ASIAN J. MATH.
Vol. 2, No. 3, pp. 443–494, September 1998

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COLLAPSING FOLIATED RIEMANNIAN MANIFOLDS*

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0. Introduction. We develop here a foliated version of the Cheeger-Fukaya-Gromov theory [4] (C-F-G theory) of "collapsing Riemannian manifolds" for Riemannian manifolds equipped with one-dimensional foliations. In a prior paper [8] the authors have examined in detail the "lowest dimensional strata" in the C-F-G theory. Our constructions in this paper combine the results of [8] (which are reviewed in section 1 below) with ideas from our paper [7]. In a separate paper [9] foliated collapsing theory is applied to obtain new topological rigidity results for some classical aspherical manifolds. We take this opportunity to clarify a point in [9]; namely the meaning of the term "diagonal action" in [9, p.257, line 34] which should be interpreted as follows: let $a_1, a_2, \ldots, a_s \in \Gamma$ be a complete list of coset representatives for π in Γ. Then

$$\alpha(y_1, \ldots, y_s) = (a_1 \alpha a_1^{-1}(y_1), \ldots, a_s \alpha a_s^{-1}(y_s))$$

where $y_i \in \tilde{M}$ and $\alpha \in \pi$.

In this introductory section the main results of foliated collapsing theory are formulated (cf. 0.5,0.6).

A complete Riemannian manifold (M, g) is said to be *A-regular*, for some sequence of positive numbers $A = \{A_i\}$, if we have

$$|\nabla^i R| \leq A_i$$

where the indices i vary over the natural numbers and $\nabla^i R$ is the i-th covariant derivative of the curvature tensor (cf. [4;pg. 334]). Note that the 0-th condition means the sectional curvatures are pinched; i.e., bounded away from $\pm\infty$. The C-F-G-theory [4] applies to any complete A-regular Riemannian manifold M.

A one-dimensional smooth foliation $\mathfrak{F}$ of a complete Riemannian manifold (M, g) is said to be *A-regular*, for some sequence of positive numbers $A = \{A_i\}$, if for each locally defined unit cross section F of $\mathfrak{F}$ we have

$$|\nabla^i F| \leq A_i$$

where the indices vary over all natural numbers and where $\nabla^i F$ is the i-th covariant derivative of F. Foliated collapsing theory applies to any complete A-regular Riemannian manifold M with one-dimensional A-regular foliation $\mathfrak{F}$. The foliated theory gives a covering of a portion of M by "long and thin" open submanifolds E_i, $i \in I$ ("long" in the direction tangent to $\mathfrak{F}$ and "thin" in the direction perpendicular to $\mathfrak{F}$) each of which is the domain of a fiber bundle projection $E_i \longrightarrow B_i$ whose fiber is a product of a Euclidean space with a closed aspherical manifold having an infrasolv fundamental group and whose base space is a rectangle in a Euclidean space. The collection of all these projections overlap one another in a well-behaved manner.

In the remainder of this paper (M, g) will denote a complete A-regular Riemannian manifold and $\mathfrak{F}$ will denote a smooth one-dimensional A-regular foliation for (M, g). For any numbers $\alpha, \beta > 0$ let $M(\alpha, \beta)$ denote the subset of M described in 0.1 below.

* Received February 11, 1998; accepted for publication March 12, 1998.
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Both authors were supported in part by the National Science Foundation.

Our first foliated collapsing result (cf. 0.5) states that any point $p \in M(\alpha, \beta)$ is "close" to one of the three types of "infrasolv cores" described below in 0.2,0.3.

0.1. The subsets $X^{\alpha,\beta}$ and $M(\alpha,\beta)$ of M. For any subset $X \subset M$ and any numbers $\alpha, \beta > 0$ we denote by $X^{\alpha,\beta}$ the subset of all $y \in M$ for which there is a smooth path $f : [0,1] \longrightarrow M$ of length less than α which is contained in a leaf of $\mathfrak{F}$ and which satisfies $f(0) \in X$ and $d(f(1), y) < \beta$.

For each point $x \in M$ let A_x denote the "arc" of length equal to 2α centered at x and contained in a leaf of $\mathfrak{F}$. (If the leaf L containing x has length $\leq 2\alpha$, then $Ax = L$ is a circle.) Then $M(\alpha, \beta)$ will denote the set of all $x \in M$ such that diameter $(A_x) > \beta$.

0.2. Infrasolv cores of type I and type II. An *infrasolv core (of type I or II)* for (M, g) consists of a smooth submanifold $U \subset M$ with $\partial U = \emptyset$, an open subset B of some Euclidean space, and a smooth bundle projection $r : U \longrightarrow B$, such that the following hold.

(a) The fibers of r are closed aspherical manifolds with π_1 an infrasolv group; i.e., π_1 is virtually poly-$\mathbb{Z}$.

(b) B is a product $B_1 \times B_2$, where B_2 is an open ball centered at the origin of some Euclidean space, and where B_1 is either a point or an open ball centered at the origin of $\mathbf{R}$.

(c) $U \subset M$ has a well defined open tubular neighborhood of radius equal radius (B_2). (If B_2 is a point, then radius (B_2) is defined by this property.)

Let $r = r_1 \times r_2$ denote the factorization of r corresponding to the factorization $B = B_1 \times B_2$ of B given in 0.2(b). If B_1 is a point we say that $r : U \longrightarrow B$ is of *type I*; in this case the *radius* of r is defined to be the radius of B_2. If B_1 is a ball in $\mathbf{R}$ we say that $r : U \longrightarrow B$ is of *type II*; in this case the *radius* of r is defined to be the pair (α, δ) where $\alpha =$ radius (B_1) and where $\delta =$ radius (B_2).

0.3. Infrasolv core of type III. These type of structures exist only if $T\mathfrak{F}$ is an unoriented line bundle. In this case we let $(\hat{M}, \hat{\mathfrak{F}}, \hat{g})$ denote the two fold covering of $(M, \mathfrak{F}, g)$ such that $T\hat{\mathfrak{F}}$ is the oriented line bundle. An *infrasolv core* of type III for (M, g) consists of a smooth submanifold $U \subset M$ with $\partial U = \phi$ and a map $r : U \longrightarrow B$, where $B = B_1 \times B_2$ with $B_1 = [0, \alpha)$ for some $\alpha > 0$ and B_2 is the open ball centered at the origin of some Euclidean space. We let r_1, r_2 denote the factors of r corresponding to the factors B_1, B_2 of B. The maps r, r_1, r_2 must satisfy (a)-(c) below.

(a) There is an infrasolv core of type II for $(\hat{M}, \hat{g})$ represented by a map $\hat{r} : \hat{U} \longrightarrow \hat{B}$, where $\hat{B} = \hat{B}_1 \times \hat{B}_2$ with $\hat{B}_1 = (-\alpha, \alpha)$ and $\hat{B}_2 = B_2$, and where $\hat{U}$ is the preimage of U under the covering projection $\pi : \hat{M} \longrightarrow M$.

(b) The map $\hat{r}$ is equivariant with respect to Z_2-actions $\psi_1 : Z_2 \times \hat{M} \longrightarrow \hat{M}$ and $\psi_2 : Z_2 \times \hat{B} \longrightarrow \hat{B}$, where ψ_1 is the covering space action for $\pi : \hat{M} \longrightarrow M$ and where $\psi_2(x, y) = (-x, y)$ for all $(x, y) \in \hat{B}_1 \times \hat{B}_2$.

(c) r is the quotient map of $\hat{r}$ under the actions ψ_1, ψ_2.

(d) If $\hat{E}$ denotes the open tubular neighborhood for $\hat{U}$ in $\hat{M}$ of radius equal radius $(\hat{B}_2)$ (cf. 0.2(c)), then $\pi(\tilde{E})$ is an open tubular neighborhood for U in M of radius equal radius (B_2).

We define the *radius* or r (of type III) to be equal to the radius of $\hat{r}$ (cf. 0.2).

0.4. Thickened infrasolv cores. Let $r : U \longrightarrow B$ denote an infrasolv core for (M, g) of radius δ or (α, δ). Note that properties 0.2(c), 0.3(d) assure us that U has a well defined open tubular neighborhood E of radius δ in M. This means that the exponential map exp:$T^{\perp}{}_{\delta}(U) \longrightarrow M$ is a smooth embedding onto a subset $E \subset M$, where δ=radius(B_2) and $T^{\perp}{}_{\delta}(U)$ denotes the set of all vectors $v \in TM_{|U}$ such that v is perpendicular to U and $|v| < \delta$. The orthogonal projection $\rho : E \longrightarrow U$ is just the composition of the usual fiber bundle projection $T^{\perp}{}_{\delta}(U) \longrightarrow U$ with $\exp^{-1} : E \longrightarrow T^{\perp}{}_{\delta}(U)$. We define maps $s : E \longrightarrow B$, $s_i : E \longrightarrow B_i$ for $i = 1, 2$, and $t : E \longrightarrow \mathbf{R}$ by

(a) $s = r \circ \rho$ and $s_i = r_i \circ \rho$;

(b) $t(q) = d(q, \rho(q))$.

We shall say that (s,t) is a *thickened infrasolv core* for (M, g) associated to r. The thickened infrasolv core (s,t) has the same *type* (I,II, or III) and *radius* as does the infrasolv core r.

0.5. Existence Theorem. *There is a positive integer η depending only on dimM. Let $\alpha, \beta, \varepsilon_1, \varepsilon_2 > 0$ be given numbers and let $\alpha > \alpha_1 > \alpha_2 > \ldots > \alpha_{\eta-1} > \alpha_\eta > 0$ be a given decreasing sequence satisfying $\varepsilon_1\alpha_i > 100\alpha_{i+1}$ for all i. Then there exists a number $\lambda > 1$ (which depends only on $\alpha, \beta, A = \{A_i\}$, dimM), and there exist an arbitrarily small decreasing sequence of positive numbers $\delta_1 > \delta_2 > \ldots > \delta_{\eta-1} > \delta_\eta > 0$ which have arbitrarily small quotients $\delta_i/\varepsilon_2\delta_{i-1}$; and for each $p \in M(\alpha, \beta)$ there exists a thickened infrasolv core (s,t) for (M, g) with $p \in E$, and there is an integer $j \in \{1, 2, \ldots, \eta\}$, all which satisfy the following.*

(a) Radius $(s,t) = \delta_j$ if (s,t) is of type I; radius $(s,t) = (\alpha_j, \delta_j)$ if (s,t) is of type II or III.

(b)

$$t(p) < \varepsilon_2\delta_j,$$
$$|s_2(p)| < \varepsilon_2\delta_j,$$
$$|s_1(p)| < \varepsilon_1\alpha_j.$$

(c) For any $x \in E$ we have that

$$E \subset \{x\}^{7\alpha_j, \lambda\delta_j}.$$

(d) Suppose we have that

$$y \in \{x\}^{\nu,\mu},$$

for $x, y \in E$ and for $(\nu, \mu) \in [0, 7\alpha_j] \times [0, \lambda\delta_j]$. Then we also have that

$$|s_1(x) - s_1(y)| \leq \nu + \varepsilon_1\alpha_j,$$
$$|s_2(x) - s_2(y)| \leq \lambda\mu + \lambda t(x)\nu + \varepsilon_2\delta_j,$$
$$|t(x) - t(y)| \leq \mu + \lambda t(x)\nu + \varepsilon_2\delta_j.$$

0.5.1. Remark. The second inequality of 0.5(d) (in the special case when $\mu = 0$ and $x \in U$) suggests that along each fiber L of $r_2 : U \longrightarrow B_2$ the tangents $T\mathfrak{F}$ must be $\varepsilon_2\delta_j$-close to the tangents TL. In fact (for infrasolv cores of type II and III) the foliation $\mathfrak{F}$ must be tangent to each fiber L (cf. 2.5.1(a) and 0.3), and

$$|Ds_1(v)| = |v| \ , \ |Ds(w)| \leq \lambda|w| \ , \ |Dt(w)| \leq |w|$$

must hold for all $v \in T\mathfrak{F}_{|U}$ and all $w \in T(E - U)$ (cf. 0.3,0.4,2.4.1-2.4.4). These facts, together with certain "curvature" conditions established in §2 for infrasolv cores of type II and III (cf. 2.5.1(a),1.1), imply the three inequalities of 0.5(d). For infrasolv cores of type I the foliation $\mathfrak{F}$ is not necessarily tangent to each fiber L of $r : U \longrightarrow B$, but the angular distance from $T\mathfrak{F}_{|L}$ to TL is much less than $\varepsilon_2\delta_j$ (cf. §5); and

$$|Ds(w)| \leq \lambda|w| \ , \ |Dt(w)| \leq |w|$$

hold for all $w \in T(E - U)$. These properties, together with certain "curvature" conditions established in §5 for infrasolv cores of type I (cf. 5.1.13,5.1.23.2(b),1.1), imply the three inequalities of 0.5(d) for thickened infrasolv cores of type I.

Our next theorem describes the relation between two thickened infrasolv cores associated to points $p, p' \in M$ by Theorem 0.5 provided p and p' are sufficiently close.

0.6. Comparison Theorem. *The thickened infrasolv cores given by 0.5 can be chosen to satisfy the additional properties listed below. Let (s,t) and (s',t') be two such thickened infrasolv cores for (M,g) which have radii (α_j, δ_j) (or δ_j) and $(\alpha_{j'}, \delta_{j'})$ (or $\delta_{j'}$) respectively, which are associated to the points $p, p' \in M(\alpha, \beta)$ by 0.5. Suppose $E \cap E' \neq \phi$ and that $j = j'$. Then the following hold.*

(a) (s,t) and (s',t') are of the same type, $dimU = dimU'$, and $B = B'$.

(b) For all $x \in E \cap E'$ we have $|t(x) - t'(x)| < \varepsilon_2\delta_j$.

(c) There is an affine isomorphism $\mathbf{A}_2 : \mathbf{R}^k \longrightarrow \mathbf{R}^k$ (where $k = dimB_2$) which satisfies

$$|\mathbf{A}_2| < \lambda \ , \ |\mathbf{A}_2^{-1}| < \lambda,$$

and

$$|\mathbf{A}_2 \circ s_2(x) - s'_2(x)| < \varepsilon_2\delta_j \ \textit{for all } x \in E \cap E'$$

where $\lambda > 1$ depends only on $\alpha, \beta, A = \{A_i\}, dimM$. Moreover if (s,t) is of type II or type III then there is an isometry $\mathbf{A}_1 : \mathbf{R} \longrightarrow \mathbf{R}$ such that

$$|\mathbf{A}_1 \circ s_1(x) - s'_1(x)| < \varepsilon_1\alpha_j$$

for all $x \in E \cap E'$; and if (s,t) is of type III then $\mathbf{A}_1$ =identity.

0.6.1. Remark. In the event that $j < j'$ much can also be said about how the projections (s,t) and (s',t') overlap. These details are not given here since they are not needed for the application of foliated collapsing theory carried out by the authors in [9].

Outline of the paper.

In §1 we review the main results of our paper [8].

In §2 we use the results of §1 to verify Theorems 0.5,0.6 for infrasolv cores of type II and of type III.

In §3 we state and prove two lemmas (from linear algebra) which will be used in §5 to aid in the construction of infrasolv cores of type I.

In §4 we introduce the set of smooth embeddings $H_p(\lambda\tilde{\delta}_c)$ and formulate and prove several lemmas about these embeddings. These lemmas, in conjunction with those formulated in §3, will be used in the construction of infrasolv cores of type I in §5.

In §5 we use the results of §1,§3,§4, together with ideas from the authors' paper [7], to verify Theorems 0.5,0.6 for infrasolv cores of type I.

There is also one appendix at the end of this paper, in which we examine the A-regularity condition placed on the foliation $\mathfrak{F}$ for M in terms of special local coordinate systems for M.

1. Local collapsing theory. We let (M, g) denote a complete Riemannian manifold which is A-regular for some sequence $A = \{A_i\}$ of positive numbers. In this section we review the main results of our paper [8] (cf. Theorems 1.3 and 1.5 below). Our Existence Theorem 1.3 states that near any $p \in M$ there is an "infranil core" as described in 1.2 and 1.3; this theorem was verified in [8] by considering the "lowest dimensional strata" in the local C-F-G theory. Our Comparison Theorem 1.5 (also proven in [8]) formulates a "stability" property for infranil cores which satisfy the conclusions of 1.3. This "stability" is of essential importance in the sections 2 and 5 for verifying Theorems 0.5, 0.6.

In 1.1-1.3 below we let $\tilde{r} : \tilde{U} \to \tilde{B}$ denote a smooth mapping from a submanifold $\tilde{U} \subset M$ (with $\partial\tilde{U} = \phi$) onto a Riemannian manifold $(\tilde{B}, \tilde{g})$.

1.1. The curvature $\mathbf{K(\tilde{r}; M)}$. We let $K(\tilde{r}; M)$ denote the glb of all numbers $\sigma > 0$ which satisfy the following properties for any smooth path $f : [0,1] \to \tilde{U}$:

(a) $\Theta(T\tilde{U}_{f(1)}, P_f(T\tilde{U}_{f(0)})) < \sigma(\text{length}(f))$;

(b) $||D\tilde{r}_{f(1)} - P_f(D\tilde{r}_{f(0)})|| < \sigma(\text{length}(f))$.

Here $\Theta(V, W)$ denotes the angular distance between planes V and W (i.e. the maximum of the angular distance from each vector of V to W and from each vector of W to V); and P_f denotes parallel translation along f in (M, g) in part (a). In part (b) $P_f(D\tilde{r}_{f(0)})$ is defined to be the composite map

$$P_{\tilde{r}\circ f} \circ D\tilde{r}_{f(0)} \circ \pi \circ P_{\bar{f}} : T\tilde{U}_{f(1)} \to T\tilde{B}_{r\circ f(1)}$$

where $P_{\bar{f}}$ is parallel translation in (M, g) along the path $\bar{f}(t) = f(1-t)$, and $\pi : TM_{f(0)} \to T\tilde{U}_{f(0)}$ is orthogonal projection, and $P_{\tilde{r}\circ f}$ denotes parallel translation along $\tilde{r} \circ f$ in $(\tilde{B}, \tilde{g})$.

1.2. Infranil cores. An *infranil core* for (M, g) consists of a smooth submanifold $\tilde{U} \subset M$ with $\partial\tilde{U} = \phi$, an open subset of some Euclidean space $\tilde{B} \subset \mathbf{R}^n$ with $\tilde{g}$ denoting the Euclidean metric, and a smooth bundle projection $\tilde{r} : \tilde{U} \to \tilde{B}$, such that the following properties hold.

1.2.1. (a) The fibers of $\tilde{r}$ are infranil manifolds; in particular they are closed aspherical manifolds with π_1 an infranil group.

(b) $\tilde{B}$ is an open ball centered at the origin of $\mathbf{R}^n$.

(c) $\tilde{U} \subset M$ has a well defined tubular neighborhood of radius equal radius $(\tilde{B})$. (If $\tilde{B}$ is a point, then radius $(\tilde{B})$ is defined by this property.)

The *radius* of an infranil core $\tilde{r}$ is defined to be the radius of $\tilde{B}$.

Now let $\tilde{r} : \tilde{U} \to \tilde{B}$ denote an infranil core of radius $\tilde{\delta}$. We shall say that $\tilde{r} : \tilde{U} \to \tilde{B}$ is $(\tilde{\varepsilon}, \tilde{\vartheta})$-*rigid*, for numbers $\tilde{\varepsilon}, \tilde{\vartheta} > 0$, if the following properties hold.

1.2.2. (a) $K(\tilde{r}; M) \leq \tilde{\varepsilon}(\tilde{\delta}^{-1})$.

(b) diameter $(\tilde{r}^{-1}(x)) < \tilde{\varepsilon}\tilde{\delta}$, for all $x \in \tilde{B}$. (This refers to the diameter of the manifold $\tilde{r}^{-1}(x)$ with respect to its Riemannian metric inherited from (M, g).)

(c) For any $w \in T\tilde{U}$ which is perpendicular to the fibers of $\tilde{r}$ we have that

$$(1 - \tilde{\varepsilon})|w| \leq |D\tilde{r}(w)| \leq (1 + \tilde{\varepsilon})|w|.$$

(d) For each $v \in TM_{|\tilde{U}}$ which is perpendicular to $\tilde{U}$ there is a smooth path $f : [0,1] \to \tilde{U}$, which starts and ends at the foot of v, and which satisfies

$$\text{length } (f) < \tilde{\varepsilon}\tilde{\delta} \text{ and } \tilde{\vartheta} < \Theta(v, P_f(v)).$$

The following theorem has been proven in [8; Theorem 0.3].

1.3. Existence Theorem. *There is an integer $\tilde{\eta} > 0$ and a number $\tilde{\vartheta} \in (0,1)$ which depend only on dimM. For any given $\tilde{\varepsilon} \in (0,1)$ there is an arbitrarily small decreasing sequence of positive numbers $\tilde{\delta}_1 > \tilde{\delta}_2 > \tilde{\delta}_3 > \ldots$ having arbitrarily small quotients $\tilde{\delta}_j/\tilde{\varepsilon}\tilde{\delta}_{j-1}$. There is, for each integer $n \geq 0$ and for each $p \in M$, an infranil core $\tilde{r} : \tilde{U} \to \tilde{B}$ for (M, g) and a point $p' \in \tilde{U}$ which satisfy the following properties.*

(a) *The radius of $\tilde{r}$ is equal $\tilde{\delta}_c$ for some $c \in (n, n + \tilde{\eta})$.*

(b) $\tilde{r}$ is $(\tilde{\varepsilon}, \tilde{\vartheta})$-rigid.

(c) $d(p, p') < \tilde{\varepsilon}\tilde{\delta}_c$ and $|\tilde{r}(p')| = 0$.

1.4. Thickened infranil cores. Let $\tilde{r} : \tilde{U} \to \tilde{B}$ denote an $(\tilde{\varepsilon}, \tilde{\vartheta})$-rigid infranil core of radius $\tilde{\delta} > 0$. Note that property 1.2.1(c) assures us that $\tilde{U}$ has a well defined open tubular neighborhood $\tilde{E}$ of radius $\tilde{\delta}$ in (M, g) and that the orthogonal projection $\tilde{\rho} : \tilde{E} \to \tilde{U}$ is a well defined bundle projection map. (The meaning of 1.2.1(c) is that the exponential map exp:$T^{\perp}_{\tilde{\delta}}(\tilde{U}) \to M$ is a smooth embedding with image $\tilde{E}$, where $\tilde{\delta} = \text{radius}(\tilde{r})$ and where $T^{\perp}_{\tilde{\delta}}(\tilde{U})$ denotes the set of all $v \in TM_{|\tilde{U}}$ which are perpendicular to $\tilde{U}$ and which satisfy $|v| < \tilde{\delta}$; the orthogonal projection $\tilde{\rho} : \tilde{E} \to \tilde{U}$ is just the composite of the usual fiber bundle projection $T^{\perp}_{\tilde{\delta}}(\tilde{U}) \to \tilde{U}$ with $\exp^{-1}$: $\tilde{E} \to T^{\perp}_{\tilde{\delta}}(\tilde{U})$.) Define $\tilde{s} : \tilde{E} \to \tilde{B}$ to be the composite $\tilde{r} \circ \tilde{\rho}$; and define $\tilde{t} : \tilde{E} \to \mathbf{R}$ by $\tilde{t}(x) = d(x, \tilde{\rho}(x))$ for all $x \in \tilde{E}$. The pair of maps $(\tilde{s}, \tilde{t})$ represent a *thickened infranil core* of diameter $\tilde{\delta}$ which is the "thickening" for the infranil core $\tilde{r}$. For each $x \in (0, \tilde{\delta}]$ we let $\tilde{B}(x)$ denote the open ball of radius x centered at the origin of $\tilde{B}$, and we set

$$\tilde{E}(x) = \tilde{s}^{-1}(\tilde{B}(x)) \cap \tilde{t}^{-1}([0, x)) \text{ and } \tilde{U}(x) = \tilde{r}^{-1}(\tilde{B}(x)).$$

The following theorem has been proven in [8; Theorem 0.5].

1.5. Comparison Theorem. *Given $\tilde{\vartheta} > 0$ we let $\tilde{\varepsilon}, \tilde{\delta} > 0$ denote sufficiently small numbers, where how small is sufficient depends only on $\tilde{\vartheta}, A = \{A_i\}, dimM$. Let $\tilde{r}_i : \tilde{U}_i \to \tilde{B}_i, i = 1, 2$, denote $(\tilde{\varepsilon}, \tilde{\vartheta})$-rigid infranil cores both of radius $\tilde{\delta}$. If $\tilde{E}_1(\tilde{\delta}/9) \cap \tilde{E}_2(\tilde{\delta}/9) \neq \phi$ then there is an isometry $\tilde{\mathbf{A}} : \mathbf{R}^k \to \mathbf{R}^k$ (where $k = dim\tilde{B}_1$) such that the following properties hold for each $x \in \tilde{E}_1 \cap \tilde{E}_2$.*

(a) $dim\tilde{U}_1 = dim\tilde{U}_2$ and $\tilde{B}_1 = \tilde{B}_2$.

(b) $|\tilde{t}_1(x) - \tilde{t}_2(x)| < \mathcal{O}(\tilde{\varepsilon})\tilde{\delta}$.

(c) $|\tilde{\mathbf{A}} \circ s_1(x) - s_2(x)| < \mathcal{O}(\tilde{\varepsilon})\tilde{\delta}$.

1.5.1. Remark. The notation "$\mathcal{O}(\tilde{\varepsilon})$" appearing in the inequalities of 1.5(b)(c) means that there is a C^{∞}-function $g : \mathbf{R} \to \mathbf{R}$ with $g(0) = 0$, which is independent of the infranil cores $\tilde{r}_1, \tilde{r}_2$ and of the numbers $\tilde{\varepsilon}, \tilde{\delta}$, such that when $\mathcal{O}(\tilde{\varepsilon})$ is replaced by the number $|g(\tilde{\varepsilon})|$ then the resulting inequality is actually true.

The following remark has been verified in [8; Remark 0.5.2].

1.5.2. Remark. It is a consequence of the $(\tilde{\varepsilon}, \tilde{\vartheta})$-rigidity for $\tilde{r}_1, \tilde{r}_2$, and of the inequality $\tilde{E}_1(\tilde{\delta}/9) \cap \tilde{E}_2(\tilde{\delta}/9) \neq \phi$, that properties 1.5(a) -(c) are equivalent to the following three properties. For each $x \in \tilde{U}_2 \cap \tilde{E}_1$ let $f_x : [0, 1] \to \tilde{\rho}_1^{-1}(\tilde{\rho}_1(x))$ denote the geodesic with $f_x(0) = x, f_x(1) = \tilde{\rho}_1(x)$; and let $\tilde{\mathcal{G}}_i$ denote the foliation of $\tilde{U}_i$ by the fibers of $\tilde{r}_i$.

(a) length $(f_x) < \mathcal{O}(\tilde{\varepsilon})\tilde{\delta}$.

(b) $\Theta(P_{f_x}(T\tilde{U}_{2|x}), T\tilde{U}_{1|\rho_1(x)}) < \mathcal{O}(\tilde{\varepsilon})$.

(c) $\Theta(P_{f_x}(T\tilde{\mathcal{G}}_{2|x}), T\tilde{\mathcal{G}}_{1|\rho_1(x)}) < \mathcal{O}(\tilde{\varepsilon})$.

2. Construction of infrasolv cores of type II, III. Let $\alpha, \beta, M(\alpha, \beta)$ be as in Theorem 0.5. There are subsets $M_i(\alpha, \beta) \subset M(\alpha, \beta), i = 1, 2, 3$, defined in 2.3 below. The main result of this section (cf. 2.4, 2.5) is the construction for each $p \in M_2(\alpha, \beta)$ (or $p \in M_3(\alpha, \beta)$) of an infrasolv core $r_p : U_p \to B_p$ of type II (or of type III) such that the collection $\{r_p\}$ satisfies all the conclusions of Theorems 0.5 and 0.6, and in particular satisfies the statements of 0.5(b) with respect to our present choice of p. In section five below we will construct for each $p \in M_1(\alpha, \beta)$ an infrasolv core of type I.

Before giving a precise statement of the main results of this section we state two lemmas (cf. 2.1, 2.2) which will be needed in their formulation and proof. These lemmas will be proven at the end of the section.

The first of these lemmas is concerned with a map $f_p : (-5\alpha, 5\alpha) \times B(p, \beta) \to M$ defined as follows. Let $B(p, \beta)$ denote the set of all vectors $v \in TM_p$ with $|v| < \beta$ which are perpendicular to $\mathfrak{F}$, and define $f_p|0 \times B(p, \beta)$ to be the restriction to $B(p, \beta)$ of the exponential map exp: $TM \to M$. For each $x \in B(p, \beta)$ choose a unit speed parametrization $h_x : \mathbf{R} \to M$ for the leaf of $\mathfrak{F}$ which contains $f_p(0, x)$ such that $h_x(0) = f_p(0, x)$ and such that $\dot{h}_x(0)$ is smooth in x. Now for each $(s, x) \in (-5\alpha, 5\alpha) \times B(p, \beta)$ set $f_p(s, x) = h_x(s)$. Note that for β sufficiently small f_p will be a smooth immersion. How small is sufficient depends only on dim$M$ and $A = \{A_i\}$, where both M and $\mathfrak{F}$ are A-regular for the same sequence A; cf. A.1 and [8, A.1.2]. Let $\frac{\partial}{\partial t}$ denote a unit length vector field on $(-5\alpha, 5\alpha) \times B(p, \beta)$ tangent to the first factor; note that if $f_p(s, x) = f_p(s', x')$ then $Df_p(\frac{\partial}{\partial t}(s, x)) = \pm Df_p(\frac{\partial}{\partial t}(s', x'))$.

