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MULTIPLICITIES OF EQUIFOCAL HYPERSURFACES IN SYMMETRIC SPACES*

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Abstract. Some necessary and sufficient conditions on the multiplicities for which there exist equifocal hypersurfaces in symmetric spaces of rank two, or in the Cayley projective plane, are obtained.

1. Introduction. We will assume that N is a compact, rank-k symmetric space of semi-simple type. Let $i : M \to N$ be an immersion, and $\mathcal{V}(M)$ the normal bundle of M. The end point map $\eta : \mathcal{V}(M) \to N$ of M is the restriction of the exponential map exp to $\mathcal{V}(M)$. A normal vector $v \in \mathcal{V}_x(M)$ is called a multiplicity m focal normal and $\exp(v)$ is called a multiplicity m focal point of M with respect to M in N if v is a singular point of η and $\dim \operatorname{Ker}(d\eta_v)$ is equal to m. The focal data, $\Gamma(M)$, is defined to be the set of all pairs (v, m) such that v is a multiplicity m focal normal of M.

A connected, compact, immersed submanifold M in a symmetric space N is called equifocal if (a) $\mathcal{V}(M)$ is globally flat and abelian, and (b) the focal data $\Gamma(M)$ is invariant under normal parallel translation. For the details, the reader is referred to [TT], a fundamental and important paper in this field. The definition of weakly equifocal submanifold is also given in [TT].

When the ambient space N is the space form S^n, R^n or H^n, equifocal and weakly equifocal hypersurfaces have been extensively studied. In fact, in these space forms, they are just the isoparametric and proper Dupin hypersurfaces respectively.

Isoparametric hypersurfaces, i.e., hypersurfaces with constant principal curvatures, are in some sense the simplest examples for the theory of hypersurfaces. The study of isoparametric hypersurfaces in S^n has a long history. Assume that M is an isoparametric hypersurface of S^n with g distinct constant principal curvatures $\lambda_1 > \cdots > \lambda_g$ with multiplicities $m_1, \ldots, m_g$. E. Cartan [Ca] considered isoparametric hypersurfaces first, and solved completely the classification problem for $g = 1, 2$ or 3.

Using cohomological arguments, Münzner [Mu] proved the following:

(a) g must be 1, 2, 3, 4 or 6;

(b) $m_k = m_1$ if k is odd, and $m_k = m_2$ if k is even;

(c) S^n admits a disc bundles decomposition: $S^n = D_1 \cup D_2$, where D_k is the normal disc bundle of the focal submanifold M_k, and $D_1 \cap D_2 = M$.

It was shown by Thorbergsson in [Th1] that proper Dupin hypersurfaces have the properties (a), (b), (c) above.

Without loss of generality, we can assume that $m_1 \leq m_2$ throughout this paper.

In the case $g = 4$ or $g = 6$, many significant results have been obtained. For example, the construction of the isoparametric hypersurfaces of Clifford types [FKM], the restrictions on the multiplicities m_k [Ab, Ta], as well as the calculations of the

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Brouwer degrees of the gradient maps of the isoparametric polynomials [PT]. Quite recently, by using delicate homotopy theory, Stolz [St] showed that there exists a proper Dupin hypersurface in S^n with $g = 4$ and the multiplicities (m_1, m_2) if and only if $(m_1, m_2) = (2,2), (4,5)$ or $2^{\phi(m_1-1)}$ divides $m+1$, where $m = m_1 + m_2$ and $\phi(x) = \#\{0 < s \leq x | s \equiv 0, 1, 2, \text{ or } 4 \bmod 8\}$.

In [TT], Terng and Thorbergsson generalized results on isoparametric hypersurfaces in spheres to equifocal hypersurfaces in simply connected, compact symmetric space N. More precisely, they established the following:

THEOREM ([TT]). *Let M be an immersed, compact, equifocal hypersurface in a simply connected, compact, symmetric space N. Then the following hold:*

(a) There exist integers m_1, m_2, an even number $2g$, and $0 < \theta < l/2g$ (where l denotes the length of normal geodesics) such that the focal points on the normal circle $T_x = \exp(\mathcal{V}_x(M))$ are

$$x(j) = \exp\left(\left(\theta + \tfrac{(j-1)l}{2g}\right)\mathcal{V}(x)\right),$$

$1 \leq j \leq 2g$, and their multiplicities are m_1 if j is odd, and m_2 if j is even;

(b) Let $\eta_{tv} : M \to N$ be the end point map, and $M_t = \eta_{tv}(M)$ the set parallel to M at distance t. Then M_t is an equifocal hypersurface and η_{tv} maps M diffeomorphically onto M_t if $t \in (-l/2g + \theta, \theta)$;

(c) $M_1 = M_\theta$ and $M_2 = M_{-l/2g+\theta}$ are embedded submanifolds of codimension $m_1 + 1$, $m_2 + 1$ in N. The map $\eta_{\theta v} : M \to M_1$ and $\eta_{(-l/2g+\theta)v} : M \to M_2$ produce S^{m_1} and S^{m_2} bundles respectively;

(d) $N = D_1 \cup D_2$ and $M = D_1 \cap D_2$, where D_1 and D_2 are diffeomorphic to the normal disc bundles of M_1 and M_2 respectively.

When the ambient space N is a complex or quaternionic projective space, one can get a new isoparametric hypersurface in the Euclidean sphere by using the Hopf fibration. It was proved in [Wu] that g has to be 1, 2 or 3 if $N = \mathbb{C}P^n$ or $\mathbb{H}P^n$.

The case of $N = \mathbb{Q}P^2$, the Cayley projective plane, is more interesting since there one cannot use a Hopf fibration. A necessary and sufficient condition for the multiplicities of equifocal hypersurfaces in $\mathbb{Q}P^2$ will be given in section 5.

It is then left to consider equifocal hypersurfaces in symmetric spaces of rank more than one, and we will work mainly on rank two in this paper.

Recall that any cohomogeneity one action on a symmetric space G/K is hyperpolar (cf. [HPTT]). Moreover, it is stated in [TT] that if H is a closed subgroup of G that acts hyperpolarly on G/K, then the principal H-orbits are equifocal.

In order to construct equifocal hypersurfaces, we have not only the method of cohomogeneity one action, but also the way of Riemannian submersion. Suppose that $\pi : \tilde{N} \to N$ is a Riemannian submersion from a Riemannian manifold $\tilde{N}$ to a simply connected, compact, symmetric space N, and we are given an equifocal hypersurface M in N with multiplicities (m_1, m_2). Define

$$\tilde{M} = \pi^{-1}(M),$$

$\tilde{M}_k = \pi^{-1}(M_k)$, $\tilde{D}_k = \pi^{-1}(D_k)$, where $M_k \subset N$ are the focal submanifolds of the equifocal hypersurface M.

PROPOSITION 1.1. *There are two fibrations*

$$S^{m_k} \hookrightarrow \tilde{M} \xrightarrow{\tilde{p}_k} \tilde{M}_k$$

with $\pi \circ \tilde{p}_k = p_k \circ \pi$ *and a disc bundles decomposition:*

$$\tilde{N} = \tilde{D}_1 \cup \tilde{D}_2, \quad \tilde{M} = \tilde{D}_1 \cap \tilde{D}_2, \quad \tilde{D}_k \simeq \tilde{M}_k.$$

Furthermore, if $\tilde{N}$ *is a symmetric space then* $\tilde{M}$ *is an equifocal hypersurface of* $\tilde{N}$.

Proof. Let v be a parallel normal field on M, and $\tilde{v}$ the horizontal lift of v to $\tilde{M}$. since π is a Riemannian submersion, $\tilde{v}$ is a parallel normal field of $\tilde{M}$. Observing that the map $f : \pi^{-1}(p) \to \pi^{-1}(q)$ for $q = \exp^N(tv(p))$, given by sending x to $\exp^{\tilde{N}}(t\tilde{v}(x))$, is a diffeomorphism (see 5.10 lemma in [TT]), it defines the maps $\tilde{p}_k : \tilde{M} \to \tilde{M}_k$ satisfying $\pi \circ \tilde{p}_k = p_k \circ \pi$.

It is clear that the maps $\tilde{p}_k$ are all fibrations.

When $\tilde{N}$ is a symmetric space, we make use of transformal map. According to [Wa], a smooth real function $f : N \to R$ defined on a Riemannian manifold N is called transnormal if there is a smooth function ϕ such that $\|df\|^2 = \phi(f)$. It is not difficult to see that in a symmetric space N, the regular level set of a transnormal function corresponds to the equifocal hypersurface. Finally, we observe that the composition $f \circ \pi : \tilde{N} \to R$ is a transnormal function, since π is a Riemannian submersion. These complete the proof of the proposition. □

REMARK 1.2. In a sense, this proposition can be regarded as a generalization of Theorem 1.9 in [TT] which was used to construct inhomogeneous equifocal hypersurfaces in $SO(n+1)$ by the Riemannian submersion: $SO(n+1) \to SO(n+1)/SO(n) \cong S^n$. It should be interesting to investigate the properties of the equifocal hypersurfaces in $SU(n)$ by the Riemannian submersion:

$$SU(n) \to SU(n)/SU(n-1) \cong S^{2n-1}.$$

Note that $\big(SU(n), SU(n-1)\big)$ is not a symmetric pair.

Thorbergsson in [Th2] established a remarkable equality involving g, m_1, and m_2, for an equifocal hypersurface M in a simply connected symmetric space N.

PROPOSITION ([Th2]). *Let* i *denote the index of* $\gamma|_{[0,2\pi]}$ *as a critical point of the energy functional* E *in the path space* Ω_{pp} *and let* v *denote its nullity. Then we have*

$$g(m_1 + m_2) = i + v.$$

REMARK 1.3. For $N = S^n$, $\mathbb{C}P^n$, HP^n, the equality of Thorbergsson will give well-known formulas $g(m_1 + m_2) = 2(n-1)$ for S^n, $g(m_1 + m_2) = 2n$ for CP^n and $g(m_1 + m_2) = 4n + 2$ for HP^n. More interesting, it gives $g(m_1 + m_2) = 22$ for the Cayley projective plane QP^2.

Unfortunately, for the symmetric space of rank more than one, we don't know how to use this formula.

Some necessary and sufficient conditions for the multiplicities of equifocal hypersurfaces in symmetric spaces of rank two, or in the Cayley projective plane were obtained in this paper. However, we are not able to get any estimates of the number g, except in the Cayley projective plane. The problem for symmetric spaces of rank more than two is still open. In order to recognize the number g, it would be helpful to recall the fact ([TT]) that the dihedral group W with $2g$ elements acts on M freely. We don't know how to calculate the α-invariants $\alpha(M^{2n-1}, x)$, introduced by Atiyah and Singer [AS]. Note that in most cases, the equifocal hypersurface is odd dimensional.

REMARK 1.4. As mentioned in [TT], Lie sphere geometry of S^n (cf. [CC]) should be naturally extended to compact symmetric spaces. Several interesting questions in these directions were posed in [TT].

2. In complex Grassmann spaces. Let $M \subset N$ be an equifocal hypersurface with multiplicities (m_1, m_2) in a simply connected, compact, symmetric space $N (m_1 \le m_2)$; M_1, M_2 the focal submanifolds. Let

$$\xi_k : S^{m_k} \hookrightarrow M \xrightarrow{p_k} M_k \quad (k = 1, 2)$$

be the normal sphere bundle of M_k in N, and $D_k \to M_k$ the normal disc bundle associated to ξ_k. According to [TT], there exists a disc bundles decomposition of N:

$$D_1 \cup D_2 = N, \quad D_1 \cap D_2 = M, \quad D_k \simeq M_k.$$

As consequences, we have immediately the following two exact cohomology sequences ($\mathbb{Z}_2$ as the coefficients): the Meyey-Vietoris (M-V for short) sequence

(2.0a)
$$\cdots \to H^i(N;\mathbb{Z}_2) \to H^i(M_1;\mathbb{Z}_2) \oplus H^i(M_2;\mathbb{Z}_2) \xrightarrow{f} H^i(M;\mathbb{Z}_2) \xrightarrow{\delta} H^{i+1}(N;\mathbb{Z}_2) \to \cdots$$

where $f(x,y) = p_1^* x - p_2^* y$ for $x \in H^i(M_1;\mathbb{Z}_2)$ and $y \in H^i(M_2;\mathbb{Z}_2)$; and the Gysin sequence

(2.0b)
$$\cdots H^{i-m_k-1}(M_k;\mathbb{Z}_2) \xrightarrow{\psi_k} H^i(M_k;\mathbb{Z}_2) \xrightarrow{p_k^*} H^i(M;\mathbb{Z}_2) \xrightarrow{\partial} H^{i-m_k}(M_k;\mathbb{Z}_2) \to \cdots$$

where $\psi_k(x) = x \cdot e(k)$ for $x \in H^{i-m_k-1}(M_k;\mathbb{Z}_2)$, $e(k) \in H^{m_k+1}(M_k;\mathbb{Z}_2)$ is the mod 2 Euler class of the bundle ξ_k, i.e., the top Stiefel-Whitney characteristic class of the sphere bundle ξ_k.

If the dimension of N is even, we state

PROPOSITION 2.1. *Suppose $N = N^{2n}$, then the following statements hold:*
(i) When $H^{n+1}(N;\mathbb{Z}_2) \cong 0$ and $H^n(N:\mathbb{Z}_2) \ncong 0$, we have $m_1 \le n-1$;
(ii) When $H^{n+1}(N;\mathbb{Z}_2) \ncong 0$ and $H^n(N;\mathbb{Z}_2) \cong 0$, we have $m_1 \le n$.
(*The statements remain true for any coefficients.*)

Proof. (i) Suppose that $m_1 > n-1$, then $\dim M_2 \le \dim M_1 < n$, in particular $H^n(M_1;\mathbb{Z}_2) \cong H^n(M_2;\mathbb{Z}_2) \cong 0$. The M-V sequence

$$\begin{aligned} \cdots &\to H^{n-1}(M;\mathbb{Z}_2) \to H^n(N;\mathbb{Z}_2) \to H^n(M_1;\mathbb{Z}_2) \oplus H^n(M_2;\mathbb{Z}_2) \\ &\to H^n(M;\mathbb{Z}_2) \to H^{n+1}(N;\mathbb{Z}_2) \to \cdots \end{aligned}$$

together with the hypothesis imply that $H^n(M;\mathbb{Z}_2) \cong 0$. It follows from the Poincaré duality theorem that

$$H^{n-1}(M;\mathbb{Z}_2) \cong H^n(M;\mathbb{Z}_2) \cong 0.$$

The M-V sequence, mentioned above, implies finally that $H^n(N;\mathbb{Z}_2) \cong 0$, a contradiction to the assumption.

