & R_{0i} \arrow[u] \arrow[r] & \frac{R_{0i}\oplus A_{NW}}{A^{<1}_{NW}} \arrow[u] \arrow[r] & \cdots \arrow[r] & \frac{R_{0n}\oplus A_{NW}}{A^{<1}_{NW}} \arrow[u] \arrow[r] & \frac{S_{00}\oplus A_{p0}}{A^{<1}_{NW}} \arrow[u] \arrow[r] & \cdots \arrow[r] & \frac{S_{00}\oplus A_{pq}}{A^{<1}_{NW}} \arrow[u] \\
 & & & & & & A_{p0} \arrow[u] \arrow[r] & \cdots \arrow[r] & A_{pq} \arrow[u] \\
 & & & & & & \vdots \arrow[u] & & \vdots \arrow[u] \\
 & & & & & & A_{00} \arrow[u] \arrow[r] & \cdots \arrow[r] & A_{0q} \arrow[u]
\end{tikzcd}
$$

We deduce that the diagram has a lifting.

Every time we use t–structure truncations in the proof, the Group 2 reader needs to go through an argument similar to the one we have just given. Fortunately, we use t–structure truncations only four times; once in the proof of Theorem 1.2, as we have just seen. The next time we will see a homotopy using the truncation is in Lemma 2.1. Then it occurs in Lemma 3.2. The fourth and last time we will see such a homotopy will be in Lemma 3.4. Hopefully the reader will not object too vigorously, if I leave him to check for himself the other three occurrences of such an argument.

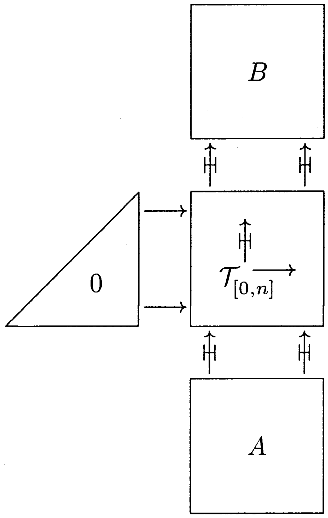
Continuation of the Proof of Theorem 1.2. We are now finished proving that the homotopy



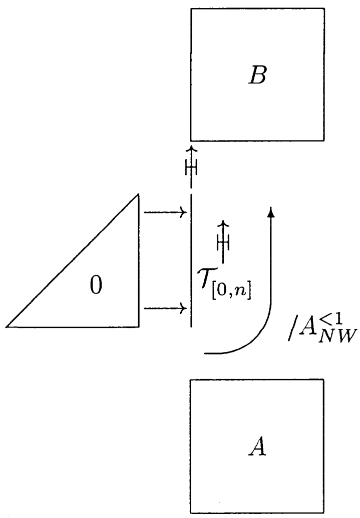
is well defined, even for the simplicial set that the Group 2 reader has been working with. But this is not yet enough. The well–defined homotopy above shows the identity is homotopic to some other map. We have to show that this other map is, in turn, homotopic to the null map.

Warning for the Group 2 Reader. As far as you are concerned, the next two paragraphs contain brazen, unmitigated lies. Remark 1.4 will clarify what the lies are, and how to fix them.

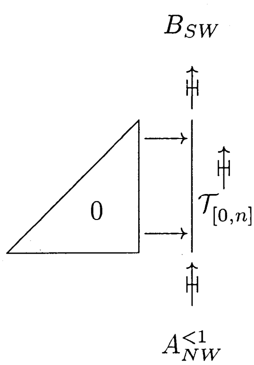
We have shown that the identity on



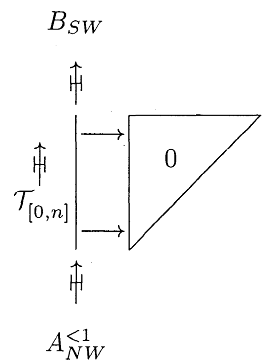
is homotopic to the simplicial map



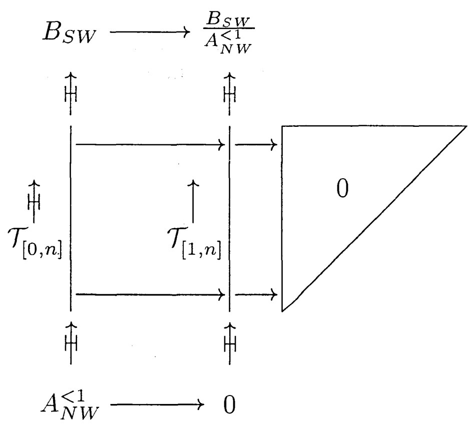
and this map factors through the simplicial set



But because kernels and cokernels agree, the above simplicial set is equal to



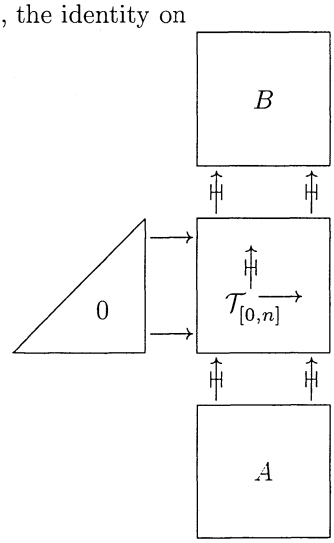
and now, by the functoriality of the truncation, this set agrees with



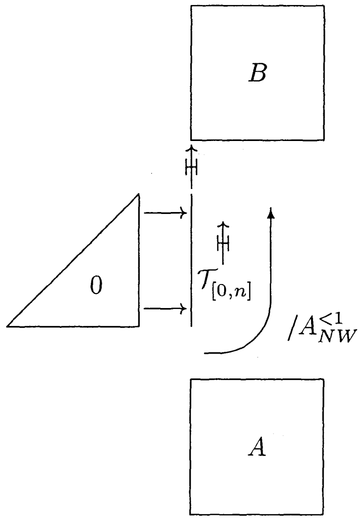
and this makes it clear that the contraction to the initial object contracts the set. □

REMARK 1.4. As I said, as far as a Group 2 reader goes, this was a brazen lie. What was wrong about it, and how can it be fixed?