2.1. Lemma. *Suppose that $f_p(s, x) = f_p(s', x')$ for $(s, x) \neq (s', x')$ in $(-\alpha, \alpha) \times B(p, \beta^2)$. Then one of (a), (b), or (c) must hold, provided $\beta > 0$ is sufficiently small. (How small is sufficient depends only on $\alpha, A = \{A_i\}, dimM$.)*

(a) $Df_p(\frac{\partial}{\partial t}(s, x)) = Df_p(\frac{\partial}{\partial t}(s', x'))$ and $\beta^2 < |s - s'|$.

(b) $Df_p(\frac{\partial}{\partial t}(s, x)) = Df_p(\frac{\partial}{\partial t}(s', x'))$ and $\beta^2|s - s'| < |x - x'|$.

(c) $Df_p(\frac{\partial}{\partial t}(s, x)) = -Df_p(\frac{\partial}{\partial t}(s', x'))$.

The second of these lemmas refers to a selection of infranil cores $\tilde{r}_p : \tilde{U}_p \to \tilde{B}_p$, $p \in M(\alpha, \beta)$, which satisfy the conclusions of Theorem 1.3 with respect to some selection of numbers $\tilde{\varepsilon}, n, \{\tilde{\delta}_j\}$ as in 1.3, and in particular satisfies 1.3(c) for our present choice of p. Let $\tilde{E}_p$ denote the thickening of $\tilde{U}_p$ described in 1.4; let $\tilde{\mathfrak{G}}_p$ be the foliation of $\tilde{U}_p$ by the fibers of $\tilde{r}_p$; and let $\mathfrak{F}$ be one dimensional foliation of M refered to in 0.5.

2.2. Lemma. *Properties (a)-(c) hold provided β is sufficiently small (how small is sufficient depends only on $\alpha, A = \{A_i\}, dimM$) and provided the $\{\tilde{\delta}_j\}$ are sufficiently small (how small is sufficient depends only on $\alpha, \beta, A = \{A_i\}, dimM$).*

(a) The angles between the leaves of $\mathfrak{F}$ and of $\tilde{\mathfrak{G}}_p$ are bounded below by β^3.

(b) If the restricted bundle $T\mathfrak{F}_{|\tilde{E}_p}$ is not orientable then the angles between the leaves of $\mathfrak{F}$ and $\tilde{U}_p$ are bounded below by $\beta^3 - \mathcal{O}(\tilde{\varepsilon})$.

(c) If the restricted bundle $T\mathfrak{F}_{|\tilde{E}_p}$ is orientable then the angles between the leaves of $\mathfrak{F}$ and $\tilde{U}_p$ is bounded above by $\mathcal{O}(\tilde{\varepsilon})$.

2.3. The subsets $\mathbf{M}_i(\boldsymbol{\alpha},\boldsymbol{\beta}) \subset \mathbf{M}(\boldsymbol{\alpha},\boldsymbol{\beta}), \mathbf{i}=1,2,3$. A point $p \in M(\alpha,\beta)$ is in $M_2(\alpha,\beta)$ iff neither 2.1.(a) nor 2.1.(c) hold for any $(s,x),(s',x')$ in $(-\alpha_y,\alpha_y)\times B(p,\tilde{\delta}_y)$ where $y=(\frac{4}{5}\eta)-2$. A point $p\in M(\alpha,\beta)$ is in $M_3(\alpha,\beta)$ iff 2.1(a) doesn't hold for any $(s,x),(s',x')$ in $(-\alpha_z,\alpha_z)\times B(p,\tilde{\delta}_z)$, where $z=(\frac{3}{5}\eta)-2$, but 2.1(c) does hold for some $(s,x),(s',x')$ in $(-\alpha_y,\alpha_y)\times B(p,\tilde{\delta}_y)$. Set

$$M_1(\alpha,\beta) = M(\alpha,\beta) - (M_2(\alpha,\beta)\cup M_3(\alpha,\beta)).$$

Note that the sets $M_i(\alpha,\beta), i=1,2,3$, are well defined provided η is divisible by 5.

Now we can formulate the two main results of this section. In both of the following propositions we assume that $\beta>0$ is small enough to satisfy the hypothesis of Lemmas 2.1,2.2. Note that this assumption will cause no loss of generality in our statement of Theorems 0.5,0.6. In fact if $\beta'>\beta>0$ then we have $M(\alpha,\beta')\subset M(\alpha,\beta)$ (cf. 0.1); so if 0.5 and 0.6 have been proven for β, they also hold true for β'.

2.4. Proposition. *If $p\in M_2(\alpha,\beta)$ (or if $p\in M_3(\alpha,\beta)$) then there is an infrasolv core $r_p: U_p\to B_p$ of type II (or of type III) which satisfies 0.5(a)-(d).*

2.5. Proposition. *The collection of all infrasolv cores $\{r_p : p\in M_2(\alpha,\beta)\cup M_3(\alpha,\beta)\}$ from 2.4 also satisfy 0.6(a)-(c).*

The idea for the proof of 2.4 is quite simple in the special case that $T\mathfrak{F}$ is orientable. We use Lemma 2.2 to construct a portion of the infranil core $\tilde{r}_p:\tilde{U}_p\to\tilde{B}_p$ which is "transverse" to $\mathfrak{F}$. Then we "flow" this transverse portion of $\tilde{r}_p$ in the direction of $T\mathfrak{F}$ over the time interval $(-\alpha_c,\alpha_c)$ to obtain the infrasolv core $r_p:U_p\to B_p$, where $c>0$ is the integer subscript for $\tilde{\delta}$ associated to $\tilde{r}_p$ in 1.3. Because this idea is referred to again later (in greater detail) it will prove convenient to carry out this idea before beginning the proof for 2.4.

2.6. Flowing the transverse part of $\tilde{r}_p$. There are the following two cases to consider: $T\mathfrak{F}_{|\tilde{E}_p}$ is orientable; $T\mathfrak{F}_{|\tilde{E}_p}$ is not orientable. Here $\tilde{E}_p$ is the thickening for the domain $\tilde{U}_p$ of $\tilde{r}_p$ (cf. 1.4).

Case I: $T\mathfrak{F}_{|\tilde{E}_p}$ is orientable.

In this case $\tilde{U}_p$ is close to tangent to $\mathfrak{F}$ (cf. 2.2(c)) and $\tilde{\mathfrak{G}}_p$ is transversal to $\mathfrak{F}$; cf. 2.2(a). Let $(\tilde{s}_p,\tilde{t}_p)$ denote the thickening for $\tilde{r}_p$ described in 1.4, and let $c>0$ denote the integer subscript for $\tilde{\delta}$ in 1.3. Recall that $\tilde{B}_p$ is an open ball centered at the origin in some Euclidean space $\mathbf{R}^k$. Let $q\in\tilde{r}_p^{-1}(0)$ be any point, and let V_p denote all vectors $v\in T(\tilde{U}_p)_q$ which are perpendicular to $\tilde{r}_p^{-1}(0)$ and to $\mathfrak{F}$ and which satisfy $|D\tilde{r}_p(v)|<\frac{1}{2}\tilde{\delta}_c$ where $|\ |$ denotes the Euclidean norm. We set

$$\begin{aligned}\mathfrak{B}_p &= \exp\circ D\tilde{r}_p(V_p),\\ \mathfrak{U}_p &= \tilde{r}_p^{-1}(\mathfrak{B}_p),\\ \mathfrak{E}_p &= \tilde{s}_p^{-1}(\mathfrak{B}_p)\cap\tilde{t}_p^{-1}([0,\frac{1}{2}\tilde{\delta}_c)),\end{aligned}$$

where $\exp: T\mathbf{R}^k\to\mathbf{R}^k$ is the exponential map for Euclidean space. We also have mappings

$$\mathfrak{r}_p:\mathfrak{U}_p\to\mathfrak{B}_p, \mathfrak{s}_p:\mathfrak{E}_p\to\mathfrak{B}_p, \mathfrak{t}_p:\mathfrak{E}_p\to\mathbf{R},$$

defined simply as the restriction maps $\tilde{r}_p|\mathfrak{U}_p, \tilde{s}_p|\mathfrak{E}_p, \tilde{t}_p|\mathfrak{E}_p$ respectively. We call $\mathfrak{r}_p,\mathfrak{s}_p,\mathfrak{t}_p$

the "portions of $\tilde{r}_p, \tilde{s}_p, \tilde{t}_p$ transversal to $\mathfrak{F}$". For each $y \in \mathfrak{E}_p$ choose a unit speed parameterization $u_y : \mathbf{R} \to M$ for the leaf of $\mathfrak{F}$ containing y, such that $u_y(0) = y$ and such that $\dot{u}_y(0)$ is a smooth vector field along $\mathfrak{E}_p$. Then define a map

$$\mathfrak{f}_p : \mathbf{R} \times \mathfrak{E}_p \to M$$

by $\mathfrak{f}_p(s, y) = u_y(s)$. Note it follows from 2.2 that $\mathfrak{f}_p$ is a smooth immersion. Note that $\mathfrak{f}_p$ accomplishes the "flowing" of the transverse portions of $\tilde{r}_p, \tilde{s}_p, \tilde{t}_p$ in the direction of $\mathfrak{F}$. We will also have use for the following notation:

$$\mathfrak{B}_p(t) = \mathfrak{B}_p \cap \tilde{B}_p(t), \mathfrak{U}_p(t) = \mathfrak{U}_p \cap \tilde{U}_p(t), \mathfrak{E}_p(t) = \mathfrak{E}_p \cap \tilde{E}_p(t)$$

for any $t \in (0, \frac{1}{2}\tilde{\delta}_c]$, where the sets $\tilde{B}_p(t), \tilde{U}_p(t), \tilde{E}_p(t)$ have been defined in 1.4.

Case II: $T\mathfrak{F}_{|\tilde{E}_p}$ is not orientable.

In this case $\tilde{U}_p$ is already transverse to $\mathfrak{F}$ (by 2.2(b)), and we may set

$$\mathfrak{B}_p = \tilde{B}_p(\frac{1}{2}\tilde{\delta}_c) \text{ and } \mathfrak{U}_p = \tilde{U}_p(\frac{1}{2}\tilde{\delta}_c).$$

To get $\mathfrak{E}_p$ we let τ_p denote all vectors $v \in TM_{|\mathfrak{U}_p}$ with $|v| < \frac{1}{2}\tilde{\delta}_c$ which are perpendicular to both $\mathfrak{U}_p$ and $\mathfrak{F}$. Then we set

$$\mathfrak{E}_p = \exp(\tau_p),$$

where $\exp : TM \to M$ is the exponential map. The maps

$$\mathfrak{r}_p : \mathfrak{U}_p \to \mathfrak{B}_p, \mathfrak{s}_p : \mathfrak{E}_p \to \mathfrak{B}_p, \mathfrak{t}_p : \mathfrak{E}_p \to \mathbf{R},$$

are defined simply as the restriction maps $\tilde{r}_p|\mathfrak{U}_p, \tilde{s}_p|\mathfrak{E}_p, \tilde{t}_p|\mathfrak{E}_p$ respectively. For each $t \in (0, \frac{1}{2}\tilde{\delta}_c]$ sets $\mathfrak{B}_p(t), \mathfrak{U}_p(t), \mathfrak{E}_p(t)$ are defined by

$$\mathfrak{B}_p(t) = \tilde{B}_p(t), \ \mathfrak{U}_p(t) = \tilde{U}_p(t), \ \mathfrak{E}_p(t) = \mathfrak{E}_p \cap \tilde{E}_p(t).$$

Note that there is a smooth immersion

$$\mathfrak{f}_p : T\mathfrak{F}_{|\mathfrak{E}_p} \to M$$

which is just the inclusion $\mathfrak{E}_p \subset M$ on the zero section of the bundle $T\mathfrak{F}_{|\mathfrak{E}_p}$ and which maps each fiber of $T\mathfrak{F}_{|\mathfrak{E}_p}$ locally isometrically onto a leaf of $\mathfrak{F}$.

Proof of Proposition 2.4. As we have indicated above we wish to obtain the infrasolv core r_p by "flowing" the "transversal portion" of $\tilde{r}_p$ in the direction of $\mathfrak{F}$ over the time interval $(-\alpha_c, \alpha_c)$. In order to fill in the details for this argument it is clearly necessary to first understand the relation between the numbers $\beta, \varepsilon_1, \varepsilon_2, \eta, \{\delta_j\}$ of 0.5 and the numbers $\tilde{\varepsilon}, \tilde{\eta}, \{\tilde{\delta}_j\}$ of 1.3. In the remainder of this proof we shall assume that these numbers are related as follows. For any numbers $a, b \in (0, 1)$ we let $a << b$ denote that the ratio $\frac{b}{a}$ is much greater than 1.

2.4.1. (a) $100\tilde{\eta} < \eta$.

(b) $\delta_j << \tilde{\delta}_j$ but $\tilde{\varepsilon}\tilde{\delta}_j << \varepsilon_2\delta_j$ for all integers $\frac{3}{5}\eta \leq j \leq \eta$.

(c) $\tilde{\delta}_j << \beta$ for all j.

The proof naturally breaks into the two cases $p \in M_2(\alpha, \beta)$ and $p \in M_3(\alpha, \beta)$.

Case I: $\mathbf{p \in M_2(\alpha, \beta)}$.

In this case we select the positive integer n of 1.3 as follows.

2.4.2. $n = \frac{4}{5}\eta$.
Now we can deduce the following important property of $\mathfrak{f}_p$ from 2.1,2.3,2.6,2.4.1, 2.4.2 and from the inequalities $n < c < n + \bar{\eta}$ of 1.3. (See also the results A.2 and A.3 in the Appendix to this paper.)

2.4.3. There is $\lambda \in (0, \frac{1}{2})$, which depends only on $\alpha, \beta, A = \{A_i\}, \dim M$, such that $\mathfrak{f}_p : (-\alpha_c, \alpha_c) \times \mathfrak{C}_p(\lambda\tilde{\delta}_c) \to M$ is an embedding.

Select the infrasolv core $r_p : U_p \to B_p$ of type II and the integer j of 0.5 as follows.

2.4.4. (a) $j = c$.
(b) $U_p = \mathfrak{f}_p((-\alpha_j, \alpha_j) \times \mathfrak{U}_p(\delta_j))$ and $B_p = (-\alpha_j, \alpha_j) \times \mathfrak{B}_p(\delta_j)$.
(c) $r_p : U_p \to B_p$ is equal to the composition map

$$U_p \xrightarrow{\mathfrak{f}_p^{-1}} (-\alpha_j, \alpha_j) \times \mathfrak{U}_p(\delta_j) \xrightarrow{id \times \mathfrak{r}_p} (-\alpha_j, \alpha_j) \times \mathfrak{B}_p(\delta_j).$$

Let (s_p, t_p) denote the thickened infrasolv core associated to the infrasolv core r_p of 2.4.4 by 0.4. We can deduce from 1.3,2.6,2.4.1-2.4.3, and from results A.2 and A.3 in the Appendix to this paper, that the thickened infrasolv core (s_p, t_p) and the integer j satisfy 0.5(a)-(d).

Before proceeding to the next case we remark that there is an "extension" of each of the infrasolv cores $r_p : U_p \to B_p$ just constructed to a larger infrasolv core $\bar{r}_p : \bar{U}_p \to \bar{B}_p$ of type II for $\mathfrak{F}$ which satisfies the following properties.

2.4.5. (a) $U_p \subset \bar{U}_p, B_p \subset \bar{B}_p, \bar{r}_p | U_p = r_p$.
(b) The $\bar{r}_p$ have radius $(2\alpha_j, \bar{\delta}_j)$ where $\delta_j << \bar{\delta}_j << \tilde{\delta}_j$.

The $\bar{r}_p$ shall be referred to in the proof for Proposition 2.5. They are obtained by "extending" the preceding construction for the infrasolv cores by simply replacing the
$\{\delta_j : 1 \le j \le \eta\}$ in 2.4.4 by positive numbers $\{\bar{\delta}_j : 1 \le j \le \eta\}$ which satisfy

$$\delta_j << \bar{\delta}_j << \tilde{\delta}_j$$

and replacing the numbers $\{\alpha_j : 1 \le j \le \eta\}$ in 2.4.4 by the numbers $\{2\alpha_j : 1 \le j \le \eta\}$.

Case II: $\mathbf{p} \in \mathbf{M}_3(\boldsymbol{\alpha}, \boldsymbol{\beta})$.
In this case we select the integer n of 1.3 as follows.

2.4.6. $n = \frac{3}{5}\eta$.
We may choose $(s, x), (s', x') \in (-\alpha_{n+\bar{\eta}+3}, \alpha_{n+\bar{\eta}+3}) \times B_p(\tilde{\delta}_{n+\bar{\eta}+3})$ such that

$$f_p(s, x) = f_p(s', x') \text{ and } df_p\left(\frac{\partial}{\partial t}(s, x)\right) = -df_p\left(\frac{\partial}{\partial t}(s', x')\right)$$

(cf. 2.1,2.3,2.4.1,2.4.6). Note that there is a smooth embedding

$$h : X \to (-\alpha_{n+\bar{\eta}}, \alpha_{n+\bar{\eta}}) \times B_p(\tilde{\delta}_{n+\bar{\eta}})$$

where

$$X = (-\alpha_{n+\bar{\eta}+2}, \alpha_{n+\bar{\eta}+2}) \times B_p(\tilde{\delta}_{n+\bar{\eta}+2})$$

uniquely determined by properties 2.4.7(a)(b), which also satisfies properties 2.4.7(c)-(e). (See result A.3 in the Appendix to this paper; and use the smallness of the $\tilde{\delta}_i$ and

$\tilde{\delta}_i/\tilde{\delta}_{i-1}$ (cf. 1.3), and use the inequalities $\varepsilon_1\alpha_i > 100\alpha_{i+1}$ of 0.5.)

2.4.7. (a) $h(s, x) = (s', x')$.
(b) $f_p \circ h(t, y) = f_p(t, y)$ for all $(t, y) \in X$.
(c) $dh(\frac{\partial}{\partial t}(t, y)) = -\frac{\partial}{\partial t}(h(t, y))$ for all $(t, y) \in X$.
(d) $|h(0, y)| < 10\alpha_{n+\tilde{\eta}+3}$ for all $y \in B_p(\tilde{\delta}_{n+\tilde{\eta}+2})$.
(e) There is an orientation reversing isometry $\mathbf{I} : \mathbf{R} \to \mathbf{R}$ such that

$$|(I(t), 0) - h(t, y)| < \tilde{\delta}_{n+\tilde{\eta}+1}$$

for all $(t, y) \in X$.

Now we can use 2.4.7(c)-(e) to derive the following property.

2.4.8. $I(a) = a$ for some $a \in (-30\alpha_{n+\tilde{\eta}+3}, 30\alpha_{n+\tilde{\eta}+3})$.

Set

2.4.9. $q = f_p(a, 0)$.

Consider now the infranil core $\tilde{r}_q : \tilde{U}_q \to \tilde{B}_q$ of radius $\tilde{\delta}_c$ given by 1.3. Since $n < c < n + \tilde{\eta}$ (cf. 1.3) it follows from 2.4.7-2.4.9 that $T\mathfrak{F}_{|\tilde{E}_q}$ is unoriented. (See also A.2 and A.3.) Thus case II of 2.6 may be applied to get a smooth immersion $\mathfrak{f}_q : T\mathfrak{F}_{|\mathfrak{E}_q} \to M$. Now we can deduce the following important property of $\mathfrak{f}_q$ from 2.1,2.3,2.6,2.4.1,2.4.6-2.4.9 and from the inequality $n < c < n + \eta$. (See also results A.2 and A.3 in the Appendix to this paper.)

2.4.10. There is $\lambda \in (0, 1/2)$, which depends only on $\alpha, \beta, A = \{A_i\}, \dim M$, such that

$$\mathfrak{f}_q : T_{\alpha_c}(\mathfrak{F})_{|\mathfrak{E}_q(\lambda\tilde{\delta}_c)} \longrightarrow M$$

is a smooth embedding. (Where for each $a > 0$, we let $T_a(\mathfrak{F})$ denote the collection of all vectors $v \in T(\mathfrak{F})$ with $|v| < a$.)

Select the infrasolv core $r_p : U_p \to B_p$ of type III and the integer j of 0.5 as follows.

2.4.11. (a) $j = c$.
(b) $U_p = \mathfrak{f}_q(T_{\alpha_j}(\mathfrak{F})_{|\mathfrak{U}_q(\delta_j)})$ and $B_p = [0, \alpha_j) \times \mathfrak{B}_q(\delta_j)$.
(c) $r_p : U_p \to B_p$ is equal to the composition map

$$U_p \xrightarrow{\mathfrak{f}_p^{-1}} T_{\alpha_j}(\mathfrak{F})_{|\mathfrak{U}_q(\delta_j)} \xrightarrow{\pi_1 \times \pi_2} [0, \alpha_j) \times \mathfrak{U}_q(\delta_j) \xrightarrow{id \times \mathfrak{r}_p} [0, \alpha_j) \times \mathfrak{B}_q(\delta_j),$$

where $\pi_1(v) = |v|$ and $\pi_2 : T\mathfrak{F} \to M$ is the standard projection.

Let (s_p, t_p) denote the thickened infrasolv core associated by 0.4 to the infrasolv core r_p of 2.4.11. Now one can deduce from 1.3,2.6,2.4.6-2.4.10,2.4.1, and from the results A.2 and A.3 of the Appendix to this paper, that the $(s_p, t_p), j$ satisfy 0.5(a)-(d). Note in particular that we must use the restriction of 0.5 that $\varepsilon_1\alpha_i > 100\alpha_{i+1}$ for all i, in conjunction with 2.4.8 and 2.4.9, in order to derive the inequality $|s_{p,1}(x)| < \varepsilon_1\alpha_j$ of 0.5(b).

Before concluding this proof we remark that there is an "extension" of each of the infrasolv cores $r_p : U_p \to B_p$ constructed in this step to a larger infrasolv core $\bar{r}_p : \bar{U}_p \to \bar{B}_p$ of type III which satisfies the following properties.

2.4.12. (a) $U_p \subset \bar{U}_p, B_p \subset \bar{B}_p, \bar{r}_p|U_p = r_p$.
(b) The $\bar{r}_p$ have radius $(2\alpha_j, \bar{\delta}_j)$ where $\delta_j << \bar{\delta}_j << \tilde{\delta}_j$.

The $\bar{r}_p$ shall be referred to in the proof of Proposition 2.5. They are obtained by simply "extending" the construction for the infrasolv cores r_p by a procedure similar to that described in Case I.

This completes the proof for Proposition 2.4.

Proof of Proposition 2.5. We use Theorem 1.5 to verify that the infrasolv cores

$$r_p : U_p \to B_p,\ p \in M_2(\alpha,\beta) \cup M_3(\alpha,\beta)$$

of type II and of type III constructed in the proof for 2.4 satisfy the conclusions of Theorem 0.6.

Case I: $\mathbf{p} \in \mathbf{M_2}(\boldsymbol{\alpha},\boldsymbol{\beta})$.

We must verify properties 0.6(a)(b)(c). Towards this end we first note that any infrasolv core $r_p : U_p \to B_p$ constructed in Case I of the proof for 2.4 satisfies (in addition to properties 0.5(a)-(d)) the following properties. We let $\mathfrak{G}_p$ denote the foliation of U by the fibers of r_p; $B_p = B_{p,1} \times B_{p,2}$ denote the two factors of B_p (cf. 2.4.4(b)) and we let $r_{p,2} : U_p \to B_{p,2}$ be the composite of r_p with projection onto the second factor.

2.5.1. (a) $\mathfrak{F}$ is tangent to each fiber of $r_{p,2}$.

(b) $K(r_p; M) < \varepsilon_2(\delta_j)^{-1}$.

(c) diameter$(L) < \varepsilon_2\delta_j$ for each $L \in \mathfrak{G}_p$.

(d) For any $v \in TU_{p|r_p^{-1}(0\times B_{p,2})}$ which is perpendicular to $T\mathfrak{G}_p$ we have

$$(1-\varepsilon_2)|v| \le |Dr_p(v)| \le (1+\varepsilon_2)|v|.$$

(e) There is $\vartheta > 0$ which depends only on $\alpha, A = \{A_i\}, \dim M$. For each $v \in TM_{|U_p}$ which is perpendicular to U_p, there is a smooth path $f : [0,1] \to U_p$ which starts and ends at the foot of v and satisfies length$(f) < \varepsilon_2\delta_j$ and $\vartheta < \Theta(v, P_f(v))$.

Note that the infranil core $\tilde{r}_p : \tilde{U}_p \to \tilde{B}_p$ used in Case I of the proof for 2.4 to construct $r_p : U_p \to B_p$ satisfies properties 1.3(a)-(c); properties 2.5.1(a)-(e) are simply a reflection of those properties and of the relations in 2.4.1. (See also 2.2(c), and results A.2 and A.4 in the Appendix to this paper.)

For example to verify 2.5.1(b) we argue as follows. From 1.2.2 (which holds by 1.3(b)) and from 2.6, it follows that

$$K(\mathfrak{r}_p : M) < \mathcal{O}(\tilde{\varepsilon})\tilde{\delta}_c^{-1}.$$

This last inequality, together with Theorem A.4 in the Appendix to this paper, implies that

$$K(\hat{\mathfrak{r}}_p; (-5\alpha, 5\alpha) \times B(p,\beta)) < \mathcal{O}(\tilde{\varepsilon})\tilde{\delta}_c^{-1}$$

where $\hat{\mathfrak{r}}_p : \hat{\mathfrak{U}} \to \hat{\mathfrak{B}}_p$ is defined by $\hat{\mathfrak{B}}_p = \mathfrak{B}_p, \hat{\mathfrak{U}}_p = f_p^{-1}(\mathfrak{U}_p)$, and $\hat{\mathfrak{r}}_p = \mathfrak{r}_p \circ f_p$. It also follows from 2.6 (case I) and from Theorem A.2 in the Appendix to this paper, that the angle between $\hat{\mathfrak{U}}_p$ and the first factor of $(-5\alpha, 5\alpha) \times B(p, \mathfrak{B})$ is everywhere bounded below by a positive number which depends only on $\alpha, A = \{A_i\}, \dim M$. We may deduce from the existence of this lower angular bound, and from the preceding curvature inequality, that

$$K(\hat{r}_p; (-5\alpha, 5\alpha) \times B(p,\beta)) < \mathcal{O}(\tilde{\varepsilon})\tilde{\delta}_c^{-1}$$

where $\hat{r}_p : \hat{U}_p \to \hat{B}_p$ is defined by $\hat{B}_p = B_p, \hat{U}_p = f_p^{-1}(U_p)$, and $\hat{r}_p = r_p \circ f_p$. Now this last inequality, together with 2.4.1 and A.4, imply property 2.5.1(b).

2.5.2. Remark. We note that the extensions $\bar{r}_p : \bar{U}_p \to \bar{B}_p$ of 2.4.5 also satisfy 2.5.1(a)-(d).

We let $r : U \to B$ and $r' : U' \to B'$ denote any two infrasolv cores constructed in Case I of the proof for 2.4 whose thickenings (s,t) and (s',t') satisfy $E \cap E' \neq \phi$. We shall first deduce from 2.5.1, 2.5.2 and Claim 2.5.3 (stated below) that (s,t) and (s',t') satisfy properties 0.6(a)(b)(c). Then we shall complete the proof of Case I of 2.5 by verifying 2.5.3. Let $\bar{r}, \bar{r}'$ denote the extensions of r, r' given by 2.4.5, and let $(\bar{s},\bar{t}), (\bar{s}',\bar{t}')$ denote their thickenings. And let $\mathfrak{G}'$ and $\bar{\mathfrak{G}}$ denote the foliations of U' and $\bar{U}$ by the fibers of r' and $\bar{r}$, respectively.

2.5.3. Claim. *We have that $E' \subset \bar{E}$. For each $x \in U'$, let $f_x : [0,1] \to M$ denote the geodesic in $\rho^{-1}(\rho(x))$ with $f_x(0) = x$ and $f_x(1) = \rho(x)$, where $\rho : \bar{E} \to \bar{U}$ is the orthogonal projection map. The following properties also hold.*

(a) length $(f) < \mathcal{O}(\varepsilon_2)\delta_j$.

(b) $\Theta(P_f(TU'_{|f(0)}), T\bar{U}_{|f(1)}) < \mathcal{O}(\varepsilon_2)$.

(c) $\Theta(P_f(T\mathfrak{G}'_{|f(0)}), T\bar{\mathfrak{G}}_{|f(1)}) < \mathcal{O}(\varepsilon_2)$.

First note that if the angular distance between two planes is sufficiently small then the planes must have equal dimension. Thus 2.5.3(b)(c) imply 0.6(a).

Now we construct the isometry $\mathbf{A}_1 : \mathbf{R} \to \mathbf{R}$ and the affine map $\mathbf{A}_2 : \mathbf{R}^k \to \mathbf{R}^k$ of 0.6 as follows. Choose $x \in E \cap E'$ and let $\mathbf{A}_1$ denote the translation which maps $s_1(x)$ to $s_1'(x)$, where s_1, s_1' denote the first coordinates for s, s'. Let V denote all $v \in TM_{|x}$ which are perpendicular to both $s^{-1}(s(x))$ and $\mathfrak{F}$. Note that it follows from 2.5.1 and 2.5.2 that each derivative $Ds_2 : V \to \mathbf{R}^k$ and $Ds_2' : V \to \mathbf{R}^k$ is an invertible linear transformation, where s_2, s_2' denote the second coordinates for s, s'; let $L : \mathbf{R}^k \to \mathbf{R}^k$ denote the composition $L = (Ds_2') \circ (Ds_{2|V})^{-1}$. Now set $\mathbf{A}_2$ equal the affine map which maps $s_2(x)$ to $s_2'(x)$ and whose derivative is equal L. It is straightforward to argue now, based on 2.5.1-2.5.3 and on Theorems A.2 and A.4 in the Appendix to this paper, that $\mathbf{A}_1$ and $\mathbf{A}_2$ satisfy the properties listed in 0.6(c), and that 0.6(b) also holds, provided ε_2 is replaced by $\mathcal{O}(\varepsilon_2)$ in 0.6(b)(c). Now choosing $\varepsilon_2' = \max\{\varepsilon_2, \mathcal{O}(\varepsilon_2)\}$, and replacing ε_2 in 0.5 and 0.6 by $\varepsilon_2{}'$, we see that all the infrasolv cores $r_p : U_p \to B_p, p \in M_2(\alpha,\beta)$, constructed in the proof of Proposition 2.4, and their associated thickcned infrasolv cores (s_p,t_p), $p \in M_2(\alpha,\beta)$, satisfy all the properties listed in 0.5 and 0.6.