(ii) Suppose that $m_1 > n$, then $\dim M_2 \le \dim M_1 < n-1$, in particular

$$H^{n-1}(M_k;\mathbb{Z}_2) \cong H^n(M_k;\mathbb{Z}_2) \cong H^{n+1}(M_k;\mathbb{Z}_2) \cong 0 \quad \text{for } k = 1 \text{ or } 2.$$

The M-V sequence (2.0a) yields that

$$H^{n-1}(M;\mathbb{Z}_2) \cong H^n(N;\mathbb{Z}_2) \cong 0$$

and

$$H^n(M;\mathbb{Z}_2) \cong H^{n+1}(N;\mathbb{Z}_2) \not\cong 0,$$

a contradiction to the Poincaré duality theorem: $H^n(M;\mathbb{Z}_2) \cong H^{n-1}(M;\mathbb{Z}_2)$. □

REMARK. The manifolds $N = CP^{2m}$, $\mathbb{H}P^{2m}$, $G_p(\mathbb{C}^{p+m})$, $G_p(\mathbb{H}^{p+m})$ with even pm and $\mathbb{Q}P^2$ satisfy the assumption (i) of Proposition 2.1; and the manifolds

$$N = \mathbb{C}P^{2m+1}, \quad G_p(\mathbb{C}^{p+m})$$

with odd pm satisfy the assumption (ii) of Proposition 2.1.

Recall that a symmetric space G/K is called inner if $\operatorname{rank} G = \operatorname{rank} K$([He]), that is if K contains a maximal torus of G. Inner symmetric spaces have always even dimensions.

PROPOSITION 2.2. *Let $N = G/K$ be an inner symmetric space, then m_1 and m_2 are not all even.*

Proof. It is well known that an inner symmetric space has positive Euler number. More precisely, the Euler number $\chi(G/K)$ is equal to the quotient $\operatorname{ord} W(G)$ by $\operatorname{ord} W(K)$ (see [BC]). Combining the well known equality

$$\begin{array}{ccccccc} \chi(D_1 \cup D_2) & + & \chi(D_1 \cap D_2) & = & \chi(D_1) & + & \chi(D_2) \\ \| & & \| & & \| & & \| \\ \chi(N) & & \chi(M) & & \chi(M_1) & & \chi(M_2) \end{array}$$

with the fact that the Euler number of an odd dimensional manifold is always zero, we imply the conclusion. □

We are now in a position to consider $N = G_2(\mathbb{C}^{n+2})$, the complex Grassmann manifold of real dimension $4n$. M denotes an equifocal hypersurface with multiplicities (m_1, m_2).

Let us consider the complex Stiefel manifold

$$\tilde{N} = V_2(\mathbb{C}^{n+2}) = U(n+2)/U(n).$$

$K = U(2)$ acts on the Riemannian manifold $\tilde{N}$, and the action is isometric, has only one orbit type. Then there exists a unique metric on

$$\tilde{N}/K = U(n+2)/\big(U(n) \times U(2)\big) = G_2(\mathbb{C}^{n+2}),$$

and a Riemannian submersion:

$$\pi : \tilde{N} = V_2(\mathbb{C}^{n+2}) \to N = G_2(\mathbb{C}^{n+2}).$$

Applying Proposition 1.1, we get a disc bundles decomposition:

$$\tilde{N} = \tilde{D}_1 \cup \tilde{D}_2, \quad \tilde{M} = \tilde{D}_1 \cap \tilde{D}_2, \quad \tilde{D}_k \simeq \tilde{M}_k \quad (k = 1, 2).$$

LEMMA 2.3.

(i) The inequality $m_1 \leq 2n - 1$ holds;

(ii) m_1 and m_2 are not all even.

Proof. (i) It is well known that the complex Grassmannian $G_2(\mathbb{C}^{n+2})$ has the Poincaré polynomial (cf. [BT], p. 292)

$$P\big(G_2(\mathbb{C}^{n+2}), t\big) = \frac{(1-t^2)\cdots\cdots(1-t^{2n+4})}{(1-t^2)(1-t^4)(1-t^2)\cdots(1-t^{2n})}.$$

Particularly $H^{2n}(G_2(\mathbb{C}^{n+2});\mathbb{Z}_2) \not\cong 0$, and $H^{2n+1}(G_2(\mathbb{C}^{n+2});\mathbb{Z}_2) \cong 0$. Then the assertion (i) follows directly from Proposition 2.1.

(ii) Note that the complex Grassmannians are all inner symmetric space. The conclusion follows immediately from Proposition 2.2. □

Now let us restrict attention to the disc bundles decomposition of $\tilde{N}$. Let

$$e(k) \in H^{m_k+1}(\tilde{M}_k;\mathbb{Z}_2)$$

be the mod 2 Euler class of the sphere bundle

$$S^{m_k} \hookrightarrow \tilde{M} \xrightarrow{\tilde{p}_k} \tilde{M}_k.$$

When $m_k > 2n+1$, we have $m_k + 1 > 4n + 3 - m_k = \dim \tilde{M}_k$, then

$$e(k) \in H^{m_k+1}(\tilde{M}_k;\mathbb{Z}_2) \cong 0.$$

LEMMA 2.4. *There exist isomorphisms* $H^i(\tilde{M};\mathbb{Z}_2) \cong H^i(\tilde{M}_1;\mathbb{Z}_2) \oplus H^i(\tilde{M}_2;\mathbb{Z}_2)$ *for* $i \neq 0,\ 2n,\ 2n+1,\ 2n+2,\ 2n+3,\ 4n+3$, *and an exact cohomology sequence:*

$$\begin{array}{cccccccc}
0 & \longrightarrow & H^{2n}(\tilde{M}_1;\mathbb{Z}_2) & \oplus & H^{2n}(\tilde{M}_2;\mathbb{Z}_2) & \longrightarrow & H^{2n}(\tilde{M};\mathbb{Z}_2) & \longrightarrow \\
\mathbb{Z}_2 & \xrightarrow{\phi} & H^{2n+1}(\tilde{M}_1;\mathbb{Z}_2) & \oplus & H^{2n+1}(\tilde{M}_2;\mathbb{Z}_2) & \longrightarrow & H^{2n+1}(\tilde{M};\mathbb{Z}_2) & \longrightarrow \\
0 & \longrightarrow & H^{2n+2}(\tilde{M}_1;\mathbb{Z}_2) & \oplus & H^{2n+2}(\tilde{M}_2;\mathbb{Z}_2) & \longrightarrow & H^{2n+2}(\tilde{M};\mathbb{Z}_2) & \longrightarrow \\
\mathbb{Z}_2 & \xrightarrow{\psi} & H^{2n+3}(\tilde{M}_1;\mathbb{Z}_2) & \oplus & H^{2n+3}(\tilde{M}_2;\mathbb{Z}_2) & \longrightarrow & H^{2n+3}(\tilde{M};\mathbb{Z}_2) & \longrightarrow \\
0. & & & & & & &
\end{array}$$

Proof. It is sufficient to notice that the cohomology of the complex Stiefel manifold $\tilde{N} = V_2(\mathbb{C}^{n+2})$ is given by (cf. [Wh], p. 348)

$$H^*(V_2(\mathbb{C}^{n+2});\mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2, & \text{for } * = 0, 2n+1, 2n+3, 4n+4; \\ 0, & \text{otherwise.} \end{cases}$$

The results we want to prove follow immediately from the M-V sequence (2.0a). □

LEMMA 2.5. *The Euler class* $e(1)$ *vanishes, and hence*

$$H^i(\tilde{M};\mathbb{Z}_2) \cong H^i(\tilde{M}_1;\mathbb{Z}_2) \oplus H^{i-m_1}(\tilde{M}_1;\mathbb{Z}_2)$$

for every i.

Proof. It is evident to see from Lemma 2.4 that, the map

$$\tilde{p}_1^* : H^i(\tilde{M}_1;\mathbb{Z}_2) \to H^i(\tilde{M};\mathbb{Z}_2)$$

is injective for $i \leq 2n$. On the other hand, by Lemma 2.3, we have $m_1 \leq 2n-1$. Thus, $e(1)$ must be trivial and the map

$$\psi_1 : H^i(\tilde{M}_1;\mathbb{Z}_2) \to H^{i+m_1+1}(\tilde{M}_1;\mathbb{Z}_2)$$

in the Gysin sequence (2.0b) of the sphere bundle $S^{m_1} \hookrightarrow \tilde{M} \to \tilde{M}_1$ vanishes for every i. □

LEMMA 2.6. *The Euler class* $e(2)$ *vanishes, and hence* $H^i(\tilde{M};\mathbb{Z}_2) \cong H^i(\tilde{M}_2;\mathbb{Z}_2) \oplus H^{i-m_2}(\tilde{M}_2;\mathbb{Z}_2)$ *for every* i.

Proof. It is obvious to see that the Euler class $e(2)$ has to be trivial if $m_2 \neq 2n$, $2n+2$. When $m_2 = 2n+2$, then $e(2) \in H^{m_2+1}(\tilde{M}_2;\mathbb{Z}_2) \cong 0$, since $m_2 + 1 > \dim \tilde{M}_2$. So we are left to consider the case $m_2 = 2n$. By Lemma 2.3, m_1 has to belong to $\{1, 3, 5, \ldots, 2n-1\}$.

(i) For $m_1 = 1$. We have $\dim \tilde{M}_1 = 4n+2$, and $\dim \tilde{M}_2 = 2n+3$. It follows from Lemma 2.4, 2.5, and the Gysin sequence of the sphere bundle $S^{m_2} \hookrightarrow \tilde{M} \to \tilde{M}_2$ that:

$$\begin{array}{rcccccc} H^1(\tilde{M};\mathbb{Z}_2) & \cong & H^1(\tilde{M}_2;\mathbb{Z}_2) & \cong & H^1(\tilde{M}_1;\mathbb{Z}_2) & \oplus & H^1(\tilde{M}_2;\mathbb{Z}_2) \\ H^2(\tilde{M};\mathbb{Z}_2) & \cong & H^2(\tilde{M}_2;\mathbb{Z}_2) & \cong & H^2(\tilde{M}_1;\mathbb{Z}_2) & \oplus & H^2(\tilde{M}_2;\mathbb{Z}_2) \\ & \cong & H^2(\tilde{M}_1;\mathbb{Z}_2) & \oplus & H^1(\tilde{M}_1;\mathbb{Z}_2) & & \end{array}$$

which imply that $H^2(\tilde{M}_1;\mathbb{Z}_2) \cong 0$. Thus

$$H^2(\tilde{M}_2;\mathbb{Z}_2) \cong H^2(\tilde{M}_1;\mathbb{Z}_2) \oplus H^1(\tilde{M}_1;\mathbb{Z}_2) \cong 0,$$

and hence the Euler class

$$e(2) \in H^{2n+1}(\tilde{M}_2;\mathbb{Z}_2) \cong H^2(\tilde{M}_2;\mathbb{Z}_2) \cong 0.$$

(ii) For $m_1 \geq 3$. We have $\dim \tilde{M}_2 = 2n+3$. Similarly, we have

$$\begin{array}{l} H^2(\tilde{M};\mathbb{Z}_2) \cong H^2(\tilde{M}_1;\mathbb{Z}_2) \oplus H^2(\tilde{M}_2;\mathbb{Z}_2) \\ H^2(\tilde{M};\mathbb{Z}_2) \cong H^2(\tilde{M}_1;\mathbb{Z}_2) \oplus H^{2-m_1}(\tilde{M}_1;\mathbb{Z}_2) \cong H^2(\tilde{M}_1;\mathbb{Z}_2) \end{array}$$

which imply that $H^2(\tilde{M}_2;\mathbb{Z}_2) \cong 0$. □

THEOREM 2.7. *For $N = G_2(\mathbb{C}^{n+2})$, we have either*
(i) (A) $(m_1+m_2)d = 2n$, or
(ii) (B) $(m_1+m_2)d = 2n+2$,
where $d = 1/4 \sum \dim H^i(\tilde{M}_1;\mathbb{Z}_2)$ is an integer. Moreover if n is odd, only (B) occurs.

Proof. Given an equifocal hypersurface M in $N = G_2(\mathbb{C}^{n+2})$. As already noted we have a disc bundles decomposition of

$$\tilde{N} = V_2(\mathbb{C}^{n+2}).$$

Let

$$P(\tilde{M},t) = \sum a_i t^i,\, P(\tilde{M}_1,t) = \sum b_j t^j \quad \text{and} \quad P(\tilde{M}_2,t) = \sum c_k t^k$$

be the Poincaré polynomials of

$$\tilde{M}, \quad \tilde{M}_1, \quad \tilde{M}_2,$$

with $a_i = \dim H^i(\tilde{M};\mathbb{Z}_2)$, $b_j = \dim H^j(\tilde{M}_1;\mathbb{Z}_2)$, $c_k = \dim H^k(\tilde{M}_2;\mathbb{Z}_2)$ respectively.

It follows from Lemma 2.5 and 2.6 that

$$P(\tilde{M},t) = (1+t^{m_1})\,P(\tilde{M}_1,t) \tag{2.1}$$

and

$$P(\tilde{M},t) = (1+t^{m_2})\,P(\tilde{M}_2,t). \tag{2.2}$$

Since $\tilde{M}$ is a closed manifold, satisfying the Poincaré duality theorem, we claim:

$$P(\tilde{M},t) = t^{4n+3}P(\tilde{M},t^{-1}). \tag{2.3}$$

On the other hand, it follows from Lemma 2.4 that

$$a_i = b_i + c_i \quad \text{for } 0 < i \le 2n-1, \text{ or } 2n+4 \le i < 4n+3. \tag{2.4}$$

(2.5)

$$\begin{array}{lllllll} \text{Either} & a_{2n} & = & b_{2n} + c_{2n} + 1, & a_{2n+1} & = & b_{2n+1} + c_{2n+1}, \\ \text{or} & a_{2n} & = & b_{2n} + c_{2n}, & a_{2n+1} & = & b_{2n+1} + c_{2n+1} - 1; \\ \text{either} & a_{2n+2} & = & b_{2n+2} + c_{2n+2} + 1, & a_{2n+3} & = & b_{2n+3} + c_{2n+3}, \\ \text{or} & a_{2n+2} & = & b_{2n+2} + c_{2n+2}, & a_{2n+3} & = & b_{2n+3} + c_{2n+3} - 1. \end{array}$$

We have

$$\begin{aligned} &(1+t^{m_2})P(\tilde{M},t) + (1+t^{m_1})P(\tilde{M},t) \\ &= (1+t^{m_1})(1+t^{m_2})\big(P(\tilde{M}_1,t) + P(\tilde{M}_2,t)\big) \qquad (\text{by}(2.1),(2.2)) \\ &= (1+t^{m_1})(1+t^{m_2})\big(P(\tilde{M},t) + 1 - t^{4n+3} + F\big) \quad (\text{by}(2.4),(2.5)) \end{aligned}$$

where $F = \sum_{i=2n}^{2n+3}(b_i + c_i - a_i)t^i$.
It is equivalent to

$$P(\tilde{M},t)(1 - t^{m_1+m_2})) = (1+t^{m_1})(1+t^{m_2})(1 - t^{4n+3} + F). \tag{2.6}$$

Moreover, by Lemma 2.4, we can deduce

$$F = t^{2n}(-1-t^2), t^{2n}(-1+t^3), t^{2n}(t-t^2) \text{ or } t^{2n}(t+t^3). \tag{2.7}$$

Applying (2.3), we have only two cases

$$F = t^{2n}(-1+t^3) \text{ or } F = t^{2n}(t-t^2). \tag{2.8}$$

(A) For $F = t^{2n}(-1+t^3)$, (2.6) becomes

$$P(\tilde{M},t)(1 - t^{m_1+m_2}) = (1+t^{m_1})(1+t^{m_2})(1-t^{2n})(1+t^{2n+3}). \tag{2.9}$$

As an immediate consequence, we see

$$P(\tilde{M},1)\cdot(m_1+m_2) = 8\cdot 2n. \tag{2.10}$$

Recall Lemma 2.3, which says that m_1 and m_2 are not all even. If m_1+m_2 is odd, then (2.10) yields

$$P(\tilde{M},1) \equiv 0 \bmod 16$$

and $P(\tilde{M}_1,1) = \frac{1}{2}P(\tilde{M},1) \equiv 0 \bmod 8$. If m_1 and m_2 are all odd, we can use the following lemma to show that

$$2n \equiv 0 \bmod (m_1+m_2)$$

and then $P(\tilde{M}_1,1) \equiv 0 \bmod 4$.