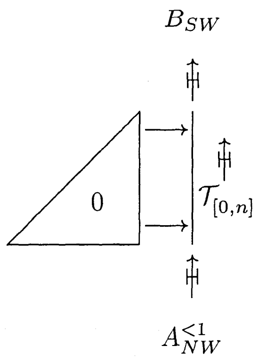
As we correctly showed, the identity on



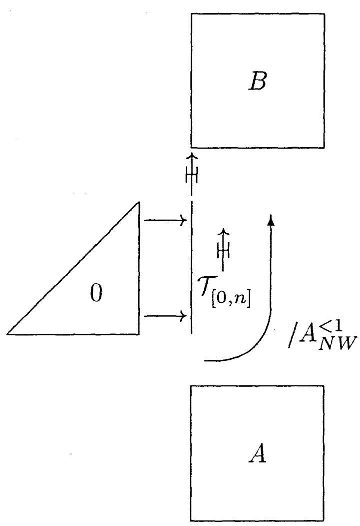
is homotopic to the simplicial map



But when the author cavalierly suggested that the map factors through the simplicial set



he was blatantly lying, at least from the point of view of a Group 2 reader. The simplicial map



takes a simplex

$$
s_n = \left(
\begin{tikzcd}
 & & & B_{r0} \arrow[r] & \cdots \arrow[r] & B_{rq} \\
 & & & \vdots \arrow[u, tail] & & \vdots \arrow[u, tail] \\
 & & & B_{00} \arrow[u, tail] \arrow[r] & \cdots \arrow[r] & B_{0q} \arrow[u, tail] \\
 & & 0 \arrow[r] & S_{n0} \arrow[u, tail] \arrow[r] & \cdots \arrow[r] & S_{nq} \arrow[u, tail] \\
 & \iddots & \vdots \arrow[u] & \vdots \arrow[u, tail] & & \vdots \arrow[u, tail] \\
0 \arrow[r] & \cdots \arrow[r] & R_{0n} \arrow[u] \arrow[r] & S_{00} \arrow[u, tail] \arrow[r] & \cdots \arrow[r] & S_{0q} \arrow[u, tail] \\
 & & & A_{p0} \arrow[u, tail] \arrow[r] & \cdots \arrow[r] & A_{pq} \arrow[u, tail] \\
 & & & \vdots \arrow[u, tail] & & \vdots \arrow[u, tail] \\
 & & & A_{00} \arrow[u, tail] \arrow[r] & \cdots \arrow[r] & A_{0q} \arrow[u, tail]
\end{tikzcd}
\right)
$$

to the simplex

$$
\begin{tikzcd}
 & & & B_{r0} \arrow[r] & \cdots \arrow[r] & B_{rq} \\
 & & & \vdots \arrow[u, tail] & & \vdots \arrow[u, tail] \\
 & & & B_{00} \arrow[u, tail] \arrow[r] & \cdots \arrow[r] & B_{0q} \arrow[u, tail] \\
 & & 0 \arrow[r] & \frac{S_{n0}\oplus A_{p0}}{A^{<1}_{NW}} \arrow[u, tail] \arrow[r] & \cdots \arrow[r] & \frac{S_{n0}\oplus A_{pq}}{A^{<1}_{NW}} \arrow[u, tail] \\
 & \iddots & \vdots \arrow[u] & \vdots \arrow[u, tail] & & \vdots \arrow[u, tail] \\
0 \arrow[r] & \cdots \arrow[r] & R_{0n} \arrow[u] \arrow[r] & \frac{S_{00}\oplus A_{p0}}{A^{<1}_{NW}} \arrow[u, tail] \arrow[r] & \cdots \arrow[r] & \frac{S_{00}\oplus A_{pq}}{A^{<1}_{NW}} \arrow[u, tail] \\
 & & & A_{p0} \arrow[u, tail] \arrow[r] & \cdots \arrow[r] & A_{pq} \arrow[u, tail] \\
 & & & \vdots \arrow[u, tail] & & \vdots \arrow[u, tail] \\
 & & & A_{00} \arrow[u, tail] \arrow[r] & \cdots \arrow[r] & A_{0q} \arrow[u, tail]
\end{tikzcd}
$$

Of course, this simplex is obtained from

$$
\begin{tikzcd}
 & & & B_{r0}\oplus B_{00} \arrow[r] & \cdots \arrow[r] & B_{rq}\oplus B_{00} \\
 & & & \vdots \arrow[u, tail] & & \vdots \arrow[u, tail] \\
 & & & B_{00}\oplus B_{00} \arrow[r] \arrow[u, tail] & \cdots \arrow[r] & B_{0q}\oplus B_{00} \arrow[u, tail] \\
 & & 0 \arrow[r] & S_{n0}\oplus A_{p0} \arrow[r] \arrow[u, tail] & \cdots \arrow[r] & S_{n0}\oplus A_{pq} \arrow[u, tail] \\
 & \iddots & \vdots \arrow[u] & \vdots \arrow[u, tail] & & \vdots \arrow[u, tail] \\
0 \arrow[r] & \cdots \arrow[r] & R_{0n} \arrow[r] \arrow[u] & S_{00}\oplus A_{p0} \arrow[r] \arrow[u, tail] & \cdots \arrow[r] & S_{00}\oplus A_{pq} \arrow[u, tail] \\
 & & & A_{p0} \arrow[r] \arrow[u] & \cdots \arrow[r] & A_{pq} \arrow[u] \\
 & & & \vdots \arrow[u, tail] & & \vdots \arrow[u, tail] \\
 & & & A_{00} \arrow[r] \arrow[u, tail] & \cdots \arrow[r] & A_{0q} \arrow[u, tail]
\end{tikzcd}
$$