Verification of Claim 2.5.3. Now we wish to apply 2.5.1,2.5.2 and 1.5 to verify that the thickened infrasolv cores (s,t) and (s',t') satisfy 2.5.3. We will be applying 1.5 to two new infranil cores, which will be denoted by $\tilde{r}_i : \tilde{U}_i \to \tilde{B}_i, i = 1,2$, with thickenings denoted by $(\tilde{s}_i,\tilde{t}_i), i = 1,2$. The new infranil cores will be constructed from the "extended" infrasolv cores $\bar{r}, \bar{r}'$ of 2.4.5 by taking small pieces of these infrasolv cores. (Note: these new infranil cores $\tilde{r}_i, i = 1,2$, will not be the infranil cores used in the proof for 2.4 to construct the infrasolv cores r, r'.) These new infranil cores will both be $(\tilde{\varepsilon},\tilde{\vartheta})$-rigid and of radius equal $\tilde{\delta}$, for $\tilde{\varepsilon} = \lambda\varepsilon_2$, $\tilde{\delta} = \lambda\delta_j$, and $\tilde{\vartheta} = \vartheta$, where $\varepsilon, \delta_j, \vartheta$ come from 2.5.1, and where $\lambda >> 1$ depends only on $\alpha, A = \{A_i\}, \dim M$. Then 1.5 may be applied to the $\tilde{r}_i, i = 1,2$; and 2.5.3 will be an immediate consequence of 1.5(a)-(c) and of 1.5.2. Towards constructing the $\tilde{r}_i, i = 1,2$, we let $x \in U'$ be as in 2.5.3, and let $x_1 \in U$ and $x_2 \in U'$ denote the image of x under orthogonal projections $E \to U$

and $E' \to U'$ (thus $x_2 = x$). Note that there are affine maps $T_i : \mathbf{R}^{k_i} \to \mathbf{R}^{k_i}, i = 1,2$, such that $T_1(\bar{r}(x_1)) = 0$ and $T_2(r'(x_2)) = 0$, and such that $D(T_1 \circ \bar{r}) : T\bar{\mathfrak{G}}^{\perp}_{|x_1} \to T\mathbf{R}^{k_1}_{|0}$ and $D(T_2 \circ \bar{r}') : (T\bar{\mathfrak{G}}')^{\perp}_{|x_2} \to T\mathbf{R}^{k_2}_{|0}$ are both linear isometries (where $T\bar{\mathfrak{G}}^{\perp}$ denotes the orthogonal complement to $T\bar{\mathfrak{G}}$ in $T\bar{U}$, and where $(T\bar{\mathfrak{G}}')^{\perp}$ denotes the orthogonal complement to $T\bar{\mathfrak{G}}'$ in TU'). Let $\tilde{B}_i, i = 1,2$, denote the open ball in $\mathbf{R}^{k_i}$ of radius $\tilde{\delta}$ which is centered at $0 \in \mathbf{R}^{k_i}$. Set $\tilde{U}_1 = (T_1 \circ r)^{-1}(\tilde{B}_1)$ and define $\tilde{r}_1 : \tilde{U}_1 \to \tilde{B}_1$ to be the restriction $T_1 \circ r|\tilde{U}_1$; also set $\tilde{U}_2 = (T_2 \circ \bar{r}')^{-1}(\tilde{B}_2)$ and define $\tilde{r}_2 : \tilde{U}_2 \to \tilde{B}_2$ to be the restriction $T_2 \circ \bar{r}'|\tilde{U}_2$. Note that $\tilde{E}_1(1/9) \cap \tilde{E}_2(1/9) \neq \phi$ (because $d(x_i, x) < 3\delta_j$ for $i = 1,2$, and $\tilde{\delta} = \lambda\delta_j$ with $\lambda >> 1$). Note also that the truth of 2.5.1(a)-(e) for the $\bar{r}, \bar{r}'$ (cf. 2.5.2) implies immediately that the $\tilde{r}_i, i = 1,2$, are both $(\tilde{\varepsilon}, \tilde{\vartheta})$-rigid and of radius equal $\tilde{\delta}$, where $\tilde{\varepsilon}, \tilde{\delta}, \tilde{\vartheta}$ have been defined in the beginning of this paragraph. Thus we may apply Theorem 1.5 (cf. Remark 1.5.2) to the $\tilde{r}_i, i = 1,2$, to complete the verification for Claim 2.5.3.

This completes the verification of Proposition 2.5 when $p \in M_2(\alpha, \beta)$.

Case II: $\mathbf{p} \in \mathbf{M}_3(\boldsymbol{\alpha}, \boldsymbol{\beta})$.

Properties 0.6(a)-(c) have already been verified in the preceding case for infrasolv cores of type II. So in the remainder of this proof we will let $r : U \to B$ and $r' : U' \to B'$ denote any two infrasolv cores of type III constructed in Case II for the proof of 2.4 whose thickenings (s,t) and (s',t') satisfy $E \cap E' \neq \phi$, and we will show that r, r' satisfy 0.6(a)-(c).

We have the following claim where j comes from 2.4.11.

2.5.4. Claim. $|s_1(x) - s'_1(x)| << \tilde{\delta}_j$ *for all* $x \in E \cap E'$.

Putting aside for a moment the verification of this claim, we note that it may be used to complete the proof for Proposition 2.5 (for $p \in M_3(\alpha, \beta)$) as follows. The assumption $E \cap E' \neq \phi$, and 2.5.4,2.4.1, together imply that $\tilde{E}_1(\tilde{\delta}_j/9) \cap \tilde{E}_2(\tilde{\delta}_j/9) \neq \phi$, where $\tilde{r}_1, \tilde{r}_2$ denote the infranil cores of radius $\tilde{\delta}_j$ used in Case II for the proof of Proposition 2.4 from which r, r' are constructed. Thus we may apply 1.5 to $\tilde{r}_1, \tilde{r}_2$ to conclude that they are related as in 1.5(a)-(c), where the number $\tilde{\delta}$ of 1.5 is equal to $\tilde{\delta}_j$. Now it follows from 1.3 and 1.5 (as applied to $\tilde{r}, \tilde{r}_2$), from 2.4.1 and Case II of 2.6, and from 2.4.11 and 2.5.4, that r, r' satisfy 0.6(a)-(c). (See also results [8;A.1.1 and A.1.2] and see Theorem A.2 in the Appendix to this paper.)

Verification of Claim 2.5.4. Let $\rho : \hat{M} \to M$ denote the two fold covering for M such that the corresponding two fold cover $\hat{\mathfrak{F}}$ for $\mathfrak{F}$ has an orientable tangent bundle $T\hat{\mathfrak{F}}$; let $\hat{r} : \hat{U} \to \hat{B}$ and $\hat{r}' : \hat{U}' \to \hat{B}'$ be the corresponding two fold coverings for the r, r' (cf. 0.3). The extension $\bar{r}$ for r given by 2.4.12 lifts to an extension $\hat{\bar{r}}$ for $\hat{r}$. Note that $\hat{r}, \hat{\bar{r}}, \hat{r}'$ are all infrasolv cores of type II. Let $(\hat{\bar{s}}, \hat{\bar{t}})$ and $(\hat{s}', \hat{t}')$ denote the thickenings for $\hat{\bar{r}}, \hat{r}'$. Note that $\hat{E}' \subset \hat{\bar{E}}$ (cf. Claim 2.5.3). To verify 2.5.4 it will suffice to show that for each $x \in \hat{s}'^{-1}_1(0)$ we have that $|\hat{\bar{s}}_1(x)| << \tilde{\delta}_j$ (cf. 0.2,0.3). Let $\psi : Z_2 \times \hat{M} \to \hat{M}$ denote the group action by the covering transformations for the covering $\rho : \hat{M} \to M$. We note first that

$$x, \psi(1,x) \in \rho^{-1}(\tilde{E}_2(2\delta_j))$$

since by 2.6 and 2.4.11 we have that $s'^{-1}_1(0) \subset \tilde{E}_2(2\delta_j)$ and since the ψ-action leaves $\rho^{-1}(\tilde{E}_2(2\delta_j))$ invariant. Also, because diameter $(\tilde{E}_2(2\delta_j)) < 10\delta_j$ (cf. 1.2,1.3 as applied to $\tilde{r}_2$) it follows from this fact, 2.4.1 and the path connectivity of $\rho^{-1}(\tilde{E}_2(2\delta_j))$ that

$d(x, \psi(1,x)) << \tilde{\delta}_j$. We conclude from this inequality that

2.5.5. $|\hat{\bar{s}}_1(x) - \hat{\bar{s}}_1(\psi(1,x))| << \tilde{\delta}_j$.

On the other hand, we have for all $y \in \hat{\bar{E}}$ that

2.5.6. $\hat{\bar{s}}_1(y) = -\hat{\bar{s}}_1(\psi(1,y))$.

Now the desired inequality $|\hat{\bar{s}}_1(x)| << \tilde{\delta}_j$ follows from 2.5.5 and 2.5.6.

This completes the proof of Proposition 2.5.

Proof of Lemma 2.1. We assume that none of the conditions 2.1(a)(b)(c) hold, and we complete the proof of 2.1 by deriving a contradiction. We shall assume in the following proof (and also in the proof for Lemma 2.2) that

2.1.0. $\beta << 1$.

We will need the following properties concerning the immersion

$$f_p : (-5\alpha, 5\alpha) \times B(p, \beta) \to M$$

of 2.1. For each pair of points $(s,x), (s',x')$ as in 2.1 there is a smooth embedding

$$h : (-2\alpha, 2\alpha) \times B(p, \beta^{3/2}) \to (-5\alpha, 5\alpha) \times B(p, \beta)$$

which is uniquely determined by properties 2.1.1(a)(b), and which also satisfies property 2.1.1(d) in which h^i denotes the i-fold composite of h with itself. (See A.2 and A.3 in the Appendix.)

2.1.1. (a) $h(s,x) = (s',x')$.

(b) $f_p(t,y) = f_p \circ h(t,y)$ and $h(t,y) = (h_1(t,y), h_2(y))$ both hold for all $(t,y) \in (-2\alpha, 2\alpha) \times B(p, \beta^{3/2})$, where $h_1(\ ,y)$ is an isometry in the t-variable.

(c) There is $\kappa > 1$ which depends only on $\alpha, A = \{A_i\}, \dim M$, such that $\frac{1}{\kappa} < ||Df_{p|(t,y)}|| < \kappa$ for all $(t,y) \in (-2\alpha, 2\alpha) \times B(p, \beta^{3/2})$.

(d) For each integer $i \geq 0$, and for each (t,y) for which all of $\{h^j(t,y) : 0 \leq j \leq i\}$ are well defined and lie in $(-\alpha, \alpha) \times B(p, \beta^{3/2})$, we have $\frac{1}{\kappa} < ||Dh^i_{|(t,y)}|| < \kappa$.

For $i = 0, 1, 2, 3, \ldots$ we set $h^i(s,x) = (s_i, x_i)$ whenever all of $\{h^j(s,x) : 0 \leq j \leq i\}$ are well defined and lie in $(-\alpha, \alpha) \times B(p, \beta^{3/2})$. We let

$$h^{-1} : (-2\alpha, 2\alpha) \times B(p, \beta^{3/2}) \to (-5\alpha, 5\alpha) \times B(p, \beta)$$

be the map given by 2.1.1 when the roles of (s,t) and (s',t') are reversed; and we set $(h^{-1})^i(s,x) = (s_{-i}, x_{-i})$ whenever all of $\{(h^{-1})^j(s,x) : 0 \leq j \leq i\}$ are well defined and lie in $(-\alpha, \alpha) \times B(p, \beta^{3/2})$. We deduce from the failure of 2.1(a)-(c), and from 2.1.1, that the (s_i, x_i) satisfy the following properties provided β is sufficiently small (cf. 2.1.0) and $s < s'$. (Here 2.1.0 is interpreted so as to imply that $\kappa\beta << 1$.)

2.1.2. (a) $0 < s_{i+1} - s_i \leq 2\kappa\beta^2$.

(b) $|x_{i+1} - x_i| \leq 2\kappa\beta^2(s_{i+1} - s_i)$.

Since $|x_{i+k} - x_i| \leq \sum_{j=1}^{k} |x_{i+j} - x_{i+j-1}|$ for all i, k, it follows from 2.1.2 that

2.1.3. $|x_{i+k} - x_i| < 2\kappa\beta^2(s_{i+k} - s_i) < 8\kappa\beta^2\alpha$.

We conclude from 2.1.3, and from $(s_0, x_0) \in (-\alpha, \alpha) \times B(p, \beta^2)$, that the following holds.

2.1.4. $|x_i| < 8\kappa\beta^2(\alpha + 1)$.
Now we conclude from 2.1.0,2.1.2(a) and from 2.1.4 that (s_{i+1}, x_{i+1}) and (s_{i-1}, x_{i-1}) are both defined and lie in $(-\alpha, \alpha) \times B(p, \beta^{3/2})$, provided $s_i \in (-\alpha + 3\kappa\beta^2, \alpha - 3\kappa\beta^2)$. (Here 2.1.0 is interpreted so as to imply that $8\kappa\beta^2(\alpha + 1) << \beta^{3/2}$.) Using this last fact, together with 2.1.2(a), 2.1.4, and $s_0 \in (-\alpha, \alpha)$, we deduce the following.

2.1.5. For each $t \in (-\alpha, \alpha)$ there is (s_i, x_i) such that

$$|t - s_i| + |x_i| < 12\kappa\beta^2(\alpha + 1).$$

Since $f_p(s_{i+k}, x_{i+k}) = f_p(s_i, x_i)$ for all i, k, it follows from 2.1.1(c) and 2.1.5 that

$$\text{diameter } (f_p((-\alpha, \alpha) \times 0)) < 50\kappa^2\beta^2(\alpha + 1).$$

This last inequality implies that

2.1.6. $p \notin M(\alpha, \beta')$ if $\beta' > 50\kappa^2\beta^2(\alpha + 1)$.

Note that $\beta > 50\kappa^2\beta^2(\alpha + 1)$ follows from 2.1.0; so we conclude from 2.1.6 that $p \notin M(\alpha, \beta)$. Since our original assumption is that $p \in M(\alpha, \beta)$, we have arrived at the desired contradiction which completes the proof of Lemma 2.1.

Proof of Lemma 2.2. First we will verify 2.2(a). Suppose that the angle between $\mathfrak{F}$ and $\tilde{\mathfrak{G}}_p$ is less than β^3 at some point of $\tilde{U}_p$. It follows from 1.3(b) that the next property holds.

2.2.1. The angle between $\mathfrak{F}$ and $\tilde{\mathfrak{G}}_p$ is less than $2\beta^3$ everywhere.

Now set $V = f_p^{-1}(\tilde{U}_p)$ and set $\mathfrak{H} = f_p^{-1}(\tilde{\mathfrak{G}}_p)$. All the following properties are a consequence of 2.1.1(c) and 2.2.1.

2.2.2. (a) V is a smooth submanifold of $(-5\alpha, 5\alpha) \times B(p, \beta)$.
(b) $\mathfrak{H}$ is a smooth foliation of V; each leaf is a closed subset of $(-5\alpha, 5\alpha) \times B(p, \beta)$.
(c) The angle between any leaf of $\mathfrak{H}$ and $\frac{\partial}{\partial t}$ is less than $6\kappa^2\beta^3$ everywhere.

Now we complete the proof of Lemma 2.2(a) as follows. Our strategy is to choose (s, x) and (s', x') in $(-\alpha, \alpha) \times B(p, \beta^2)$ which satisfy the hypothesis of 2.1 but don't satisfy any of the conclusions of 2.1. This contradiction could be traced back to our assumption that 2.2(a) doesn't hold, and thus would complete the proof of Lemma 2.2(a). First we use 1.3(c) and 2.1.1(c) to choose $(s, x) \in V$ satisfying

2.2.3. $|x| + |s| < 3\kappa\tilde{\varepsilon}\tilde{\tilde{\delta}}_c$.
Let L denote the leaf of $\mathfrak{H}$ which contains (s, x). We conclude from 2.2.2(b)(c) (as applied to L) and from 2.2.3, that for any number $t > 0$ satisfying 2.2.4(a) there is $(t, y) \in L$ which satisfies 2.2.4(b).

2.2.4. (a) $0 < t - s < \alpha - 3\kappa\tilde{\varepsilon}\tilde{\tilde{\delta}}_c$.
(b) $\frac{|y-x|}{t-s} < 12\kappa^2\beta^3$.
We appeal to 1.3(b) and 1.2.2(b), and to 2.1.1(c), to choose $(s', x') \in L$ such that

2.2.5. (a) $f_p(s,x) = f_p(s',x')$;
(b) $|t-s'| + |y-x'| < 3\kappa\tilde{\varepsilon}\tilde{\delta}_c$.

Note that 2.2.3, 2.2.4(b), 2.2.5(b) imply that (s,x) and (s',x') satisfy 2.2.6(a)(b)(c), provided the number t of 2.2.4 is chosen to be $t = \beta^3 + s$, and provided β and $\frac{\tilde{\delta}_c}{\beta}$ are sufficiently small (cf. 2.1.0 and the hypotheses of 2.2).

2.2.6. (a) $|s-s'| < \frac{1}{4}\beta^2$.
(b) $\frac{|x-x'|}{|s-s'|} < \frac{1}{4}\beta^2$.
(c) $(s,x),(s',x') \in (-\alpha,\alpha) \times B(p,\beta^2)$.

We note that 2.2.6 would contradict the conclusions of Lemma 2.1 provided that (s,x) and (s',x') do not satisfy property 2.1(c). Thus the proof of lemma 2.2(a) is completed by this contradiction.

If our present (s,x) and (s',x') do satisfy 2.1(c) then we must continue our argument as follows. Set $\bar{t} = \frac{1}{2}\beta^3 + s$. Note that there is $(\bar{t},\bar{y}) \in L$ related to (s,x) as in 2.2.4(b). Note also that there is $(\bar{s}',\bar{x}') \in L$ which is related to $(\bar{t},\bar{y})$ and to (s,x) as in 2.2.5 and 2.2.6. If the pairs (s,x) and $(\bar{s}',\bar{x}')$ don't satisfy property 2.1(c), then we arrive at the desired contradiction as in the preceding paragraph. However if the two pairs (s,x) and $(\bar{s}',\bar{x}')$, as well as the two pairs (s,x) and (s',x'), both satisfy property 2.1(c), then we conclude that the two pairs (s',x') and $(\bar{s}',\bar{x}')$ do not satisfy property 2.1(c). Moreover we conclude from 2.2.6 (first as applied to the pairs $(s,x),(s',x')$, and then as applied to the pairs $(s,x),(\bar{s}',\bar{x}')$) and from 2.2.5(b) (first as applied to the pairs $(t,y),(s',x')$, and then as applied to the pairs $(\bar{t},\bar{y}),(\bar{s}',\bar{x}')$) that the two pairs (s',x') and $(\bar{s}',\bar{x}')$ also do not satisfy 2.1(a)(b).

This completes the proof of Lemma 2.2(a).

In order to complete the proof for Lemma 2.2(b)(c) we use some of the ideas from the proof of Theorem 1.5; cf. [8;§1]. We let $f : \mathbf{R}^m \to M$ denote the composition of a linear isometry $\mathbf{R}^m \to TM_p$ with the exponential map $TM_p \to M$. Note that for $\tilde{\delta}_c > 0$ sufficiently small the restricted map $f : B^m(\tilde{\delta}_c) \to M$ is a smooth immersion, where $B^m(\tilde{\delta}_c)$ denotes the open ball of radius $\tilde{\delta}_c$ centered at the origin in $\mathbf{R}^m$. Set $\hat{U} = f^{-1}(\tilde{U}_p) \cap B^m(\tilde{\delta}_c/3)$ and set $\hat{\mathfrak{G}} = f^{-1}(\tilde{\mathfrak{G}}_p)|B^m(\tilde{\delta}_c/3)$ and set $\hat{\mathfrak{F}} = f^{-1}(\mathfrak{F})|B^m(\tilde{\delta}_c/3)$. We note that it follows from 1.3(a)-(c) (See also [8;A.1.1 and A.1.2].) that for $\tilde{\delta}_c > 0$ sufficiently small there are vector subspaces $L \subset T\mathbf{R}^m_0$ and $H \subset V \subset T\mathbf{R}^m_0$ which satisfy the following properties. For any $x \in \mathbf{R}^m$ and $v \in T\mathbf{R}^m_x$ we let $P(v) \in T\mathbf{R}^m_0$ denote the (Euclidean) parallel translate of v.

2.2.7. For each $x \in \hat{U}$ we have

$$\Theta(V, P(T\hat{U}_x)) < \mathcal{O}(\tilde{\varepsilon}),$$
$$\Theta(H, P(T\hat{\mathfrak{G}}_x)) < \mathcal{O}(\tilde{\varepsilon}),$$
$$\Theta(L, P(T\hat{\mathfrak{F}}_x)) < \mathcal{O}(\tilde{\varepsilon}).$$

To complete the proof of 2.2(b)(c) it will therefore suffice to verify their following "linearized" versions.

2.2.8. (a) $\beta^3 - \mathcal{O}(\tilde{\varepsilon}) < \Theta^{\mathrm{u}}(L,V)$ ("linearized" version of 2.2(b)).
(b) $\Theta^{\mathrm{u}}(L,V) < \mathcal{O}(\tilde{\varepsilon})$ ("linearized" version of 2.2(c)).

Here $\Theta^{\mathrm{u}}(L,V)$ denotes the (unsymmetrical) angular distance from L to V, i.e. $\Theta^{\mu}(L,V)$ denotes the maximum of all the angular distances from vectors $v \in L$ to V.

As a first step towards verifying 2.2.8 we remark that the "linearization" of properties 1.2.2(a)-(d) given in [8;1.3-1.5] still hold. That is for each $x, y \in B^m(\frac{1}{6}\tilde{\delta}_c)$ with $f(x) = f(y)$ there is a smooth embedding $h : B^m(\frac{1}{6}\tilde{\delta}_c) \to B^m(\tilde{\delta}_c)$ which is uniquely determined by property [8;1.3(a)] and there is an isometry $\bar{h} : \mathbf{R}^m \to \mathbf{R}^m$ which approxiates h as in [8;1.3(b)]. Furthermore properties [8;1.4(a)(b), 1.5(a)(b)] hold when H_i, V_i in [8;1.4] are replaced by our present H, V.

To verify 2.2.8(a) (when $T\mathfrak{F}_{|\tilde{E}_p}$ is not orientable) we first choose h as in [8;1.3,1.4] such that $dh(\frac{\partial}{\partial t}(x)) = -\frac{\partial}{\partial t}(h(x))$ holds for all $x \in B^m(\frac{1}{6}\tilde{\delta}_c)$, where $\frac{\partial}{\partial t}$ is a unit vector field on $B^m(\frac{1}{6}\tilde{\delta}_c)$ tangent to $\hat{\mathfrak{F}}$. It then follows from [8;1.3(b)] and from 2.2.7 that the rotational part of $\bar{h}$, denoted by $\bar{h}_r$, satisfies the following property where v denotes a fixed unit vector in L.

2.2.9. $\Theta(-v, \bar{h}_r(v)) < \mathcal{O}(\tilde{\varepsilon})$.

We write $v = v_1 + v_2 + v_3$ where $v_1 \in H, v_2 \in V$ and is perpendicular to H, v_3 is perpendicular to V in $\mathbf{R}^m$. To complete the verification of 2.2.8(a) it will suffice (by 2.2(a) and 2.2.7) to show that $|v_2| < \mathcal{O}(\tilde{\varepsilon})$. Note that it follows from [8;1.4(a)] that

2.2.10. $\Theta(v_2, \bar{h}_r(v_2)) < \mathcal{O}(\tilde{\varepsilon})$.

Note that 2.2.9 and 2.2.10 together imply the desired inequality $|v_2| < \mathcal{O}(\tilde{\varepsilon})$.

This completes the verification for 2.2.8(a).

To verify 2.2.8(b) it will suffice to show that $|v_3| < \mathcal{O}(\tilde{\varepsilon})$, where v, v_1, v_2, v_3 are as in the preceding paragraph. We use [8;1.4(b)] to choose h of [8;1.3] such that

2.2.11. $\Theta(v_3, \bar{h}_r(v_3)) > \tilde{\vartheta}/4$.

Note that it follows from the hypothesis of 2.2(c) that $dh(\frac{\partial}{\partial t}(x)) = \frac{\partial}{\partial t}(h(x))$ holds for all $x \in B^m(\frac{1}{6}\tilde{\delta}_c)$. Thus from 2.2.7 and from [8;1.3(b)] we deduce that

2.2.12. $\Theta(v, \bar{h}_r(v)) < \mathcal{O}(\tilde{\varepsilon})$.

Finally, using the fact that h leaves $\hat{U}$ invariant (cf. [8;1.3(c)]) in conjunction with 2.2.7 and [8;1.3(b)] we get that

2.2.13. (a) $\Theta^u(\bar{h}_r(v_1 + v_2), V) < \mathcal{O}(\tilde{\varepsilon})$.
(b) $\Theta^u(\bar{h}_r(v_3), V) > \frac{1}{2}\pi - \mathcal{O}(\tilde{\varepsilon})$.

Note that 2.2.11-2.2.13 together with the original properties of v, v_1, v_2, v_3 imply the desired inequality $|v_3| < \mathcal{O}(\tilde{\varepsilon})$.

This completes the proof for Lemma 2.2.

3. Two lemmas from linear algebra. In this section we state and prove two lemmas concerned with collections of affine isomorphisms $f : \mathbf{R}^k \to \mathbf{R}^k$. These results will be needed (in addition to 1.3 and 1.5 of §1, and in addition to 4.2 and 4.3 of §4) in section 5 below to carry out the construction of infrasolv cores of type I.

Recall that an *affine isomorphism* $f : \mathbf{R}^k \to \mathbf{R}^k$ is the composition of a linear isomorphism ${}_lf : \mathbf{R}^k \to \mathbf{R}^k$ with translation by a vector ${}_tf \in \mathbf{R}^k$, i.e. $f(x) = {}_lf(x) + {}_tf$; we set $||f|| = \max\{|x - f(x)| : |x| \leq 1\}$. For any finite collection of affine isomorphisms A we set ${}_lA = \{{}_lf : f \in A\}$, ${}_tA = \{{}_tf : f \in A\}$, and $||A|| = \max\{||f|| : f \in A\}$. We shall say that A is (ω, d)-*cyclic*, for some numbers $\omega, d > 0$, if there is an integer $I > 0$ and an element $g \in A$ such that $A = \{f_{-I}, f_{-I+1}, \cdots, f_{I-1}, f_I\}$ and for each $i \in \{-I, -I+1, \ldots, I-1, I\}$ we have that $|{}_l(g^i) - {}_l f_i| < \omega$ and $|{}_t(g^i) - {}_t f_i| < \omega d$; the element $g \in A$ is called an (ω, d)-*generator* for A.

In the following two lemmas, we let $A_1 \subset A_2 \subset \ldots \subset A_{k+2}$ denote a given increasing sequence of finite collections of affine isomorphisms of $\mathbf{R}^k$; $0 < a_1 < a_2 < \ldots < a_x < 1$, with $x = (k+4)^{k+4}$, and $0 < d_1 < d_2 < \ldots < d_{k+2} < 1$ are given increasing sequences of numbers; $\nu > 1$ and $\omega > 0$ are given numbers. All of these sets and numbers satisfy the following hypotheses.

3.0. Hypotheses.

(a) Each $A_i, i \in \{1, 2, \ldots, k+2\}$, is (ω, d_i)-cyclic with an (ω, d_i)-generator $g_i \in A_i$. Moreover we have for each $i \in \{1, 2, \ldots, k+2\}$ that

$$\frac{1}{\nu} < |{}_lA_i| < \nu \text{ and } |{}_tA_i| < d_i.$$

That is $\frac{1}{\nu} < |B(x)| < \nu$ for all $B \in_l A_i$, $x \in \mathbf{R}^k$ with $|x| = 1$; and $|u| < d_i$ for all $u \in {}_tA_i$.

(b) We have, for all $i \in \{1, 2, \ldots, x-1\}$ and all $j \in \{1, 2, \ldots, k+1\}$, that

$$1000ka_i < (a_{i+1})^2,$$
$$d_j < \frac{a_1 d_{j+1}}{\nu + 1}; \text{ and}$$
$$\omega < \frac{a_1}{2(\nu+1)}.$$

(c) The cardinality of A_{k+2} has an upper bound independent of the $\{a_i\}$ and of the $\{d_j\}$.

3.1. Existence Lemma. *For some integers $y \in \{1, 2, \ldots, x-1\}$ and $z \in \{1, 2, \ldots, k+1\}$ there is a vector subspace $V \subset \mathbf{R}^k$ and a point $q \in \mathbf{R}^k$, with $|q| < d_{z+1}$, which satisfy properties (a) – (c). Moreover the (ω, d_{z+1})-generator g_{z+1} for A_{z+1} satisfies property (d).*

(a) For each unit vector $v \in V$ and each $f \in A_{z+1}$ we have that $|{}_lf(v) - v| \le a_y$.

(b) For each unit vector $u \in \mathbf{R}^k$ which is perpendicular to V, there exists $f \in A_z$ such that $|{}_lf(u) - u| > a_{y+1}$.

(c) For each $f \in A_{z+1}$ there is $v \in V$ such that $|f(q) - (q+v)| < \mathcal{O}(\frac{a_y}{a_{y+1}})d_{z+1}$.

(d) $|g_{z+1}(q) - q| < \mathcal{O}(\frac{a_y}{a_{y+1}})d_{z+1}$.