(B) For $F = t^{2n}(t - t^2)$, the proof is analogous.
If n is odd, observing that the cohomology algebra $H^*(U(n+2)/U(n); \mathbb{Z}_2)$ is generated by $u_{n+1} \in H^{2n+1}$ and $u_{n+2} \in H^{2n+3}$, and

$$\mathrm{sq}^2\, u_{n+1} = u_{n+2} \quad \text{(cf. [Wh], p. 400)}$$

we can see clearly that the case (A) does not occur. The reason is that the relation

$$\psi(\mathrm{sq}^2\, u_{n+1}) = \mathrm{sq}^2\big(\phi\,(u_{n+1})\big)$$

yields that $\phi = 0$ implies $\psi = 0$. □

LEMMA 2.8. *If m_1, m_2 are all odd, and $P(t) \in \mathbb{Z}[t]$, such that*

$$P(t)(1 - t^{m_1+m_2}) = (1 + t^{m_1})(1 + t^{m_2})(1 - t^{2n})(1 + t^{2n+3})$$

then $2n \equiv 0 \bmod (m_1 + m_2)$.

Proof. Note that $\mathbb{Z}[t]$ is a Gauss ring, and there is a decomposition

$$t^m - 1 = \prod_{d|m} \phi_d(t)$$

where

$$\phi_d(t) = \prod_{\xi:\ \text{primitive } d\text{-th root of } 1} (t - \xi).$$

Since we can write

$$P(t) = \frac{(1 - t^{2m_1})(1 - t^{2m_2})(1 - t^{2n})(t - t^{4n+6})}{(1 - t^{m_1})(1 - t^{m_2})(1 - t^{m_1+m_2})(1 - t^{2n+3})} \in \mathbb{Z}[t],$$

decomposing every item and noting that m_1, m_2 and $2n + 3$ are all odd, we see that $(m_1 + m_2)$ divides $2n$. □

Furthermore by using the methods of Münzner [Mu], we can obtain some restrictions on the number d.

THEOREM 2.9. *For $N = G_2(\mathbb{C}^{n+2})$.*
(A) $(m_1 + m_2)d = 2n$. If $m_2 > m_1 + 3 > 6$, then $d = 1, 2$ or 3.
(B) $(m_1 + m_2)d = 2n + 2$. If $m_2 > m_1 + 1 > 2$, then $d = 1, 2$ or 3.

In order to prove this theorem, we need some preliminaries.

LEMMA 2.10. *Let $M \subset N$ be an equifocal hypersurface with multiplicities (m_1, m_2), $m_1 \le m_2$, in a simply connected symmetric space. If $m_1 > 1$, then M, M_1 and M_2 are all simply connected.*

Proof. For a fixed point $p \in N$, let $P(N, M \times p)$ denote the space of H^1-paths $\gamma : [0,1] \to N$ such that $\big(\gamma(0), \gamma(1)\big) \in M \times \{p\}$. Let

$$E : P(N, M \times p) \to R$$

given by $E(\gamma) = \int_0^1 \|\gamma'(t)\|^2\, dt$ be the energy functional. According to Theorem 1.6 in [TT], if p is not a focal point of M then the map E is a perfect Morse function, and the index set of the critical points is

$$\{m_1, m_2, m_1 + m_2, \dots\},$$

thus E has no point of index one. We conclude that the space $P(N, M\times p)$ is simply connected.

Now the fibration

$$\Omega N \hookrightarrow P(N, M\times p) \to M$$

has an exact homotopy sequence

$$\pi_1\Omega N \to \pi_1 P(N, M\times p) \to \pi_1 M \to \pi_o\Omega N.$$

Since $\pi_0\Omega N \cong \pi_1 N \cong 0$, we have that $\pi_1 M = 0$.

Finally, considering the exact homotopy sequence of the sphere bundle

$$S^{m_k} \hookrightarrow M \to M_k,$$

we prove that $\pi_1 M_k = 0$ for $k = 1$ or 2 as required. □

LEMMA 2.11. *For* $N = G_2(\mathbb{C}^{n+2})$, $\tilde{N} = V_2(\mathbb{C}^{n+2})$. *If* $m_1 > 1$, *then* $\tilde{M}$, $\tilde{M}_1$ *and* $\tilde{M}_2$ *in* $\tilde{N}$ *are all orientable.*

Proof. By the preceding lemma, M is simply connected. Note that we have a decomposition

$$\begin{aligned}\tilde{N} &= SU(n+2)/SU(n) \xrightarrow{\pi_1} \bar{N}\\ &= SU(n+2)/SU(n)\times SU(2) \xrightarrow{\pi_2} N = SU(n+2)/S\big(U(n)\times U(2)\big)\end{aligned}$$

satisfying $\pi_2\circ\pi_1 = \pi : \tilde{N} \to N$, where π_1 and π_2 are orientable sphere bundles with fiber S^3 and S^1 respectively. Denoting $\bar{M} = \pi_2^{-1}(M)$, then we have an isomorphism of bundles

$$T\bar{M}\oplus 1 \cong \pi_2^* TM \oplus \pi_2^*\xi$$

where ξ is the associated vector bundle of $\pi_2 : \bar{N} \longrightarrow N$. It follows that $\bar{M}$ is orientable, and hence $\tilde{M}$ is orientable by the same reason. The conclusions are also true for $\tilde{M}_k (k = 1, 2)$ by an analogous argument. □

We recall the definition of type $(*)$ by Münzner [Mu].

DEFINITION 2.12. Let $R = \mathbb{Z}_2$ or $\mathbb{Z}$, $G = \sum_{i=0}^{\infty} G_i$ be an associate graded R-algebra. G is called to be type $(*)$ if the following conditions are satisfied:

(a) There exists a natural number $g \geq 2$, such that

$$G_i = \begin{cases} R, & \text{for } i = 0, g;\\ 0, & \text{for } i > g;\\ G_i^+ \oplus G_i^- & \text{for } 1 \leq i \leq g-1.\end{cases}$$

where $G_i^k = R$ for $k = \pm 1$, $1 \leq i \leq g-1$.

(b) $G^k = G_o + \sum_{i=1}^{g-1} G_i^k$ is a subalgebra of G.

(c) There exists an R-generator of G_1^+ (resp. G_1^-), say x_1 (y_1 resp.), such that G is a free G^+-modul (G^--modul) with basis $\{1, y_1\}$ ($\{1, x_1\}$, resp.).

(d) For $u \in G_i$, $w \in G_j$, $uw = \pm wu \in G_{i+j}$.

The following theorem was used to prove that $g \in \{1, 2, 3, 4, 6\}$ for isoparametric hypersurfaces in spheres.

THEOREM ([Mu]).

(1) If $R = \mathbb{Z}$, G *is of type* $(*)$, *then* $g \in \{2, 3, 4, 6\}$;

(2) If $R = \mathbb{Z}_2$, G is of type $(*)$, then $g = 2^r \cdot s$ with $s \in \{2, 3, 4\}$.

Let us apply this theorem to an equifocal hypersurface M in $G_2(\mathbb{C}^{n+2})$ with multiplicities (m_1, m_2), $m_1 \leq m_2$.

Following Theorem 2.7 and Theorem 2.9, it is convenient to denote

$$\mu = m_1 + m_2, \quad d = 1/4\, P(\tilde{M}_1, 1) \in \mathbb{Z},$$

then we have either (A) $\mu d = 2n$, or (B) $\mu d = 2n + 2$. ¿From now on, we will restrict attention to case (B). By Lemma 2.11, $\tilde{M}$, $\tilde{M}_1$ and $\tilde{M}_2$ are all orientable, thus it enables us to make us of the Gysin sequences and the M-V sequences with coefficient $\mathbb{Z}$.

Recall that

$$P(\tilde{M}_1, t) = (1 + t^{m_2})\frac{1 - t^{\mu d}}{1 - t^{\mu}}(1 + t^{2n+1}) = (1 + t^{m_2})(1 + t^{2n+1})(1 + t^{\mu} + \cdots + t^{\mu(d-1)})$$

and similarly

$$P(\tilde{M}_2, t) = (1 + t^{m_1})(1 + t^{2n+1})(1 + t^{\mu} + \cdots + t^{\mu(d-1)}).$$

Since $m_1 + \mu(d - 1) = \mu d - m_2 < 2n - 1$, when $* \leq 2n - 2$, we have

$$H^*(\tilde{M}_2; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & * = j\mu + \epsilon m_1,\ \epsilon = 0 \text{ or } 1,\ 0 \leq j \leq d - 1; \\ 0, & \text{otherwise.} \end{cases} \tag{2.11}$$

By the Gysin sequence of $S^{m_2} \hookrightarrow \tilde{M} \to \tilde{M}_2$ (with coefficient $\mathbb{Z}$), we see

$$H^*(\tilde{M}; \mathbb{Z}) \cong H^*(\tilde{M}_2; \mathbb{Z}) \oplus H^{*-m_2}(\tilde{M}_2; \mathbb{Z}) \quad \text{for } * \leq d\mu. \tag{2.12}$$

By using the Gysin sequence of $S^{m_1} \hookrightarrow \tilde{M} \to \tilde{M}_1$, when $* \leq 2n - 2$ we have

$$H^*(\tilde{M}_1; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & * = j\mu + \epsilon m_2, \epsilon = 0 \text{ or } 1, 0 \leq j \leq d - 1. \\ 0, & \text{otherwise.} \end{cases} \tag{2.13}$$

LEMMA 2.13. *We assume that $m_2 > m_1 + 1 > 2$ and $m_{-1} = m_2$, $m_{-2} = m_1$. For $x, y \in H^*(\tilde{M}_k; \mathbb{Z})$, $0 < * < (d-1)\mu + m_{-k}$, if $\deg(x \cdot y) > (d-1)\mu + m_{-k}$, then $x \cdot y = 0$.*

Proof. For $k = 2$. If $\deg x = a_1\mu, \deg y = a_2\mu$ with $a_1, a_2 \leq (d-1)$.

(1) If $a_1 + a_2 \leq d - 1$, then

$$a_1\mu + a_2\mu \leq (d-1)\mu = 2n + 2 - \mu < 2n - 1,$$

and hence $x \cdot y = 0$.

(2) If $a_1 + a_2 \geq d$, then by defining $a_3 = (a_1 + a_2) - d$, we have

$$d\mu - 1 + a_3\mu < a_1\mu + a_2\mu < d\mu - 1 + a_3\mu + m_1,$$

thus $x \cdot y = 0$. For the other cases or $k = 1$, the proofs are similar and straightforward. □

In fact, we can easily prove that for $x, y \in H^*(\tilde{M}; \mathbb{Z})$ with $0 < * \leq d\mu$,

(1) if $\deg(xy) > d\mu$, then $x \cdot y = 0$

(2) $\deg(xy) \neq d\mu - 1$.

Now, let $R = \mathbb{Z}$. Define

$$G^k_{2j+\epsilon} = \begin{cases} \tilde{p}_k^*(H^{j\mu+\epsilon m_{-k}}(\tilde{M}_k;\mathbb{Z})), & \text{for } k=\pm 1,\ \epsilon = 0 \text{ or } 1,\ 1 \le 2j+\epsilon \le 2d-1; \\ 0, & \text{for } 2j+\epsilon > 2d-1; \\ \mathbb{Z}, & \text{for } 2j+\epsilon = 0. \end{cases}$$

and

$$\begin{cases} G_{2j+1} = H^{j\mu+m_+}(\tilde{M};\mathbb{Z}) \oplus H^{j\mu+m_-}(\tilde{M};\mathbb{Z}), & \text{for } 0 < j \le (d-1); \\ G_{2j} = \tilde{p}_1^*(H^{j\mu}(\tilde{M}_1;\mathbb{Z})) \oplus \tilde{p}_2^*(H^{j\mu}(\tilde{M}_2;\mathbb{Z})), & \text{for } 0 < j \le (d-1); \\ G_0 = \mathbb{Z}; G_{2d} = \mathbb{Z}. \end{cases}$$

LEMMA 2.14. *G, G^+ and G^- are type $(*)$ for $R = \mathbb{Z}$.*

Proof. We prove the lemma for case (B), and for case (A) the proof is completely similar. By the previous lemma, (a), (b) and (d) of type $(*)$ are clearly satisfied. Choose $x_1(y_1)$ a generator of $H^{m_2}(\tilde{M}_1;\mathbb{Z}) \cong \mathbb{Z}$ ($H^{m_1}(\tilde{M}_2;\mathbb{Z}) \cong \mathbb{Z}$, resp.). We need to show that for $x \neq 0 \in H^i(\tilde{M}_1;\mathbb{Z})$,

$$\tilde{p}_1^* x \cdot y_1 \neq 0 \in H^{i+m_-}(\tilde{M};\mathbb{Z}).$$

It is only necessary to observe that

$$\lambda_1(\tilde{p}_1^* x \cdot y_1) = x \cdot \lambda_1(y_1) = x \cdot 1 = x,$$

where λ_1 originates in the Gysin sequence of $S^{m_1} \hookrightarrow \tilde{M} \to \tilde{M}_1$,

$$0 \to H^j(\tilde{M}_1;\mathbb{Z}) \xrightarrow{\tilde{p}_1^*} H^j(\tilde{M};\mathbb{Z}) \xrightarrow{\lambda_1} H^{j-m_1}(\tilde{M}_1;\mathbb{Z}) \to 0. \quad \square$$

We are now in a position to give a

Proof of Theorem 2.9. Applying the theorem of Münzner, mentioned above, to Lemma 2.14, we have $2d \in \{2,3,4,6\}$ and hence $d \in \{1,2,3\}$, The proof is complete. $\square$

EXAMPLE 2.15. Now notice that

$$G_2(\mathbb{C}^{n+2}) = SU(n+2)/S(U(2) \times U(n))$$

and $S(U(1) \times U(n+1))$ is a closed subgroup of $G = SU(n+2)$. Clearly the action of $S(U(1) \times U(n+1))$ on $G_2(\mathbb{C}^{n+2})$ is of cohomogeneity one. Thus we get an equifocal hypersurface in $G_2(\mathbb{C}^{n+2})$ with two focal submanifolds (the singular orbits of the action), one is the complex projective space $\mathbb{C}P^n$, and the other one is the complex Grassmann manifold $G_2(\mathbb{C}^{n+1})$. Therefore we have an equifocal hypersurface in $G_2(\mathbb{C}^{n+2})$ with multiplicities $(m_1, m_2) = (3, 2n-1)$. This example belongs to case (B) of Theorem 2.7 and $d = 1$.