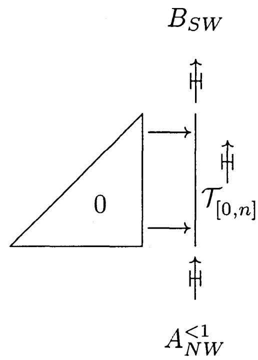
by dividing every object in the top rectangle by the (diagonal) inclusion of B_{00}, and everything in the middle rectangle by the diagonal inclusion of $A_{NW}^{<1}$. In other words, we obtain it from the direct sum of the fixed simplex

$$
\begin{array}{ccccccccc}
 & & & & & & B_{r0} & \longrightarrow \cdots \longrightarrow & B_{rq} \\
 & & & & & & \uparrow & & \uparrow \\
 & & & & & & \vdots & & \vdots \\
 & & & & & & \uparrow & & \uparrow \\
 & & & & & & B_{00} & \longrightarrow \cdots \longrightarrow & B_{0q} \\
 & & & & & & \uparrow & & \uparrow \\
 & & & & 0 & \longrightarrow & A_{p0} & \longrightarrow \cdots \longrightarrow & A_{pq} \\
 & & & & \uparrow & & \uparrow & & \uparrow \\
 & & \ddots & & \vdots & & \vdots & & \vdots \\
 & & & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & A_{p0} & \longrightarrow \cdots \longrightarrow & A_{pq} \\
 & & & & & & \uparrow & & \uparrow \\
 & & & & & & A_{p0} & \longrightarrow \cdots \longrightarrow & A_{pq} \\
 & & & & & & \uparrow & & \uparrow \\
 & & & & & & \vdots & & \vdots \\
 & & & & & & \uparrow & & \uparrow \\
 & & & & & & A_{00} & \longrightarrow \cdots \longrightarrow & A_{0q}
\end{array}
$$

and the simplex

$$
\begin{array}{ccccccccccc}
 & & & & & & B_{00} & \longrightarrow & \cdots & \longrightarrow & B_{00} \\
 & & & & & & \uparrow & & & & \uparrow \\
 & & & & & & \vdots & & & & \vdots \\
 & & & & & & \uparrow & & & & \uparrow \\
 & & & & & & B_{00} & \longrightarrow & \cdots & \longrightarrow & B_{00} \\
 & & & & & & \uparrow & & & & \uparrow \\
 & & & & 0 & \longrightarrow & S_{n0} & \longrightarrow & \cdots & \longrightarrow & S_{n0} \\
 & & & & \uparrow & & \uparrow & & & & \uparrow \\
 & & & \iddots & \vdots & & \vdots & & & & \vdots \\
 & & & & \uparrow & & \uparrow & & & & \uparrow \\
0 & \longrightarrow & \cdots & \longrightarrow & R_{0n} & \longrightarrow & S_{00} & \longrightarrow & \cdots & \longrightarrow & S_{00} \\
 & & & & & & \uparrow & & & & \uparrow \\
 & & & & & & 0 & \longrightarrow & \cdots & \longrightarrow & 0 \\
 & & & & & & \uparrow & & & & \uparrow \\
 & & & & & & \vdots & & & & \vdots \\
 & & & & & & \uparrow & & & & \uparrow \\
 & & & & & & 0 & \longrightarrow & \cdots & \longrightarrow & 0
\end{array}
$$

This latest simplex, which contains all the information that varies as we vary the integer n, does indeed lie in the simplicial set



So the reader may well wonder what all the fuss is about. Surely the map does factor up to homotopy. All the variable information is contained in the smaller simplicial set.

But there is a problem. Precisely, the problem is with the lifting to model categories. The simplices

$$
\begin{tikzcd}
 & & & B_{r0} \arrow[r] & \cdots \arrow[r] & B_{rq} \\
 & & & \vdots \arrow[u] & & \vdots \arrow[u] \\
 & & & B_{00} \arrow[u] \arrow[r] & \cdots \arrow[r] & B_{0q} \arrow[u] \\
 & & 0 \arrow[r] & A_{p0} \arrow[u] \arrow[r] & \cdots \arrow[r] & A_{pq} \arrow[u] \\
 & \iddots & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] \\
0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[u] \arrow[r] & A_{p0} \arrow[u] \arrow[r] & \cdots \arrow[r] & A_{pq} \arrow[u] \\
 & & & A_{p0} \arrow[u] \arrow[r] & \cdots \arrow[r] & A_{pq} \arrow[u] \\
 & & & \vdots \arrow[u] & & \vdots \arrow[u] \\
 & & & A_{00} \arrow[u] \arrow[r] & \cdots \arrow[r] & A_{0q} \arrow[u]
\end{tikzcd}
$$

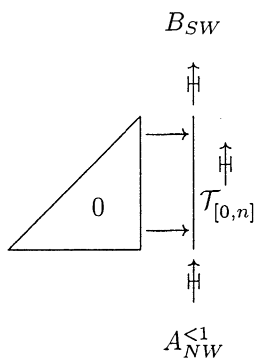
and

$$
\begin{array}{ccccccc}
 & & & & B_{00} & \longrightarrow \cdots \longrightarrow & B_{00} \\
 & & & & \uparrow & & \uparrow \\
 & & & & \vdots & & \vdots \\
 & & & & \uparrow & & \uparrow \\
 & & & & B_{00} & \longrightarrow \cdots \longrightarrow & B_{00} \\
 & & & & \uparrow & & \uparrow \\
 & & 0 & \longrightarrow & S_{n0} & \longrightarrow \cdots \longrightarrow & S_{n0} \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & \iddots & \vdots & & \vdots & & \vdots \\
 & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow \cdots \longrightarrow & R_{0n} & \longrightarrow & S_{00} & \longrightarrow \cdots \longrightarrow & S_{00} \\
 & & & & \uparrow & & \uparrow \\
 & & & & 0 & \longrightarrow \cdots \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow \\
 & & & & \vdots & & \vdots \\
 & & & & \uparrow & & \uparrow \\
 & & & & 0 & \longrightarrow \cdots \longrightarrow & 0
\end{array}
$$

split as direct sums of diagrams having liftings to model categories. But in fact, if we assume that