Remark. Actually $q = 0$ will work in the above lemma, for the right choice of x, y.

The preceding lemma may be viewed as a linear-affine analog of Theorem 1.3: instead of the infranil core provided by 1.3 there is the plane $P = q + V$ provided by 3.1. The next lemma may be viewed as a linear-affine analog of Theorem 1.5: instead of the closeness of two infranil cores provided by 1.3 there is the closeness of the two planes $P_1 = q_1 + V_1$ and $P_2 = q_2 + V_2$ provided by 3.1.

3.2. Comparison Lemma. *Suppose that ω is sufficiently small (how small is sufficient depends only on the cardinality of A_{z+1} and on the $\{a_i\}$). Let $V_1, V_2 \subset \mathbf{R}^k$ be subspaces and let $q_1, q_2 \in \mathbf{R}^k$ be points, with $|q_i| < d_{z+1}$, such that both of the pairs $(V_i, q_i), i = 1, 2$, satisfy properties 3.1(a)-(d). Then the pairs (V_1, q_1) and (V_2, q_2) are close in the following sense.*

(a) $\Theta(V_1, V_2) < \mathcal{O}(\frac{a_y}{a_{y+1}})$.

(b) $|q_1 + v_1 - q_2| < \mathcal{O}(\frac{a_y}{(a_{y+1})^2})d_{z+1}$ for some $v_1 \in V_1$.

Proof of Lemma 3.1. We will first construct for each $s \in \{1, 2, \ldots, k+2\}$ a vector subspace $V_s \subset \mathbf{R}^k$ satisfying the following assertions.

3.1.1. There are integers $x_1, x_2, \ldots, x_{k+2} \in \{1, \ldots, k+2\}$ such that statements (a) and (b) below are satisfied when we set $y_s = \sum_{t=1}^{s} x_t(k+4)^{k+4-t}$ and $z_s = y_s + (k+4)^{k+4-s}$. The subspace $V_s \subset \mathbf{R}^k$ satisfies:

(a) For each unit vector $v \in V_s$ and each $f \in A_s$ we have that $|_l f(v) - v| \leq k a_{y_s}$.

(b) For each unit vector $u \in \mathbf{R}^k$ which is perpendicular to V_s, there exists $f \in A_s$ such that $|_l f(u) - u| > a_{z_s}$.

The construction of V_s, y_s proceeds by induction. Set $w_{s,r} = y_{s-1} + r(k+4)^{k+4-s}$ for each $r \in \{1, 2, \ldots, k+2\}$ (where $y_0 = 0$). If there is no unit vector $v \in \mathbf{R}^k$ such that $|_l f(v) - v| \leq a_{w_{s,2}}$ holds for all $f \in A_s$, then we may set $V_s = \{0\}$ and $y_s = w_{s,1}$. Otherwise there is a unit vector $v_1 \in \mathbf{R}^k$ such that $|_l f(v_1) - v_1| \leq a_{w_{s,2}}$ holds for all $f \in A_s$, and we set $V_{s,1} = \mathrm{span}\{v_1\}$. If there is no unit vector $v \in V_{s,1}^{\perp}$ (where $V_{s,1}^{\perp}$ denotes the orthogonal complement for $V_{s,1}$ in $\mathbf{R}^k$) such that $|_l f(v) - v| \leq a_{w_{s,3}}$ holds for all $f \in A_s$, then we set $V_s = V_{s,1}$ and $y_s = w_{s,2}$. Otherwise there is a unit vector $v_2 \in \mathbf{R}^k$ perpendicular to $V_{s,1}$ such that $|_l f(v_2) - v_2| \leq a_{w_{s,3}}$ holds for all $f \in A_s$, and we set $V_{s,2} = \mathrm{span}\{v_1, v_2\}$. We proceed in this way until we arrive at the following situation: $V_{s,r} = \mathrm{span}\{v_1, v_2, \ldots, v_r\}$ where $\{v_1, v_2, \ldots, v_r\}$ is an orthonormal set such that $|_l f(v_i) - v_i| \leq a_{w_{s,i+1}}$ for all $i \in \{1, 2, \ldots, r\}$ and all $f \in A_s$; for each unit vector $v \in \mathbf{R}^k$ which is perpendicular to $V_{s,r}$ we have that $|_l f(v) - v| > a_{w_{s,r+2}}$ for some $f \in A_s$. Then we set $V_s = V_{s,r}$ and $y_s = w_{s,r+1}$.

This completes the verification for 3.1.1.

Next we verify the following relation between V_s and V_{s+1}.

3.1.2. $\Theta^{u}(V_{s+1}, V_s) < 4k\frac{a_{y_{s+1}}}{a_{z_s}}$.

Towards verifying 3.1.2 we let $u \in V_{s+1}$ denote a unit vector and v, w its components in $V_s, V_s^{\perp}$ respectively. Note we have (from 3.1.1 for $s, s+1$ and from the triangle inequality) that

$$a_{z_s}|w| - k a_{y_s} \leq |_l f(w) - w| - |_l f(v) - v| \leq |_l f(u) - u| \leq k a_{y_{s+1}}$$

for some $f \in A_s$; from which we deduce that $|w| \leq 2k\frac{a_{y_{s+1}}}{a_{z_s}}$. This last inequality implies 3.1.2 because $\Theta^{u}(u, V_s) < 2|w|$.

We note that as a consequence of the inequality $1000 k a_y < a_{y+1}$ (assumed in 3.0 for all y) we have that the following relations exist.

3.1.3. (a) $1000 k a_{y_{s+1}} < a_{z_s}$.
(b) $k a_{y_{s+1}} < a_{1+y_{s+1}}$.
(c) $a_{2+y_{s+1}} < \frac{1}{2} a_{z_s}$.

As a consequence of 3.1.2 and 3.1.3(a), we see that $\dim(V_s) \geq \dim(V_{s+1})$ for all $s \in \{1, 2, \ldots, k+1\}$. So we may choose $s \in \{1, 2, \ldots, k+1\}$ such that $\dim(V_s) = \dim(V_{s+1})$. It then follows from 3.1.1(a)(b), 3.1.2, 3.1.3(a) that V_{s+1} satisfies the following.

3.1.4. For each unit vector $v \in V_{s+1}^{\perp}$, there is $f \in A_s$ such that $|_l f(v) - v| > \frac{1}{2} a_{z_s}$.

We can now define the subspace $V \subset \mathbf{R}^k$ and the integers $y \in \{1, 2, \ldots, x-1\}$

and $z \in \{1, 2, \ldots, k+1\}$ of 3.1 as follows:

$$V = V_{s+1}, z = s, y = y_{s+1} + 1.$$

That the (V, y, z) satisfy 3.1(a) is immediate from 3.1.1(a) (as applied to $s+1$) and from 3.1.3(b). That the (V, y, z) satisfy 3.1(b) is immediate from 3.1.4 and 3.1.3(c).

Now we will use 3.1(a)(b) to deduce 3.1(c). Note that in proving 3.1.(a)(b) we have not used the hypothesis (from 3.0(a)) that each A_i is (ω, d_i)-cyclic. However in proving 3.1.(c) we shall need the following weak form of this hypothesis (easily deduced from 3.0(a)).

3.1.5. For each $g, f \in A_{z+1}$ we have that $|g({}_tf) - f({}_tg)| < 2(1+\nu)\omega d_{z+1}$.

We shall also need (in proving 3.1(c)) the following relations between the numbers $\omega, a_y, a_{y+1}, d_z, d_{z+1}$ which are an immediate consequence of the properties of these numbers assumed in 3.0.

3.1.6. (a) $\frac{a_y + 2(v+1)\omega}{a_{y+1}} < 2\frac{a_y}{a_{y+1}}$.
(b) $\frac{(v+1)d_z}{a_{y+1}} < (\frac{a_y}{a_{y+1}})d_{z+1}$.

Now to complete the proof of 3.1(c) we choose $q = 0$ in 3.1; consequently we have that

3.1.7. ${}_tf = f(q) - q$

holds for any $f \in A_{z+1}$. We write ${}_tf = v + w$ where $v \in V$ and $w \in V^\perp$. To verify 3.1(c) it will suffice (by 3.1.7) to show that $|w| < 3(\frac{a_y}{a_{y+1}})d_{z+1}$. Using 3.1(b) we choose $g \in A_z$ such that $|{}_lg(w) - w| > a_{y+1}|w|$. Using 3.1(a) (as applied to g and v), and this last inequality, and the triangle inequality, and the fact that $|v| \leq |{}_tf| \leq d_{z+1}$ (cf. hypothesis 3.0), we get:

3.1.8. $a_{y+1}|w| - a_y d_{z+1} \leq |{}_lg(w) - w| - |{}_lg(v) - v| \leq |{}_lg({}_tf) - {}_tf| \leq |g({}_tf) - {}_tf| + |{}_tg|$.

On the other hand we may deduce from the conditions $|{}_tg| < d_z$ and $|{}_lf| < \nu$ imposed by hypothesis 3.0, and from 3.1.5, and from the triangle inequality, that the following holds.

3.1.9. $|g({}_tf) -{}_t f| + |{}_tg| \leq |g({}_tf) - f({}_tg)| + |f({}_tg) -{}_t f| + |{}_tg| < 2(1+\nu)\omega d_{z+1} + (\nu+1)d_z$.

By combining 3.1.6, 3.1.8, 3.1.9, we get $|w| < 3(\frac{a_y}{a_{y+1}})d_{z+1}$ as desired.

Now to complete the proof of Lemma 3.1 it remains to verify 3.1(d). Towards this end we first apply 3.1(a)(c) to the(ω, d_{z+1})-generator g_{z+1} of A_{z+1} to conclude that

3.1.10. $|(g^s_{z+1}(q) - q) - s(g_{z+1}(q) - q))| < \tau_s \mathcal{O}(\frac{a_y}{a_{y+1}})d_{z+1}$

holds for all $s \in \{-I, -I+1, \ldots, I\}$, where $A_{z+1} = \{f_i : -I \leq i \leq I\}$ (cf. the paragraph preceding 3.0) and where $\tau_s > 1$ depends only on s, ν. Since the cardinality of A_{z+1} (and hence also the cardinality of I) is bounded above by a number independent of the $\{a_i\}$ and the $\{d_j\}$ (cf. 3.0(c)) it follows from 3.1.10 that

3.1.11. $|(g_{z+1}^s(q) - q) - s(g_{z+1}(q) - q))| < \mathcal{O}(\frac{a_y}{a_{y+1}})d_{z+1}$.

On the other hand, there is (by 3.1(b) and 3.0(b) and by our above choice $q = 0$) an integer $s \neq 0, s \in \{-I, -I+1, \dots, I\}$ such that

3.1.12. $|f_s(q) - q| < d_z$.

By 3.0(a) and our choice $q = 0$, we have that

3.1.13. $|f_s(q) - g_{z+1}^s(q)| < \omega d_{z+1}$.

Now by combining 3.1.11-3.1.13 with the inequalities

$$d_z < \frac{a_y}{a_{y+1}} d_{z+1} \text{ and } \omega d_{z+1} < \frac{a_y}{a_{y+1}} d_{z+1}$$

(cf. 3.0(b)) we get

3.1.14. $|s(g_{z+1}(q) - q))| < \mathcal{O}(\frac{a_y}{a_{y+1}})d_{z+1}$.

Finally 3.1(d) is a consequence of 3.1.14.

This completes the proof of Lemma 3.1.

Proof of Lemma 3.2. First we will prove 3.2(a). For any unit vector $u \in V_1$ we write $u = v + w$ where $v \in V_2$ and $w \in V_2^{\perp}$. It will suffice (in verifying 3.2(a)) to show that $|w| \leq 2\frac{a_y}{a_{y+1}}$. Using 3.1(b) (as applied to $w \in V_2^{\perp}$) we may choose $g \in A_z$ such that $|\iota g(w) - w| > a_{y+1}|w|$. This last inequality, together with 3.1(a) (as applied to g and $v \in V_2$) and the triangle inequality, imply that

3.2.1. $a_{y+1}|w| - a_y|v| \leq |\iota g(w) - w| - |\iota g(v) - v| \leq |\iota g(u) - u|$.

On the other hand, by applying 3.1(a) to g and $u \in V_1$, we get that

3.2.2. $|\iota g(u) - u| \leq a_y$.

Now, by combining 3.2.1, 3.2.2 with the inequality $|v| \leq 1$, we get the desired inequality $|w| \leq 2\frac{a_y}{a_{y+1}}$.

Now we will verify 3.2(b). Set $q_1 - q_2 = v + w$ where $v \in V_1$ and $w \in V_1^{\perp}$; then it will suffice (in verifying 3.2(b)) to show that $|w| < \mathcal{O}(\frac{a_y}{(a_{y+1})^2})d_{z+1}$. Towards this end we first note that as a consequence of 3.0 and 3.1(a)(b) we have that

3.2.3. (a) $|\iota g_{z+1}(v) - v| < a_y|v|$
(b) $|\iota g_{z+1}(w) - w| > \frac{2(\nu-1)}{\nu^\sigma - 1} a_{y+1}|w|$

where g_{z+1} denotes the (ω, d_{z+1})-generator for A_{z+1} (cf. 3.0(a)) and σ denotes the upper bound for the cardinality of A_{z+1} posited in 3.0(c). By applying 3.1(d) to both of $q_i, i = 1, 2$, we get that

3.2.4. $|g_{z+1}(q_i) - q_i| < \mathcal{O}(\frac{a_y}{a_{y+1}})d_{z+1}$.

Next we note that

$$\iota g_{z+1}(w) - w = (g_{z+1}(q_1) - q_1) - (g_{z+1}(q_2) - q_2) - (\iota g_{z+1}(v) - v);$$

by applying the traingle inequality to this equality, in conjunction with 3.2.3(a) and 3.2.4, we get that

3.2.5. $|_\iota g_{z+1}(w) - w| < \mathcal{O}(\frac{a_y}{a_{y+1}})d_{z+1} + a_y|v|$.
Note that $|q_i| < d_{z+1}$ for $i = 1, 2$ (cf. the hypothesis of 3.1) and thus $|v| < 2d_{z+1}$; hence 3.2.5 implies that

3.2.6. $|_\iota g_{z+1}(w) - w| < \mathcal{O}(\frac{a_y}{a_{y+1}})d_{z+1}$.
Finally 3.2.3(b) together with 3.2.6 imply the desired inequality $|w| < \mathcal{O}(\frac{a_y}{(a_{y+1})^2})d_{z+1}$.
This completes the proof of Lemma 3.2.

4. The set of embeddings $\mathbf{H_p}(\lambda\tilde{\delta}_c)$. We let $\alpha, \beta, A = \{A_i\}$ be as in 0.5, and let $\tilde{\varepsilon}, \tilde{\delta}_c$ be as in 1.3,2.6; let $\tilde{r}_p : \tilde{E}_p \to \tilde{B}_p$ denote the (thickened) infranil core referred to in 2.2 and 2.6. In this section we always assume that

4.0. $T\mathfrak{F}$ is orientable.

Thus there is the smooth immersion $\mathfrak{f}_p : \mathbf{R} \times \mathfrak{C}_p \to M$ described in case I of 2.6. For each $t \in (0, \frac{1}{2}\tilde{\delta}_c)$ there is the subset $\mathfrak{C}_p(t) \subset \mathfrak{C}_p$ described in 2.6. If $\lambda \in (0, \frac{1}{2})$ is sufficiently small (how small is sufficient depends only on $\alpha, \beta, A = \{A_i\}, \dim M$) then for each pair of points (t_1, y_1) and (t_2, y_2) in $(-\alpha, \alpha) \times \mathfrak{C}_p(\lambda\tilde{\delta}_c)$ which satisfy $\mathfrak{f}_p(t_1, y_1) = \mathfrak{f}_p(t_2, y_2)$, there is a smooth embedding

$$h : (-\alpha, \alpha) \times \mathfrak{C}_p(\lambda\tilde{\delta}_c) \to (-4\alpha, 4\alpha) \times \mathfrak{C}_p(\frac{1}{4}\tilde{\delta}_c)$$

which is uniquely determined by the following properties.

4.1. (a) $h(t_1, y_1) = h(t_2, y_2)$.
(b) $\mathfrak{f}_p \circ h = \mathfrak{f}_p$.

We denote the collection of all such embeddings by

(c) $H_p(\lambda\tilde{\delta}_c)$

If for $g, h \in H_p(\lambda\tilde{\delta}_c)$ and for some $(t, y) \in (-\alpha, \alpha) \times \mathfrak{C}_p(\lambda\tilde{\delta}_c)$ we have that $g(h(t, y)) = (t, y)$, we will say that g is the inverse of h and write $h^{-1} = g$; note that for sufficiently small λ and $\tilde{\delta}_c$ the inverse h^{-1} is always well defined. If for $g, h, h' \in H_p(\lambda\tilde{\delta}_c)$ and for some $(t, y) \in (-\alpha, \alpha) \times \mathfrak{C}_p(\lambda\tilde{\delta}_c)$ we have that $g(t, y) = h'(h(t, y))$, we will say that g is the composition of h' with h and write $g = h' \circ h$; note that the composition $h' \circ h$ need not exist; note also that for sufficiently small $\lambda, \tilde{\delta}_c$ the composition $h' \circ h$ (if it exists) is uniquely determined.
We shall prove the following two lemmas concerning $H_p(\lambda\tilde{\delta}_c)$.

4.2. Lemma. *There is a number $\kappa > 1$ which depends only on $\alpha, \beta, A = \{A_i\}, dimM$. Suppose that $\lambda, \tilde{\delta}_c$ are sufficiently small (how small is sufficient depends only on $\alpha, \beta, A = \{A_i\}, dimM$). Then there is an integer $I \in (0, 8\frac{\alpha}{\beta^2} + 4)$ and an element $g \in H_p(\kappa\lambda\tilde{\delta}_c)$ such that the following hold*
(a) For each integer $-I \le i \le I$ the power g^i is a well defined element of $H_p(\kappa\lambda\tilde{\delta}_c)$.
(b) For each $h \in H_p(\lambda\tilde{\delta}_c)$, there is an integer $-I \le i \le I$ such that $h = g^i_{|(-\alpha,\alpha)\times\mathfrak{C}_p(\lambda\tilde{\delta}_c)}$.

In the next lemma we let $\mathcal{T}$ denote the foliation of $(-4\alpha, 4\alpha) \times \mathfrak{U}_p$ by the fibers of $\mathrm{id}\times\mathfrak{r}_p : (-4\alpha, 4\alpha) \times \mathfrak{U}_p \to (-4\alpha, 4\alpha) \times \mathfrak{B}_p$, where $\mathfrak{r}_p : \mathfrak{U}_p \to \mathfrak{B}_p$ comes from 2.6. For each $h \in H_p(\lambda\tilde{\delta}_c)$ set $V_h = h((-\alpha, \alpha) \times \mathfrak{U}_p(\lambda\tilde{\delta}_c))$ and set $\mathcal{T}_h = h(\mathcal{T}|(-\alpha, \alpha) \times \mathfrak{U}_p(\lambda\tilde{\delta}_c))$. All the geometric constructions in the next lemma refer to the pulled back metric $\mathfrak{f}_p^*(g)$ on $\mathbf{R} \times \mathfrak{E}_p$, where g denotes the given metric on M.

4.3. Lemma. *Suppose that $\lambda, \tilde{\delta}_c$ are sufficiently small (how small is sufficient depends only on $\alpha, \beta, A = \{A_i\}, dimM$). Then for any $h \in H_p(\lambda\tilde{\delta}_c)$ and any $z \in V_h$, there is a path $u : [0, 1] \to \mathbf{R} \times \mathfrak{E}_p$ with $u(0) = z$ and $u(1) \in \mathbf{R} \times \mathfrak{U}_p$ which satisfies properties (a) - (e).*

(a) length $(u) < \mathcal{O}(\tilde{\varepsilon})\tilde{\delta}_c$.
(b) $\Theta(T(\mathbf{R} \times \mathfrak{U}_p)_{u(1)}, P_u(T(V_h)_{u(0)})) < \mathcal{O}(\tilde{\varepsilon})$.
(c) $\Theta(T(\mathcal{T})_{u(1)}, P_u(T(\mathcal{T}_h)_{u(0)})) < \mathcal{O}(\tilde{\varepsilon})$.
(d) $K(id \times \mathfrak{r}_p; \mathbf{R} \times \mathfrak{E}_p) + K((id \times \mathfrak{r}_p) \circ h^{-1}; \mathbf{R} \times \mathfrak{E}_p) < \frac{\mathcal{O}(\tilde{\varepsilon})}{\tilde{\delta}_c}$.
(e) diameter $(L) < \mathcal{O}(\tilde{\varepsilon})\tilde{\delta}_c$ for all $L \in \mathcal{T}$ (or for all $L \in \mathcal{T}_h$).

Proof for Lemma 4.2. For any $h \in H_p(\lambda\tilde{\delta}_c)$ we let $h = (h_1, h_2)$ denote the two components of h corresponding to the first and second factor of $\mathbf{R} \times \mathfrak{E}_p$ and set

4.2.1. (a) $h_{1,1}(x) = h_1(x, y_1)$

for all $x \in (-\alpha, \alpha)$, where y_1 comes from 4.1. Note that $h_{1,1}$ extends to a translation $h_{1,1} : \mathbf{R}\to\mathbf{R}$ (cf. 4.0). Note also that both $h_2(x_1, x_2)$ and $h_1(x_1, x_2) - h_{1,1}(x_1)$ depend only on x_2, for $(x_1, x_2) \in (-\alpha, \alpha) \times \mathfrak{E}_p(\lambda\tilde{\delta}_c)$; thus we may set

(b) $h_{1,2}(x_2) = h_1(x_1, x_2) - h_{1,1}(x_1)$
(c) $h_2(x_2) = h_2(x_1, x_2)$

for all $x_2 \in \mathfrak{E}_p(\lambda\tilde{\delta}_c)$. We have the following claim.

4.2.2. Claim. *There is a number $\rho > 1$ which depends only on $\alpha, \beta, A = \{A_i\}, dimM$. For sufficiently small $\tilde{\delta}_c$ (in 1.3) and for all $(x_1, x_2) \in (-\alpha, \alpha) \times \mathfrak{E}_p(\lambda\tilde{\delta}_c)$ the following properties hold.*

(a) $h(x_1, x_2) = (h_{1,1}(x_1) + h_{1,2}(x_2), h_2(x_2))$.
(b) $|h_{1,2}(x_2)| < \rho\lambda\tilde{\delta}_c$ and $h_2(x_2) \in \mathfrak{E}_p(\rho\lambda\tilde{\delta}_c)$.
(c) If h is not the identity imbedding then $h_{1,1}$ is a translation satisfying

$$|h_{1,1}(0)| > \frac{1}{2}\beta^2.$$

We shall first use Claim 4.2.2 to complete the proof of Lemma 4.2, and then we will verify Claim 4.2.2.

Set $J = 4[50\frac{\alpha}{\beta^2}] + 2^8$, where $[x]$ denotes the least integer greater then x, and let $\lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_J$ denote a sequence of numbers which satisfy the following.

4.2.3. (a) Each λ_i depends only on $\alpha, \beta, A = \{A_i\}, dimM$; $\lambda_1 = \lambda$.
(b) $\lambda_{i+1} > \rho^2\lambda_i$ for all i, where $\rho > 1$ comes from 4.2.2.
(c) $H_p(\lambda_i\tilde{\delta}_c)$ is well defined when λ_i replaces λ in 4.1.
(d) Properties 4.2.2(a)-(c) hold when λ is replaced by any λ_i.

Note that for each $h \in H_p(\lambda_i\tilde{\delta}_c)$ there is a unique $\bar{h} \in H_p(\lambda_{i+1}\tilde{\delta}_c)$ such that

$$\bar{h}|(-\alpha,\alpha) \times \mathfrak{E}_p(\lambda_i\tilde{\delta}_c) = h$$

(cf. 4.1 as applied to λ_i, λ_{i+1}); thus by identifying h with its extension $\bar{h}$ we get an inclusion $H_p(\lambda_i\tilde{\delta}_c) \subset H_p(\lambda_{i+1}\tilde{\delta}_c)$ for each i.

The reader can deduce the following properties directly from 4.2.1-4.2.3, and from the hypothesis (placed on $\tilde{\delta}_c$ by 4.2 and 4.2.2) that $\tilde{\delta}_c$ is sufficiently small.

4.2.4. For any $i = 1, 2, \ldots, J-1$, and for any $h, h' \in H_p(\lambda_i\tilde{\delta}_c)$, the following hold.

(a) $|h_{1,1}(0)| < 2\alpha + \lambda_{i+1}\tilde{\delta}_c$.
(b) $|h_{1,1}^{-1}(0) + h_{1,1}(0)| < 2\lambda_{i+1}\tilde{\delta}_c$.
(c) $|h'_{1,1}(0) - h_{1,1}(0)| < \frac{1}{3}\beta^2$ implies $h = h'$.
(d) $|(h' \circ h)_{1,1}(0) - h'_{1,1}(0) - h_{1,1}(0)| < 2\lambda_{i+1}\tilde{\delta}_c$, provided $h' \circ h$ is well defined in $H_p(\lambda_{i+1}\tilde{\delta}_c)$.
(e) $h' \circ h$ is well defined in $H_p(\lambda_{i+1}\tilde{\delta}_c)$ if $|h'_{1,1}(0) + h_{1,1}(0)| < \frac{3}{2}\alpha$, or if there is $g \in H_p(\lambda_{i+1}\tilde{\delta}_c)$ such that

$$|g_{1,1}(0) - h'_{1,1}(0) - h_{1,1}(0)| < \frac{1}{4}\beta^2$$

(in which case $g = h' \circ h$).

For example, to verify 4.2.4(e) we proceed as follows. First suppose that the inequality $|h'_{1,1}(0) + h_{1,1}(0)| < \frac{3}{2}\alpha$ of 4.2.4(e) holds. Then it follows from 4.2.2(a)(b) and 4.2.3(b), and from the preceding inequality, that there is a point $(t, y) \in (-\alpha,\alpha) \times \mathfrak{E}_p(\lambda_i\tilde{\delta}_c)$ such that $h(t,y) \in (-\alpha,\alpha) \times \mathfrak{E}_p(\lambda_i\tilde{\delta}_c)$ and such that $h'(h(t,y)) \in (-\alpha',\alpha') \times \mathfrak{E}_p(\lambda_{i+1}\tilde{\delta}_c)$ where $\alpha' = 5\lambda_{i+1}\tilde{\delta}_c + \frac{3}{4}\alpha$; so $h' \circ h \in H_p(\lambda_{i+1}\tilde{\delta}_c)$ as claimed in 4.2.4(e). Next suppose that the inequality $|g_{1,1}(0) - h'_{1,1}(0) - h_{1,1}(0)| < \frac{1}{4}\beta^2$ of 4.2.4(e) holds for some $g \in H_p(\lambda_{i+1}\tilde{\delta}_c)$. Since each of $g_{1,1}^{-1}, h_{1,1}, h'_{1,1}$ extends to a translation $\mathbf{R} \to \mathbf{R}$, we may use the formulae in 4.2.2(a) (and the remark at the end of the preceding paragraph) to extend each of g^{-1}, h, h' to maps $\mathbf{R} \times \mathfrak{E}_p(\lambda_{i+1}\tilde{\delta}_c) \to \mathbf{R} \times \mathfrak{E}_p(\frac{1}{4}\tilde{\delta}_c)$; we denote these extended maps by $\bar{g}^{-1}, \bar{h}, \bar{h}'$. It follows from the inequality $|g_{1,1}(0) - h'_{1,1}(0) - h_{1,1}(0)| < \frac{1}{4}\beta^2$, from 4.2.4(b) as applied to g, from 4.2.3(b), and from 4.2.2(b) as applied to g, h, h', that the compositon $\bar{g}^{-1} \circ \bar{h}' \circ h$ is well defined on $(-\alpha,\alpha) \times \mathfrak{E}_p(\lambda_i\tilde{\delta}_c)$ and satisfies $|(\bar{g}^{-1} \circ \bar{h}' \circ h)_{1,1}(0)| < \frac{1}{3}\beta^2$. Thus, by 4.2.4(c) (as applied to $\bar{g}^{-1} \circ \bar{h}' \circ h$), we conclude that $\bar{g}^{-1} \circ \bar{h}' \circ h$ is equal the identity map on $(-\alpha,\alpha) \times \mathfrak{E}_p(\lambda_i\tilde{\delta}_c)$; from which we deduce that $\bar{g} = \bar{h} \circ h$ on $(-\alpha,\alpha) \times \mathfrak{E}_p(\lambda_i\tilde{\delta}_c)$. It follows from this last equality that $h' \circ h$ is well defined in $H_p(\lambda_{i+1}\tilde{\delta}_c)$, and is in fact equal to g. This completes the verification of 4.2.4(e).

To complete the proof of Lemma 4.2 we need now appeal only to 4.2.4. For any given $i \leq \frac{1}{2}J$ we choose, from among all the ordered pairs (h, h') of elements in $H_p(\lambda_i\tilde{\delta}_c)$ with $h \neq h'$, that ordered pair (h, h') for which the number $|h'_{1,1}(0) - h_{1,1}(0)|$ is minimal. Then it follows from 4.2.4, and from the sufficiently small hypothesis for $\tilde{\delta}_c$, that we have only the following three possibilities:

(1) h is the identity embedding and $H_p(\lambda_i\tilde{\delta}_c) = \{h, h', h'^{-1}\}$;
(2) $h' \circ h^{-1} \in H_p(\lambda_{i+1}\tilde{\delta}_c)$ and every element of $H_p(\lambda_i\tilde{\delta}_c)$ can be written as a power $(h' \circ h^{-1})^j$ for some $j \in \{-I, -I+1, -I+2, \ldots, I\}$ and for some

$I \in (0, 8\frac{\alpha}{\beta^2}+4)$, where $(h' \circ h^{-1})^j \in H_p(\lambda_{i+1+j}\tilde{\delta}_c)$ for each such j;

(3) $h' \circ h^{-1} \in H_p(\lambda_{i+1}\tilde{\delta}_c)$ and for some $f \in H_p(\lambda_i\tilde{\delta}_c)$ we have that $(h' \circ h^{-1}) \circ f \notin H_p(\lambda_i\tilde{\delta}_c)$ but $(h' \circ h^{-1}) \circ f \in H_p(\lambda_{i+2}\tilde{\delta}_c)$.