EXAMPLE 2.16. When $n = 2m$ is even, $K_1 = \mathrm{Sp}(m+1)$ is a closed subgroup of $G = SU(2m+2)$. Let K_1 act on $G/K_2 = SU(2m+2)/S(U(2) \times U(2m))$. Clearly the action is of cohomogeneity one. Thus we get an equifocal hypersurface in $G_2(\mathbb{C}^{2m+2})$ with two focal submanifolds $\mathrm{Sp}(m+1)/U(2) \times \mathrm{Sp}(m-1)$ and HP^m. Therefore we have multiplicities $(m_1, m_2) = (1, 2n-1)$. This example belongs to case (A) of Theorem 2.7, and $d = 1$.

PROBLEM. Find more equifocal hypersurfaces in complex Grassmannians.

3. In quaternionic and real Grassmannians. First suppose we start with an equifocal hypersurface M in $N = G_2(H^{n+2})$, the quaternionic Grassmann manifold of rank two. There is a well-known fibration

$$\pi : \tilde{N} = V_2(H^{n+2}) = \mathrm{Sp}(n+2)/\mathrm{Sp}(n) \to N = G_2(H^{n+2}) = \frac{\mathrm{Sp}(n+2)}{\mathrm{Sp}(n) \times \mathrm{Sp}(2)}$$

with fiber Sp(2). Using the lifts of π, we get a decomposition of $\tilde{N}$:

$$\tilde{N} = \tilde{D}_1 \cup \tilde{D}_2, \quad \tilde{M} = \tilde{D}_1 \cap \tilde{D}_2, \quad \tilde{D}_k \cong \tilde{M}_k \tag{3.1}$$

and two fibrations $S^{m_k} \hookrightarrow \tilde{M} \xrightarrow{\tilde{p}_k} \tilde{M}_k$ $(k = 1, 2)$ with mod 2 Euler class

$$e(k) \in H^{m_k+1}(\tilde{M}_k; \mathbb{Z}_2).$$

An analogous statement of Lemma 2.3 is

LEMMA 3.1. (i) *The inequality* $m_1 \leq 4n - 1$ *holds;* (ii) m_1 *and* m_2 *are not all even.*

Proof.

(i) By [MT], it follows that

$$H^{4n}(G_2(H^{n+2}); \mathbb{Z}_2) \not\cong 0 \quad \text{and} \quad H^{4n+1}(G_2(H^{n+2}); \mathbb{Z}_2) \cong 0.$$

Applying Proposition 2.1, we have $m_1 \leq 4n - 1$.

(ii) Note that the quaternionic Grassmannians are inner symmetric space. The conclusion follows directly from Proposition 2.2. □

LEMMA 3.2. *There exist isomorphisms* $H^i(\tilde{M}; \mathbb{Z}_2) \cong H^i(\tilde{M}_1; \mathbb{Z}_2) \oplus H^i(\tilde{M}_2; \mathbb{Z}_2)$ *for* $i \neq 0,\ 4n+2,\ 4n+3,\ 4n+6,\ 4n+7,\ 8n+9$, *and an exact cohomology sequence:*

$$\begin{array}{lllllll}
0 & \longrightarrow & H^{4n+2}(\tilde{M}_1, \mathbb{Z}_2) & \oplus & H^{4n+2}(\tilde{M}_2; \mathbb{Z}_2) & \longrightarrow & H^{4n+2}(\tilde{M}; \mathbb{Z}_2) \longrightarrow \\
\mathbb{Z}_2 & \xrightarrow{\phi} & H^{4n+3}(\tilde{M}_1; \mathbb{Z}_2) & \oplus & H^{4n+3}(\tilde{M}_2; \mathbb{Z}_2) & \longrightarrow & H^{4n+3}(\tilde{M}; \mathbb{Z}_2) \longrightarrow \\
0 & \longrightarrow & H^{4n+6}(\tilde{M}_1; \mathbb{Z}_2) & \oplus & H^{4n+6}(\tilde{M}_2; \mathbb{Z}_2) & \longrightarrow & H^{4n+6}(\tilde{M}; \mathbb{Z}_2) \longrightarrow \\
\mathbb{Z}_2 & \xrightarrow{\psi} & H^{4n+7}(\tilde{M}_1; \mathbb{Z}_2) & \oplus & H^{4n+7}(\tilde{M}_2; \mathbb{Z}_2) & \longrightarrow & H^{4n+7}(\tilde{M}; \mathbb{Z}_2) \longrightarrow \\
0 & & & & & &
\end{array}$$

Proof. The proof is analogous to that of Lemma 2.4. It suffices to notice that the cohomology of the quaternionic Stiefel manifold $\tilde{N} = V_2(H^{n+2})$ is given by (cf. [Wh], p. 348)

$$H^*(V_2(H^{n+2}); \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2, & \text{for } * = 0, 4n+3, 4n+7, 8n+10 \\ 0, & \text{otherwise.} \end{cases}$$

□

An analogous result of Lemma 2.5 states

LEMMA 3.3. *The Euler class* $e(1)$ *vanishes, and hence*

$$H^i(\tilde{M}; \mathbb{Z}_2) \cong H^i(\tilde{M}_1; \mathbb{Z}_2) \oplus H^{i-m_1}(\tilde{M}_1; \mathbb{Z}_2) \quad \textit{for every } i.$$

Proof. It follows from Lemma 3.1 (i) that $m_1 + 1 \leq 4n$. It is evident to see from the previous lemma that the map

$$\tilde{p}_1^* : H^i(\tilde{M}_1; \mathbb{Z}_2) \to H^i(\tilde{M}; \mathbb{Z}_2)$$

is injective for $i \leq 4n+2$. Thus we have $e(1)=0$ and hence the lemma. □

LEMMA 3.4. *The Euler class* $e(2)$ *vanishes, and hence*

$$H^i(\tilde{M};\mathbb{Z}_2) \cong H^i(\tilde{M}_2;\mathbb{Z}_2) \oplus H^{i-m_2}(\tilde{M}_2;\mathbb{Z}_2) \quad \textit{for every } i.$$

Proof. Clearly the statement is true for $m_2 \neq 4n+2,\ 4n+6$.

(i) If $m_2 = 4n+6$, then

$$e(2) \in H^{4n+7}(\tilde{M}_2^{8n+9-m_2};\mathbb{Z}_2) \cong 0.$$

(ii) If $m_2 = 4n+2$. By Lemma 3.1 (ii), we see that m_1 must be odd. Since $e(2) \in H^{4n+3}(\tilde{M}_2;\mathbb{Z}_2) \cong H^4(\tilde{M}_2;\mathbb{Z}_2)$ by the Poincaré duality theorem. However,

$$\begin{aligned} H^4(\tilde{M};\mathbb{Z}_2) &\cong H^4(\tilde{M}_2;\mathbb{Z}_2) && \text{(by the Gysin Sequence)}\\ &\cong H^4(\tilde{M}_1;\mathbb{Z}_2) \oplus H^4(\tilde{M}_2;\mathbb{Z}_2) && \text{(by the M-V sequence)}\\ &\cong H^4(\tilde{M}_1;\mathbb{Z}_2) \oplus H^{4-m_1}(\tilde{M}_1;\mathbb{Z}_2) && \text{(by the Gysin sequence)} \end{aligned}$$

These imply that $H^4(\tilde{M}_2;\mathbb{Z}_2) \cong H^{4-m_1}(\tilde{M}_1;\mathbb{Z}_2)$. For $m_1 \geq 5$, $e(2) \in H^4(\tilde{M}_2;\mathbb{Z}_2) \cong H^{4-m_1}(\tilde{M}_1;\mathbb{Z}_2) \cong 0$. For $m_1 = 1, 3$, it is easy to see that $H^{4-m_1}(\tilde{M}_1;\mathbb{Z}_2) \cong 0$. □

THEOREM 3.5. *For* $N = G_2(H^{n+2})$, *we have either*

(A) $(m_1+m_2)\,d = 4n+2$, *or*

(B) $(m_1+m_2)\,d = 4n+6$,

where $d = 1/4 \sum \dim H^i(\tilde{M}_1;\mathbb{Z}_2) \in \mathbb{Z}$. *Moreover, if n is odd, only (B) occurs.*

Proof. By Lemma 3.2, 3.3 together with 3.4, we make use of Poincaré polynomials to get

$$P(\tilde{M},t)(1-t^{m_1+m_2}) = (1+t^{m_1})(1+t^{m_2})(1-t^{8n+9}+F)$$

where $F = -t^{4n+2}+t^{4n+7}$ or $F = -t^{4n+3}+t^{4n+6}$. The proof of this theorem is analogous to that of Theorem 2.7, the details will be omitted. However, it is worth noting that the cohomology algebra $H^*(\mathrm{Sp}\,(n+2)/\,\mathrm{Sp}\,(n);\mathbb{Z}_2)$ is generated (cf. [Wh], p. 400) by $u_{n+1} \in H^{4n+3}$ and $u_{n+2} \in H^{4n+7}$ with a relation

$$\mathrm{sq}^{4i}\, u_{n+1} = \binom{n}{i} u_{n+1+i} \quad \text{for } i + (n+1) \leq n+2.$$

When n is odd, we have $\mathrm{sq}^4\, u_{n+1} = u_{n+2}$. □

Furthermore, we can show

THEOREM 3.6. *For* $N = G_2(H^{n+2})$,

(A) $(m_1+m_2)\,d = 4n+2$. *If* $m_2 > m_1+5 > 10$, *then* $d = 1, 2$ *or* 3;

(B) $(m_1+m_2)\,d = 4n+6$. *If* $m_2 > m_1+3 > 6$, *then* $d = 1, 2$ *or* 3.

Proof. Similar to that of Theorem 2.9. □

EXAMPLE 3.7. Notice that $\mathrm{Sp}(1)\times\mathrm{Sp}(n+1)$ is a closed subgroup of $G = \mathrm{Sp}(n+2)$. Clearly that action of $\mathrm{Sp}(1)\times\mathrm{Sp}(n+1)$ on $G_2(H^{n+2}) = \mathrm{Sp}(n+2)/\,\mathrm{Sp}(2)\times\mathrm{Sp}(n)$ is of cohomogeneity one. Thus we get an equifocal hypersurface in $G_2(H^{n+2})$ with two focal submanifolds $G_2(H^{n+1})$ and HP^n with multiplicities $(m_1, m_2) = (7, 4n-1)$. This example belongs to case (B) of Theorem 3.5 and Theorem 3.6.

PROBLEM. Find examples belonging to case (A) of Theorem 3.5.

The rest of this section will be concerned with the study of multiplicities (m_1, m_2) of equifocal hypersurface in real oriented Grassmann manifold $N = \widetilde{G_2}(R^{n+2})$, which can be regarded as the complex quadric.

First we need the following results (cf. [BH]).

LEMMA 3.8. *Let* $N = \widetilde{G_2}(R^{n+2})$.

(1) If n is odd, then

$$H^*(N;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{for } * = 0,2,4,\dots,2n; \\ 0, & \text{otherwise.} \end{cases}$$

(2) If n is even, then

$$H^*(N;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{for } * = 0,2,4,\dots,\hat{n},\dots,2n; \\ \mathbb{Z}\oplus\mathbb{Z}, & \text{for } * = n; \\ 0, & \text{otherwise.} \end{cases}$$

As an immediate consequence, we have

LEMMA 3.9.

(1) $m_1 \le n$ *if* n *is odd;* $m_1 \le n-1$ *if* n *is even.*

(2) m_1 *and* m_2 *are not all even.*

Proof.

(1) The statements follow from Proposition 2.1 together with the previous lemma.

(2) Note that $\widetilde{G_2}(R^{n+2})$ is an inner symmetric space, thus the lemma follows from Proposition 2.2. □

Now suppose we are given an equifocal hypersurface M in N. Recall that there is a well-known fibration:

$$\pi : \tilde{N} = V_2(R^{n+2}) \to N = \widetilde{G_2}(R^{n+2})$$

from the Stiefel manifold.

Using the lifts by the fibration π, we get a decomposition of $\tilde{N} = V_2(R^{n+2})$,

$$\tilde{N} = \tilde{D}_1 \cup \tilde{D}_2, \quad \tilde{M} = \tilde{D}_1 \cup \tilde{D}_2, \quad \tilde{D}_k \cong \tilde{M}_k$$

and two fibrations:

$$S^{m_k} \hookrightarrow \tilde{M} \xrightarrow{\tilde{p}_k} \tilde{M}_k \quad \text{for } k = 1,2.$$

An analogous statement of Lemma 2.4 is

LEMMA 3.10. *There exist isomorphisms* $H^i(\tilde{M};\mathbb{Z}_2) \cong H^i(\tilde{M}_1;\mathbb{Z}_2) \oplus H^i(\tilde{M}_2;\mathbb{Z}_2)$ *for* $i \ne 0,\ n-1,\ n,\ n+1,\ 2n,$ *and an exact sequence*

$$\begin{array}{llccccc} 0 & \longrightarrow & H^{n-1}(\tilde{M}_1;\mathbb{Z}_2) & \oplus & H^{n-1}(\tilde{M}_2;\mathbb{Z}_2) & \longrightarrow & H^{n-1}(\tilde{M};\mathbb{Z}_2) & \longrightarrow \\ \mathbb{Z}_2 & \xrightarrow{\phi} & H^{n}(\tilde{M}_1;\mathbb{Z}_2) & \oplus & H^{n}(\tilde{M}_2;\mathbb{Z}_2) & \longrightarrow & H^{n}(\tilde{M};\mathbb{Z}_2) & \longrightarrow \\ \mathbb{Z}_2 & \xrightarrow{\psi} & H^{n+1}(\tilde{M}_1;\mathbb{Z}_2) & \oplus & H^{n+1}(\tilde{M}_2;\mathbb{Z}_2) & \longrightarrow & H^{n+1}(\tilde{M};\mathbb{Z}_2) & \longrightarrow \\ 0 & . \end{array}$$

Proof. The lemma follows from the M-V sequence. It suffices to give the cohomology of the Stiefel manifold (cf. [Wh], p. 348)

$$H^*\left(V_2(R^{n+2});\mathbb{Z}_2\right) \cong \begin{cases} \mathbb{Z}_2, & \text{for } * = 0,n,n+1,2n+1; \\ 0, & \text{otherwise.} \end{cases}$$

□

An analogous result of Lemma 2.5 states

LEMMA 3.11. *The Euler class* $e(1)$ *vanishes, and hence*

$$H^i(\tilde{M};\mathbb{Z}_2) \cong H^i(\tilde{M}_1;\mathbb{Z}_2) \oplus H^{i-m_1}(\tilde{M}_1;\mathbb{Z}_2) \quad \text{for every } i.$$

Proof. By Lemma 3.10, the statement is true for $m_1 \le n-2$. By Lemma 3.9, we are left to consider the case of $m_1 = n$ or $m_1 = n-1$.