$$
\begin{tikzcd}
 & & & B_{00} \arrow[r] & \cdots \arrow[r] & B_{00} \\
 & & & \vdots \arrow[u] & & \vdots \arrow[u] \\
 & & & B_{00} \arrow[u] \arrow[r] & \cdots \arrow[r] & B_{00} \arrow[u] \\
 & & 0 \arrow[r] & S_{n0} \arrow[u] \arrow[r] & \cdots \arrow[r] & S_{n0} \arrow[u] \\
 & \cdot^{\cdot^{\cdot}} & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] \\
0 \arrow[r] & \cdots \arrow[r] & R_{0n} \arrow[u] \arrow[r] & S_{00} \arrow[u] \arrow[r] & \cdots \arrow[r] & S_{00} \arrow[u] \\
 & & & 0 \arrow[u] \arrow[r] & \cdots \arrow[r] & 0 \arrow[u] \\
 & & & \vdots \arrow[u] & & \vdots \arrow[u] \\
 & & & 0 \arrow[u] \arrow[r] & \cdots \arrow[r] & 0 \arrow[u]
\end{tikzcd}
$$

is just an arbitrary simplex in



then these liftings will in general be to unrelated model categories, and the careful reader will notice that in constructing the simplex

$$
\begin{tikzcd}
 & & & B_{r0} \arrow[r] & \cdots \arrow[r] & B_{rq} \\
 & & & \vdots \arrow[u] & & \vdots \arrow[u] \\
 & & & B_{00} \arrow[u] \arrow[r] & \cdots \arrow[r] & B_{0q} \arrow[u] \\
 & & 0 \arrow[r] & \frac{S_{n0}\oplus A_{p0}}{A^{<1}_{NW}} \arrow[u] \arrow[r] & \cdots \arrow[r] & \frac{S_{n0}\oplus A_{pq}}{A^{<1}_{NW}} \arrow[u] \\
 & \cdot\ \ \cdot\ \ \cdot & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] \\
0 \arrow[r] & \cdots \arrow[r] & R_{0n} \arrow[u] \arrow[r] & \frac{S_{00}\oplus A_{p0}}{A^{<1}_{NW}} \arrow[u] \arrow[r] & \cdots \arrow[r] & \frac{S_{00}\oplus A_{pq}}{A^{<1}_{NW}} \arrow[u] \\
 & & & A_{p0} \arrow[u] \arrow[r] & \cdots \arrow[r] & A_{pq} \arrow[u] \\
 & & & \vdots \arrow[u] & & \vdots \arrow[u] \\
 & & & A_{00} \arrow[u] \arrow[r] & \cdots \arrow[r] & A_{0q} \arrow[u]
\end{tikzcd}
$$

we had to assume a compatibility of liftings. Precisely, we needed to assume that

$$\begin{tikzcd}
 & & & B_{r0} \arrow[r] & \cdots \arrow[r] & B_{rq} \\
 & & & \vdots \arrow[u] & & \vdots \arrow[u] \\
 & & & B_{00} \arrow[u] \arrow[r] & \cdots \arrow[r] & B_{0q} \arrow[u] \\
 & & 0 \arrow[r] & A_{p0} \arrow[u] \arrow[r] & \cdots \arrow[r] & A_{pq} \arrow[u] \\
 & \iddots & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] \\
0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[u] \arrow[r] & A_{p0} \arrow[u] \arrow[r] & \cdots \arrow[r] & A_{pq} \arrow[u] \\
 & & & A_{p0} \arrow[u] \arrow[r] & \cdots \arrow[r] & A_{pq} \arrow[u] \\
 & & & \vdots \arrow[u] & & \vdots \arrow[u] \\
 & & & A_{00} \arrow[u] \arrow[r] & \cdots \arrow[r] & A_{0q} \arrow[u]
\end{tikzcd}$$

and

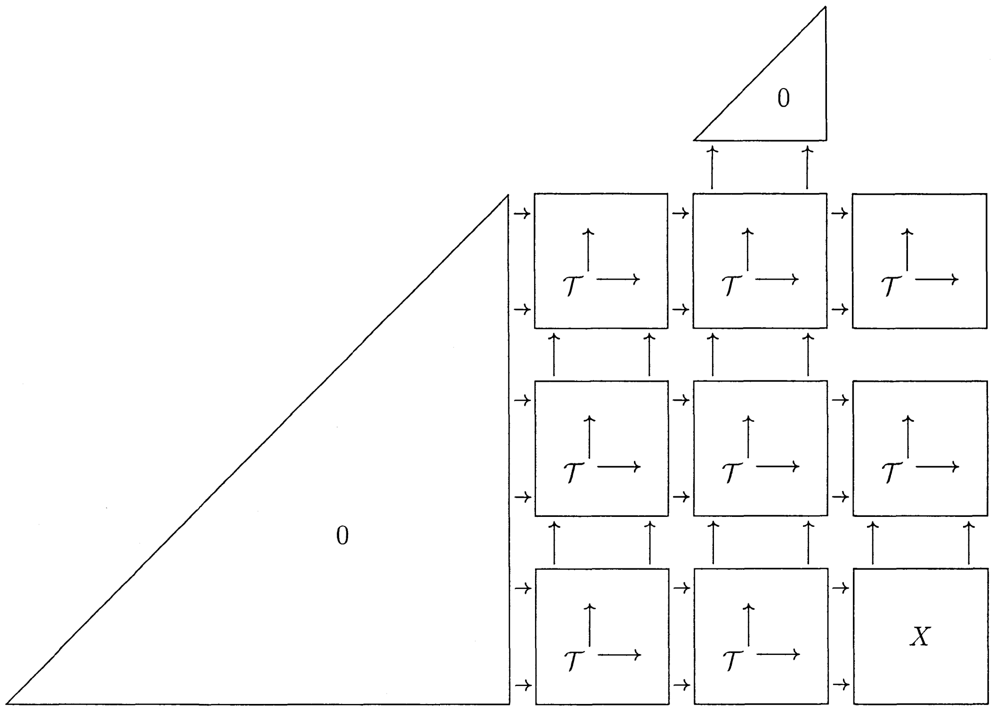
$$
\begin{tikzcd}
 & & & B_{00} \arrow[r] & \cdots \arrow[r] & B_{00} \\
 & & & \vdots \arrow[u, "\wr"'] & & \vdots \arrow[u, "\wr"'] \\
 & & & B_{00} \arrow[r] \arrow[u, "\wr"'] & \cdots \arrow[r] & B_{00} \arrow[u, "\wr"'] \\
 & & 0 \arrow[r] & S_{n0} \arrow[r] \arrow[u, "\wr"'] & \cdots \arrow[r] & S_{n0} \arrow[u, "\wr"'] \\
 & \ddots & \vdots \arrow[u] & \vdots \arrow[u, "\wr"'] & & \vdots \arrow[u, "\wr"'] \\
0 \arrow[r] & \cdots \arrow[r] & R_{0n} \arrow[r] \arrow[u] & S_{00} \arrow[r] \arrow[u, "\wr"'] & \cdots \arrow[r] & S_{00} \arrow[u, "\wr"'] \\
 & & & 0 \arrow[r] \arrow[u] & \cdots \arrow[r] & 0 \arrow[u] \\
 & & & \vdots \arrow[u, "\wr"'] & & \vdots \arrow[u, "\wr"'] \\
 & & & 0 \arrow[r] \arrow[u, "\wr"'] & \cdots \arrow[r] & 0 \arrow[u, "\wr"']
\end{tikzcd}
$$