Note that if (1) or (2) occurs for some $i \in \{1, 2, \ldots, \frac{1}{2}J\}$ then we can complete the proof of 4.2 by defining κ and g of 4.2 by

$$\kappa = \lambda_J/\lambda_1,$$
$$g = h' \circ h^{-1},$$

and recalling that $\lambda = \lambda_1$ (cf. 4.2.3(a)). If (1) or (2) is never satisfied for any $i \in \{1, 2, \ldots, \frac{1}{2}J\}$, then (3) must hold for all $i \in \{1, 2, \ldots, \frac{1}{2}J\}$; from which it follows that the cardinality of $H_p(\lambda_{\frac{1}{2}J}\tilde{\delta}_c)$ is greater than $(\frac{1}{4}J)-1$. On the other hand it follows from 4.2.4(a)(c), and from the equality $J = 4[50\frac{\alpha}{\beta^2}] + 2^8$, that the cardinality of $H_p(\lambda_{\frac{1}{2}J}\tilde{\delta}_c)$ is less than $\frac{1}{4}J - 1$. This contradiction shows that there is $i \in \{1, 2, \ldots, \frac{1}{2}J\}$ for which (1) or (2) holds.

This completes the proof of Lemma 4.2 modulo the verification of Claim 4.2.2.

Verification of 4.2.2. Property 4.2.2(a) is an immediate consequence of 4.2.1(a)-(c). Property 4.2.2(b) follows after some argument from 1.3,2.2,2.6,4.2.1. (See also [8; Appendix 1] and the Appendix to this paper.)

Towards verifying 4.2.2(c) we first wish to translate the possible conclusions of Lemma 2.1 into statements concerning any map $h \in H_p(\lambda\tilde{\delta}_c)$. We start by noticing that 2.1(c) never occurs when 4.0 is asusmed to hold. By appealing to 1.3,2.1,2.2,2.6, we conclude that each $h \in H_p(\lambda\tilde{\delta}_c)$ must satisfy at least one of the following two properties. (See also [8; Appendix 1] and the Appendix to this paper.) Note that properties 4.2.5(a)(b) below correspond to properties 2.1(a)(b) respectively.

4.2.5. (a) $h_{1,1}$ is a translation satisfying $|h_{1,1}(0)| > \frac{1}{2}\beta^2$.
(b) $h_{1,1}$ is a translation satisfying $|h_{1,1}(0)| < \frac{\mathcal{O}(\lambda)}{\beta^2}\tilde{\delta}_c$.

Thus to complete the verification of 4.2.2(c) it will suffice to show that if $h_{1,1}$ satisfies 4.2.5(b), then h must be the identity embedding. Note that the "sufficiently small" hypothesis placed on λ in 4.2, together with properties 4.2.2(b) and 4.2.5(b), imply that the distance in $\mathbf{R} \times \mathfrak{E}_p$ from z to $h(z)$ satisfies

$$d(z, h(z)) << \tilde{\delta}_c$$

for all $z \in (-\alpha, \alpha) \times \mathfrak{E}_p(\lambda\tilde{\delta}_c)$. On the other hand the restricted map

$$\mathfrak{f}_p : (-\frac{1}{4}\tilde{\delta}_c, \frac{1}{4}\tilde{\delta}_c) \times \mathfrak{E}_p(\frac{1}{4}\tilde{\delta}_c) \to M$$

must be an embedding (cf. 1.3,2.6). So if h were not the identity embedding, then the preceding inequality would lead to a contradiction since we must have that $\mathfrak{f}_p(z) = \mathfrak{f}_p(h(z))$ by 4.1(b).

Proof of Lemma 4.3. We use an argument similar to that used in Case I of the proof given for Proposition 2.5. The following properties replace properties 2.5.1(b)-(e) in that argument. In the following we let $r : U \to B$ denote either

$$\text{id} \times \mathfrak{r}_p : (-4\alpha, 4\alpha) \times \mathfrak{U}_p \to (-4\alpha, 4\alpha) \times \mathfrak{B}_p$$

or

$$\text{id } \times \mathfrak{r}_p \circ h^{-1} : V_h \to (-\alpha, \alpha) \times \mathfrak{B}_p$$

where h, V_h are as in 4.3; and we let $\mathfrak{G}$ denote the foliation of U by the fibers of r. Recall that in 4.3 the product $\mathbf{R} \times \mathfrak{C}_p$ is equipped with the metric pulled back from M along $\mathfrak{f}_p : \mathbf{R} \times \mathfrak{C}_p \to M$.

4.3.1. (a) $K(r : \mathbf{R} \times \mathfrak{C}_p) < \frac{\mathcal{O}(\bar{\varepsilon})}{\tilde{\delta}_c}$.

(b) diameter$(L) < \mathcal{O}(\bar{\varepsilon})\tilde{\delta}_c$ for each $L \in \mathfrak{G}$.

(c) For each $v \in TU_{|r} - 1_{(0 \times \mathfrak{B}_p)}$ which is perpendicular to $T\mathfrak{G}$ we have that $(1 - \mathcal{O}(\bar{\varepsilon}))|v| < |Dr(v)| < (1 + \mathcal{O}(\bar{\varepsilon}))|v|$.

(d) There is $\vartheta > 0$ which depends only on $\alpha, A = \{A_i\}$, dim$M$. For each $v \in T(\mathbf{R} \times \mathfrak{C}_p)_{|U}$ which is perpendicular to TU there is a smooth path $f : [0,1] \to \mathbf{R} \times \mathfrak{C}_p$ which starts and ends at the foot of v and which satisfies

$$\text{length}(f) < \mathcal{O}(\bar{\varepsilon})\tilde{\delta}_c \text{ and } \Theta(v, P_f(v)) > \vartheta .$$

Properties 4.3.1(a)-(d) follow immediately from 1.3(a)(b), as applied to the infranil core of 2.2 used in the construction of $\mathfrak{f}_p : \mathbf{R} \times \mathfrak{C}_p \to M$ in 2.6. (See also [8; Appendix 1] and the Appendix to this paper.)

Note that properties 4.3(d)(e) are an immediate consequence of property 4.3.1 (a)(b). Thus to complete the proof of Lemma 4.3 it remains to deduce properties 4.3(a)-(c) from 4.3.1 by simply repeating the argument used in the proof for 2.5 that deduced 2.5.3 from 2.5.1(b)-(e). Here is an outline of that argument. Let the path $u : [0,1] \to \mathbf{R} \times \mathfrak{C}_p$ in 4.3 be given by $u(t) = (z_1, u_2(t))$, where $z = (z_1, z_2)$ and where $u_2 : [0,1] \to \mathfrak{C}_p$ is the geodesic in the fiber of the orthogonal projection map $\mathfrak{C}_p \to \mathfrak{U}_p$ which connects z_2 to its image in $\mathfrak{U}_p$. We take the images under $\mathfrak{f}_p$ of a small piece of $((-4\alpha, 4\alpha) \times \mathfrak{C}_p, \mathcal{T})$ near $u(1)$ and of a small piece of $(V_h, \mathcal{T}_h)$ near $u(0)$ to get two infranil cores which (by 4.3.1(a)-(d)) satisfy the hypothesis of 1.5. By applying 1.5 and 1.5.2 to these infranil cores we can deduce properties 4.3(a)-(c).

This completes the proof of Lemma 4.3.

5. Construction of infrasolv cores of type I. In this section we complete the verification of Theorems 0.5 and 0.6 by proving the following two results.

5.1. Proposition. *For each $p \in M_1(\alpha, \beta)$ there is an infrasolv core $r_p : U_p \to B_p$ of type I,II, or III which satisfies properties 0.5(a)-(d).*

5.2. Proposition. *The collection of all infrasolv cores $\{r_p : p \in M(\alpha, \beta)\}$ constructed in 2.4 and 5.1 satisfy properties 0.6(a)-(c).*

We shall first carry out the proofs for 5.1 and 5.2 in the special case that $T\mathfrak{F}$ is orientable. Then we will use these special cases of 5.1,5.2 and some additional arguments to prove 5.1,5.2 in the case that $T\mathfrak{F}$ is not orientable.

Proof of 5.1 when $\mathbf{T}\mathfrak{F}$ *is orientable.* In this proof we assume that the integers $n, \tilde{\eta}$ of 1.3 are related by

5.1.1. (a) $30\tilde{\eta} \leq n \leq 60\tilde{\eta}$.

Thus the subscript c for $\tilde{\delta}$ in 1.3 satisfies

(b) $c \in \{30\tilde{\eta} + 1, 30\tilde{\eta} + 2, \ldots, 61\tilde{\eta} - 1\}$.

Let $\tilde{r}_p : \tilde{U}_p \to \tilde{B}_p$ denote the infranil core of 2.2, and let $\mathfrak{r}_p : \mathfrak{U}_p \to \mathfrak{B}_p$ denote the map associated to $\tilde{r}_p$ in 2.6 (Case I). In this proof we assume that the integer k of 3.0 and 3.1 is given by

(c) $k = \dim \mathfrak{B}_p$;

the number $\nu > 1$ of 3.0 and 3.1 is given in 5.1.3(b) below, and is dependent only on $\alpha, \beta, A = \{A_i\}, \dim M$; and the number $\omega > 0$ of 3.0 and 3.1 is given by

(d) $\omega = \frac{a_1}{4(1+\nu)}$.

Note that ω given by 5.1.1(d) is consistent with the restriction placed on ω by 3.0(b).

We also assume that the $\varepsilon_2, \eta, \{\delta_j : 1 \leq j \leq \eta\}$ of 0.5, and the $\tilde{\varepsilon}, \tilde{\eta}, \{\tilde{\delta}_j : 1 \leq j\}, c$ of 1.3, and the $\{d_r : 1 \leq r \leq k+2\}, \{a_y : 1 \leq y \leq (k+4)^{k+4}\}$ of 3.0, are related as follows.

5.1.2. (a) $200(k+4)^{k+5}\tilde{\eta} = \eta$.

(b) $\delta_{j'-l} << \tilde{\delta}_j$ but $\tilde{\varepsilon}\tilde{\delta}_j << \frac{a_y}{a_{y+1}}\delta_{j'-l}$ hold for all $2 \leq j \leq 100\tilde{\eta}$ and all $1 \leq l \leq (k+4)^{k+5}$ and all $1 \leq y \leq (k+4)^{k+4} - 1$, where $j' = j(k+4)^{k+5}$.

(c) $d_r = \delta_{x_r}$ where $x_r = c(k+4)^{k+5} - r(k+4)^{k+4}$, for $r = 1, 2, \ldots, k+1$.

(d) $\frac{a_y}{(a_{y+1})^2} d_{r+1} << \varepsilon_2 \delta_j$ for all $j \in \{x_{r+1}, x_{r+1}+1, x_{r+1}+2, \ldots, x_r - 1\}$ and for all r, y.

(e) Note that $\frac{a_y}{(a_{y+1})^2} << \varepsilon_2$ for all $1 \leq y \leq (k+4)^{k+4} - 1$ follows from (d) above; note also that $\tilde{\varepsilon} << \frac{a_y}{a_{y+1}}$ for all $1 \leq y \leq (k+4)^{k+4} - 1$ follows from (b) above.

We note that properties 5.1.2(a)-(e) are consistent with properties 2.4.1(a)(b), and with the restrictions placed on the $\nu, \{d_r : 1 \leq r \leq k+2\}, \{a_y : 1 \leq y \leq (k+4)^{k+4}\}$ by the hypothesis 3.0. The reader should keep in mind that for the duration of this proof that $\tilde{r}_p$ of 2.2 satisfies 1.3(a)-(c) for $\tilde{\varepsilon}, \tilde{\delta}, \tilde{\eta}, n, c$ as in 5.1.1 and 5.1.2; $\tilde{s}_p, \tilde{t}_p$ are associated to $\tilde{r}_p$ by 1.4; and $\mathfrak{r}_p, \mathfrak{s}_p, \mathfrak{t}_p$ are the maps associated to the $\tilde{r}_p, \tilde{s}_p, \tilde{t}_p$ by 2.6 (Case I).

Since we are assuming that 4.0 holds in this proof we may use all of the facts verified in §4 concerning $H_p(\lambda\tilde{\delta}_c)$. In particular we would like to investigate the map

$$h_2 : \mathfrak{E}_p(\lambda\tilde{\delta}_c) \to \mathfrak{E}_p(\frac{1}{4}\tilde{\delta}_c)$$

of 4.2.2 in more detail. In particular we note that (by Lemma 4.3 and Claim 4.2.2) h_2 "almost" permutes the fibers of $\mathfrak{s}_p$; thus a quotient map $\mathfrak{h}_2 : \mathfrak{B}_p(\lambda\tilde{\delta}_c) \to \mathfrak{B}_p(\frac{1}{4}\tilde{\delta}_c)$ for h_2 should "almost" be defined which satisfies $\mathfrak{s}_p \circ h_2 = \mathfrak{h}_2 \circ \mathfrak{s}_p$. We define $\mathfrak{h}_2$ in 5.1.3(a) below; since h_2 only approximately maps each fiber of $\mathfrak{s}_p|\mathfrak{E}_p(\lambda\tilde{\delta}_c)$ into another fiber of $\mathfrak{s}_p$, we must replace the desired equality $\mathfrak{s}_p \circ h_2 = \mathfrak{h}_2 \circ \mathfrak{s}_p$ by its approximation in 5.1.3(c).

Recall that $\mathfrak{B}_p$ is the open ball of radius $\frac{1}{2}\tilde{\delta}_c$ centered at the origin of some Euclidean space $\mathbf{R}^k$. In what follows we will identify $T(\mathfrak{B}_p)_0$ with $\mathbf{R}^k$ via the Euclidean exponential map, and we shall also identify each $T(\mathfrak{B}_p)_x, x \in \mathfrak{B}_p$, with $T(\mathfrak{B}_p)_0$ via Euclidean parallel translation. Now choose $q \in \mathfrak{r}_p^{-1}(0)$, and let $V \subset T\mathfrak{U}_q$ denote all vectors in $T\mathfrak{U}_q$ which are perpendicular to $T\mathfrak{r}_p^{-1}(0)$. Note (by 1.2.2 and 2.6) that

$D\mathfrak{r}_p : V \to T(\mathfrak{B}_p)_0$ is an isomorphism, whose inverse we denote by

$$L : \mathbf{R}^k \to V.$$

Note (by 1.2.2,2.6, and 4.3,4.2.2) that the composition map

$$\mathbf{R}^k \xrightarrow{L} V \xrightarrow{Dh_2} T(\mathfrak{C}_p(\frac{1}{4}\tilde{\delta}_c))_{h_2(q)} \xrightarrow{D\mathfrak{s}_p} \mathbf{R}^k$$

is also a linear isomorphism. Thus we define an affine isomorphism $\mathfrak{h}_2 : \mathbf{R}^k \to \mathbf{R}^k$ by

5.1.3. (a) $\mathfrak{h}_2(0) = \mathfrak{s}_p \circ h_2(q)$ and $D\mathfrak{h}_2 = D\mathfrak{s}_p \circ Dh_2 \circ L$.

We claim that $\mathfrak{h}_2$ satisfies

(b) $||D\mathfrak{h}_2|| < \nu$ and $|\mathfrak{h}_2(0)| < \nu\lambda\tilde{\delta}_c$, where $\nu > 1$ depends only on $\alpha, \beta, A = \{A_i\}, \dim M$.
(c) $|\mathfrak{h}_2 \circ \mathfrak{s}_p(y) - \mathfrak{s}_p \circ h_2(y)| < \mathcal{O}(\tilde{\varepsilon})\tilde{\delta}_c$ for all $y \in \mathfrak{C}_p(\lambda\tilde{\delta}_c)$.

REMARK. Throughout this section the norm of a linear transformation A is denoted by $||A||$.

Note that second inequality of 5.1.3(b) is an immediate consequence of 5.1.3(a) and of 4.2.2(b) (See also A.2 and A.3 in the Appendix.) Towards verifying the first inequality of 5.1.3(b) we first note (by 1.2.2 and 2.6) that there is a linear isometry $I : \mathbf{R}^k \to V$ such that

$$||I - L|| < \mathcal{O}(\tilde{\varepsilon}),$$

where L is the linear map of 5.1.3(a). Next we note (by 1.2.2,2.6,4.3,4.2.2, and A.2,A.3 in the Appendix) that the linear map $D\mathfrak{s}_p \circ Dh_2|V$ of 5.1.3(a) satisfies

$$||D\mathfrak{s}_p \circ Dh_2|| < \mu,$$

where $\mu > 0$ is a number which depends only on $\alpha, \beta, A = \{A_i\}, \dim M$. Now these last two inequalities, together with 5.1.3(a), imply that the first inequality in 5.1.3(b) is true.

Towards verifying 5.1.3(c) we first note that (by 1.2.2,2.6, and [8;A.1.6]) we have

5.1.4. (a) $K(\mathfrak{s}_p; M) < \frac{\mathcal{O}(\tilde{\varepsilon})}{\tilde{\delta}_c}$.

Next we note that it follows from 5.1.3(a), and from 1.2.2 and 2.6 and 4.2.2, that $\mathfrak{h}_2$ satisfies the following properties. (See also A.2 and A.3 in the Appendix.)

(b) $\mathfrak{h}_2 \circ \mathfrak{s}_p(q) - \mathfrak{s}_p \circ h_2(q) = 0$.
(c) $||D(\mathfrak{h}_2 \circ \mathfrak{s}_p)_{|q} - D(\mathfrak{s}_p \circ h_2)_{|q}|| < \mathcal{O}(\tilde{\varepsilon})$.

Now property 5.1.3(c) follows easily from 5.1.4(a)(b)(c). (See also A.1-A.4 in the Appendix and [8;A.1.1,A.1.7].)

Our plan now is to apply Lemmas 3.1,4.2,4.3, together with 5.1.3, to complete the proof for Proposition 5.1 when $T\mathfrak{F}$ is assumed orientable. First we will use 4.2 and 5.1.3 to define the sets of affine maps $\mathbf{A}_1 \subset \mathbf{A}_2 \subset \ldots \subset \mathbf{A}_{k+2}$ to which we will apply Lemma 3.1. For any given $r \in \{1, 2, \ldots, k+2\}$ we choose λ in 4.2 to satisfy

5.1.5. (a) $\kappa\lambda\tilde{\delta}_c = \frac{1}{4\nu}d_r$

where $\kappa > 1$ is also described in 4.2. Let g, I be as in 4.2 for this choice of λ; and set

(b) $\mathbf{A}_r = \{\mathfrak{h}_2 : h \in H_p(\kappa\lambda\tilde{\delta}_c) \text{ and } h = g^i \text{ with } -I \leq i \leq I\}$.

In order to apply Lemma 3.1 to the collections $\{\mathbf{A}_r\}$ of 5.1.5 the following hypotheses (of 3.0) must hold true for all $r = 1, 2, \ldots, k+2 : |_l\mathbf{A}_r| < \nu$; $|_t\mathbf{A}_r| < d_r$; each collection $\mathbf{A}_r$ is (ω, d_r)-cyclic; the cardinality of $\mathbf{A}_{k+2}$ has an upper bound independent of the $\{a_i\}$ and the $\{d_j\}$. The first two hypotheses are implied by 5.1.3(b) and 5.1.5. The third hypothesis is immediate from 5.1.1(d), 5.1.2, 5.1.3, 5.1.5(b), and from the fact that $I < 8\frac{\alpha}{\beta^2} + 4$ where I comes from 4.2 and 5.1.5(b). The last hypothesis is a consequence of 5.1.5(b) and the inequality $I < 8\frac{\alpha}{\beta^2} + 4$ of 4.2.

Thus we may apply Lemma 3.1 (see also Remark following 3.1) to get integers

$$y \in \{1, 2, \ldots, (k+4)^{k+4} - 1\} \text{ and } z \in \{1, 2, \ldots, k+1\}$$

and a vector subspace $V_p \subset \mathbf{R}^k$ which satisfy 3.1(a)-(d). For each $t \in (0, \frac{1}{2}\tilde{\delta}_c)$ let $V_p(t)$ denote the open ball in V_p of radius t centered at the origin, and set $W_p(t) = \mathfrak{r}_p^{-1}(V_p(t))$. Here we are identifying $V_p(\frac{1}{2}\tilde{\delta}_c)$ with a subspace of $\mathfrak{B}_p$ via the composition map

$$V_p(\frac{1}{2}\tilde{\delta}_c) \subset \mathbf{R}^k = T(\mathfrak{B}_p)_0 \xrightarrow{\exp} \mathfrak{B}_p.$$

Let g denote the map of 4.2 and 5.1.5(b) when $r = z+1$ in 5.1.5, and set

$$X_p = g((-\alpha, \alpha) \times W_p(d_{z+1})).$$

Let $g_{1,1}$ and $\mathfrak{g}_2$ denote the maps associated to g in 4.2.1 and 5.1.3. Note that for each $q \in X_p$ there is a unique geodesic $u_q : [0,1] \to \mathbf{R} \times \mathfrak{E}_p$ with $u_q(0) = q$ which meets $\mathbf{R} \times W_p(\frac{1}{2}\tilde{\delta}_c)$ perpendicularly at $u_q(1)$. Now we have the following crucial claim, from which we can complete the proof of Proposition 5.1.

5.1.6. Claim.
(a) $\beta^3 << g_{1,1}(0) << \alpha_j$, *where* $j = c(k+4)^{k+5} - (z+1)(k+4)^{k+4} + y$.
(b) $|\mathfrak{g}_2(0)| < \mathcal{O}(\frac{a_y}{a_{y+1}})d_{z+1}$
(c) *length* $(u_q) < \mathcal{O}(\frac{a_y}{a_{y+1}})d_{z+1}$.
(d) $\Theta(T(\mathbf{R} \times W_p(\frac{1}{2}\delta_c))_{u_{q(1)}}, P_{u_q}(T(X_p)_{u_q(0)}) < \mathcal{O}(\frac{a_y}{a_{y+1}})$.

We will first use this claim to help us complete the proof of Proposition 5.1. Then we will verify Claim 5.1.6.

Choose a smooth function $f : \mathbf{R} \to \mathbf{R}$ which satisfies:

5.1.7. (a) $f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x \geq \frac{1}{2}\beta^3 \end{cases}$
(b) $0 \leq f'(t) \leq \frac{C}{\beta^3}$ and $|f''(t)| < \frac{C}{\beta^6}$
where C is a positive constant independent of β.

Define a map $F : \mathfrak{X}_p \to \mathbf{R} \times \mathfrak{E}_p$ by

5.1.8. (a) $\mathfrak{X}_p = X_p \cap [(-\frac{1}{4}\beta^3, g_{1,1}(0) - \frac{1}{8}\beta^3) \times \mathfrak{E}_p]$
(b) $F(q_1, q_2) = u_q(1 - f(q_1))$

where $X_p, q \in X_p, u_q$ come from 5.1.6, and $(q_1, q_2) \in \mathbf{R} \times \mathfrak{E}_p$ are the coordinates of q, and f comes from 5.1.7. Note it follows from 5.1.6-5.1.8 (see also 5.1.2) that F is a well defined one-one smooth embedding. We also have two smooth maps p_1, p_2: Image $(F) \to V_p$ defined by

5.1.9. $p_1 = \pi_2 \circ \mathfrak{s}_p \circ \pi_1 | \text{Image}(F)$
$p_2 = \pi_2 \circ \mathfrak{s}_p \circ \pi_1 \circ g^{-1} \circ F^{-1}$

where $\pi_1 : \mathbf{R} \times \mathfrak{E}_p \to \mathfrak{E}_p$ denotes projection onto the second factor and $\pi_2 : \mathfrak{B}_p \to V_p$ denotes orthogonal projection in $\mathbf{R}^k$. We claim that the following relations exist between the maps p_1, p_2, where in these relations we use the following notation: $\tau > 1$ is a number which depends only on $\alpha, \beta, A = \{A_i\}, \dim M$; $x \in \text{Image}(F)$ and $v \in T(\text{Image}(F))_x$; also $w \in T(\text{Image}(F))_x$ is any vector which is perpendicular to the fibers of the composite projection $\mathbf{R} \times \mathfrak{E}_p \xrightarrow{\pi_1} \mathfrak{E}_p \xrightarrow{\mathfrak{s}_p} \mathfrak{B}_p$.

5.1.10. (a) $|p_1(x) - p_2(x)| < \mathcal{O}(\frac{a_y}{a_{y+1}}) d_{z+1}$.
(b) $|Dp_1(v) - Dp_2(v)| < \mathcal{O}(\frac{a_y}{a_{y+1}}) |v|$.
(c) $\frac{1}{\tau}|w| < |Dp_i(w)| < \tau |w|$.

These properties can be deduced from 5.1.5-5.1.9, and from 3.1(a)(d) as applied to $\mathfrak{g}_2$ and V_p; note that 3.1(d) as applied to $\mathfrak{g}_2$ and V_p is just property 5.1.6(b), with $q = 0$ in 3.1(d) (cf. Remark following 3.1). (See also 4.2.2, 5.1.2, and 5.1.3 as applied to g.)

We define a third smooth map $p_3 : \text{Image}(F) \to V_p$ by

5.1.11. $p_3(x) = f(x_1) p_2(x) + (1 - f(x_1)) p_1(x)$

for each $x \in \text{Image}(F)$, where $x = (x_1, x_2)$ are the components of x corresponding to the first and second factors of $\mathbf{R} \times \mathfrak{E}_p$.

Finally we can define j and $r_p : U_p \to B_p$ of 0.5 as follows.

5.1.12. (a) $j = c(k+4)^{k+5} - (z+1)(k+4)^{k+4} + y$.
(b) $B_p = V_p(\delta_j)$.
(c) $U_p = \mathfrak{f}_p(p_3^{-1}(B_p))$, where $\mathfrak{f}_p$ comes from 2.6.
(d) For each $x \in U_p$, we set $r_p(x) = p_3(x')$ for any $x' \in \mathfrak{f}_p^{-1}(x)$.

Note it follows from 5.1.1-5.1.12 (see in particular 5.1.9-5.1.11) that $r_p : U_p \to B_p$ is a well defined smooth fiber bundle projection each fiber of which is diffeomorphic to a mapping torus for a self diffeomorphism of a fiber of $\mathfrak{r}_p : \mathfrak{U}_p \to \mathfrak{B}_p$ of 2.6. (See also 1.3 and 2.6.) Thus each fiber of r_p is an aspherical manifold with infrasolv fundamental group, as required in 0.5. In more detail, we set

$$C = g^{-1}(p_3^{-1}(B_p)) \cap p_3^{-1}(B_p)$$

and note that $g^2(C) \cap p_3^{-1}(B_p) = \phi$ and that C is a tubular neighborhood for

$$\text{Image } (F) \cap (-\frac{1}{6}\beta^3 \times \mathfrak{E}_p)$$

in Image(F) (cf. 5.1.7, 5.1.8). Thus the quotient space $p_3^{-1}(B_p)/\sim$ (where $x \sim y$ for $x, y \in p_3^{-1}(B_p)$ if $x \in C$ and $g(x) = y$) is a mapping torus for a self diffeomorphism

$$\text{Image } (F) \cap (-\frac{1}{6}\beta^3 \times \mathfrak{E}_p) \to \text{Image}(F) \cap (-\frac{1}{6}\beta^3 \times \mathfrak{E}_p).$$

Now by the construction of p_3 (cf. 5.1.9-5.1.11) it follows that $p_3 \circ g(x) = p_3(x)$ holds for all $x \in C$; thus p_3 induces a map

$$p_3/\sim : (p_3^{-1}(B_p)/\sim) \to B_p.$$

Note that $p_3 : \text{Image}(F) \to B_p$ is a fiber bundle with each fiber $p_3^{-1}(x), x \in B_p$, diffeomorphic to $(-1,1) \times \mathfrak{r}_p^{-1}(x)$ (cf. 5.1.9-5.1.11); note also that $g(C \cap p_3^{-1}(x)) \subset p_3^{-1}(x)$ and that $C \cap p_3^{-1}(x)$ is a tubular neighborhood for $p_3^{-1}(x) \cap (-\frac{1}{6}\beta^3 \times \mathfrak{E}_p)$ in $p_3^{-1}(x)$. Thus each fiber $(p_3/\sim)^{-1}(x)$ of the map $p_3/\sim$ is diffeomorphic to the mapping torus for a self diffeomorphism $\mathfrak{r}_p^{-1}(x) \to \mathfrak{r}_p^{-1}(x)$. Finally, we let

$$\mathfrak{f}_p/\sim : (p_3^{-1}(B_p)/\sim) \to M$$

denote the map induced by $\mathfrak{f}_p|p_3^{-1}(B_p)$. Note that by 5.1.2(c)(d), 5.1.5, 5.1.12(a)(d) the map $\mathfrak{f}_p/\sim$ is a one-one smooth immersion and $r_p = (p_3/\sim) \circ (\mathfrak{f}_p/\sim)^{-1}$; thus each fiber of r_p is also a mapping torus for a self diffeomorphism $\mathfrak{r}_p^{-1}(x) \to \mathfrak{r}_p^{-1}(x)$ as claimed.

Note that r_p satisfies properties 0.2(a)-(b). Note also that property 0.2(c) for r_p can be deduced in part (that is locally) from the following curvature property for r_p.