(1) If $m_1 = n$, then $e(1) \in H^{m_1+1}(\tilde{M}_1^n; \mathbb{Z}_2) \cong 0$.
(2) If $m_1 = n-1$, then

$$e(1) \in H^n(\tilde{M}_1; \mathbb{Z}_2) \cong H^1(\tilde{M}_1; \mathbb{Z}_2)$$

by the Poincaré duality theorem. By Lemma 3.10 and the Gysin sequence of $S^{m_2} \hookrightarrow \tilde{M} \to \tilde{M}_2$ we have

$$H^1(\tilde{M}; \mathbb{Z}_2) \cong H^1(\tilde{M}_1; \mathbb{Z}_2) \oplus H^1(\tilde{M}_2; \mathbb{Z}_2),$$
$$H^1(\tilde{M}; \mathbb{Z}_2) \cong H^1(\tilde{M}_2; \mathbb{Z}_2).$$

Consequently $H^1(\tilde{M}_1; \mathbb{Z}_2) \cong 0$, hence $e(1) = 0$. □

LEMMA 3.12. *If $(m_1, m_2) \neq (1, n-1)$, then $e(2)$ vanishes, and hence*

$$H^i(\tilde{M}; \mathbb{Z}_2) \cong H^i(\tilde{M}_2; \mathbb{Z}_2) \oplus H^{i-m_2}(\tilde{M}_2; \mathbb{Z}_2) \quad \text{for every } i.$$

Proof. Case (1), $m_2 \leq n-2$. We have $m_2 + 1 \leq n-1$, and the conclusion follows from Lemma 3.10

Case (2), $m_2 \geq n$. We have

$$e(2) \in H^{m_2+1}(\tilde{M}_2; \mathbb{Z}_2) \cong 0,$$

since $m_2 + 1 > \dim \tilde{M}_2 = 2n - m_2$.

Case (3), $m_2 = n-1$, and $m_1 > 1$. Combining Lemma 3.10 together with the Gysin sequence of $S^{m_1} \hookrightarrow \tilde{M} \to \tilde{M}_1$, we obtain

$$H^1(\tilde{M}; \mathbb{Z}_2) \cong H^1(\tilde{M}_1; \mathbb{Z}_2) \oplus H^1(\tilde{M}_2; \mathbb{Z}_2),$$
$$H^1(\tilde{M}; \mathbb{Z}_2) \cong H^1(\tilde{M}_1; \mathbb{Z}_2).$$

Consequently $H^1(\tilde{M}_2; \mathbb{Z}_2) \cong 0$, and hence

$$e(2) \in H^n(\tilde{M}_2; \mathbb{Z}_2) \cong H^1(\tilde{M}_2; \mathbb{Z}_2) \cong 0. \square$$

THEOREM 3.13. *For $N = \widetilde{G_2}(R^{n+2})$, if $(m_1, m_2) \neq (1, n-1)$, then we have one of the following conditions:*

(A_1) $(m_1 + m_2)\, d = (n-1)$ with $2d \in \mathbb{Z}$, for $m_1 + m_2 \equiv 2 \bmod 4$ and n even;
(A_2) $(m_1 + m_2) d = n-1$ with $d \in \mathbb{Z}$, for the other cases;
(B) $(m_1 + m_2) 2d = 2n$ with $2d \in \mathbb{Z}$.

Where $d = 1/4 \sum \dim H^i(\tilde{M}_1; \mathbb{Z}_2)$. Moreover, if n is odd, any (B) occurs.

Proof. The proof is similar to that of Theorem 2.7. Assuming that $(m_1, m_2) \neq (1, n-1)$, by Lemma 3.10-3.12, we have

$$P(\tilde{M}, t) + P(\tilde{M}_2, t) - P(\tilde{M}, t) = 1 - t^{2n} + F$$

where $F = -t^{n-1} + t^{n+1}$ or $F = 0$. Finally we need to note that the cohomology algebra of $V_2(R^{n+2})$ is generated by $y_n \in H^n$ and $y_{n+1} \in H^{n+1}$. Moreover if n is odd, then $\mathrm{sq}^1 y_n = y_{n+1}$ (cf. [Wh], p.400), hence the case (A) does not occur. □

EXAMPLE 3.14. Let $SO(n+1)$ act on $\widetilde{G_2}(R^{n+2}) = SO(n+2)/SO(2) \times SO(n)$. It is of cohomogeneity one, hence the principal orbit is an equifocal hypersurface. In fact, the focal manifolds are S^n and $\widetilde{G_2}(R^{n+1})$, thus $(m_1, m_2) = (1, n-1)$. This example belongs to case (B) of Theorem 3.13 and $d = 1$.

EXAMPLE 3.15. If $n = 2m$ is even. Let $SU(m+1)$ act on $\widetilde{G_2}(R^{2m+2})$, it is of cohomogeneity one. The principal orbit is $SU(m+1)/SO(2) \times SU(m-1)$, and two focal submanifolds are $\mathbb{C}P^m$, thus $(m_1, m_2) = (n-1, n-1)$. This example belongs to case (A_1) of Theorem 3.13, and $2d = 1$.

4. In other symmetric spaces of rank two. This section will carry out an analogous study of the multiplicities of equifocal hypersurfaces in the left irreducible, compact, simply connected, symmetric space of rank two, which may be listed below:

$$SU(3), \quad SU(3)/SO(3), \quad \mathrm{Sp}(2) \cong \mathrm{Spin}(5),$$

$$G_2/SO(4), \quad G_2, \quad SU(6)/\mathrm{Sp}(3), \quad SO(8)/U(4),$$

$$SO(10)/U(5), \quad E_6/F_4, \quad E_6/(SO(2)\cdot \mathrm{Spin}(10))$$

whose dimensions are $8, 5, 10, 8, 14, 14, 12, 20, 26, 32$ respectively. Note that

$$\mathrm{Sp}(2)/U(2)$$

is isometric to $\widetilde{G_2}(R^5)$ which has been studied in the previous section.

THEOREM 4.1. *For $N = SU(3)$, there exists an equifocal hypersurface in N with multiplicities (m_1, m_2) if and only if $(m_1, m_2) \in \{(1,1), (1,3), (2,2), (4,4)\}$.*

Proof. Recall that there is a well-known fibration:

$$SU(2) \hookrightarrow SU(3) \xrightarrow{\pi} SU(3)/SU(2) = S^5.$$

Clearly we have isoparametric hypersurfaces in S^5 (cf. [CR]) with $(m_1, m_2) = (1,1)$, $(1,3)$, $(2,2)$ or $(4,4)$. Therefore the first part of the theorem follows from Proposition 1.1.

Conversely, suppose now that M^7 is an equifocal hypersurface in $SU(3)$ with multiplicities (m_1, m_2). Recall that (cf. [Wh], p. 342)

$$H^*(SU(3); \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2, & \text{for } * = 0, 3, 5, 8; \\ 0, & \text{otherwise.} \end{cases}$$

It follows immediately from Proposition 2.1 that $m_1 \le 4$.

For $m_1 = 4$. We have $\dim M_1 = 3 \ge \dim M_2$. Since $m_2 \ge m_1 = 4$, M_1 and M_2 are simply connected, thus $\dim M_2 \ge 2$. It follows from the M-V sequence with integral coefficients that $m_2 = 4$ and

$$H^*(M_k; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{for } * = 0, 3; \\ 0, & \text{otherwise.} \end{cases}$$

For $m_1 = 3$. We have $\dim M_1 = 4$, and $\dim M_2 = 2, 3$ or 4. The M-V sequence will produce a contradiction.

For $m_1 = 2$. Suppose that $m_2 \ge 3$, then $\dim M_1 = 5 > \dim M_2$. It follows from the Gysin sequence that

$$H^2(M; \mathbb{Z}_2) \cong H^2(M_2; \mathbb{Z}_2).$$

Combining this with the M-V sequence, we get

$$H^1(M_1; \mathbb{Z}_2) \cong 0, \quad H^2(M_1; \mathbb{Z}_2) \cong 0,$$

and hence $H^3(M_1; \mathbb{Z}_2) \cong 0$ by the Poincaré duality theorem. Applying the Gysin sequence of $S^2 \hookrightarrow M \to M_1$, we find

$$H^4(M; \mathbb{Z}_2) \cong 0, \quad H^5(M; \mathbb{Z}_2) \cong \mathbb{Z}_2.$$

Putting these results together with the M-V sequence, we obtain an exact sequence

$$\begin{aligned}&H^4(M;\mathbb{Z}_2)\cong 0\to H^5(N;\mathbb{Z}_2)\cong\mathbb{Z}_2\\&\to H^5(M_1;\mathbb{Z}_2)\oplus H^5(M_2;\mathbb{Z}_2)\cong\mathbb{Z}_2\to H^5(M;\mathbb{Z}_2)\cong \mathbb{Z}_2\\&\to H^6(N;\mathbb{Z}_2)\cong 0,\end{aligned}$$

which is clearly impossible.

For $m_1 = 1$. First suppose $m_2 \geq 4$. Then $\dim M_1 = 6$, $\dim M_2 \leq 3$. By the Gysin sequence, we have isomorphisms

$$H^i(M;\mathbb{Z}_2)\cong H^i(M_2;\mathbb{Z}_2)\quad \text{for } i\leq 3.$$

Thus by the M-V sequence and the Poincaré duality theorem we have finally

$$H^*(M_1^6;\mathbb{Z}_2)\cong\begin{cases}\mathbb{Z}_2, & \text{for } *=0,3,6;\\ 0, & \text{otherwise},\end{cases}$$

which contradicts a well-known theorem by Adams [Ad].

It remains to consider $m_1 = 1$ and $m_2 = 2$. Thus we have $\dim M_1 = 6$, and $\dim M_2 = 5$. The Gysin sequence of $S^2 \hookrightarrow M \to M_2$ implies that $H^1(M;\mathbb{Z}_2) \cong H^1(M_2;\mathbb{Z}_2)$. Thus by the M-V sequence, we have $H^1(M_1;\mathbb{Z}_2) \cong 0$. On the other hand, the M-V sequence follows that $H^6(M;\mathbb{Z}_2)\cong\mathbb{Z}_2$, and hence $H^1(M;\mathbb{Z}_2)\cong\mathbb{Z}_2$ by the Poincaré duality theorem. These arguments yield that the Euler class $e(1)$ must be trivial by analysing the Gysin sequence of $S^1 \hookrightarrow M \to M_1$. Consequently we have $H^i(M;\mathbb{Z}_2)\cong H^i(M_1;\mathbb{Z}_2)\oplus H^{i-1}(M_1;\mathbb{Z}_2)$ for every i. Applying again the M-V sequence, we will get a contradiction, and hence the theorem. □

THEOREM 4.2. *For $N = SU(3)/SO(3)$, there exists an equifocal hypersurface in N with multiplicities (m_1, m_2) if and only if $m_1 = m_2 = 1$.*

Proof. We consider the action of $S\big(U(1)\times U(2)\big)$ on $N = SU(3)/SO(3)$, and will get an equifocal hypersurface in N with multiplicities $(m_1, m_2) = (1, 1)$.

Conversely suppose we are given an equifocal hypersurface M^4 in N with multiplicities (m_1, m_2). By the fibration

$$\pi : SU(3)\to SU(3)/SO(3),$$

we get an equifocal hypersurface in $SU(3)$ with the same multiplicities. By the previous theorem, we have

$$(m_1,m_2)\in\{(1,1),(2,2),(4,4),(1,3)\}.$$

To complete the proof of the theorem, it is only necessary to exclude the late three cases. Let us recall the cohomology of $N = SU(3)/SO(3)$ (cf. [MT], p. 150),

$$H^*(N;\mathbb{Z}_2)\cong\begin{cases}\mathbb{Z}_2, & \text{for } *=0,2,3,5;\\ 0, & \text{otherwise}.\end{cases}$$

For $(m_1, m_2) = (2, 2)$. From the fibration $SO(3) \hookrightarrow SU(3) \to N$, we see that

$$\pi_1 N = 0 \quad \text{and} \quad \pi_2 N\cong\pi_1 SO(3)\cong\mathbb{Z}_2.$$

Thus the Hurwicz theorem implies that

$$H_1(N;\mathbb{Z})\cong 0 \quad \text{and } H_2(N;\mathbb{Z})\cong\mathbb{Z}_2.$$

By universal coefficient theorem and the Poincaré duality theorem, we get

$$H^1(N;\mathbb{Z}) \cong H^2(N;\mathbb{Z}) \cong 0, \quad H^3(N;\mathbb{Z}) \cong \mathbb{Z}_2.$$

Now, since $m_1, m_2 > 1$, it follows that $M_1 \cong M_2 \cong S^2$. Let $a \in H^2(S^2;\mathbb{Z})$ be a generator, by the Gysin sequence

$$0 \to H^2(S^2;\mathbb{Z}) \xrightarrow{p_k^*} H^2(M;\mathbb{Z}) \to H^0(S^2;\mathbb{Z}) \to 0,$$

we know that $p_k^*(a)$ is a generator of $H^2(M;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$ for $k = 1$ or 2. From the M-V sequence

$$0 \to H^2(S^2;\mathbb{Z}) \oplus H^2(S^2;\mathbb{Z}) \underset{p_1^*-p_2^*}{\xrightarrow{\Phi}} H^2(M;\mathbb{Z}) \to H^3(N;\mathbb{Z}) \cong \mathbb{Z}_2 \to 0,$$

since Φ is injective, $p_1^*a \neq \pm p_2^*a$, this yields that Φ is an isomorphism, a contradiction.

For $(m_1, m_2) = (4, 4)$, $M_1 = M_2 = \{pt\}$, and $M \cong S^4$. There will be a contradiction in the M-V sequence.

For $(m_1, m_2) = (1, 3)$, $M_2 \cong S^1$. It follows from the Gysin sequence that

$$H^*(M;\mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2, & \text{for } * = 0, 1, 3, 4; \\ 0, & \text{otherwise.} \end{cases}$$

There will be a contradiction in the M-V sequence. □

THEOREM 4.3. *For $N = \mathrm{Sp}(2)$, there exist an equifocal hypersurface in N with multiplicities (m_1, m_2) if and only if*

$$(m_1, m_2) \in \{(1,1), (1,2), (1,5), (2,2), (2,4), (3,3), (6,6)\}.$$

Proof. Recall that there is a well known fibration

$$S^3 \cong \mathrm{Sp}(1) \hookrightarrow \mathrm{Sp}(2) \to \mathrm{Sp}(2)/\mathrm{Sp}(1) \cong S^7.$$

We consider isoparametric hypersurfaces in S^7 with $g/2(m_1 + m_2) = 6$. Clearly the following pairs are available (cf. [CR]):

$$\begin{array}{llllllll} g = 1, & (6,6); & g = 2, & (1,5), & (2,4), & (3,3); \\ g = 3, & (2,2); & g = 4, & (1,2); & g = 6 & (1,1). \end{array}$$

In order to show the necessary condition, we recall the cohomology of Sp(2) (cf. [MT], p. 148),

$$H^*(\mathrm{Sp}(2);\mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2, & \text{for } * = 0, 3, 7, 10; \\ 0, & \text{otherwise.} \end{cases}$$

Suppose $m_1 \geq 7$, then $\dim M_2 \leq \dim M_1 \leq 2$. The M-V sequence implies that

$$H^7(M;\mathbb{Z}_2) \cong H^8(M;\mathbb{Z}_2) \cong 0,$$

and the Poincaré dualities

$$H^1(M;\mathbb{Z}_2) \cong H^8(M;\mathbb{Z}_2) \cong 0, \quad H^2(M;\mathbb{Z}_2) \cong H^7(M;\mathbb{Z}_2) \cong 0$$

together with the M-V sequence imply that $H^3(\mathrm{Sp}(2);\mathbb{Z}_2) \cong 0$, a contradiction.