both split compatibly, that is one can choose a splitting of each so that the summands correspond in pairs, and each pair lifts to the same model category. One furthermore needs that the homology $H^0(S^r_{i0})$ agree with the homology $H^0(A^r_{p0})$ for the r^{th} summands A^r_{p0} of A_{p0} and S^r_{i0} of S_{i0}.

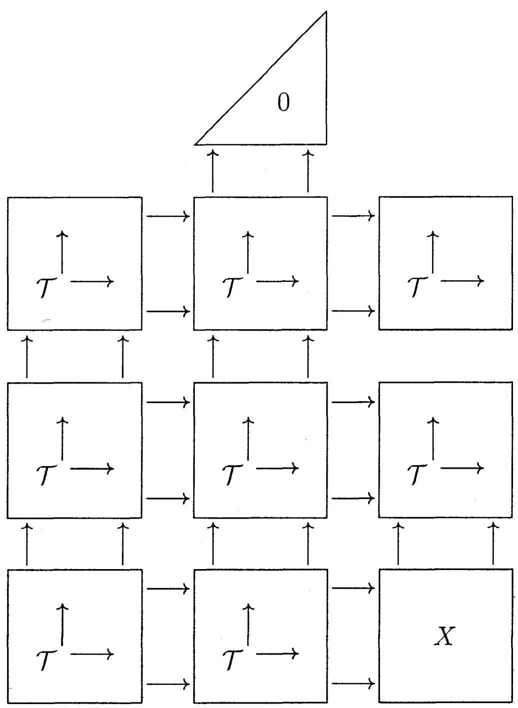
Having said what the difficulty is, it is also clear how to fix it. The point is that the contraction to the initial object preserves this direct sum decomposition into pairs of simplices with liftings to the same model category.

There is only one place in the remainder of the article where we will allow ourselves a similar lie. The reader is given notice that the same problem, and the same solution, apply to the proof of Lemma 3.2.

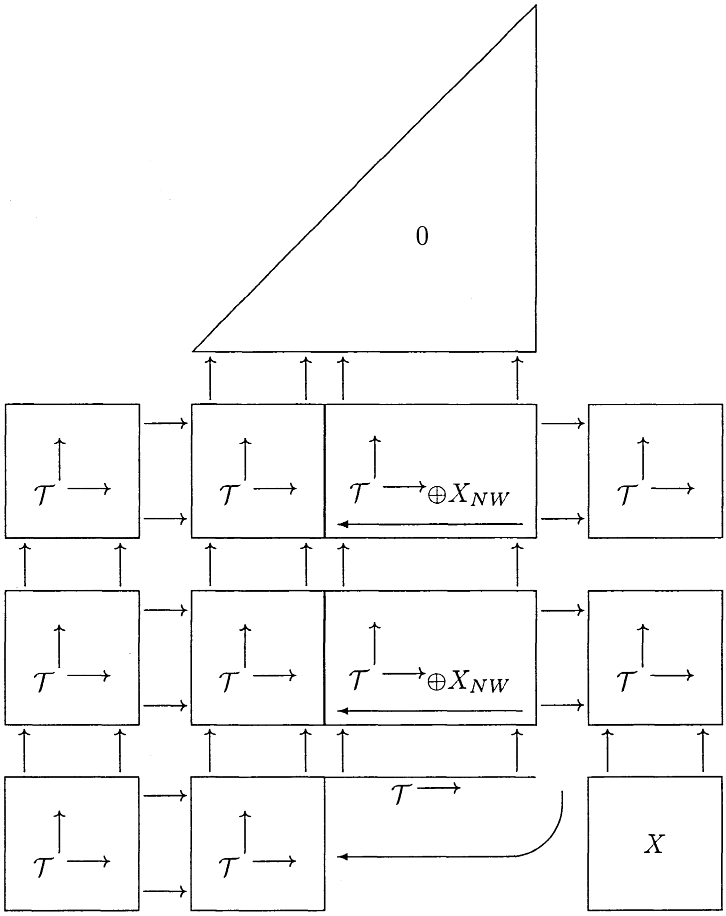
Now may be a good time to divulge a small secret. The blueprint simplicial set