5.1.13. Claim. $K(r_p; M) << \frac{\varepsilon_2}{\delta_j}$.

We will verify Claim 5.1.13 (along with Claim 5.1.6) at the end of this proof. To deduce that 0.2(c) is satisfied globally we need (in addition to 5.1.13) to appeal to the following properties: to 1.2.1(c) and 1.3, as applied to $\tilde{r}_p$; to 2.6 Case I, for the relation between $\mathfrak{r}_p$ and $\tilde{r}_p$; and to 4.2, 5.1.2(b)(d)(e), 5.1.5, 5.1.6(a)(b), 5.1.12(a), from which we deduce that g of 5.1.6 satisfies$g \in H_p(\varepsilon_2\delta_j)$ and $H_p(\varepsilon_2\delta_j) = H_p(\frac{\delta_{j-1}}{\kappa})$, where g is as in 5.1.5(b) and κ comes from 4.2 and 5.1.5(b). (See also 5.1.6-5.1.12.)

Let $s_p : E_p \to B_p$ and $t_p : E_p \to \mathbf{R}$ be the thickening for the infrasolv core r_p (cf. 0.4). We leave the deduction of properties 0.5(a)-(d) for the thickened infrasolv core (s_p, t_p) as an exercise for the reader (cf. 5.1.1-5.1.13 and 1.3 and 2.6).

This completes the verification of Proposition 5.1 in the case that $T\mathfrak{F}$ is orientable, modulo the proof of Claims 5.1.6 and 5.1.13.

Verification of Claim 5.1.6. First we verify 5.1.6(a). If $g_{1,1}(0) < 0$, then we replace g by g^{-1}. Then we get $\beta^3 << g_{1,1}(0)$ from 4.2.2(c), assuming that $\beta << \min\{1, \alpha\}$ and that $g \neq \text{id}$. There is no loss in assuming that $\beta << \min\{1, \alpha\}$ in 0.5, since $M(\alpha, \beta') \subset M(\alpha, \beta)$ holds for all $\beta' \geq \beta$. On the other hand if $g = \text{id}$, then $H_p(\lambda\tilde{\delta}_c) = \{\text{id}\}$; which would contradict the fact that $p \in M_1(\alpha, \beta)$ (cf. 2.1, 2.3, 5.1.1, 5.1.2, and recall that $T\mathfrak{F}$ is orientable).

To get the second half of the inequality in 5.1.6(a), $g_{1,1}(0) << \alpha_j$, we first choose $h \in H_p(d_z)$ such that $0 < h_{1,1}(0) << \alpha_j$ (cf. 2.1, 2.3, 5.1.1, 5.1.2, 5.1.12(a), and use the properties $p \in M_1(\alpha, \beta)$ and that $T\mathfrak{F}$ is orientable). Use 4.2 and 5.1.5 to write $h = g^i$ for some $i \in \{-I, -I+1, \ldots, I\}$. By applying 4.2.2(a)(b) to g we conclude that $g_{1,1}(0) << \alpha_j / i$.

Since $\mathfrak{g}_2$ is the (ω, d_{z+1})-generator for $\mathbf{A}_{z+1}$ we may apply 3.1(d) to $\mathfrak{g}_2$ (with $q = 0$ as in Remark following 3.1) to conclude that 5.1.6(b) holds.

Finally we verify 5.1.6(c)(d) by applying Lemmas 3.1 and 4.3. Since $\mathfrak{g}_2 \in A_{z+1}$ (cf. 5.1.5(b)) we may apply Lemma 3.1 to $\mathfrak{g}_2$ and V_p to conclude that 3.1(a)-(d) hold. Now recall that $g \in H_p(\bar{\lambda}\tilde{\delta}_c)$, where $\bar{\lambda} = \frac{d_{z+1}}{4\nu\tilde{\delta}_c}$ (cf. 5.1.5(a)); thus we may apply Lemma 4.3 to conclude that V_g and $\mathcal{T}_g$ satisfy properties 4.3(a)-(e) when $\bar{\lambda}$ replaces λ in 4.3. Now 5.1.6(c)(d) are a consequence of 3.1(a)-(d), 4.3(a)-(d). (See also 5.1.2 and 5.1.4.)

Verification of Claim 5.1.13. First we note it follows from 1.3 and 2.6 that

5.1.13.1. (a) $K(\mathfrak{r}_p \circ \pi_1 | (-3\alpha, 3\alpha) \times \mathfrak{U}_p; \mathbf{R} \times \mathfrak{E}_p) < \frac{\mathcal{O}(\bar{\varepsilon})}{\tilde{\delta}_c}$

where $\pi_1 : \mathbf{R} \times \mathfrak{E}_p \to \mathfrak{E}_p$ is projection onto the second factor. (See also the Appendix to this paper.) Note that it follows from 5.1.13.1(a), and from the relation between $\mathfrak{s}_p$ and $\mathfrak{r}_p$ given in 2.6 Case I, that

5.1.13.1. (b) $K(\mathfrak{s}_p \circ \pi_1 | (-3\alpha, 3\alpha) \times \mathfrak{E}_p(\frac{1}{4}\tilde{\delta}_c); \mathbf{R} \times \mathfrak{E}_p) < \frac{\mathcal{O}(\bar{\varepsilon})}{\tilde{\delta}_c}$.

Since $W_p(d_{z+1}) = \mathfrak{r}_p^{-1}(V_p(d_{z+1}))$ it follows from 5.1.13.1(a) that

5.1.13.2. $K((-3\alpha, 3\alpha) \times W_p(d_{z+1}); \mathbf{R} \times \mathfrak{E}_p) < \frac{\mathcal{O}(\bar{\varepsilon})}{\tilde{\delta}_c}$

where for any submanifold $N \subset \mathbf{R} \times \mathfrak{E}_p$ we denote by $K(N; \mathbf{R} \times \mathfrak{E}_p)$ the curvature $K(\tau; \mathbf{R} \times \mathfrak{E}_p)$ of the constant map $\tau : N \to \{1\}$, as described in 1.1. Since

$$X_p = g((-\alpha, \alpha) \times W_p(d_{z+1}))$$

it follows from 5.1.13.2 that

5.1.13.3. $K(X_p; \mathbf{R} \times \mathfrak{E}_p) < \frac{\mathcal{O}(\bar{\varepsilon})}{\tilde{\delta}_c}$.

Now it follows from 5.1.2, 5.1.6-5.1.8, and from 5.1.13.2 and 5.1.13.3, that

5.1.13.4. $K(\text{Image}(F); \mathbf{R} \times \mathfrak{E}_p) < \mathcal{O}(\frac{a_y}{a_{y+1}}) \frac{d_{z+1}}{\beta^6} + \frac{\mathcal{O}(\bar{\varepsilon})}{\tilde{\delta}_c}$.

We note that property 5.1.13.4 is the first of the two properties (cf. 1.1(a)(b)) which define the inequality

5.1.13.5. $K(p_i; \mathbf{R} \times \mathfrak{E}_p) < \mathcal{O}(\frac{a_y}{a_{y+1}}) \frac{d_{z+1}}{\beta^6} + \frac{\mathcal{O}(\bar{\varepsilon})}{\tilde{\delta}_c}$

for $i = 1, 2, 3$. The second of the two properties which defines the inequalities of 5.1.13.5 (cf. 1.1(b)) is deduced for $i = i, 2$ from 5.1.13.1(b) and 5.1.13.4 and from 5.1.9; and the second of these properties is deduced for $i = 3$ from 5.1.7,5.1.11, and from 5.1.13.5 where $i = 1, 2$. Finally we note that Claim 5.1.13 is a consequence of 5.1.2, 5.1.13.5 (for $i = 3$), and of the facts that $U_p \subset M$ is locally isometric to Image$(F) \subset \mathbf{R} \times \mathfrak{E}_p$ via $\mathfrak{f}_p$ and that $p_3 = r_p \circ (\mathfrak{f}_p | \text{Image}(F))$.

This completes the verification of Claim 5.1.13.

Proof of 5.2 when $\mathbf{T}\mathfrak{F}$ *is orientable.* Let $r_{p_i} : U_{p_i} \to B_{p_i}, i = 1, 2$, denote two infrasolv cores of type I of radius δ_j as constructed in the proof of 5.1 when $T\mathfrak{F}$ is orientable (where $p_1, p_2 \in M_1(\alpha, \beta)$); and let $(s_{p_i}, t_{p_i}), i = 1, 2$, be the thickened infrasolv cores associated to the $r_{p_i}, i = 1, 2$, as in 0.4. To complete the proof of 5.2 we must verify properties 0.6(a)-(c) for (s_{p_1}, t_{p_1}) and (s_{p_2}, t_{p_2}) when $E_{p_1} \cap E_{p_2} \neq \phi$.

Towards this end we first note that there is a number $0 < \tau < 1$, which depends only on $\alpha, A = \{A_i\}, \dim M$, and there is a smooth embedding

$$e : (-4\alpha, 4\alpha) \times \mathfrak{E}_{p_2}(\tau\tilde{\delta}_c) \to \mathbf{R} \times \mathfrak{E}_{p_1}$$

which satisfies the following properties. (To verify 5.2.1(b) we use that $E_{p_1} \cap E_{p_2} \neq \phi$, and refer to the construction of r_{p_1}, r_{p_2} in 5.1.1-5.1.12.)

5.2.1. (a) $\mathfrak{f}_{p_1} \circ e = \mathfrak{f}_{p_2} | (-4\alpha, 4\alpha) \times \mathfrak{E}_{p_2}(\tau\tilde{\delta}_c)$.
(b) $e(0 \times \mathfrak{E}_{p_2}(\delta_j)) \subset (0, 5\alpha_j) \times \mathfrak{E}_{p_1}(\frac{1}{\tau}\delta_j)$.

We define a subset $W \subset \mathfrak{E}_{p_1}$ by requiring that

$$(-\alpha,\alpha)\times W = e((-4\alpha,4\alpha)\times W_{p_2}(\delta_j))\bigcap(-\alpha,\alpha)\times\mathfrak{E}_{p_1}$$

where the sets $W_{p_i}(t), i=1,2$, are defined prior to 5.1.6. Let

$$\mathfrak{r}_{p_i}(t): W_{p_i}(t)\to V_{p_i}(t)$$

$i=1,2$, denote the restricted map $\mathfrak{r}_{p_i}|W_{p_i}(t)$, where the $V_{p_i}(t)$ are also defined just prior to 5.1.6. Let $\mathfrak{H}_{p_i}(t), i=1,2$, denote the foliation for $W_{p_i}(t)$ whose leaves are the fibers of the $\mathfrak{r}_{p_i}(t)$. We let $\mathfrak{H}$ denote the foliation for W whose leaves $L\in\mathfrak{H}$ are defined by the equations

$$(-\alpha,\alpha)\times L = e((-4\alpha,4\alpha)\times L')\bigcap(-\alpha,\alpha)\times\mathfrak{E}_{p_1}$$

where L' is a leaf of $\mathfrak{H}_{p_2}$. In the following claim the geoemtric measurements are all made with respect to the metrics on $\mathfrak{E}_{p_1}$ and $\mathfrak{E}_{p_2}$ that they inherit as a subsets of M (cf. 2.6).

5.2.2. Claim. *For each $x\in W$ there is a smooth path $f_x:[0,1]\to\mathfrak{E}_{p_1}$ satisfying the following properties*

(a) $f_x(0)=x, f_x(1)\in W_{p_1}(\tau\tilde{\delta}_c)$, *and length* $(f_x) << \varepsilon_2\delta_j$.

(b) $\Theta(T(W_{p_1}(\tau\tilde{\delta}_c))_{f_x(1)}, P_{f_x}(T(W)_{f_x(0)})) << \varepsilon_2$.

(c) $\Theta(T(\mathfrak{H}_{p_1}(\tau\tilde{\delta}_c))_{f_x(1)}, P_{f_x}(T(\mathfrak{H})_{f_x(0)})) << \varepsilon_2$.

(d) $K(\mathfrak{r}_{p_i}(\delta_j);\mathfrak{E}_{p_i}) << \frac{\varepsilon_2}{\delta_j}$ *for* $i=1,2$.

By examining the details of the preceding proof (cf. 5.1.6-5.1.12) and reviewing the relations in 5.1.1 and 5.1.2, the reader can see that properties 0.6(a)-(c) follow directly from Claim 5.2.2. Note that, in 0.6(c), $\mathbf{A}_2$ may be defined in a manner similar to that given in the proof of Proposition 2.5 Case I. (See the two paragraphs proceeding Claim 2.5.3.)

Thus to complete the proof for Proposition 5.2 (when $T\mathfrak{F}$ is orientable and $p_1,p_2\in M_1(\alpha,\beta)$) it will suffice to verify Claim 5.2.2.

Verification of Claim 5.2.2.

First we note that 5.2.2(d) is a consequence of Claim 5.1.13 and of 5.1.1 and 5.1.2. (See also 5.1.6-5.1.12.)

We shall employ Theorem 1.5 and Lemma 3.2 in carrying out the verification of 5.2.2(a)-(c).

We introduce the following notation in anticipation of applying Theorem 1.5. For each $t\in(0,\frac{1}{2}\tilde{\delta}_c)$ let $\mathfrak{G}_{p_i}(t), i=1,2$, denote the foliations for $\mathfrak{U}_{p_i}(t), i=1,2$, by the fibers of $\mathfrak{r}_{p_i}|\mathfrak{U}_{p_i}(t), i=1,2$. Let $\mathfrak{U}\subset\mathfrak{E}_{p_1}$ denote the subset defined by the equation

$$(-\alpha,\alpha)\times\mathfrak{U} = e((-4\alpha,4\alpha)\times\mathfrak{U}_{p_2}(\tau\tilde{\delta}_c))\bigcap(-\alpha,\alpha)\times\mathfrak{E}_{p_1}$$

and let $\mathfrak{G}$ denote the foliation for $\mathfrak{U}$ whose leaves $L\in\mathfrak{G}$ are defined by the equations

$$(-\alpha,\alpha)\times L = e((-4\alpha,4\alpha)\times L')\bigcap(-\alpha,\alpha)\times\mathfrak{E}_{p_1}$$

where $L'\in\mathfrak{G}_{p_2}(\tau\tilde{\delta}_c)$. Now for any positive number t sufficiently small (i.e. for $t\le\tau'\tilde{\delta}_c$ where $\tau'\in(0,\tau)$ depends only on $\alpha,\tau,A=\{A_i\},\dim M$) we can define a fiber bundle projection $\mathfrak{r}:\mathfrak{U}(t)\to\mathfrak{B}(t)$ as follows. Choose $q\in\mathfrak{U}$ such that $0\times q\in e((-4\alpha,4\alpha)\times\mathfrak{r}_{p_2}^{-1}(0))$, and let $\mathfrak{B}(t)$ denote the set of all vectors $v\in T(\mathfrak{U})_q$ which are perpendicular to $T(\mathfrak{G})_q$ and satisfy $|v|<t$. Let $B(t)$ denote the image of $\mathfrak{B}(t)$ under the exponential

map exp:$\mathfrak{B}(t) \to M$. Recall that $\mathfrak{E}_{p_1} \subset M$. Note that the orthogonal projection $\rho_t : B(t) \to \mathfrak{U}$ is a well defined embedding with image denoted by $T(t)$. Let $\mathfrak{U}(t)$ denote the union of all leaves of $\mathfrak{G}$ which intersect with $T(t)$; and set $\mathfrak{G}(t) = \mathfrak{G}|\mathfrak{U}(t)$. Now define $\mathfrak{r} : \mathfrak{U}(t) \to \mathfrak{B}(t)$ to send each leaf $L \in \mathfrak{G}(t)$ to $\exp^{-1}(\rho_t^{-1}(L \cap T(t)))$. Note the following properties can be deduced from 1.3,2.6 (as applied to r_{p_1} and $\mathfrak{r}_{p_2}$) and from the preceding construction of $\mathfrak{r}$.

5.2.3. For $t = \tau'\tilde{\delta}_c$, each of the bundle maps $\mathfrak{r} : \mathfrak{U}(t) \to \mathfrak{B}(t)$ and $\mathfrak{r}_{p_1} : \mathfrak{U}_{p_1}(t) \to \mathfrak{B}_{p_1}(t)$ is $(\mathcal{O}(\tilde{\varepsilon},)\tilde{\vartheta}')$-rigid (cf. 1.2.2), where $\tilde{\vartheta}' > 0$ depends only on $\tilde{\vartheta}$ of 1.2.2 and on $\alpha, \beta, A = \{A_i\}, \dim M$.

For each $t \in (0, \tau'\tilde{\delta}_c]$ we let $\mathfrak{E}(t)$ denote the tubular neighborhood of radius t for $\mathfrak{U}(t)$ in $\mathfrak{E}_{p_1}$; $\mathfrak{E}_{p_1}(t)$ has been defined in 2.6 as the tubular neighborhood of radius t for $\mathfrak{U}_{p_1}(t)$ in $\mathfrak{E}_{p_1}$. It follows from the hypothesis $E_{p_1} \cap E_{p_2} \neq \phi$ of this proof that

5.2.4. (a) $\mathfrak{E}(t) \subset \mathfrak{E}_{p_1}$ is well defined for all $t \in [0, \tau'\tilde{\delta}_c]$.
(b) $\mathfrak{E}(\frac{1}{9}\tau'\tilde{\delta}_c) \cap \mathfrak{E}_{p_1}(\frac{1}{9}\tau'\tilde{\delta}_c) \neq \phi$.

Now we want to use Theorem 1.5 and Remark 1.5.2 (as applied to $\mathfrak{r} : \mathfrak{U}(\tau'\tilde{\delta}_c) \to \mathfrak{B}(\tau'\tilde{\delta}_c)$ and $\mathfrak{r}_{p_1} : \mathfrak{U}_{p_1}(\tau'\tilde{\delta}_c) \to \mathfrak{B}_{p_1}(\tau'\tilde{\delta}_c)$), in conjunction with 5.2.3 and 5.2.4, to conclude that the following properties hold.

5.2.5. For each $x \in \mathfrak{U}(\tau'\tilde{\delta}_c)$ there is a smooth path $f_x : [0,1] \to \mathfrak{E}_{p_1}$ which satisfies:
(a) $f_x(0) = x, f_x(1) \in \mathfrak{U}_{p_1}$, and length $(f_x) < \mathcal{O}(\tilde{\varepsilon})\tilde{\delta}_c$.
(b) $\Theta(P_{f_x}(T(\mathfrak{U}(\tau'\tilde{\delta}_c))_{f_x(0)}), T(\mathfrak{U}_{p_1})_{f_x(1)}) < \mathcal{O}(\tilde{\varepsilon})$.
(c) $\Theta(P_{f_x}(T(\mathfrak{G}(\tau'\tilde{\delta}_c))_{f_x(0)}), T(\mathfrak{G}_{p_1})_{f_x(1)}) < \mathcal{O}(\tilde{\varepsilon})$.

[Note that $\mathfrak{E}_{p_1}$ may not be an A-regular Riemannian manifold, nor is it complete, with respect to the metric it inherits from (M, g). Even though A-regularity and completeness are both implicit assumptions of 1.5, we can nevertheless still apply 1.5 and 1.5.2 to yield 5.2.5. One way to see this is to "thicken" the relevant maps

$$\mathfrak{r} : \mathfrak{U}(t) \to \mathfrak{B}(t) \text{ and } \mathfrak{r}_{p_1} : \mathfrak{U}_{p_1}(t) \to \mathfrak{B}_{p_1}(t)$$

by flowing $\mathfrak{U}(t), \mathfrak{U}_{p_1}(t)$ in the direction of the foliation $\mathfrak{F}$ of M over the time interval $(-t, t)$ to get thickenings $\bar{\mathfrak{U}}(t), \bar{\mathfrak{U}}_{p_1}(t)$ for $\mathfrak{U}(t), \mathfrak{U}_{p_1}(t)$ and by defining thickenings $\bar{\mathfrak{B}}(t), \bar{\mathfrak{B}}_{p_1}(t)$ for $\mathfrak{B}(t), \mathfrak{B}_{p_1}(t)$ by

$$\bar{\mathfrak{B}}(t) = (-t, t) \times \mathfrak{B}(t) \text{ and } \bar{\mathfrak{B}}_{p_1}(t) = (-t, t) \times \mathfrak{B}_{p_1(t)}.$$

Now a thickening $\bar{\mathfrak{r}}$ for $\mathfrak{r}$ is defined by $\bar{\mathfrak{r}}(\psi(s, q)) = (s, \mathfrak{r}(q))$ for all $s \in (-t, t)$ and all $q \in \mathfrak{U}(t)$, where $\psi : \mathbf{R} \times M \to M$ denotes the unit speed flow in the direction of the leaves of $\mathfrak{F}$. And a thickening $\bar{\mathfrak{r}}_{p_1}$ for $\mathfrak{r}_{p_1}$ is defined in a similar manner. Finally, we define infranil cores $\mathfrak{r}' : \mathfrak{U}'(t) \to \mathfrak{B}'(t)$ and $\mathfrak{r}'_{p_1} : \mathfrak{U}'_{p_1}(t) \to \mathfrak{B}'_{p_1}(t)$ as follows: Let $\mathfrak{B}'(t)$ and $\mathfrak{B}'_{p_1}(t)$ denote the open balls of radius t centered at the origins of $\bar{\mathfrak{B}}(t)$ and $\bar{\mathfrak{B}}_{p_1}(t)$; set $\mathfrak{U}'(t) = (\bar{\mathfrak{r}})^{-1}(\mathfrak{B}'(t))$ and set $\mathfrak{U}'_{p_1}(t) = (\bar{\mathfrak{r}}_{p_1})^{-1}(\mathfrak{B}'_{p_1}(t))$; finally, we set $\mathfrak{r}' = \bar{\mathfrak{r}}|\mathfrak{U}'(t)$ and set $\mathfrak{r}'_{p_1} = \bar{\mathfrak{r}}_{p_1}|\mathfrak{U}'_{p_1}(t)$. Note that $\mathfrak{r}'$ and $\mathfrak{r}'_{p_1}$ are infranil cores in (M, g) which have radius t and which are both $(\mathcal{O}(\tilde{\varepsilon}), \tilde{\vartheta}')$-rigid when we set $t = \tau'\tilde{\delta}_c$ (cf. 5.2.3). Moreover, we have that $\mathfrak{E}'(\frac{1}{9}\tau'\tilde{\delta}_c) \cap \mathfrak{E}'_{p_1}(\frac{1}{9}\tau'\tilde{\delta}_c) \neq \phi$ (cf. 5.2.4(b)). Thus we may apply Theorem 1.5 to $\mathfrak{r}' : \mathfrak{U}'(\tau'\tilde{\delta}_c) \to \mathfrak{B}'(\tau'\tilde{\delta}_c)$ and $\mathfrak{r}'_{p_1} : \mathfrak{U}'_{p_1}(\tau'\tilde{\delta}_c) \to \mathfrak{B}'_{p_1}(\tau'\tilde{\delta}_c)$ to derive 5.2.5(a)-(c).]

Now we use Lemma 3.2, in conjunction with 5.2.3-5.2.5 (and 5.1.1,5.1.2), to complete the verification of Claim 5.2.2(a)-(c). We define V_1, V_2, q_1, q_2 of 3.2 as follows. For the V_1 of 3.2 we take V_{p_1} (which was defined just prior to 5.1.6), and we define $q_1 = 0$. We get the V_2, q_2 of 3.2 as follows. Let $q \in \mathfrak{U}$ be as in the construction of $\mathfrak{r}$ in the preceding few paragraphs and set $q_2 = \mathfrak{s}_{p_1}(q)$. Note that $q \in W$, where W is defined just prior to 5.2.2; let P denote the subplane of $T(W)_q$ which is perpendicular to $T(\mathfrak{H})$ ($\mathfrak{H}$ is also defined just prior to 5.2.2), and let V_2 denote the Euclidean parallel translation to the origin of the image of P under the derivative map $D\mathfrak{s}_{p_1} : T(\mathfrak{E}_{p_1})_q \to T(\mathbf{R}^k)_{q_2}$.

We are just about ready to apply Lemma 3.2 to complete the verification of Claim 5.2.2(a)-(d). In this application of 3.2, we let the numbers $\{a_i\}, \{d_i\}$ be as in 3.0-3.2 and 5.1.2; and we let the collection of affine maps $\{\mathbf{A}_i\}$ (of 3.0-3.2) be the same as used in the preceding proof (cf. 5.1.5) when in that proof we set $p = p_1$. We note that the hypotheses of 3.2 do not necessarily hold for both (V_1, q_1) and (V_2, q_2) with respect to the numbers $a_y, a_{y+1}, d_z, d_{z+1}$ and the collections $\mathbf{A}_z, \mathbf{A}_{z+1}$ (although these hypotheses do hold for (V_1, q_1)). However we do have the following claim.

5.2.6. Claim. *There is a number $\tau'' \in (0,1)$ which depends only on $\alpha, \beta, A = \{A_i\}, dim M$. All the hypotheses of 3.2 are satisfied for the V_1, q_1, V_2, q_2 just described with respect to $a'_y, a'_{y+1}, d'_z, d'_{z+1}, \mathbf{A}'_z, \mathbf{A}'_{z+1}$ given as follows:*

$$\begin{gathered} a'_y = \frac{1}{\tau''} a_y \text{ and } a'_{y+1} = \tau'' a_{y+1}; \\ d'_z = \frac{1}{\tau''} d_z \text{ and } d'_{z+1} = \frac{1}{\tau''} d_{z+1}; \\ \mathbf{A}'_i = \mathbf{A}_i \text{ for } i = z, z+1. \end{gathered}$$

The verification for Claim 5.2.6 for (V_1, q_1) is immediate, since (as we have just pointed out) the pair (V_1, q_1) satisfies the hypothesis of 3.2 for the numbers $a_y, a_{y+1}, d_z, d_{z+1}$ and for the collections of affine maps $\mathbf{A}_z, \mathbf{A}_{z+1}$. The verification of Claim 5.2.6 for (V_2, q_2), which appeals to 5.1.1-5.1.6 and to 5.2.3-5.2.5, is left as an exercise for the reader. Thus we may apply 3.2 to conclude that V_1, q_1, V_2, q_2 are related by

5.2.7. (a) $\Theta(V_1, V_2) < \mathcal{O}(\frac{a'_y}{a'_{y+1}})$,

(b) $|q_1 + v_1 - q_2| < \mathcal{O}(\frac{a'_y}{(a'_{y+1})^2}) d'_{z+1}$ for some $v_1 \in V_1$.

Now we can deduce Claim 5.2.2(a)-(c) from 5.2.5 and 5.2.7. (See also 5.1.1-5.1.6.) This completes the proof for Proposition 5.2 when $T\mathfrak{F}$ is assumed to be orientable.

Proof of 5.1 when $\mathbf{T}\mathfrak{F}$ *is not orientable.* We begin by introducing the following two subsets of $M_1(\alpha, \beta)$.

5.1.14. $\mathbf{M_{1,i}(\alpha, \beta) \subset M_1(\alpha, \beta), i = 1, 2}$. Note (by 2.3) we have that for any $p \in M_1(\alpha, \beta)$ there are points $(s, x), (s', x') \in (-\alpha_z, \alpha_z) \times B(p, \tilde{\delta}_z)$ which satisfy 2.1(a), where $z = \frac{3}{5}\eta - 2$. A point $p \in M_1(\alpha, \beta)$ is in $M_{1,1}(\alpha, \beta)$ iff there are no points

$$(s, x), (s', x') \in (-\alpha_{z'}, \alpha_{z'}) \times B(p, \tilde{\delta}_{z'})$$

which satisfy 2.1(c), where $z' = 45(k+4)^{k+5}\tilde{\eta}$. (Note by 5.1.2(a) we have that $z' < \frac{1}{2}\eta - 15$.) Set $M_{1,2}(\alpha, \beta) = M_1(\alpha, \beta) - M_{1,1}(\alpha, \beta)$.

Now we divide the proof into two cases $p \in M_{1,1}(\alpha, \beta)$ or $p \in M_{1,2}(\alpha, \beta)$.

Case I: $p \in M_{1,1}(\alpha, \beta)$.

In this case we define the integer n of 1.3 by

5.1.15. (a) $n = 58\tilde{\eta}$.

Thus the integer c of 1.3 satisfies

$$\text{(b)}\ c \in \{58\tilde{\eta} + 1, 58\tilde{\eta} + 2, \ldots, 59\tilde{\eta} - 1\}.$$

Note that 5.1.15 is consistent with 5.1.1(a)(b). We let $\pi : \hat{M} \to M$ denote the two fold covering for M such that the corresponding two fold covering $\hat{\mathfrak{F}}$ for $\mathfrak{F}$ has an orientable tangent bundle $T\hat{\mathfrak{F}}$; and choose $\hat{p} \in \hat{M}$ such that $\pi(\hat{p}) = p$. Let $\hat{g}$ denote the pull back of the metric g along π, and let $\phi : \mathbb{Z}_2 \times \hat{M} \to \hat{M}$ denote the group action by the covering transformations for $\pi : \hat{M} \to M$. Since $T\hat{\mathfrak{F}}$ is orientable we may apply to $(\hat{p}, \hat{M}, \hat{g})$ the special case already proven of Proposition 5.1 to get an infrasolv core $r_{\hat{p}} : U_{\hat{p}} \to B_{\hat{p}}$ for $(\hat{M}, \hat{g})$ of type I and of radius δ_j, where j is given by 5.1.12(a) and where c of 5.1.12(a) comes from 5.1.15. (The more specific stipulation of n, c in 5.1.15(a)(b) now replaces their less specific description in 5.1.1(a)(b).) Note that it follows from 5.1.14,5.1.15 (see also 5.1.1-5.1.12) that $\phi(1, E_{\hat{p}}) \cap E_{\hat{p}} = \phi$. Thus we may define the desired infrasolv core $r_p : U_p \to B_p$ by

$$\begin{aligned} &B_p = B_{\hat{p}}, \\ &U_p = \pi(U_{\hat{p}}), \\ &r_p = r_{\hat{p}} \circ (\pi | U_{\hat{p}})^{-1}. \end{aligned}$$

We note that r_p is an infrasolv core for (M, g) whose associated thickening (s_p, t_p) satisfies the conclusions of 0.5.

Case II: $p \in M_{1,2}(\alpha, \beta)$.

In this case we define the integer n of 1.3 by

5.1.16. (a) $n = 32\tilde{\eta}$.