If $m_1 = 6$, we prove $m_2 = 6$. Otherwise $m_2 \geq 7$, then $\dim M_1 = 3$, $\dim M_2 \leq 2$. The M-V sequence follows that

$$H^6(M;\mathbb{Z}_2) \cong \mathbb{Z}_2, \quad H^7(M;\mathbb{Z}_2) \cong 0,$$

and hence $H^2(M;\mathbb{Z}_2) \cong 0$, $H^3(M;\mathbb{Z}_2) \cong \mathbb{Z}_2$, by the Poincaré duality theorem. Again using the M-V sequence, we get an exact sequence

$$\begin{aligned}&H^2(M;\mathbb{Z}_2) \cong 0 \to H^3(N;\mathbb{Z}_2) \cong \mathbb{Z}_2 \to \\ &H^3(M_1;\mathbb{Z}_2) \oplus H^3(M_2;\mathbb{Z}_2) \to H^3(M;\mathbb{Z}_2) \cong \mathbb{Z}_2 \to H^4(N;\mathbb{Z}_2) \cong 0,\end{aligned}$$

a contradiction to the fact that $\dim M_2 \leq 2$.

Suppose $m_1 = 5$, then $\dim M_2 \leq \dim M_1 = 4$. The M-V sequence implies that

$$H^5(M;\mathbb{Z}_2) \cong 0.$$

Consequently $H^4(M;\mathbb{Z}_2) \cong 0$ by the Poincaré duality theorem. Again using the M-V sequence, we get

$$H^4(N;\mathbb{Z}_2) \cong 0 \to H^4(M_1;\mathbb{Z}_2) \oplus H^4(M_2;\mathbb{Z}_2) \to H^4(M;\mathbb{Z}_2) \cong 0,$$

and hence $H^4(M_1;\mathbb{Z}_2) = 0$ which is impossible.

Suppose $m_1 = 4$, then $\dim M_2 \leq \dim M_1 = 5$. The M-V sequence implies that

$$H^6(M;\mathbb{Z}_2) \cong \mathbb{Z}_2, \quad H^7(M;\mathbb{Z}_2) \cong H^8(M;\mathbb{Z}_2) \cong 0,$$

and then

$$H^1(M;\mathbb{Z}_2) \cong H^2(M;\mathbb{Z}_2) \cong 0, \quad H^3(M;\mathbb{Z}_2) \cong \mathbb{Z}_2.$$

Again using the M-V sequence, we find that

$$H^1(M_2;\mathbb{Z}_2) \cong H^2(M_2;\mathbb{Z}_2) \cong 0, H^3(M_2;\mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2,$$

a contradiction to the fact that $\dim M_2 \leq 5$.

If $m_1 = 3$, we claim that $m_2 = 3$. Assume $m_2 \geq 4$. By using the M-V sequence, we have

$$H^6(M;\mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2, \quad H^7(M;\mathbb{Z}_2) \cong H^8(M;\mathbb{Z}_2) \cong 0$$

and hence

$$H^1(M;\mathbb{Z}_2) \cong H^2(M;\mathbb{Z}_2) \cong 0, \quad H^3(M;\mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

It follows from the isomorphisms

$$H^i(M;\mathbb{Z}_2) \cong H^i(M_2;\mathbb{Z}_2) \quad \text{for } i \leq 3$$

(the Gysin sequence) and the M-V sequence that

$$H^*(M_1^6;\mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2, & \text{for } * = 0,3,6; \\ 0, & \text{otherwise,} \end{cases}$$

which contradicts a well known theorem by Adams [Ad].

If $m_1 = 2$, we need to prove that $m_2 = 2$ or 4. First suppose that $m_2 = 3$. The M-V sequence, the Gysin sequence and the Poincaré duality theorem imply that

$$H^*(M_1; \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2, & \text{for } * = 0, 7; \\ 0, & \text{otherwise,} \end{cases}$$

and hence

$$H^*(M; \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2, & \text{for } * = 0, 2, 7, 9; \\ 0, & \text{otherwise,} \end{cases}$$

a contradiction to the M-V sequence. It remains to show that the case $m_1 = 2$, $m_2 \geq 5$ does not occur. Otherwise, we have the isomorphisms

$$H^i(M; \mathbb{Z}_2) \cong H^i(M_2; \mathbb{Z}_2) \oplus H^{i-m_2}(M_2; \mathbb{Z}_2) \quad \text{for every } i$$

by the Gysin sequence. It follows from the M-V sequence that

$$H^3(M_1; \mathbb{Z}_2) \cong \mathbb{Z}_2, \quad H^4(M_1; \mathbb{Z}_2) \cong 0,$$

a contradiction to the Poincaré duality theorem.

Now we assert that $m_1 = 1$ implies that $m_2 \in \{1, 2, 5\}$. Suppose that $m_1 = 1$ and $m_2 \geq 6$, then we have the isomorphisms

$$H^i(M; \mathbb{Z}_2) \cong H^i(M_2; \mathbb{Z}_2) \oplus H^{i-m_2}(M_2; \mathbb{Z}_2) \quad \text{for every } i.$$

It follows from the M-V sequence that

$$H^3(M_1; \mathbb{Z}_2) \cong \mathbb{Z}_2, \quad H^5(M_1; \mathbb{Z}_2) \cong 0,$$

a contradiction to $\dim M_1 = 8$.

Now suppose that $m_1 = 1$ and $m_2 = 3$. It follows directly from the M-V sequence that the Euler classes both $e(1)$ and $e(2)$ vanish. Then combining the Gysin isomorphisms with the M-V sequences will produce a contradiction.

Finally suppose that $m_1 = 1$ and $m_2 = 4$. There is an analogous argument to get a contradiction, we omit the details. The proof of the theorem is now complete. □

THEOREM 4.4. *For $N = G_2/SO(4)$, suppose that there exists an equifocal hypersurface in N with multiplicities (m_1, m_2), then $(m_1, m_2) = (1, 1)$ or $(2, 3)$.*

The proof will be broken up into several steps. First recalling that there is a fibration

$$\pi : \tilde{N} = G_2/SU(2) \to N = G_2/SO(4)$$

and we have

LEMMA 4.5.

$$H^*(\tilde{N}; \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2, & \textit{for } * = 0, 5, 6, 11; \\ 0, & \textit{otherwise.} \end{cases}$$

Moreover $\mathrm{sq}^1 y_5 = y_6$, where $y_5 \in H^5$ and $y_6 \in H^6$ are generators respectively.

Proof. From the fibration

$$S^5 \cong SU(3)/SU(2) \hookrightarrow G_2/SU(2) \to G_2/SU(3) \cong S^6,$$

we deduce that $\pi_j\tilde{N} \cong 0$ for $1 \leq j \leq 4$. ¿From another fibration:

$$SU(2) \cong S^3 \hookrightarrow G_2 \to G_2/SU(2)$$

we obtain that $\pi_5\tilde{N} = \mathbb{Z}_2$, here we use the fact that

$$\pi_5 G_2 \cong \pi_4 G_2 \cong 0$$

(cf. [MT], p. 360).

Thus we conclude that the mod 2 cohomology algebra $H^*(\tilde{N};\mathbb{Z}_2)$ is generated by $y_5 \in H^5$ and $y_6 \in H^6$. It remains to show that $\mathrm{sq}^1 y_5 = y_6$. Recall that (cf. [MT], p. 420)

$$H^*(G_2;\mathbb{Z}_2) \cong \mathbb{Z}_2[x_2]/(x_3^4) \otimes \Lambda(x_5) \quad \text{with } x_5 = \mathrm{sq}^2 x_3.$$

By using the Gysin sequence of the fibration

$$S^3 \hookrightarrow G_2 \xrightarrow{\pi} G_2/SU(2),$$

we obtain that

$$\pi^* \mathrm{sq}^1 y_5 = \mathrm{sq}^1 \pi^* y_5 = \mathrm{sq}^1 x_5 = \mathrm{sq}^1 \mathrm{sq}^2 x_3 = \mathrm{sq}^3 x_3 = x_3^2 = \pi^* y_6,$$

hence $\mathrm{sq}^1 y_5 = y_6$, since $\pi^* : H^i(\tilde{N} : \mathbb{Z}_2) \to H^i(G_2;\mathbb{Z}_2)$ are isomorphic for $i = 5$ and $i = 6$. □

Suppose now we are given an equifocal hypersurface M in $N = G_2/SO(4)$ with multiplicities (m_1, m_2). Since N is inner, it follows that m_1 and m_2 are not all even.

LEMMA 4.6. *For* $N = G_2/SO(4)$, $(m_1 + m_2)$ *divides* 10.

Proof. By the fibration

$$\pi : \tilde{N} = G_2/SU(2) \to N = G_2/SO(4),$$

we get a disc bundles decomposition of $\tilde{N}$.

ASSERTION. There exist isomorphisms

$$H^i(\tilde{M};\mathbb{Z}_2) \cong H^i(\tilde{M}_k;\mathbb{Z}_2) \oplus H^{i-m_k}(\tilde{M}_k;\mathbb{Z}_2)$$

for every i and $k = 1$ or 2.

When $m_k \geq 4$, then $m_k + 1 > 7 - m_k$, thus

$$\tilde{e}(k) = \pi^* e(k) \in \pi^* H^{m_k+1}(M_k^{7-m_k};\mathbb{Z}_2) \cong 0.$$

When $m_k < 4$, the assertion follows from the M-V sequence of the decomposition of $\tilde{N}$.

Since $\tilde{N}$ has the same mod 2 cohomology with that of $V_2(R^7)$, an analogous study as Theorem 3.13 will lead to the lemma we wanted. □

Proof of Theorem 4.4. By the previous lemma, it suffices to show that the cases $(m_1, m_2) = (3,7), (5,5)$ and $(1,4)$ do not occur.

Suppose that $(m_1, m_2) = (3,7)$, then $M_2 = \{pt\}$ and $M \cong S^7$. Recall that (cf. [BH], p. 529)

$$H^*(G_2/SO(4);\mathbb{Z}_2) \cong \mathbb{Z}_2[u_2, u_3]/u_2^3 = u_3^2, u_3u_2^2 = 0,$$

where $\deg u_i = i$. Clearly the M-V sequence will produce a contradiction.

Suppose $(m_1, m_2) = (5,5)$, then it follows easily that $M_1 \cong M_2 \cong S^2$, and hence that $\chi(G_2/SO(4)) = 4$. However, since G_2 has type $(3, 11)$ and $SO(4)$ has type $(3,3)$, thus the Euler number of $G_2/SO(4)$ is equal to

$$\chi(G_2/SO(4)) = |W(G_2)|/|W(SO(4))| = 12/4 = 3,$$

contradicting the result $\chi(G_2/SO(4)) = 4$.

Suppose $(m_1, m_2) = (1,4)$, then $\dim M_1 = 6$, $\dim M_2 = 3$. By using the M-V sequence and the Gysin sequence we deduce that $H^1(M_1; \mathbb{Z}_2) \cong 0$, consequently M_1 is orientable, $\chi(M_1)$ is even, thus

$$\chi(G_2/SO(4)) = \chi(M_1) + \chi(M_2)$$

is even, a contradiction. The proof of Theorem 4.4 is complete. □

EXAMPLE 4.7. G_2 is the automophism group of the Cayley algebra. Using the arguments by [Mi], we embed $SU(3)$ and $SO(4)$ into G_2 as subgroups so that

$$SU(3) \cup SO(4) = SO(3).$$

Thus let $SU(3)$ act on $G_2/SO(4)$, we get an equifocal hypersurface in $G_2/SO(4)$ with multiplicities $(m_1, m_2) = (2,3)$. In fact, we have $\mathbb{C}P^2$ and $SU(3)/SO(3)$ as the focal submanifolds. For details, see [Mi].

THEOREM 4.8. *For $N = G_2$, suppose that there exists an equifocal hypersurface in N with multiplicities (m_1, m_2), then (m_1, m_2)*

$$\in \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (5,5)\}.$$

The proof will be based on several lemmas.

LEMMA 4.9. *For $N = G_2$, we have $m_1 \leq 5$.*

Proof. Step I, we claim $m_1 \leq 7$. Recall that (cf. [MT], p. 40)

$$H^*(G_2; \mathbb{Z}_2) \cong \mathbb{Z}_2[x_3]/(x_3^4) \otimes \Lambda(x_5)$$

where $\deg x_i = i$. It follows from Proposition 2.1 that $m_1 \leq 7$.

Step II, we claim $m_1 \neq 7$. Suppose that $m_1 = 7$, then it follows from the M-V sequence and the Poincaré duality theorem that

$$H^*(M; \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2, & \text{for } * = 0,3,5,6,7,8,10,13; \\ 0, & \text{otherwise.} \end{cases}$$

As a consequence, it follows from the Gysin sequence that $H^5(M_1 \mathbb{Z}_2) \cong \mathbb{Z}_2$, and hence $H^1(M_1; \mathbb{Z}_2) \cong \mathbb{Z}_2$ by the Poincaré duality theorem. On the other hand, applying Lemma 2.10, we see that M_1 is simply connected. a contradiction.