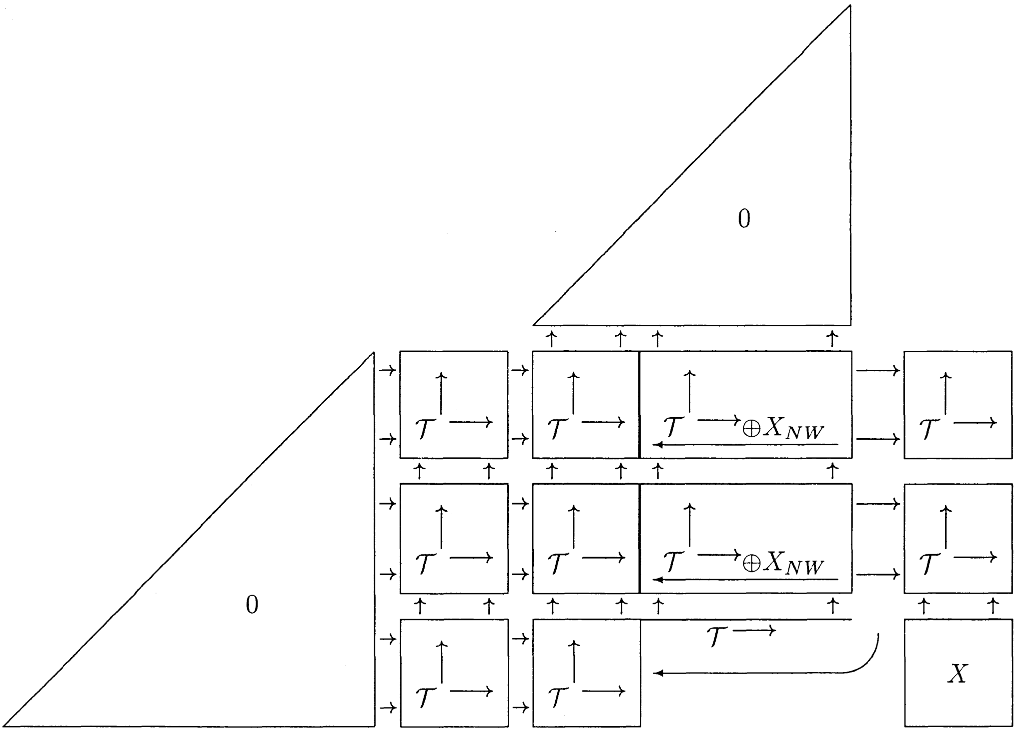
that we have been considering until now, is unnecessarily large and clumsy. Suppose we replace it by the smaller blueprint



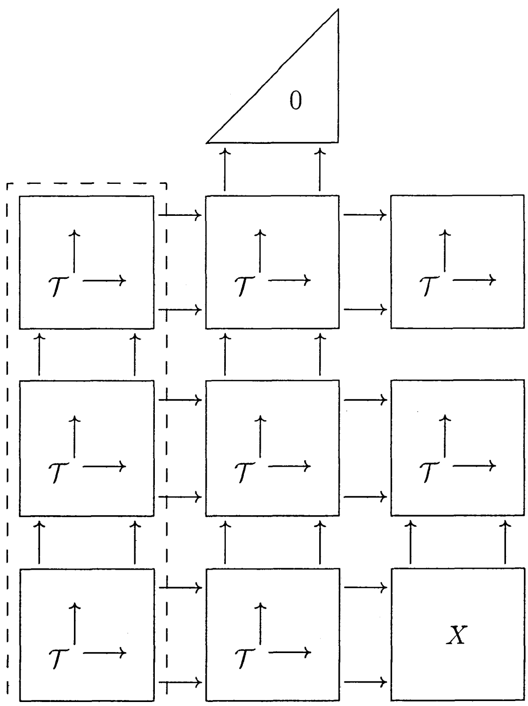
and we accept that on it the homotopy



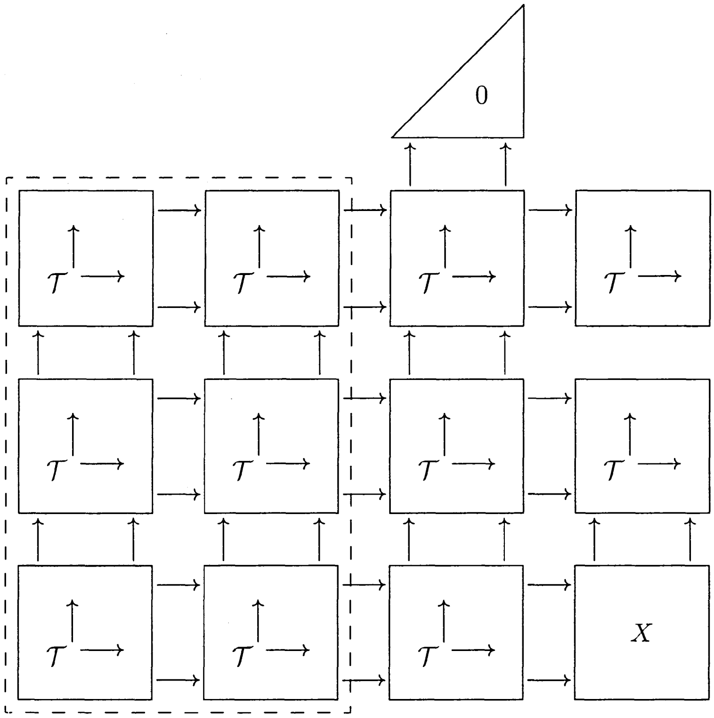
is well–defined. Let us refer to it, for now, as the "compact blueprint homotopy". Then it is a formal consequence that the blueprint homotopy



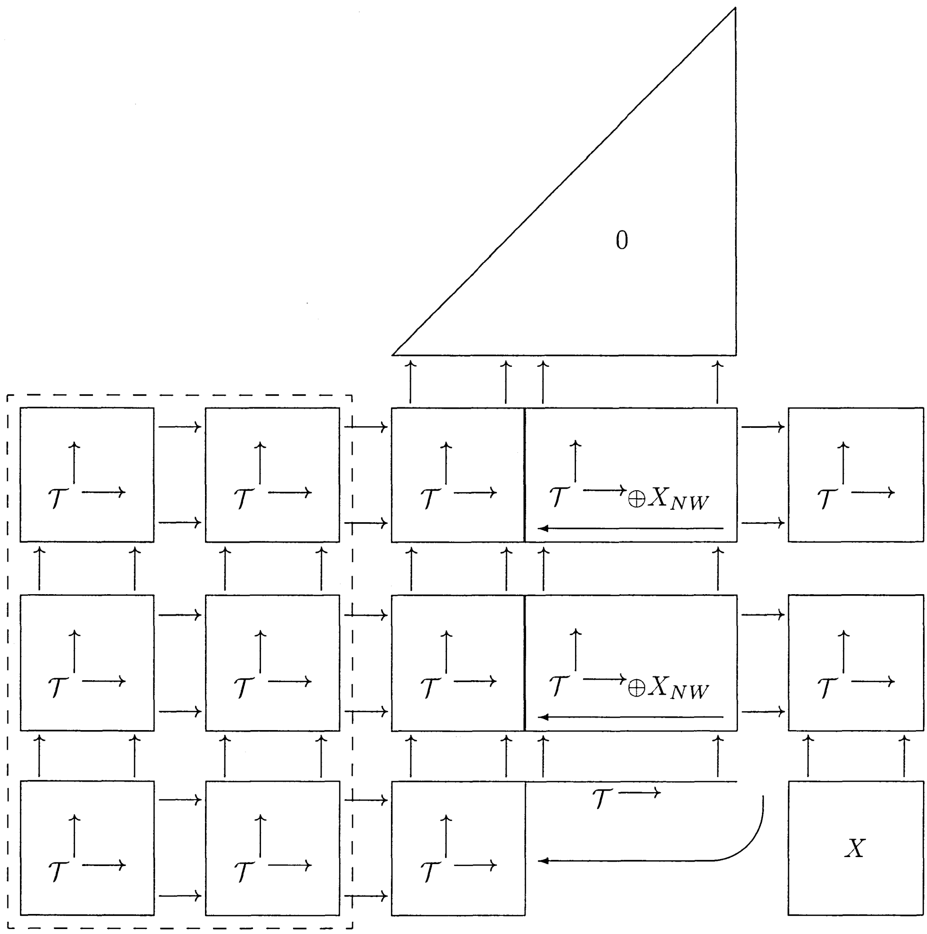
is also well defined. How does one prove this fact? Simple. In the simplicial set