Thus the integer c of 1.3 satisfies

$$\text{(b)}\ c \in \{32\tilde{\eta} + 1, 32\tilde{\eta} + 2, \ldots, 33\tilde{\eta} - 1\}.$$

Note that 5.1.16 is consistent with 5.5.1(a)(b). We also assume that the numbers $\varepsilon_2, \{\delta_i\}$ of 0.5 satisfy the following property for all i (which is consistent with 5.1.2).

5.1.17. $\delta_{i+1} << \varepsilon_2 \delta_i$.

Using the same notation as in Case I above, we have the infrasolv core $r_{\hat{p}} : U_{\hat{p}} \to B_{\hat{p}}$ for $(\hat{M}, \hat{g})$ of type I and of radius δ_j, where j is given in 5.1.12(a) and where c of 5.1.12(a) comes from 5.1.16.

There is also a second infrasolv core $r'_{\hat{p}}; U'_{\hat{p}} \to B'_{\hat{p}}$ of type I and of radius δ_j, defined by

$$B'_{\hat{p}} = B_{\hat{p}}, U'_{\hat{p}} = \phi(1, U_{\hat{p}}), r'_{\hat{p}}(x) = r_{\hat{p}}(\phi(1, x)).$$

Let $\mathfrak{G}_{\hat{p}}, \mathfrak{G}'_{\hat{p}}$ denote the foliations of $U_{\hat{p}}, U'_{\hat{p}}$ by the fibers of $r_{\hat{p}}, r'_{\hat{p}}$. We also define the thickening maps $s'_{\hat{p}} : E'_{\hat{p}} \to B'_{\hat{p}}$ and $t'_{\hat{p}} : E'_{\hat{p}} \to \mathbf{R}$ for $r'_{\hat{p}}$ (cf. 0.4) by

$$E'_{\hat{p}} = \phi(1, E_{\hat{p}}), s'_{\hat{p}}(x) = s_{\hat{p}}(\phi(1,x)), t'_{\hat{p}}(x) = t_{\hat{p}}(\phi(1,x)).$$

We note that 5.1.14,5.1.16 (see also 5.1.1-5.1.12) imply that

5.1.18. $E_{\hat{p}}(\delta_{j+1}) \cap E'_{\hat{p}}(\delta_{j+1}) \neq \phi$ where

$$E_{\hat{p}}(t) = s_{\hat{p}}^{-1}(B_{\hat{p}}(t)) \cap t_{\hat{p}}^{-1}([0,t))$$

and

$$E'_{\hat{p}}(t) = {s'}_{\hat{p}}^{-1}(B'_{\hat{p}}(t)) \cap {t'}_{\hat{p}}^{-1}([0,t))$$

with $x \in B_{\hat{p}}(t)$ or $x \in B'_{\hat{p}}(t)$ iff $|x| < t$. Thus we may apply the arguments similar to those already used in the proof given above for the special case of Proposition 5.2 when $\mathfrak{F}$ is assumed orientable (cf. 5.2.2 and its verification), together with 5.1.17 and 5.1.18, to conclude the following.

5.1.19. For each $x \in U'_{\hat{p}}$ such that $|r'_{\hat{p}}(x)| < \frac{1}{2}\delta_j$, there is a smooth path $f_x : [0,1] \to \hat{M}$ satisfying:

(a) $f_x(0) = x, f_x(1) \in U_{\hat{p}}$ and length $(f_x) << \varepsilon_2\delta_j$.
(b) $\Theta(T(U_{\hat{p}})_{f_x(1)}, P_{f_x}(T(U'_{\hat{p}})_{f_x(0)})) << \varepsilon_2$.
(c) $\Theta(T(\mathfrak{G}_{\hat{p}})_{f_x(1)}, P_{f_x}(T(\mathfrak{G}'_{\hat{p}})_{f_x(0)})) << \varepsilon_2$.

Now we define a group action $\psi : \mathbb{Z}_2 \times B_{\hat{p}} \to B_{\hat{p}}$ as follows. Choose $q \in r_{\hat{p}}^{-1}(0)$ and set $q' = r_{\hat{p}} \circ \hat{\rho} \circ \phi_1(q)$, where $\phi_1(x) = \phi(1,x)$ and where $\hat{\rho} : E_{\hat{p}} \to U_{\hat{p}}$ is the orthogonal projection map. Note that it follows from 5.1.17 and 5.1.18 that

5.1.20. (a) $|q'| << \varepsilon_2\delta_j$.

Let W denote the set of all $v \in T(U_{\hat{p}})_q$ which are perpendicular to $r_{\hat{p}}^{-1}(0)$, and let

$$\exp\colon W \to \hat{M}$$

denote the exponential map. There is a smooth embedding $I : B_{\hat{p}}(\varepsilon_2\delta_j) \to U_{\hat{p}}$ which is uniquely determined by the following requirements.

(b) $r_{\hat{p}} \circ I : B_{\hat{p}}(\varepsilon_2\delta_j) \to B_{\hat{p}}$ is equal the inclusion $B_{\hat{p}}(\varepsilon_2\delta_j) \subset B_{\hat{p}}$; and Image$(I) \subset \hat{\rho}\circ\exp(W)$.

Let $k = \dim B_{\hat{p}}$ and define a linear map $L : \mathbf{R}^k \to \mathbf{R}^k$ to be the derivative (at 0) of the composite map $r_{\hat{p}} \circ \hat{\rho} \circ \phi_1 \circ I : B_{\hat{p}} \to B_{\hat{p}}$, where $\phi_1(x) = \phi(1,x)$ and where $T(B_{\hat{p}})_0$ and $T(B_{\hat{p}})_{q'}$ (the actual domain and range of L) are identified with $\mathbf{R}^k$ under Euclidean parallel translation. Note that L satisfies the following two properties, where in 5.1.20(d) T denotes another linear map $T : \mathbf{R}^k \to \mathbf{R}^k$. (See 5.1.6 – 5.1.12 as applied to both $r_{\hat{p}}$ and $r'_{\hat{p}}$ and see 5.1.18,5.1.19,5.1.20(a)(b).)

(c) $||L|| < \mathfrak{b}$, where $\mathfrak{b}$ depends only on $\alpha, \beta, A = \{A_i\}, \dim M$.
(d) $T^2 =$ id and $||T - L|| << \varepsilon_2$.

Now if $T : \mathbf{R}^k \to \mathbf{R}^k$ is in fact a linear isometry (e.g. $T(B_{\hat{p}}) = B_{\hat{p}}$), then we may define $\psi : \mathbb{Z}_2 \times B_{\hat{p}} \to B_{\hat{p}}$ by $\psi(1,x) = T(x)$. If T is not an isometry we can choose (by 5.1.20(c)(d)) a new inner product $<,>$ on $\mathbf{R}^k$ with respect to which T is an isometry and which is related to the usual inner product $<,>_k$ on $\mathbf{R}^k$ as follows.

(e) $< v, v >< (< v, v >_k) < \tau < v, v >$ holds for all $v \in \mathbf{R}^k$ where $\tau > 1$ depends only on $\alpha, \beta, A = \{A_i\}, \dim M$.

Then we replace the original $r_{\hat{p}} : U_{\hat{p}} \to B_{\hat{p}}$ and $r'_{\hat{p}} : U'_{\hat{p}} \to B'_{\hat{p}}$ by the restrictions of $r_{\hat{p}}$ and $r'_{\hat{p}}$ to the pre-images of these maps of the $<,>_k$-ball of radius δ_j in $B_{\hat{p}} = B'_{\hat{p}}$. We note that the relevant properties of 5.1.18 and 5.1.19 (and also of 5.1.6-5.1.13) are still satisfied by these new $r_{\hat{p}}, r'_{\hat{p}}$. So in the remainder of this proof there will be no loss of generality in supposing that $T : \mathbf{R}^k \to \mathbf{R}^k$ is in fact a linear isometry (e.g. $T(B_{\hat{p}}) = B_{\hat{p}}$), and that $\psi : \mathbb{Z}_2 \times B_{\hat{p}} \to B_{\hat{p}}$ is thus well defined by $\psi(1, x) = T(x)$.

We have the following relation between the two actions

$$\psi : \mathbb{Z}_2 \times B_{\hat{p}} \to B_{\hat{p}} \text{ and } \phi : \mathbb{Z}_2 \times \hat{M} \to \hat{M}$$

where $\psi_1(x) = \psi(1, x)$ and $\phi_1(x) = \phi(1, x)$ in what follows. (See 5.1.18-5.1.20; see also 5.1.6-5.1.12 as applied to $r_{\hat{p}}$ and $r'_{\hat{p}}$.)

5.1.21. For all $x \in U_{\hat{p}}$ for which both $r_{\hat{p}} \circ \hat{\rho} \circ \phi_1(x)$ and $\psi_1 \circ r_{\hat{p}}(x)$ are well defined we have that the following inequalities hold:

(a) $|r_{\hat{p}} \circ \hat{\rho} \circ \phi_1(x) - \psi_1 \circ r_{\hat{p}}(x)| << \varepsilon_2 \delta_j$;

(b) $||D(r_{\hat{p}} \circ \hat{\rho} \circ \phi_1)_x - D(\psi_1 \circ r_{\hat{p}})_x|| << \varepsilon_2$.

Note that if $r_{\hat{p}} : U_{\hat{p}} \to B_{\hat{p}}$ were invariant under the group action $\phi : \mathbb{Z}_2 \times \hat{M} \to \hat{M}$ and the group action $\psi : \mathbb{Z}_2 \times B_{\hat{p}} \to B_{\hat{p}}$ (i.e. $\varepsilon_2 = 0$ in 5.1.20-5.1.21) then we could define the desired infrasolv core $r_p : U_p \to B_p$ of type I for (M, g) as follows: let $B_p \subset B_{\hat{p}}$ denote the fixed point set for the action $\psi : \mathbb{Z}_2 \times B_{\hat{p}} \to B_{\hat{p}}$, and define $r_p : U_p \to B_p$ to be the quotient of $r_{\hat{p}} : r_{\hat{p}}^{-1}(B_p) \to B_p$ under the $\mathbb{Z}_2$-actions $\phi : \mathbb{Z}_2 \times U_{\hat{p}} \to U_{\hat{p}}$ and $\psi : \mathbb{Z}_2 \times B_p \to B_p$.

Properties 5.1.20-5.1.21 tell us that $r_{\hat{p}} : U_{\hat{p}} \to B_{\hat{p}}$ is approximately (but in general not exactly) invariant under the two $\mathbb{Z}_2$-group actions $\phi : \mathbb{Z}_2 \times \hat{M} \to \hat{M}$ and $\psi : \mathbb{Z}_2 \times B_{\hat{p}} \to B_{\hat{p}}$. We shall use these properties to construct a "geometric average" for $r_{\hat{p}}$ and $r'_{\hat{p}}$, denoted by $r''_{\hat{p}} : U''_{\hat{p}} \to B_{\hat{p}}$, which approximates $r_{\hat{p}}$ and which is invariant under the $\mathbb{Z}_2$-actions $\phi : \mathbb{Z}_2 \times \hat{M} \to \hat{M}$ and $\psi : \mathbb{Z}_2 \times B_{\hat{p}} \to B_{\hat{p}}$. Then, replacing $r_{\hat{p}}$ by $r''_{\hat{p}}$ in the construction of the preceding paragraph, we get the desired infrasolv structure $r_p : U_p \to B_p$ for (M, g).

5.1.22. The geometric average of $\mathbf{r_{\hat{p}}}$ and $\mathbf{r'_{\hat{p}}}$. First we will construct the geometric average $U''_{\hat{p}}$ of $U_{\hat{p}}$ and $U'_{\hat{p}}$. This consists of an infinite limit process.

The first step in this process replaces $U_{\hat{p}}$ and $U'_{\hat{p}}$ by $U_{\hat{p},1}$ and $U'_{\hat{p},1}$ constructed as follows. For all $x \in U_{\hat{p}}$, let $g_x : [0, 1] \to E_{\hat{p}}$ denote the geodesic which begins at x in a direction perpendicular to $U_{\hat{p}}$ and which ends at $U'_{\hat{p}}$, and set

$$U_{\hat{p},1} = \bigcup_{x \in U_{\hat{p}}} \{g_x(\frac{1}{2})\}.$$

And for each $x' \in U'_{\hat{p}}$, let $g_{x'} : [0, 1] \to E'_{\hat{p}}$ denote the geodesic which begins at x' in a direction perpendicular to $U'_{\hat{p}}$ and which ends at $U_{\hat{p}}$, and set

$$U'_{\hat{p},1} = \bigcup_{x' \in U'_{\hat{p}}} \{g_{x'}(\frac{1}{2})\}.$$

We note that 5.1.19(a)(b), together with 5.1.13 as applied to $r_{\hat{p}}$ and $r'_{\hat{p}}$, imply that

$U_{\hat{p},1}$ and $U'_{\hat{p},1}$ are well defined smooth submanifolds of $\hat{M}$ which are C^1 close to $U_{\hat{p}}$ and $U'_{\hat{p}}$, respectively, and which are even closer to one another (in the C^1-metric) than were the $U_{\hat{p}}$ and $U'_{\hat{p}}$. (In fact by 5.1.19(a)(b) we have that the C^1-distance from $U_{\hat{p}}$ to $U'_{\hat{p}}$ is $<< \varepsilon_2$; and the C^0-distance from $U_{\hat{p}}$ to $U'_{\hat{p}}$ is $<< \varepsilon_2\delta_j$.) We also note that $U'_{\hat{p},1} = \phi(1, U_{\hat{p},1})$.

We remark that there is some difficulty with the construction of $U_{\hat{p},1}$ and $U'_{\hat{p},1}$ away from large compact subsets of $U_{\hat{p}}$ and $U'_{\hat{p}}$, i.e. "near the boundaries" $\partial U_{\hat{p}} =$ closure$(U_{\hat{p}}) - U_{\hat{p}}$ and $\partial U'_{\hat{p}} =$ closure $(U'_{\hat{p}}) - U'_{\hat{p}}$ of $U_{\hat{p}}$ and $U'_{\hat{p}}$: the geodesics g_x and $g_{x'}$ might not exist for x, x' "near" $\partial U_{\hat{p}}$ and $\partial U'_{\hat{p}}$. This difficulty can be overcome by referring, when need be, to "extensions" of $r_{\hat{p}}, r'_{\hat{p}}$ (easily constructed by taking δ_j slightly larger in 5.1.6-5.1.12).

Now in the construction just carried out if we replace $U_{\hat{p}}, U'_{\hat{p}}$ by $U_{\hat{p},1}, U'_{\hat{p},1}$ we will get the submanifolds $U_{\hat{p},2}, U'_{\hat{p},2}$ (instead of the submanifolds $U_{\hat{p},1}, U'_{\hat{p},1}$). Proceeding by induction we can use the same methods to construct the submanifolds $U_{\hat{p},r+1}, U'_{\hat{p},r+1}$ from the submanifolds $U_{\hat{p},r}, U'_{\hat{p},r}$. We can deduce from 5.1.13 (as applied to $r_{\hat{p}}$ and $r'_{\hat{p}}$), and from 5.1.19, that the following properties hold. In the following we also use the notation $U_{\hat{p},0} = U_{\hat{p}}$ and $U'_{\hat{p},0} = U'_{\hat{p}}$; note that for each integer $r \geq 0$ we have that $U'_{\hat{p},r} = \phi(1, U_{\hat{p},r})$. Recall that for any smooth submanifold $V \subset \hat{M}$ we denote by $K(V; \hat{M})$ the curvature $K(\tau; \hat{M})$ of the constant map $\tau : V \to \{1\}$ defined in 1.1.

5.1.22.1. There is $\lambda > 0$ with $\sum_{r=1}^{\infty} \lambda^r << \varepsilon_2$. For each integer $r \geq 0$ and for each $x \in U_{\hat{p},r+1}$, there are smooth paths $g_x : [0,1] \to \hat{M}$ and $h_x : [0,1] \to \hat{M}$ with $g_x(0) = h_x(0) = x$ and $g_x(1) \in U_{\hat{p},r}$ and $h_x(1) \in U'_{\hat{p},r+1}$ which satisfy the following properties.

(a) length(g_x)+length$(h_x) < \lambda^{r+1}\delta_j$.
(b) $\Theta(T(U_{\hat{p},r})_{g_x(1)}, P_{g_x}(T(U_{\hat{p},r+1})_x)) < \lambda^{r+1}$.
(c) $\Theta(T(U'_{\hat{p},r+1})_{h_x(1)}, P_{h_x}(T(U_{\hat{p},r+1})_x)) < \lambda^{r+1}$.
(d) $K(U_{\hat{p},r+1}; \hat{M}) << \varepsilon_2\delta_j^{-1}$.

We can now define the geometric average of $U_{\hat{p}}$ and $U'_{\hat{p}}$, denoted by $U''_{\hat{p}}$, to be the point set limit space of the sequence $\{U_{\hat{p},r} : r = 0, 1, 2, \ldots\}$. Note that 5.1.22.1, together with the fact that $U'_{\hat{p},r} = \phi(1, U_{\hat{p},r})$ for all r, assures that this limit is a well defined C^1-submanifold of $\hat{M}$ which satisfies the following properties.

5.1.22.2. (a) $\phi(1, U''_{\hat{p}}) = U''_{\hat{p}}$.
(b) $K(U''_{\hat{p}}; \hat{M}) << \varepsilon_2\delta_j^{-1}$.
(c) For each $x \in U''_{\hat{p}}$ there is a smooth path $f_x : [0,1] \to \hat{M}$, with $f_x(0) = x$ and $f_x(1) \in U_{\hat{p}}$, which satisfies

$$\Theta(T(U_{\hat{p}})_{f_x(1)}, P_{f_x}(T(U''_{\hat{p}})_{f_x(0)})) << \varepsilon_2$$

and length $(f_x) << \varepsilon_2\delta_j$.

And, since C^1-submanifolds can be approximated arbitrarily close (in the C^1-metric) by C^∞-submanifolds, we may assume, in fact, that $U''_{\hat{p}}$ of 5.1.22.2 is a C^∞-submanifold of $\hat{M}$. This completes our construction of the "geometric average" of $U_{\hat{p}}$ and $U'_{\hat{p}}$.

Now we will construct the geometric average of the maps $r_{\hat{p}}$ and $r'_{\hat{p}}$, denoted by

$$r''_{\hat{p}} : U''_{\hat{p}} \to B_{\hat{p}}.$$

Recall that $\hat{\rho} : E_{\hat{p}} \to U_{\hat{p}}$ denotes the orthogonal projection map; we deduce from 5.1.22.2 that $\hat{\rho} : U''_{\hat{p}} \to U_{\hat{p}}$ is well defined except possibly "near" $\partial U''_{\hat{p}} = \text{closure}(U''_{\hat{p}}) - U''_{\hat{p}}$. Now we set

5.1.22.3. $r''_{\hat{p}}(x) = \frac{1}{2}(\psi(1, r_{\hat{p}} \circ \hat{\rho}(\phi(1,x)) + r_{\hat{p}} \circ \hat{\rho}(x))),$

for each $x \in U''_{\hat{p}}$ which is not "near" $\partial U''_{\hat{p}}$. We can deduce from 5.1.22.1-5.1.22.3, and from 5.1.21, that the following properties hold if we are not close to the boundary $\partial U''_{\hat{p}}$.

5.1.22.4. (a) $r_{\hat{p}}(\phi(1,x)) = \psi(1, r_{\hat{p}}(x))$ for all $x \in U''_{\hat{p}}$.
(b) $K(r''_{\hat{p}}; \hat{M}) << \varepsilon_2 \delta_j^{-1}$.
(c) $|r''_{\hat{p}}(x) - r_{\hat{p}} \circ \hat{\rho}(x)| << \varepsilon_2 \delta_j$ and $||D(r''_{\hat{p}})_x - D(r_{\hat{p}} \circ \hat{\rho})_x|| < \varepsilon_2$ for each $x \in U''_{\hat{p}}$.

As we have remarked there is some difficulty "near" the boundary $\partial U''_{\hat{p}}$ both in the construction of $r''_{\hat{p}}$ and in assuring that properties 5.1.22.4(a)-(c) hold. To overcome these difficulties we use "extensions" of $r_{\hat{p}}$ and $r''_{\hat{p}}$. (The extension of $r_{\hat{p}}$ is easily constructed by taking δ_j slightly lager in 5.1.6-5.1.12; then the extension of $r''_{\hat{p}}$ is gotten by applying the preceding construction to this extension of $r_{\hat{p}}$.) Now the actual geometric average of the two maps $r_{\hat{p}}$ and $r'_{\hat{p}}$ is gotten by restricting the extension of $r''_{\hat{p}}$ to its preimage for $B_{\hat{p}}$. This new $r''_{\hat{p}}$ satisfies 5.1.22.4(a)-(c) even near the boundary $U''_{\hat{p}}$. Note that 5.1.22.4(b)(c) and 5.1.22.2 assure us that $r''_{\hat{p}}$ is a smooth bundle projection map with fiber diffeomorphic to the fiber of $r_{\hat{p}}$. Thus $r''_{\hat{p}}$ is an infrasolv core as desired.

Applying the fixed point construction described in the paragraph following 5.1.21 to $r''_{\hat{p}}$ completes the proof for Proposition 5.1 when $T\mathfrak{F}$ is not orientable and $p \in M_{1,2}(\alpha, \beta)$.

Proof of Proposition 5.2 when $\mathbf{T}\mathfrak{F}$ *is not orientable.*

Case I: $p \in M_{1,1}(\alpha, \beta)$.

This is just a repetition of the proof given for 5.2 in the case that $T\mathfrak{F}$ is orientable. (See Case I in the preceding proof.)

Case II: $p \in M_{1,2}(\alpha, \beta)$.

We must verify properties 0.6(a)-(c) for any two infrasolv cores r_{p_1}, r_{p_2} associated to points $p_1, p_2 \in M_{1,2}(\alpha, \beta)$ such that the thickenings $E_{p_i}, i = 1, 2$, overlap. Let

$$\pi : \hat{M} \to M$$

denote the two fold covering for M such that the pull back $\hat{\mathfrak{F}}$ of $\mathfrak{F}$ under π has oriented tangent bundle. In the notation of the preceding proof there are (for $i = 1, 2$) points $\hat{p}_i \in \pi^{-1}(p_i)$, and infrasolv cores $r_{\hat{p}_i}, r''_{\hat{p}_i}$ which are related as in 5.1.22.2 and 5.1.22.4, and group actions ${}_i\psi : \mathbb{Z}_2 \times B_{\hat{p}_i} \to B_{\hat{p}_i}$ such that the $r_{p_i} : U_{p_i} \to B_{p_i}$ are just the quotients of the maps $r''_{\hat{p}_i} : r''^{-1}_{\hat{p}_i}(B_{p_i}) \to B_{p_i}$ under the group actions ϕ and ${}_i\psi$ (where $B_{\hat{p}_i}$ is the fixed point set for ${}_i\psi$). We may apply the arguments contained in the proof of Proposition 5.2 when $\mathfrak{F}$ is orientable to the infrasolv cores $r_{\hat{p}_i}, i = 1, 2$ (where the integer n of 1.3 is now given by 5.1.16, instead of by 5.1.1 as was originally the case in the arguments referred to). We conclude (from this special case of 5.2) that the thickenings of the infrasolv cores $r_{\hat{p}_i}$, $i = 1, 2$, (denoted by $(s_{\hat{p}_i}, t_{\hat{p}_i}), i = 1, 2$) are related to one another as in 0.6(a)-(c). Now it follows from 5.1.22.2 and 5.1.22.4 that the thickenings of the infrasolv cores $r''_{\hat{p}_i}, i = 1, 2$, (denoted by $(s''_{\hat{p}_i}, t''_{\hat{p}_i}), i = 1, 2$) are also related to one another as in 0.6(a)-(c) (for a slightly larger value of ε_2 in 0.6 than that associated to the $(s_{\hat{p}_i}, t_{\hat{p}_i}), i = 1, 2$). Let ${}_i\bar{\psi} : \mathbb{Z}_2 \times \mathbf{R}^k \to \mathbf{R}^k$ denote the linear extension of the ${}_i\psi : \mathbb{Z}_2 \times B_{\hat{p}_i} \to B_{\hat{p}_i}$, and let $A_2 : \mathbf{R}^k \to \mathbf{R}^k$ be the affine isomorphism which is associated by 0.6(c) to $s''_{\hat{p}_1}$ and $s''_{\hat{p}_2}$; i.e., $A_2 \circ s''_{\hat{p}_1}$ is approximated by $s''_{\hat{p}_2}$ to within $\varepsilon_2\delta_j$. Note that it follows from 0.6(a)-(c) as applied to $(s''_{\hat{p}_1}, t''_{\hat{p}_1})$ and $(s''_{\hat{p}_2}, t''_{\hat{p}_2})$, and from the fact that $s_{\hat{p}_i} \circ \phi_1 = {}_i\bar{\psi}_1 \circ s_{\hat{p}_i}$ for $i = 1, 2$, that the following holds:

5.2.8. $|({}_2\bar{\psi}_1) \circ A_2(x) - A_2 \circ ({}_1\bar{\psi}_1)(x)| < \mathcal{O}(\varepsilon_2)\delta_j$, for all $x \in B''_{\hat{p}_1}$.

Define $A'_2 : {}_1V \to {}_2V$ as follows: let ${}_iV \subset \mathbf{R}^k$ denote the fixed point set of ${}_i\bar{\psi}$, and set

$$A'_2(x) = \frac{1}{2}({}_2\bar{\psi}_1 \circ A_2(x) + A_2(x)).$$

It follows from 5.2.8, and from 0.6(a)-(c) as applied to $r''_{\hat{p}_1}$ and $r''_{\hat{p}_2}$, that properties 0.6(a)-(c) also hold for the pair r_{p_1}, r_{p_2} when $A_2 : \mathbf{R}^k \to \mathbf{R}^k$ is replaced in 0.6(c) by $A'_2 : {}_1V \to {}_2V$ and when ε_2 is replaced by $\mathcal{O}(\varepsilon_2)$. (Note that both ${}_1V, {}_2V$ are isomorphic to the same Euclidean space $\mathbf{R}^{k'}$; thus we have $A'_2 : \mathbf{R}^{k'} \to \mathbf{R}^{k'}$ as required in 0.6(c).)

This completes the proof for Proposition 5.2 when $T\mathfrak{F}$ is not orientable.

Appendix. Let (M, g) denote an A-regular complete Riemannian manifold and let $\mathfrak{F}$ denote a smooth one-dimensional A-regular foliation for M. In this appendix we formulate the A-regular condition for $\mathfrak{F}$ in terms of the immersed normal coordinates for (M, g) (cf. Theorem A.1 below). In addition we formulate two results analogous to [8; Theorem A.1.2, Corollary A.1.3], where the normal immersed coordinates of [8; A.1.2, A.1.3] are replaced by the immersions $f_p : (-5\alpha, 5\alpha) \times B(p, \beta) \to M$ of Lemma 2.1. (See Theorem A.2 and Corollary A.3 below.)

In the following theorem we let $f : B^m_\varepsilon \to M$ be the immersion of [8; Corollary A.1.2], where $f(0)$ is randomly chosen, i.e., f is immersed normal co-ordinates. Recall that B^m_ε is the open ball of radius $\varepsilon > 0$ centered at the origin in $\mathbf{R}^m$ (where $m = \dim M$); let $x_1, x_2, \ldots, x_m$ denote the standard coordinates for $\mathbf{R}^m$, and define maps $g_{i,j} : B^m_\varepsilon \to \mathbf{R}$ by

$$g_{ij}(x) = g(Df(\frac{\partial}{\partial x_i}(x)), Df(\frac{\partial}{\partial x_j}(x))).$$

Let $\hat{\mathfrak{F}}$ denote the smooth one-dimensional foliation of B^m_ε obtained by pulling $\mathfrak{F}$ back along $f : B^m_\varepsilon \to M$, and let

$$W = \sum_{i=1}^{m} w_i \frac{\partial}{\partial x_i}$$

denote a smooth unit length vector field (measured with respect to the metric $\{g_{i,j}\}$) on B^m_ε which is everywhere tangent to $\hat{\mathfrak{F}}$.

A.1. Theorem. *There is a collection $B = \{B_i\}$ of positive numbers $B_1, B_2, B_3, \ldots$ which depend only on the $A = \{A_i\}$ and $\dim M$. For all $i, k, \{j_1, j_2, \ldots, j_k\}, x \in$*

B^m_ε *we have that*

$$|\frac{\partial^k w_i}{\partial x_{j_1}\partial x_{j_2}\dots\partial x_{j_k}}(x)| < B_k.$$

In the next theorem, and its corollary, we let $f_p : (-5\alpha, 5\alpha) \times B(p,\beta) \to M$ denote the immersions defined in the third paragraph of §2 and referred to in Lemma 2.1. Recall that, for $p \in M$, $B(p,\beta)$ denotes the open ball of radius β centered at the origin of $T\mathfrak{F}^{\perp}_{|p}$, where $T\mathfrak{F}^{\perp}$ denotes the orthogonal complement for $T\mathfrak{F}$ in TM. For each $p \in M$, we choose an orthonormal basis for $T\mathfrak{F}^{\perp}_{|p}$ and let $y_2, y_3, \dots, y_m$ denote the coordinates for $T\mathfrak{F}^{\perp}$ with respect to this choice; we let y_1 denote the standard coordinate in the interval $(-5\alpha, 5\alpha)$.

A.2. Theorem. *There is $\eta > 0$ and a collection $C = \{C_i\}$ of positive numbers $C_1, C_2, C_3, \dots$ which depend only on $\alpha, A = \{A_i\}, dimM$. For each $p \in M$ and each $\beta \in (0,\eta)$ the immersion $f_p : (-5\alpha, 5\alpha) \times B(p,\beta) \to M$ satisfies the following properties. Let $g_{p;i,j} : (-5\alpha, 5\alpha) \times B(p,\beta) \to \mathbf{R}$ be defined by*

$$g_{p;i,j}(y) = g(Df_p(\frac{\partial}{\partial y_i}(y)), Df_p(\frac{\partial}{\partial y_j}(y)))$$

for all $y \in (-5\alpha, 5\alpha) \times B(p,\beta)$. Then,
(a) $f_p(0) = p$ and $g_{p;i,j}(0) = \delta^i_j$.
(b) $|\frac{\partial^k g_{p;i,j}}{\partial y_{s_1}\partial y_{s_2}\dots\partial y_{s_k}}(y)| \le C_k$ for all $k, i, j, \{s_1, \dots, s_k\}, y \in [-4\alpha, 4\alpha] \times B(p,\beta)$.