Step III, we claim $m_1 \neq 6$. Suppose that $m_1 = 6$. then M_1 and M_2 are all simply connected, thus $H^1(M_k; \mathbb{Z}_2) \cong 0$ for $k = 1$ or 2. It follows from the M-V sequence that

$$\begin{aligned} H^7(M; \mathbb{Z}_2) &\cong \mathbb{Z}_2 \oplus H^7(M_1; \mathbb{Z}_2) \oplus H^7(M_2; \mathbb{Z}_2) \\ &\cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus H^7(M_2; \mathbb{Z}_2) \end{aligned}$$

and an onto homomorphism

$$H^6(M_1; \mathbb{Z}_2) \oplus H^6(M_2; \mathbb{Z}_2) \to H^6(M; \mathbb{Z}_2) \cong H^7(M; \mathbb{Z}_2) \to 0.$$

Since $H^6(M_1;\mathbb{Z}_2) \cong H^1(M_1;\mathbb{Z}_2) \cong 0$, and $H^6(M_2;\mathbb{Z}_2)$ is isomorphic to $\mathbb{Z}_2$ for $m_2 = 7$, isomorphic to 0 for other m_2. We get a contradiction, and hence the proof. □

LEMMA 4.10. *For $N = G_2$, the hypothesis $m_1 = 5$ implies that $m_2 = 5$.*

Proof. It follows from the Gysin sequence that

$$H^9(M;\mathbb{Z}_2) \cong H^{11}(M;\mathbb{Z}_2) \cong H^{12}(M;\mathbb{Z}_2) \cong 0$$
$$H^{10}(M;\mathbb{Z}_2) \cong H^{13}(M;\mathbb{Z}_2) \cong \mathbb{Z}_2,$$

hence

$$H^1(M;\mathbb{Z}_2) \cong H^2(M;\mathbb{Z}_2) \cong H^4(M;\mathbb{Z}_2) \cong 0,$$
$$H^3(M;\mathbb{Z}_2) \cong \mathbb{Z}_2$$

by the Poincaré duality theorem. By using the Gysin sequence we have

$$H^i(M_k;\mathbb{Z}_2) \cong H^i(M;\mathbb{Z}_2) \quad \text{for } i \leq 4, k = 1 \text{ or } 2.$$

In particular, $e(1) \in H^6(M_1;\mathbb{Z}_2) \cong H^2(M_1;\mathbb{Z}_2) \cong 0$, hence

$$H^6(M;\mathbb{Z}_2) \cong H^7(M;\mathbb{Z}_2) \cong 0,$$
$$H^8(M;\mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

Again using the M-V sequence, we have an exact sequence

$$0 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \oplus H^8(M_2;\mathbb{Z}_2) \to H^8(M;\mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to \mathbb{Z}_2 \to 0,$$

which implies that $H^8(M_2;\mathbb{Z}_2) \cong \mathbb{Z}_2$, hence $m_2 = 5$. □

LEMMA 4.11. *For $N = G_2$, we have $m_1 \neq 4$.*

Proof. Step I, we claim that $m_1 = 4$ implies $m_2 = 4$. Suppose that $m_1 = 4$ and $m_2 \geq 5$. The M-V sequence gives that

$$H^{11}(M;\mathbb{Z}_2) \cong H^{12}(M;\mathbb{Z}_2) \cong 0,$$
$$H^{10}(M;\mathbb{Z}_2) \cong H^{13}(M;\mathbb{Z}_2) \cong Z_2.$$

By using the Poincaré duality theorem and the Gysin sequences, we obtain

$$H^1(M_k;\mathbb{Z}_2) \cong H^2(M_k;\mathbb{Z}_2) \cong 0,$$
$$H^3(M_k;\mathbb{Z}_2) \cong \mathbb{Z}_2 \quad \text{for } k = 1 \text{ or } 2.$$

On the other hand, the Gysin sequence implies that

$$H^4(M;\mathbb{Z}_2) \cong H^4(M_2;\mathbb{Z}_2),$$

thus by using the M-V sequence, we obtain

$$H^4(M_1;\mathbb{Z}_2) \cong 0, \quad H^5(M_1;\mathbb{Z}_2) \cong 0,$$

and hence $e(1) \in H^5(M_1;\mathbb{Z}_2) \cong 0$. Using again the Gysin sequence, we have

$$H^8(M;\mathbb{Z}_2) \cong 0, \quad H^9(M;\mathbb{Z}_2) \cong \mathbb{Z}_2.$$

There will be a contradiction in the M-V sequence

$$\begin{array}{ccccccc} H^8(M;\mathbb{Z}_2) \to & \mathbb{Z}_2 \to \mathbb{Z}_2 & \oplus & H^9(M_2;\mathbb{Z}_2) \to & H^9(M;\mathbb{Z}_2) \to 0. \\ \| & & & \| & \| \\ 0 & & & 0 & \mathbb{Z}_2 \end{array}$$

Step II, we claim that the case $(m_1, m_2) = (4,4)$ does not occur. Suppose $m_1 = m_2 = 4$, then M_1 and M_2 are all simply connected. Note that the integral cohomology of G_2 is given by

$$H^*(G_2;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{for } * = 0,3,11,14; \\ \mathbb{Z}_2, & \text{for } * = 6,9; \\ 0, & \text{otherwise.} \end{cases}$$

Here we used the fibration $S^3 \hookrightarrow G_2 \to V_2(R^7)$ (cf. [Wh], p. 694). It follows from the M-V sequence that

$$H^{11}(M;\mathbb{Z}) \cong H^{12}(M;\mathbb{Z}) \cong 0,$$
$$H^{10}(M;\mathbb{Z}) \cong \mathbb{Z}, \quad H^9(M;\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z},$$

and thus

$$H_1(M;\mathbb{Z}) \cong H_2(M;\mathbb{Z}) \cong 0,$$
$$H_3(M;\mathbb{Z}) \cong \mathbb{Z}, \quad H_4(M;\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z},$$

then

$$H^1(M;\mathbb{Z}) \cong H^2(M;\mathbb{Z}) \cong 0,$$
$$H^3(M;\mathbb{Z}) \cong \mathbb{Z}, \quad H^4(M;\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z}.$$

Hence the Gysin sequence implies that

$$H^1(M_k;\mathbb{Z}) \cong H^2(M_k;\mathbb{Z}) \cong 0, \quad H^3(M_k;\mathbb{Z}) \cong \mathbb{Z}.$$

Since $H^8(M_k;\mathbb{Z}) \cong H^1(M_k;\mathbb{Z}) \cong 0$, the M-V sequence implies that $H^8(M;\mathbb{Z}) \cong \mathbb{Z}_2$, thus $H_5(M;\mathbb{Z}) \cong \mathbb{Z}_2$. Therefore we get $H^5(M;\mathbb{Z}) \cong 0$.

The M-V sequence will give

$$\begin{array}{ccccc} H^5(M;\mathbb{Z}) & \to H^6(N;\mathbb{Z}) \to & H^6(M_1;\mathbb{Z}) \oplus H^6(M_2;\mathbb{Z}) \\ \| & \| & \| \\ 0 & \mathbb{Z}_2 & H_3(M_1;\mathbb{Z}) \oplus H_3(M_2;\mathbb{Z}) \end{array} \tag{4.1}$$

and

$$0 \to H^4(M_1;\mathbb{Z}) \oplus H^4(M_2;\mathbb{Z}) \to H^4(M;\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z}. \tag{4.2}$$

Since

$$H^4(M_k;\mathbb{Z}) \cong \mathrm{Ext}\big(H_3(M_k;\mathbb{Z}),\mathbb{Z}\big) \oplus \mathrm{Hom}\big(H_4(M_k;\mathbb{Z})\mathbb{Z}\big),$$

we can easily find that (4.1) contradicts (4.2). □

LEMMA 4.12. *For $N = G_2$, we have $m_1 \neq 3$.*

Proof. Step I, we claim that the case $(m_1, m_2) = (3,3)$ does not occur. It is easy to prove the assertion by using the M-V, Gysin sequences with coefficients in $\mathbb{Z}$. Note that the integral coefficient cohomology of G_2 was given before.

Step II, we claim that the case $(m_1, m_2) = (3,4)$ does not occur. Suppose that $(m_1, m_2) = (3,4)$, then M_k are simply connected, we can use the exact cohomology sequences with integral coefficients. It follows from the M-V sequence, that the Euler classes both $e(1)$ and $e(2)$ vanish. It is easily seen that $H^2(M_1;\mathbb{Z}) \cong 0$ and hence $H^8(M;\mathbb{Z}) \cong 0$, thus there will be a contradiction in the M-V sequence:

$$0 \cong H^8(M;\mathbb{Z}) \to H^9(N;\mathbb{Z}) \cong \mathbb{Z}_2 \to H^9(M_1;\mathbb{Z}) \oplus H^9(M_2;\mathbb{Z}) \cong \mathbb{Z}.$$

Step III, we claim that $m_1 = 3$ implies that $m_2 \le 4$. Suppose that $m_2 \ge 5$, then we can deduce without difficulty that

$$H^1(M_2;\mathbb{Z}) \cong H^2(M_2;\mathbb{Z}) \cong H^5(M_2;\mathbb{Z}) \cong H^7(M_2;\mathbb{Z}) \cong 0$$
$$H^3(M_2;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z},$$

consequently $m_2 \in \{5, 7\}$. We can check every case to exclude it. □

We are now in a position to give a

Proof of Theorem 4.8. Putting Lemma 4.9–4.12 together, it suffices to show that:

$$\begin{aligned} m_1 &= 1 \quad \text{implies that} \quad m_2 \le 4; \\ m_1 &= 2 \quad \text{implies that} \quad m_2 = 2 \text{ or } 3. \end{aligned}$$

We prove the most difficult case $(m_1, m_2) = (2, 5)$ does not occur, and leave the other cases to the reader. Suppose that $(m_1, m_2) = (2, 5)$, first we assert that the Euler class $e(1)$ with coefficient $\mathbb{Z}$ is trivial. Since by the M-V sequence and the Gysin isomorphism that $H^2(M_2;\mathbb{Z}) \cong H^2(M;\mathbb{Z}_2)$, we have

$$0 \to \mathbb{Z} \xrightarrow{\phi} H^3(M_1;\mathbb{Z}) \oplus H^3(M_2;\mathbb{Z}) \to H^3(M;\mathbb{Z}) \to 0. \tag{4.3}$$

Note that the Gysin sequence implies that $p_1^* e(1) = 0$, thus $e(1) = \phi(k)$ for some $k \in \mathbb{Z}$. On the other hand, $e(1)$ is the Euler class of an odd dimensional vector bundle, thus $2e(1) = 0$, and hence $k = 0$, so $e(1) = 0$.

Therefore we have

$$H^3(M;\mathbb{Z}) \cong H^3(M_1;\mathbb{Z}) \oplus H^3(M_2;\mathbb{Z}),$$
$$H^2(M_2;\mathbb{Z}) \cong H^2(M_1;\mathbb{Z}) \oplus \mathbb{Z}.$$

Combining these results with (4.3), we get

$$\operatorname{rank} H^3(M_2;\mathbb{Z}) = 1, \quad \operatorname{rank} H^2(M_2;\mathbb{Z}) \ge 1,$$

hence $\chi(M_2) > 0$, a contradiction to the equality

$$\begin{array}{ccccc} \chi(M_1) & + \chi(M_2) = & \chi(M) & + & \chi(G_2) \\ \| & & \| & & \| \\ 0 & & 0 & & 0. \end{array}$$

The proof is complete. □

EXAMPLE 4.13. By the fibration $SU(3) \hookrightarrow G_2 \to G_2/SU(3) \cong S^6$, we can obtain three equifocal hypersurfaces in G_2 with

$$(m_1, m_2) \in \{(1,4), (2,3), (5,5)\}.$$

In fact, if we let $SU(3) \times SU(3)$ act on G_2, we can get also an equifocal hypersurface in G_2 with $m_1 = m_2 = 5$.

THEOREM 4.14. *For $N = SU(6)/\mathrm{Sp}(3)$, we have $m_1 = m_2 = 8$ or $(m_1 + m_2)$ divides 8.*

First we need the following

LEMMA 4.15. *$H^*(N^{14};\mathbb{Z}_2) \cong \Lambda(x_5, x_9)$ where $x_i \in H^i$, and $\mathrm{sq}^4 x_5 = x_9$.*

Proof. By ([MT], p. 149), the mod 2 cohomology of $SU(6)/\mathrm{Sp}(3)$ is given by

$$H^*(N;\mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2, & \text{for } * = 0, 5, 9, 14; \\ 0, & \text{otherwise.} \end{cases}$$

Consider the fibration

$$\pi : SU(6) \to SU(6)/\operatorname{Sp}(3).$$

Note that (cf. [Wh])

$$H^*\big(SU(6);\mathbb{Z}_2\big) \cong \Lambda(u_2, u_3, u_4, u_5, u_6)$$

where $u_i \in H^i$, and $\operatorname{sq}^4 u_3 = u_5$, $\pi^* x_5 = u_3$, $\pi^* x_9 = u_5$, we conclude that $\operatorname{sq}^4 x_5 = x_9$. □

LEMMA 4.16. *For $N = SU(6)/\operatorname{Sp}(3)$, if $m_1 \neq 4$, $m_2 \neq 4$, then we have $m_1 = m_2 = 8$ or $(m_1 + m_2)$ divides 8.*

Proof. Assume that $m_1 \neq 4$, $m_2 \neq 4$. Then it follows from the M-V sequence that the Euler classes $e(1)$ and $e(2)$ are both trivial, thus the Poincaré polynomials satisfy the relation

$$P(M,t)(1 - t^{m_1+m_2}) = (1 + t^{m_1})(1 + t^{m_2})(1 - t^{13} + F)$$

where F is easily proved to be $t^5 - t^8$ by using the previous lemma. Consequently

$$P(M,t)(1 - t^{m_1+m_2}) = (1 + t^{m_1})(1 + t^{m_2})(1 - t^8)(1 + t^5). \tag{4.4}$$

In particular, $\beta(M) \cdot (m_1 + m_2) = 64$, where $\beta(M)$ denotes the mod 2 Betti number.

Note that $1 \leq m_1 \leq m_2 \leq 13$. If $m_1 + m_2 = 16$, it follows from (4.4) that $m_1 = 8$ or $m_2 = 8$, so $m_1 = m_2 = 8$. If $m_1 + m_2 < 16$, then the relation $\beta(M) \cdot (m_1 + m_2) = 64$ implies that $(m_1 + m_2)$ divides 8. □

Proof of Theorem 4.14. By the previous lemma, it suffices to consider the cases $m_1{=}4$ and $m_2{=}4$. We claim that $m_k{=}4$ implies $m_{-k}{=}4$. For example, suppose $m_1 = 4$, $m_2 > 4$. Then $e(2) = 0$. By using the Gysin sequences and the M-V sequence, we have

$$\begin{array}{c} 0 \to \mathbb{Z}_2 \to \underset{\substack{\| \\ 0}}{H^5(M_1;\mathbb{Z}_2)} \oplus H^5(M_2;\mathbb{Z}_2) \to H^5(M;\mathbb{Z}_2) \to 0 \end{array}$$

and $H^5(M;\mathbb{Z}_2) \cong H^5(M_2;\mathbb{Z}_2) \oplus H^{5-m_2}(M_2;\mathbb{Z}_2)$, which are clearly impossible. □

LEMMA 4.17. *There exist isometric homeomorphisms*

$$SU(2n-1)/\operatorname{Sp}(n-1) \cong SU(2n)/\operatorname{Sp}(n),$$
$$SO(2n-1)/U(n-1) \cong SO(2n)/U(n).$$

Proof. Consider the embedding

$$SU(2n-1) \overset{i}{\hookrightarrow} SU(2n)$$

given by

$$i(X) = \begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix}.$$

It is clear to verify that the map $\pi \circ i : SU(2n-1) \to SU(2n) \to SU(2n)/\operatorname{Sp}(n)$ is onto. Now $SU(2n-1)$ acts transitively on $SU(2n)/\operatorname{Sp}(n)$ and the isotropy group at e is just $\operatorname{Sp}(n-1)$, hence

$$SU(2n-1)/\operatorname{Sp}(n-1) \cong SU(2n)/\operatorname{Sp}(n).$$

The proof of the other one is analogous. □

EXAMPLE 4.18. By the previous lemma, we see that $SU(6)/\operatorname{Sp}(3)$ is isometric to $SU(5)/\operatorname{Sp}(2)$, and we have a fibration

$$SU(5)/\operatorname{Sp}(2) \to SU(5)/SU(4) \cong S^9.$$

Applying Proposition 1.1, we can get several equifocal hypersurfaces in

$$N = SU(6)/\operatorname{Sp}(3).$$

Now in S^9, we have $\frac{g}{2}(m_1+m_2) = 8$. Clearly the following pairs are available (cf. [CR])

$$\begin{aligned} g=1, \quad (m_1,m_2) &= (8,8); \\ g=2, \quad (m_1,m_2) &= (1,7),(2,6),(3,5),(4,4); \\ g=4, \quad (m_1,m_2) &= (1,3),(2,2). \end{aligned}$$

THEOREM 4.19. *For $N = SO(8)/U(4)$, suppose that there exists an equifocal hypersurface in N with multiplicities (m_1,m_2), then we have*

$$(m_1,m_2) \in \{(1,1),(1,2),(1,3),(1,4),(2,3),(3,3),(5,5)\}.$$

First we need the following

LEMMA 4.20.

$$H^*(N^{12};\mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2, & \text{for } * = 0,2,4,8,10,12; \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2, & \text{for } * = 6; \\ 0, & \text{otherwise.} \end{cases}$$

and $\chi(N) = 8$.