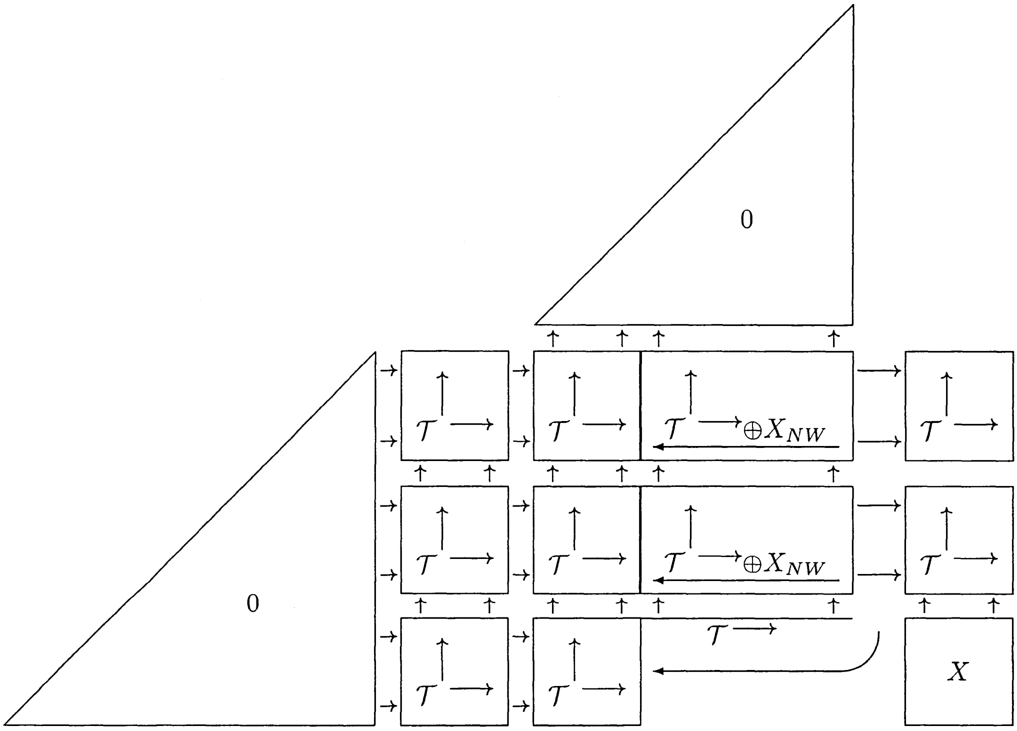
the part enclosed by a dashbox can be harmlessly subdivided. Precisely, on the simplicial set



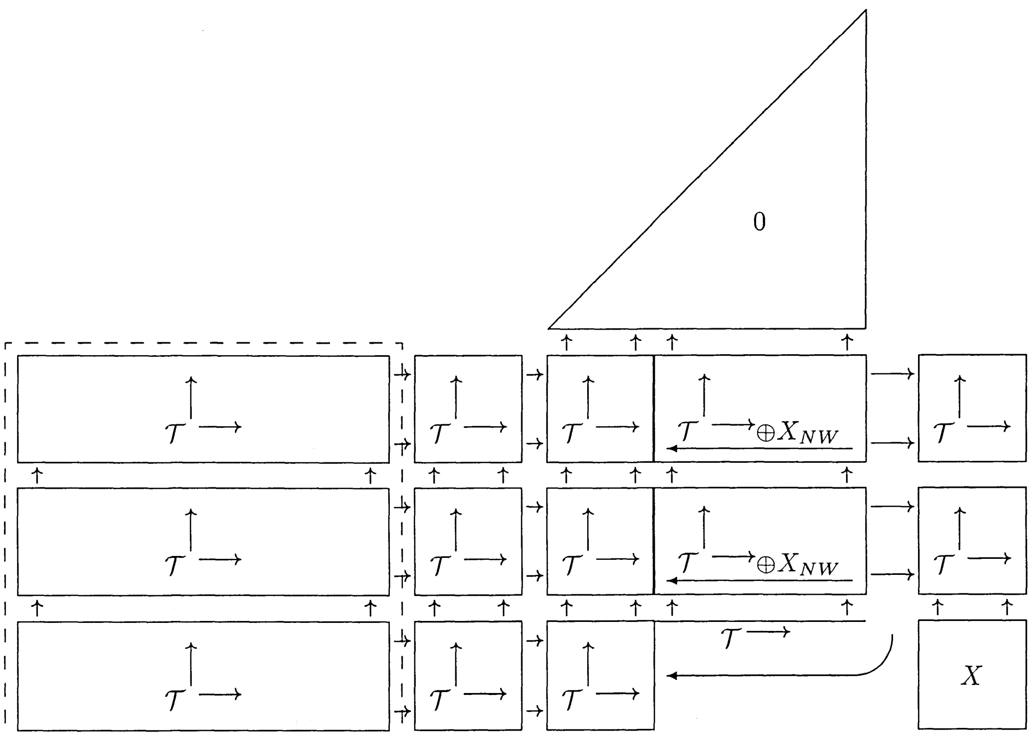
there is a homotopy whose shorthand is simply