In the following corollary of Theorem A.2, we let $f_p : (-5\alpha, 5\alpha) \times B(p,\beta) \to M$ and $f_{p'} : (-5\alpha, 5\alpha) \times B(p',\beta) \to M$ denote two of the immersions of A.2; and for any number $t \in [0,\beta)$, we let $B(p,t)$ denote the open ball of radius t centered at the origin of $B(p,\beta)$. We let $y_1, y_2, \dots, y_m$ and $y'_1, y'_2, \dots, y'_m$ denote the coordinate systems for

$$(-5\alpha, 5\alpha) \times B(p,\beta) \text{ and for } (-5\alpha, 5\alpha) \times B(p',\beta)$$

referred to in A.2.

A.3. Corollary. *There is a number $\delta > 0$ which depends only on the $\{C_i\}$ and dimM; there is also a collection of positive numbers $\{\bar{C}_i\}$ which depends only on the $\{C_i\}$ and dimM. For any two points $y \in (-\alpha, \alpha) \times B(p, \delta\beta)$ and $y' \in (-\alpha, \alpha) \times B(p', \delta\beta)$ which satisfy*

$$f_p(y) = f_{p'}(y'),$$

there is a smooth embedding

$$h : (-2\alpha, 2\alpha) \times B(p, \delta\beta) \to [-5\alpha, 5\alpha] \times B(p',\beta)$$

which is uniquely determined by the properties listed in (a) below. Moreover h also satisfies property (b) below.
(a) $h(y) = y'$, and $f_p|(-2\alpha, 2\alpha) \times B(p, \delta\beta) = f_{p'} \circ h$.
(b) Let $h = (h_1, h_2, \dots, h_m)$ denote the coordinates for h with respect to the coordinate system $y'_1, y'_2, \dots, y'_m$. For all $y \in (-2\alpha, 2\alpha) \times B(p, \delta\beta), i, k, \{j_1, j_2, \dots, j_k\}$ we have

$$|\frac{\partial^k h_i}{\partial y_{j_1}\partial y_{j_2}\dots\partial y_{j_k}}(y)| < \bar{C}_k.$$

In our final theorem of this appendix we shall need the following notation. Let $U \subset M$ denote a smooth submanifold of M without boundary, and let $r : U \to B$ denote a smooth mapping into an open subset B of some Euclidean space. For any given

$$f_p : (-5\alpha, 5\alpha) \times B(p, \beta) \to M$$

from A.2, we define a submanifold $\hat{U} \subset (-5\alpha, 5\alpha) \times B(p, \beta)$ and a map $\hat{r} : \hat{U} \to \hat{B}$ by

$$\hat{U} = f_p^{-1}(U),$$
$$\hat{B} = B,$$
$$\hat{r} = r \circ f_p.$$

Let $K(r; M)$ denote the curvature of r in (M, g) (as defined in 1.1); and let

$$K(\hat{r}; (-5\alpha, 5\alpha) \times B(p, \beta))$$

denote the curvature of $\hat{r}$ in $(-5\alpha, 5\alpha) \times B(p, \beta)$ computed (as in 1.1) with respect to the Euclidean metric $e = \delta^i_j dy_i dy_j$ on $(-5\alpha, 5\alpha) \times B(p, \beta)$.

A.4. Theorem. *Suppose that $U \subset Image(f_p)$. Then there is a number $\tau > 1$, which depends only on the $\{C_i\}$ and $dimM$, such that*

$$\frac{1}{\tau} K(r; M) - 1 < K(\hat{r}; (-5\alpha, 5\alpha) \times B(p, \beta)) < \tau K(r; M) + \tau.$$

Proof of Theorem A.1 Let ∇ denote the covariant derivative on B^m_ε with respect to the pulled back metric $\{g_{i,j}\}$, and let $\{\Gamma^k_{i,j}\}$ denote the Christoffel functions associated to ∇ and $\{g_{i,j}\}$. The A-regularity of $\mathfrak{F}$ is expressed locally by the following inequalities.

A.1.1. $|\nabla^r W| < A_r$ for all r.

We regard W as a tensor field on B^m_ε of type $(1, 0)$; then for each $r = 1, 2, 3, \ldots$ the r'th covariant derivative $\nabla^r W$ is a tensor field on B^m_ε of type $(1, r)$. Note that

A.1.2. $W(dx_i) = w_i$

follows from the definition of W.

To compute $\nabla W(dx_i, \frac{\partial}{\partial x_j})$ apply A.1.2 and the product rule

$$\frac{\partial}{\partial x_j}((W)(dx_i)) = (\nabla_{\partial/\partial x_j} W)(dx_i) + W(\nabla_{\partial/\partial x_j} dx_i)$$

together with the two equalities

$$(\nabla_{\partial/\partial x_j} W)(dx_i) = \nabla W(dx_i, \frac{\partial}{\partial x_j}))$$

and

$$W(\nabla_{\partial/\partial x_j} dx_i) = W(\sum_s -\Gamma^i_{s,j} dx_s) = \sum_s -\Gamma^i_{s,j} w_s$$

to deduce that

A.1.3. $\nabla W(dx_i, \frac{\partial}{\partial x_j}) = \frac{\partial w_i}{\partial x_j} + \sum_s \Gamma^i_{s,j} w_s$.

Likewise to compute $\nabla^2 W(dx_i, \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k})$ we appeal to A.1.2 and A.1.3, and to the product rule

$$\frac{\partial}{\partial x_k}(\nabla W(dx_i, \frac{\partial}{\partial x_j})) =$$
$$(\nabla_{\partial/\partial x_k} \nabla W)(dx_i, \frac{\partial}{\partial x_j}) + \nabla W(\nabla_{\partial/\partial x_k} dx_i, \frac{\partial}{\partial x_j}) + \nabla W(dx_i, \nabla_{\partial/\partial x_k} \frac{\partial}{\partial x_j}))$$

together with the equalities

$$(\nabla_{\partial/\partial x_k} \nabla W)(dx_i, \frac{\partial}{\partial x_j}) = \nabla^2 W(dx_i, \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k})$$

$$\nabla W(\nabla_{\partial/\partial x_k} dx_i, \frac{\partial}{\partial x_j}) = \sum_s -\Gamma^i_{s,k} \nabla W(dx_s, \frac{\partial}{\partial x_j})$$

and

$$\nabla W(dx_i, \nabla_{\partial/\partial x_k} \frac{\partial}{\partial x_j})) = \sum_s \Gamma^s_{k,j} \nabla W(dx_i, \frac{\partial}{\partial x_s}))$$

to deduce that

A.1.4.

$$\nabla^2 W(dx_i, \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}) = \frac{\partial^2 w_i}{\partial x_k \partial x_j} + \frac{\partial}{\partial x_k}(\sum_s \Gamma^i_{s,j} w_s) + \sum_s \Gamma^i_{s,k}(\frac{\partial w_s}{\partial x_j} + \sum_t \Gamma^s_{t,j} w_t)$$
$$+ \sum_s -\Gamma^s_{k,j}(\frac{\partial w_i}{\partial x_s} + \sum_t \Gamma^i_{t,s} w_t).$$

Proceeding by induction, and using the tricks of the preceding few paragraphs, it can be shown that for $r = 1, 2, 3, \ldots$, and for every choice of indices $j_1, j_2, \ldots, j_r \in \{1, 2, \ldots, m\}$ we have that

A.1.5. $\nabla^r W(dx_i, \frac{\partial}{\partial x_{j_1}}, \frac{\partial}{\partial x_{j_2}}, \ldots, \frac{\partial}{\partial x_{j_r}}) = \frac{\partial^r w_i}{\partial x_{j_1} \partial x_{j_2} \ldots \partial x_{j_r}} + E$

where E is a sum of terms which are products of lower order (i.e. less than order r) partial derivatives of the $\{\Gamma^i_{j,k}\}$ and the $\{w_i\}$. [Note that both the number of such summands in E, and the number of such factors in each summand, is bounded above by a number depending only on $\dim M = m$ and r.]

Now Theorem A.1 can be proven by using an induction argument, based on properties A.1.1 and A.1.5. (See also [8;A.1.2.1,A.1.2.2].)

This completes the proof for Theorem A.1.

Proof of Theorem A.2. We first remark that property A.2(a) is immediate from the construction given for

$$f_p : (-5\alpha, 5\alpha) \times B(p, \beta) \to M$$

in the third paragraph of §2.

Thus it will suffice to prove property A.2(b). The proof of A.2(b) hinges on the following claim. For each of the vector fields $W = \sum_{i=1}^{m} w_i \frac{\partial}{\partial x_i}$ referred to just prior to Theorem A.1 we may assume without loss of generality that

$$W(0) = \frac{\partial}{\partial x_1}(0).$$

A.2.1. Claim. *There is another set of coordinates $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m$ for $B^m_{\frac{1}{9}\varepsilon}$ and another collection of positive numbers $\{\bar{B}_i\}$ which satisfy the following properties.*

(a) $\frac{\partial}{\partial \bar{x}_1}(x) = W(x)$ for all $x \in B^m_{\frac{1}{9}\varepsilon}$.

(b) $\bar{x}_i(x) = x_i(x)$ for all i and for all $x \in B^m_{\frac{1}{9}\varepsilon}$ with $x_1(x) = 0$.

(c) The $\{\bar{B}_i\}$ depend only on the $\{B_i\}$ and on $dimM$.

(d) $|\frac{\partial^k \bar{x}_i}{\partial x_{j_1} \partial x_{j_2} \ldots \partial x_{j_k}}(x)| < \bar{B}_k$ for all k and all $\{j_1, \ldots, j_k\}$.

We shall first use the preceding claim to complete the proof for A.2(b), and then we shall verify the claim.

Let $c : \mathbf{R} \to M$ denote a unit speed parametrization of the leaf of $\mathfrak{F}$ containing $p \in M$ satisfying $c(0) = p$. For each $i \in \mathbb{Z}$, we set $t_i = i\frac{\varepsilon}{9^3}$ and we set $p_i = c(t_i)$. For each $i \in \mathbb{Z}$, we will denote the immersion $f : B^m_\varepsilon \to M$ of [8; Corollary A.1.2] which maps 0 to p_i by

$$f_i : B^m_\varepsilon \to M$$

(i.e., f_i is the immersed normal co-ordinate map sending 0 to p_i) and we denote the vector field W and the coordinates $\bar{x}_1, \ldots, \bar{x}_m$ associated to f_i by A.2.1 by

$$W_i \text{ and } \bar{x}_{i,1}, \ldots, \bar{x}_{i,m}.$$

Note that, by [8; Corollary A.1.3], there is for each $i \in \mathbb{Z}$ a smooth embedding

$$h_i : B^m_{\varepsilon/9} \to B^m_\varepsilon$$

which is uniquely determined by the following properties.

A.2.2. (a) $f_i | B^m_{\varepsilon/9} = f_{i+1} \circ h_i$.

(b) $\bar{X}_{i+1} \circ h_i(0) = (-\frac{\varepsilon}{9^3}, 0, 0, \ldots, 0)$, where $\bar{X}_{i+1} = (\bar{x}_{i+1,1}, \ldots, \bar{x}_{i+1,m})$.

We note also that for $\beta > 0$ sufficiently small (i.e. $\beta << \varepsilon$), and for $i \in \mathbb{Z}$ satisfying $|i| < 9^4 \frac{\alpha}{2\varepsilon}$, there is a smooth embedding

$$g_i : [t_{i-1}, t_i] \times B(p, \beta) \to B^m_\varepsilon$$

uniquely determined by the following properties.

A.2.3. (a) $f_p | [t_{i-1}, t_i] \times B(p, \beta) = f_i \circ g_i$.

(b) $g_i(t_i) = 0$.

For each $i \in \mathbb{Z}$ there are functions $u_i : B^{m-1}_{\frac{1}{10}\varepsilon} \to \mathbf{R}^{m-1}$ and $v_i : B^{m-1}_{\frac{1}{10}\varepsilon} \to \mathbf{R}$ which are uniquely determined by the following properties. [$B^{m-1}_{\frac{1}{10}\varepsilon}$ denotes the open ball of radius $\frac{1}{10}\varepsilon$ centered at the origin in $\mathbf{R}^{m-1}$].

A.2.4. For each $q \in B^m_{\frac{1}{11}\varepsilon}$ we have that

$$\bar{x}_{i+1,1} \circ h_i(q) = \bar{x}_{i,1}(q) - (\frac{1}{9^3}\varepsilon) + v_i(\bar{x}_{i,2}(q), \ldots, \bar{x}_{i,m}(q));$$

and

$$\bar{x}_{i+1,j+1} \circ h_i(q) = u_{i,j}(\bar{x}_{1,2}(q), \ldots, \bar{x}_{i,m}(q))$$

for all $j = 1, 2, \ldots, m-1$, where $u_i = (u_{i,1}, \ldots, u_{i,m-1})$. And for each $i \in \mathbb{Z}$ satisfying $|i| < 9^4 \frac{\alpha}{2\varepsilon}$ there are functions $r_i : B^{m-1}_\beta \to \mathbf{R}^{m-1}$ and $s_i : B^{m-1}_\beta \to \mathbf{R}$ which are uniquely determined by the following properties. [Here B^{m-1}_β denotes the open ball of radius β centered at the origin in $\mathbf{R}^{m-1}$.]

A.2.5. For each $q \in [t_{i-1}, t_i] \times B(p, \beta)$ we have that

$$\bar{x}_{i,1} \circ g_i(q) = y_1(q) - t_i + s_i(y_2(q), \ldots, y_m(q));$$

and

$$\bar{x}_{i,j+1} \circ g_i(q) = r_{i,j}(y_2(q), \ldots, y_m(q))$$

for all $j = 1, 2, \ldots, m-1$, where $r_i = (r_{i,1}, \ldots, r_{i,m-1})$.

The reader can easily check that there exists the following important relations between the $\{u_i\}, \{v_i\}, \{r_i\}, \{s_i\}$.

A.2.6. (a) For all $i \geq 1$ we have that

$$r_i = (u_{i-1} \circ u_{i-2} \circ \ldots \circ u_1) \circ r_1;$$

and for all $i < 1$ we have that

$$r_i = (u_i^{-1} \circ u_{i+1}^{-1} \circ \ldots \circ u_0^{-1}) \circ r_1.$$

(b) For all $i \geq 1$ we have

$$s_i = s_1 + \sum_{j=1}^{i-1} v_j \circ (u_{j-1} \circ u_{j-2} \circ \ldots \circ u_1) \circ r_1;$$

and for all $i < 1$ we have that

$$s_i = s_1 - \sum_{j=i}^{0} v_j \circ (u_j^{-1} \circ u_{j+1}^{-1} \circ \ldots \circ u_0^{-1}) \circ r_1.$$

Note also that there will be no loss of generality in assuming that s_1 and r_1 satisfy the following properties.

(c) $r_1 =$ identity.

(d) $s_1 = 0$.

By applying the inequalities of A.2.1(d) (to each of the coordinate systems $\bar{x}_{i,1}$, $\ldots$, $\bar{x}_{i,m}$, $i \in \mathbb{Z}$), and by applying the inequalities of [8;A.1.3(c)] (to each of h_i), it is easy to deduce that the functions u_i, v_i satisfy the following inequalities, for some collection of positive numbers $D = \{D_i\}$ which depend only on the $A = \{A_i\}$ and on $\dim M$. Let $x_1, \ldots, x_{m-1}$ denote the standard coordinates for $\mathbf{R}^{m-1}$.

A.2.7. For all $x \in B^{m-1}_{\frac{1}{10}\varepsilon}, k, \{j_1, j_2, \ldots, j_k\}$ we have that

$$|\frac{\partial^k u_i}{\partial x_{j_1} \ldots \partial x_{j_k}}(x)| < D_k$$

and

$$|\frac{\partial^k v_i}{\partial x_{j_1} \ldots \partial x_{j_k}}(x)| < D_k.$$

Now we can use A.2.6 and A.2.7 to deduce the following inequalities for the r_i, s_i.

A.2.8. There exists a collection of positive numbers $\bar{D} = \{\bar{D}_i\}$, which depend only on the $A = \{A_i\}, \alpha$, and $\dim M$, such that for all $x \in B(p,\beta), i, k, \{j_1, j_2, \ldots, j_k\}$ we have:

(a) $|\frac{\partial^k r_i}{\partial x_{j_1} \ldots \partial x_{j_k}}(x)| < \bar{D}_k$;

(b) $|\frac{\partial^k s_i}{\partial x_{j_1} \ldots \partial x_{j_k}}(x)| < \bar{D}_k$.

To see this we note that A.2.6 and A.2.7 together imply that

$$|\frac{\partial^k r_i}{\partial x_{j_1} \ldots \partial x_{j_k}}(x)| < D_{k,i}$$

and

$$|\frac{\partial^k s_i}{\partial x_{j_i} \ldots \partial x_{j_k}}(x)| < D_{k,i},$$

where $D_{k,i}$ is a positive number depending only on the $\{D_0, D_1, \ldots, D_k\}$ of A.2.7, and on the integer i, and on $\dim M$. Since r_i, s_i are only defined if $|i| < 9^4 \frac{\alpha}{2\varepsilon}$, and since $\varepsilon > 0$ need only depend on $A = \{A_i\}$ and $m = \dim M$ (cf. [8;A.1.1, A.1.2]), we may define the$\{\bar{D}_k\}$ of A.2.8 by

$$\bar{D}_k = \text{ maximum}\{D_{k,i} : |i| < 9^4 \frac{\alpha}{2\varepsilon}\}$$

and be assured (by the two preceding inequalities) that A.2.8(a)(b) both hold, and also be assured that $\{\bar{D}_k\}$ depends only on the $A = \{A_i\}, \alpha, \dim M$.

Now we can complete the verification of property A.2(b) as follows. First, by appealing to A.2.1(d) and A.2.8 (see also A.2.5), we can deduce that the $g_i : [t_{i-1}, t_i] \times B(p,\beta) \to \mathbf{R}^m$ of A.2.3 satisfy the following inequalities.

A.2.9. There exists a collection of positive numbers $E = \{E_i\}$, which depend only on $A = \{A_i\}$, α and $\dim M$, such that for all $y \in (-5\alpha, 5\alpha) \times B(p,\beta), i, k, \{j_1, j_2, \ldots, j_k\}$ we have

$$|\frac{\partial^k g_i}{\partial y_{j_1} \ldots \partial y_{j_k}}(y)| < E_k$$

where $|\ |$ denotes the Euclidean norm on B^m_ε.

Next we note that the metric $\{g_{p;i,j}\}$ of A.2 (which was defined to be the pull back along $f_p : (-5\alpha, 5\alpha) \times B(p,\beta) \to M$ of the metric g on M) can also be obtained (on $[t_{i-1}, t_i] \times B(p,\beta)$) by first pulling g back to B^m_ε along $f_i : B^m_\varepsilon \to M$ (to get the metric $\{g_{i,j}\}$ of [8;A.1.1 and A.1.2]), and then pulling back this $\{g_{i,j}\}$ of [8;A.1.1] along the map

$$g_i : [t_{i-1}, t_i] \times B(p,\beta) \to \mathbf{R}^m.$$

Finally, by appealing to [8;A.1.1(b)] and A.2.9, together with the preceding remarks, we can easily deduce property A.2(b).

This completes the proof of Theorem A.2, modulo the verification of Claim A.2.1.

Verification of Claim A.2.1. First integrate the vector field W on B^m_ε to get a partial flow $\psi : S \to B^m_\varepsilon$, where $S \subset \mathbf{R} \times B^m_\varepsilon$ is the maximal subset of $\mathbf{R} \times B^m_\varepsilon$ on which the partial flow ψ is well defined. We may deduce from A.1 that

A.2.1.1. (a) $S \subset [-2\varepsilon, 2\varepsilon] \times B^m_\varepsilon$.
(b) $[-\frac{1}{2}\varepsilon, \frac{1}{2}\varepsilon] \times B^m_{\frac{1}{8}\varepsilon} \subset S$.

We may also deduce from A.1 and A.2.1.1(a), and from the existence theory for ordinary differential equations, that ψ satisfies the following inequalities.

A.2.1.2. There is a collection of positive numbers $\hat{B} = \{\hat{B}_i\}$ which depend only on the $B = \{B_i\}$ of A.2.1 and on dim$M$. For all $(t,x) \in S, k, \{j_1, j_2, \ldots, j_k\}$ we have that

$$|\frac{\partial^k \psi}{\partial x_{j_1} \ldots \partial x_{j_k}}(t,x)| < \hat{B}_k.$$

Now we can define the new coordinates $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m$ by

A.2.1.3.

$$\bar{x}_i(\psi(t,(0,x_2,\ldots,x_m)) = \begin{cases} t & \text{if } i = 1 \\ x_i & \text{if } i > 1 \end{cases}$$

Note that properties A.2.1(a)(b) are immediate from A.1 and from A.2.1.3. Also note that it follows from A.2.1.1(b) that the coordinates $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m$ are well defined on $B^m_{\varepsilon/9}$ by A.2.1.3.
Finally, note that properties A.2.1(c)(d) follow directly from A.2.1.2 and A.2.1.3.

This completes the verification of Claim A.2.1, and also the proof of Theorem A.2.

Proof of Corollary A.3. We let $\{y_1, y_2, \ldots, y_m\}$ and $\{y'_1, y'_2, \ldots, y'_m\}$ denote the coordinates for

$$(-5\alpha, 5\alpha) \times B(p,\beta) \text{ and } (-5\alpha, 5\alpha) \times B(p',\beta)$$

provided by A.2; and let $g_{p;i,j}dy_idy_j$ and $g_{p';i,j}dy'_idy'_j$ denote the pull back of the metric g on M along the maps f_p and $f_{p'}$. Properties A.2(a)(b) hold for both of these pull backs. Thus, for sufficiently small $\mu > 0$ (how small is sufficient depends only on the $\{C_i\}$ and on $m = \dim M$), the exponential map for the metric $g_{p;i,j}dy_idy_j$

$$\exp : B(y,\mu\beta) \to (-5\alpha, 5\alpha) \times B(p,\beta)$$

is a well defined embedding of the open ball of $g_{p;i,j}dy_idy_j$-radius equal $\mu\beta$ centered at the origin in $T((-5\alpha,5\alpha) \times B(p,\beta))_{|y}$; and the exponential map for the metric $g_{p';i,j}dy'_idy'_j$

$$\exp' : B(y',\mu\beta) \to (-5\alpha, 5\alpha) \times B(p',\beta)$$

is a well defined embedding from the open ball of $g_{p';i,j}dy'_idy'_j$-radius equal $\mu\beta$ centered at the origin in $T((-5\alpha,5\alpha) \times B(p',\beta))_{|y'}$.

Let $B^m_{\mu\beta}$ denote the open ball of radius $\mu\beta$ centered at the origin of $\mathbf{R}^m$, and let

$$f : B^m_{\mu\beta} \to (-5\alpha, 5\alpha) \times B(p, \beta)$$

and

$$f' : B^m_{\mu\beta} \to (-5\alpha, 5\alpha) \times B(p', \beta)$$

denote the composition of linear isometries $B^m_{\mu\beta} \to B(y, \mu\beta)$ and $B^m_{\mu\beta} \to B(y', \mu\beta)$ with the exponential maps exp and exp′, respectively. Let $f = (f_1, \ldots, f_m)$ be the coordinates of f with respect to the $y_1, y_2, \ldots, y_k$ coordinates, and let $f' = (f'_1, \ldots, f'_m)$ be the coordinates of f with respect to the $y'_1, y'_2, \ldots, y'_k$ coordinates and let $x_1, x_2, \ldots, x_m$ denote the standard coordinates for $\mathbf{R}^m$. By appealing to properties A.2(a)(b), we can verify (as in the proof of [8;A.1.2]) that the maps f, f' satisfy the following properties.

A.3.1. (a) There are numbers $\{\tilde{C}_i\}$ which depend only on the $\{C_i\}$ and dim$M$. For all $x \in B^m_{\mu\beta}, i, k, \{j_1, \ldots, j_k\}$ we have that

$$|\frac{\partial^k f_i}{\partial x_{j_1} \ldots \partial x_{j_k}}(x)| < \tilde{C}_k$$

and

$$|\frac{\partial f'_i}{\partial x_{j_1} \ldots \partial x_{j_k}}(x)| < \tilde{C}_k.$$

Let $B^{m-1}_{\mu\beta} \subset B^m_{\mu\beta}$ denote the subset of all points $q \in B^m_{\mu\beta}$ whose first standard coordinate vanishes. There is no loss of generality in assuming that the maps f and f' also satisfy the following properties.

(b) $f(B^{m-1}_{\mu\beta})$ is perpendicular to the first factor of $(-5\alpha, 5\alpha) \times B(p, \beta)$ at y; and $f'(B^{m-1}_{\mu\beta})$ is perpendicular to the first factor of $(-5\alpha, 5\alpha) \times B(p', \beta)$ at y'

(c) $f_p \circ f = f_{p'} \circ f'$.

We may (by A.3.1(b)) define smooth embeddings

$$F : (-3\alpha, 3\alpha) \times B^{m-1}_{\mu\beta} \to (-5\alpha, 5\alpha) \times B(p, \beta)$$

and

$$F' : (-3\alpha, 3\alpha) \times B^{m-1}_{\mu\beta} \to (-5\alpha, 5\alpha) \times B(p', \beta)$$

as follows.

A.3.2. (a) $F(t, x) = f(x) + (t, 0, 0, \ldots, 0)$.

(b) $F'(t, x) = f'(x) + \sigma(t, 0, 0, \ldots, 0)$, where the number σ is defined by the equation $Df_p(\frac{\partial}{\partial y_1}(y)) = \sigma Df_{p'}(\frac{\partial}{\partial y'_1}(y'))$; note that $\sigma = \pm 1$.

It is immediate from A.3.1(a) and from A.3.2(a)(b) that F, F' satisfy the following properties. Let $F = (F_1, \ldots, F_m)$ and $F' = (F'_1, \ldots, F'_m)$ denote the coordinates of F and F' with respect to the $y_1, y_2, \ldots, y_k$ coordinates and the $y'_1, y'_2, \ldots, y'_k$ coordinates, respectively.

(c) For all $(t,x) \in (-3\alpha, 3\alpha) \times B^{m-1}_{\mu\beta}, i, k, l, \{j_1, \dots, j_{k-1}\}$ we have that

$$|\frac{\partial^k F_i}{\partial^l t \partial x_{j_1} \dots \partial x_{j_{k-l}}}(x)| < \tilde{C}_k + 1$$

and

$$|\frac{\partial^k F'_i}{\partial^l t \partial x_{j_1} \dots \partial x_{j_{k-l}}}(x)| < \tilde{C}_k + 1.$$

Now we define the map $h : (-2\alpha, 2\alpha) \times B(p, \delta\beta) \to (-5\alpha, 5\alpha) \times B(p', \delta\beta)$ of A.3 by

A.3.3. $h = F' \circ (F^{-1}|(-2\alpha, 2\alpha) \times B(p, \delta\beta))$.

Note that h is well defined by A.3.3 provided δ is chosen sufficiently small to assure that $B(p, \delta\beta) \subset \rho \circ f(B^{m-1}_{\frac{1}{2}\mu\beta})$, where $\rho : (-5\alpha, 5\alpha) \times B(p, \beta) \to B(p, \beta)$ is projection onto the second factor. Note also that property A.3(b) follows easily from A.3.2(c) and A.3.3. The verification of property A.3(a) is left as an exercise for the reader (cf. A.3.1(b)(c) and A.3.2(a)(b) and A.3.3).

This completes the proof of Corollary A.3.

Proof of Theorem A.4. Let $h : [0,1] \to (-5\alpha, 5\alpha) \times B(p, \beta)$ denote any smooth path, and we let $P_{h,e}(\ )$ and $P_{h,g}(\ )$ denote parallel translation along h with respect to the Euclidean metric $e = \delta^i_j dy_i dy_j$ and the pulled back metric $g = g_{i,j} dy_i dy_j$, respectively. Let $\text{length}_e(h)$ and $\text{length}_g(h)$ denote the lengths for h computed in terms of the metrics e and g, respectively. Theorem A.4 is an immediate consequence of the following claim.

A.4.1. Claim. *There is a number $\omega > 1$ which depends only on the $\{C_i\}$ and on $m = dim M$. For any smooth path $h : [0,1] \to (-5\alpha, 5\alpha) \times B(p, \beta)$ and any unit vector*

$$v \in T((-5\alpha, 5\alpha) \times B(p, \beta))_{|h(0)}$$

we have:

(a) $\Theta(P_{h,g}(v), P_{h,e}(v)) < \omega(\text{length}_e(h))$

(b) $\frac{1}{\omega}\ \text{length}_e(h) < \text{length}_g(h) < \omega(\text{length}_e(h))$.

Note that A.4.1(b) follows easily from A.2(a)(b).

To verify A.4.1(a) we first note that A.2(a)(b) imply the Christoffel functions $\Gamma^k_{i,j}$ associated to $g = g_{i,j} dy_i dy_j$ satisfy the following inequalities.

A.4.2. There is a positive number $\tilde{C}$ which depends only on the $\{C_i\}$ and $\dim M$. For all $y \in (-5\alpha, 5\alpha) \times B(p, \beta)$ and all i, j, k, we have that

$$|\Gamma^k_{i,j}(y)| < \tilde{C}.$$

Now A.4.2, together with the classical differential equations for parallel translation in the metric $g = g_{i,j} dy_i dy_j$, imply A.4.1(a).

This completes the proof for Theorem A.4.

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