Proof. The first part is given by [MT]. Note that $SO(2n-1)/U(n-1)$ is the total space of a fibration over S^{2n-2} with fiber $SO(2n-2)/U(n-1)$. By Lemma 4.17, it follows that

$$\chi(SO(2n)/U(n)) = \chi(SO(2n-1)/U(n-1)) = 2 \cdot \chi(SO(2n-2)/U(n-1))$$

Hence $\chi(SO(2n)/U(n)) = 2^{n-1}$ by induction. In particular, $\chi(SO(8)/U(4)) = 8$. □

In fact, the integral cohomology of $SO(2n)/U(n)$ is given in [MT].

LEMMA 4.21.

(1) m_1 and m_2 are not all even;

(2) $m_1 \leq 5$.

The proof is immediate and we omit it.

Proof of Theorem 4.19. The proof will be divided into several steps.

Step I, we claim that $m_1 = 5$ implies $m_2 = 5$. Suppose that $m_1 = 5$ and $m_2 \geq 6$, then $H^5(M;\mathbb{Z}_2) \cong H^5(M_2;\mathbb{Z}_2)$ by the Gysin sequence. Checking the M-V sequence, we find a contradiction in the following exact sequence:

$$\begin{array}{ccccccc} 0 \to & H^6(M;\mathbb{Z}_2) & \to & H^6(M_1;\mathbb{Z}_2) & \oplus & H^6(M_2;\mathbb{Z}_2) & \to \dots \\ & \| & & \| & & \| & \\ & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & & \mathbb{Z}_2 & & 0 & . \end{array}$$

Step II, we claim that $m_1 \neq 4$. Suppose $m_1 = 4$. By Lemma 4.21, we have $m_2 \in \{5,7,9,11\}$. Applying Lemma 4.20, it follows that $\chi(M_2) = 8$ hence $m_2 = 5$ or

7. If $m_2 = 5$, it follows from the M-V sequence and the Gysin sequence ($\mathbb{Z}$ coefficient) that

$$H^1(M_2;\mathbb{Z}) \cong H^3(M_2;\mathbb{Z}) \cong 0, \quad \operatorname{rank} H^2(M_2;\mathbb{Z}) = 1,$$

thus $\chi(M_2) = 4$, a contradiction. If $m_2 = 7$, a similar argument will lead to a contradiction.

Step III, we claim that $m_1 = 3$ implies $m_2 = 3$. Suppose $m_1 = 3$, then $3 \leq m_2 \leq 11$. We prove the most difficult case $(m_1, m_2) = (3,5)$ does not occur. In this case, it follows from the Gysin and the M-V sequences that

$$H^*(M_2;\mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2, & \text{for } * = 0,2,3,4,6; \\ 0, & \text{otherwise.} \end{cases}$$

Choose $x \in H^3(M_2;\mathbb{Z}_2)$ as a generator, then $x^2 \in H^6$ is a generator by the Poincaré duality theorem. However, by the Adem relation

$$x^2 = \mathrm{sq}^3\, x = \mathrm{sq}^1\, \mathrm{sq}^2\, x = \mathrm{sq}^1(0) = 0,$$

a contradiction.

Step IV, we claim that $m_1 = 2$ implies $m_2 = 3$, and $m_1 = 1$ implies $m_2 \leq 4$. The proofs are not difficult, and will be omitted.

The proof is complete. □

EXAMPLE 4.22. By Lemma 4.17, $SO(8)/U(4)$ is isometric to $SO(7)/U(3)$ which fibers over S^6. Applying Proposition 1.1, we can get several equifocal hypersurfaces in $N = SO(8)/U(4)$. Now in S^6, we have $\frac{g}{2}(m_1 + m_2) = 5$. Clearly the following pairs are available.

$$\begin{aligned} &g = 1, \quad m_1 = m_2 = 5 \quad ; \\ &g = 2, \quad (m_1, m_2) = (1,4), (2,3). \end{aligned}$$

THEOREM 4.23. *For $N = SO(10)/U(5)$, suppose that there exists an equifocal hypersurface in N with multiplicities (m_1, m_2), then $m_1 = m_2 = 7$ or $m_1 \leq 5$.*

Proof. The proof is similar to that of the previous theorem. We need to recall that (cf. [MT], p. 153)

$$H^*\big(SO(10)/U(5);\mathbb{Z}_2\big) \cong \begin{cases} \mathbb{Z}_2, & \text{for } * = 0,2,4,16,18,20; \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2, & \text{for } * = 6,8,10,12,14; \\ 0, & \text{otherwise.} \end{cases}$$

and $\chi(N) = 16$. □

EXAMPLE 4.24. By using the fibration

$$SO(10)/U(5) \cong SO(9)/U(4) \to S^8,$$

we obtain several equifocal hypersurfaces in N with multiplicities $(m_1, m_2) = (7,7)$ or $m_1 + m_2 = 7$ ($m_1 = 1,2,3$).

The remainder of this section will be concerned with the following

THEOREM 4.25. *For $N = E_6/F_4$, suppose that there exists an equifocal hypersurface in N with multiplicities (m_1, m_2), then $m_1 = m_2 = 16$ or $(m_1 + m_2)$ divides 16.*

Proof. We recall that (cf. [MT], p. 435)

$$H^*(E_6/F_4;\mathbb{Z}_2)\cong\Lambda(x_9,x_{17})\text{ with }x_{17}=\operatorname{sq}^8 x_9.$$

The proof of the theorem is similar to that of Theorem 4.14, which will be omitted. □

EXAMPLE 4.26. According to [Co], let $D_5\times T^1$ act on E_6/F_4, we get an equifocal hypersurface in E_6/F_4 with multiplicities $(m_1,m_2)=(1,15)$, and a focal submanifold is $S^1\times S^9$ of dimension 10.

CONJECTURE 4.27. The symmetric space E_6/F_4 fibers over S^{17} with fiber S^9.

REMARK 4.28. According to [Co], let F_4 act on $E_6/D_5\times T^1$, we get an equifocal hypersurface in $E_6/D_5\times T^1$ with multiplicities $(m_1,m_2)=(1,15)$, and a focal submanifold is $QP^2=F_4/\operatorname{Spin}(9)$ of dimension 16. However, for the restrictions on m_k of an equifocal hypersurface in $E_6/D_5\times T^1$, it is not so easy to get a satisfactory result. We recall only that (cf. [Is])

$$H^*(E_6/\operatorname{Spin}(10)\cdot T^1;\mathbb{Z}_2)\cong\mathbb{Z}_2[t,w]/\langle t^9=w^2t,\ w^3=w^2t^4+wt^8\rangle,$$

which is much more complicated.

5. In Cayley projective plane. After having worked on the symmetric spaces of rank two, we turn to the case of Cayley projective space QP^2 in this section.

THEOREM 5.1. *For $N=QP^2$, there exists an equifocal hypersurface in N with multiplicities (m_1,m_2), if and only if $(m_1,m_2)=(7,15)$, $g=1$; or $(m_1,m_2)=(4,7)$, $g=2$.*

Recall again that Thorbergsson gave in [Th2] an equality for $N=QP^2$,

$$g\,(m_1,m_2)=22.$$

As a consequence, $g\in\{1,2,11\}$. If $g=1$, then $m_1+m_2=22$. Since $m_1\le m_2\le 15$, we get that $m_1\ge 7$. On the other hand, by Proposition 2.1, we see that $m_1\le 7$, the reason is that (cf. [BH])

$$H^*(QP^2;\mathbb{Z}_2)\cong\begin{cases}\mathbb{Z}_2, & \text{for } *=0,8,16;\\ 0, & \text{otherwise.}\end{cases}$$

Hence we may conclude the following

LEMMA 5.2. *For $N=QP^2$, if $g=1$ then $(m_1,m_2)=(7,15)$.* □

When $g=2$, the formula $2\,(m_1+m_2)=22$ yields that $m_1+m_2=11$, thus the pair (m_1,m_2) has to belong to the set

$$\{(1,10),(2,9),(3,8),(4,7),(5,6)\}.$$

LEMMA 5.3. *For $N=QP^2$, if $g=2$, then $m_1=4$ or 5.*

Proof. Suppose that $m_1<4$, then $8\le m_2\le 10$. It follows from the dimensional reason that the Euler class $e(2)$ vanishes, thus by the M-V sequence

$$H^i(M_1;\mathbb{Z}_2)\cong 0\quad\text{for } 0<i\le 6.$$

Since $\dim M_2=15-m_2\le 7$, $H^8(M_2;\mathbb{Z}_2)\cong 0$. Furthermore

$$H^8(M_1;\mathbb{Z}_2)\cong H^{15-m_1-8}(M_1;\mathbb{Z}_2)\cong H^{7-m_1}(M_1;\mathbb{Z}_2)\cong 0$$

by the Poincaré duality. Hence the M-V sequence gives that

$$\begin{aligned}\cdots &\to H^7(M;\mathbb{Z}_2) \to \mathbb{Z}_2 \to H^8(M_1;\mathbb{Z}_2) \oplus H^8(M_2;\mathbb{Z}_2)\\ &\to H^8(M;\mathbb{Z}_2) \cong H^7(M;\mathbb{Z}_2) \to 0.\end{aligned}$$

It is easy to see a contradiction, since $H^8(M_k;\mathbb{Z}_2) \cong 0$ for $k = 1$ or 2. □

LEMMA 5.4. *For $N = QP^2$, if $g = 2$ then $m_1 \neq 5$.*

The proof is similar and straightforward.

LEMMA 5.5. *For $N = QP^2$, g has to be* 1 *or* 2.

Proof. Suppose that $g = 11$, then $m_1 = m_2 = 1$. It follows from the Gysin sequence and the M-V sequences that $e(k) = 0$ and

$$H^i(M_k;\mathbb{Z}_2) \cong \mathbb{Z}_2 \quad \text{for } i = 0,1,\dots,6, \quad k = 1 \text{ or } 2.$$

We investigate the value of ϕ given by the M-V sequence

$$\begin{array}{llll} 0 \to H^7(M_1;\mathbb{Z}_2) \oplus & H^7(M_2;\mathbb{Z}_2) \to & H^7(M;\mathbb{Z}_2) & \\ \xrightarrow{\phi} H^8(N;\mathbb{Z}_2) & \to H^8(M_1;\mathbb{Z}_2) \oplus & H^8(M_2;\mathbb{Z}_2) \to H^8(M;\mathbb{Z}_2) \to 0 & \\ \quad\ \| & & & \\ \quad\ \mathbb{Z}_2. & & & \end{array}$$

Case I, $\phi \neq 0$. We have

$$\begin{aligned} H^8(M;\mathbb{Z}_2) &\cong H^7(M_1;\mathbb{Z}_2) \oplus H^8(M_2;\mathbb{Z}_2)\\ H^7(M;\mathbb{Z}_2) &\cong H^7(M_1;\mathbb{Z}_2) \oplus H^7(M_2;\mathbb{Z}_2) \oplus \mathbb{Z}_2.\end{aligned}$$

It is easily seen that

$$H^7(M;\mathbb{Z}_2) \cong \mathbb{Z}_2 \quad \text{and} \quad H^8(M;\mathbb{Z}_2) \cong 0,$$

a contradiction to the Poincaré duality theorem.

Case II, $\phi = 0$. We have

$$\begin{aligned} &H^8(M;\mathbb{Z}_2) \oplus \mathbb{Z}_2 \cong H^8(M_1;\mathbb{Z}_2) \oplus H^8(M_2;\mathbb{Z}_2),\\ &H^7(M;\mathbb{Z}_2) \cong H^7(M_1;\mathbb{Z}_2) \oplus H^7(M_2;\mathbb{Z}_2).\end{aligned}$$

It is easily seen that

$$H^6(M_2;\mathbb{Z}_2) \cong \mathbb{Z}_2 \quad \text{and} \quad H^8(M_2;\mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2,$$

a contradiction to the Poincaré duality theorem. □

Proof of Theorem 5.1. The necessary condition can be proved by putting Lemma 5.2–5.5 together. In order to show the converse, we have to construct two equifocal hypersurfaces in QP^2. Recall that $\mathrm{Spin}(9) \subset F_4$ is a symmetric subgroup and

$$(\mathrm{Sp}(3) \times \mathrm{Sp}(1))/\mathbb{Z}_2 \subset F_4$$

is another symmetric subgroup. The Cayley projective plane QP^2 is just the symmetric space $F_4/\mathrm{Spin}(9)$. Letting $\mathrm{Spin}(9)$ act on $F_4/\mathrm{Spin}(9)$, we get an equifocal hypersurface with multiplicities $(m_1, m_2) = (7, 15)$. More precisely,

$$M_1 \cong \mathrm{Spin}(9)/\mathrm{Spin}(8) \cong S^8, \quad M_2 \cong \mathrm{Spin}(9)/\mathrm{Spin}(9) = \{pt\}$$

and $M \cong \mathrm{Spin}(9)/\mathrm{Spin}(7) \cong S^{15}$. Letting $(\mathrm{Sp}(3) \times \mathrm{Sp}(1)/\mathbb{Z}_2$ act on $F_4/\mathrm{Spin}(9)$, we get an equifocal hypersurface with multiplicities $(m_1, m_2) = (4,7)$; moreover, $M_1 \cong S^{11}$ and $M_2 \cong HP^2$.

The proof is complete. □

REMARK 5.6. Dr. Fang [Fa] got some results similar to our theorem, but under a very strong hypothesis that two focal submanifolds are all orientable.

PROPOSITION 5.7. *Let $M \subset QP^2$ be an equifocal hypersurface with multiplicities (m_1, m_2).*

(I) If $g = 2$, then M_1 is homeomorphic to S^{11};

(II) If $g = 1$, then M is homeomorphic to S^{15} and M_1 is homeomorphic to S^8.

Proof. (I) Suppose that $g = 2$, then $(m_1, m_2) = (4,7)$ by Theorem 5.1. It follows from Lemma 2.10 that M_k is simply connected for $k = 1$ or 2. We can make use of the M-V sequence and the Gysin sequences to get

$$H^*(M_1;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{for } * = 0, 11; \\ 0, & \text{otherwise.} \end{cases}$$

By using the Hurwicz isomorphism theorem and a theorem of Whitehead, we may show that M_1 has the same homotopy type as the sphere S^{11}. It follows immediately from the generalized Poincaré conjecture proved by Smale [Sm] that M_1 is homeomorphic to S^{11}.

The proof of (II) is analogous. □

CONJECTURE 5.8. Every equifocal hypersurface in QP^2 is homogeneous.

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