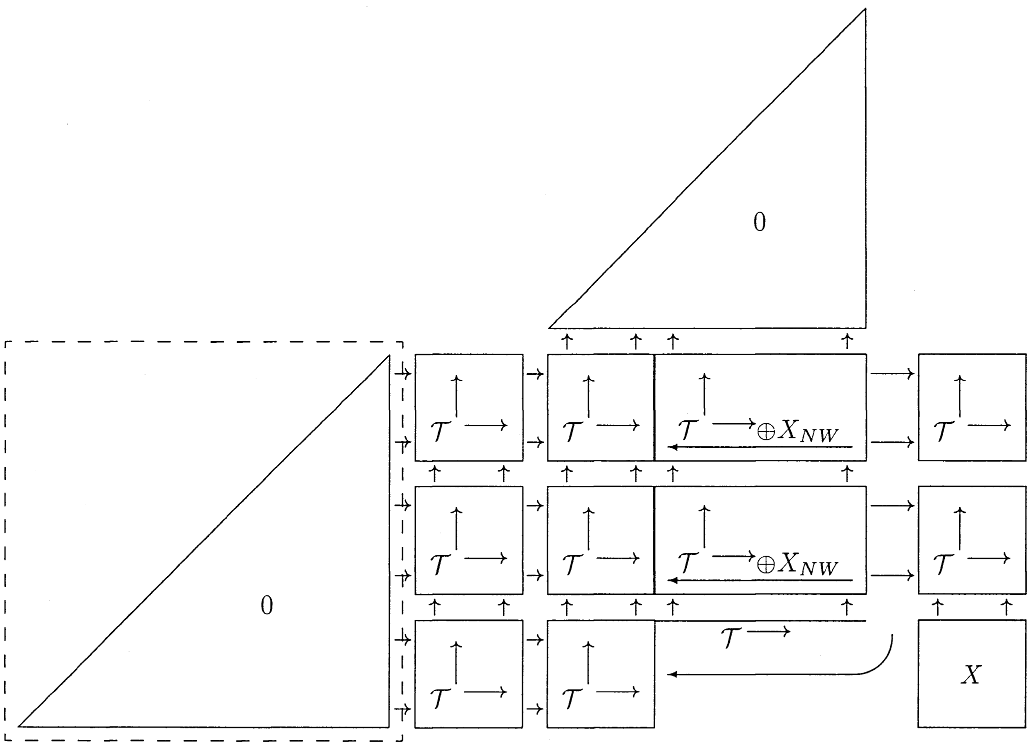
since this is just a subdivided version of the compact blueprint homotopy. But now the ordinary blueprint homotopy



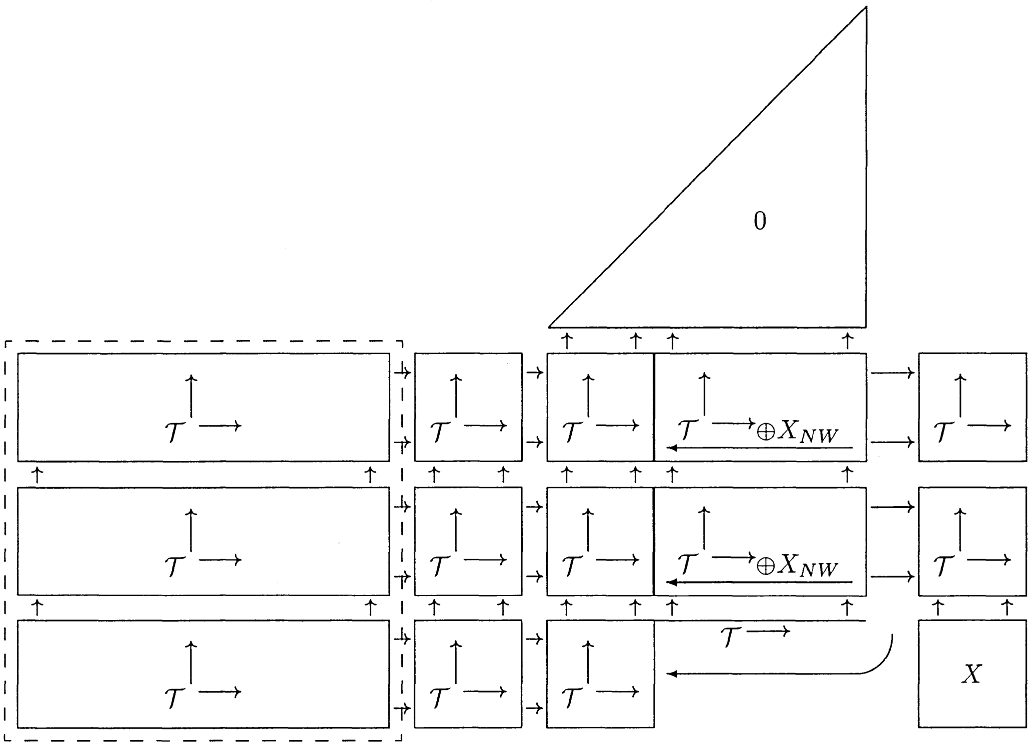
is obtained from



by deleting some of the structure inside the dashbox. If we reflect back to the proof of Theorem 1.1, it was based on the fact that all squares are naturally Mayer–Vietoris. There are fewer squares in



than in



because a triangle of squares is embedded in a rectangle. The fact that some objects are restricted to be 0 in ordinary blueprint homotopy, but are free in the compact blueprint, only shows that the ordinary blueprint is even more a special case of the compact blueprint than we might otherwise think.

There is no particularly good reason why I chose the blueprint homotopy to be the one I gave. The compact blueprint homotopy does the job just as well, and it can be made even more compact. The main point of this section is to convince the reader, that the manipulations involved, in reducing a non–trivial homotopy to a deletion of a subdivision of the blueprint, are essentially trivial. From now on, we will feel free to leave this reduction to the reader.

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