ASIAN J. MATH.
Vol. 2, No. 1, pp. 157-180, March 1998

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# ISOMETRIES OF DIRECT SUMS OF SEQUENCE SPACES*

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**Abstract.** We study isometries of direct sums (also called 1-unconditional sums) of complex and real sequence spaces. We show that if $X, Y$ are arbitrary complex symmetric sequence spaces then all surjective isometries of $X(Y)$ preserve the direct sum structure i.e. for every isometry $T : X(Y) \overset{\text{onto}}{\longrightarrow} X(Y)$ there exists a permutation $\pi$ of $\{1, \ldots, \dim X\} \subseteq \mathbb{N}$ and surjective isometries $\{S_j\}_{j \leq \dim X}$ of $Y$ so that

$$T((y_j)_{j \leq \dim X}) = (S_j y_{\pi(j)})_{j \leq \dim X}.$$

Further we show that if $X, Y$ are finite dimensional real symmetric sequence spaces then all isometries of $X(Y)$ also have the above form except when $X = \ell_p^k$ and $Y$ can be decomposed as an $\ell_p$-direct sum of two nonzero subspaces for some $1 \leq p \leq \infty$. All other possible isometries in the exceptional case are also characterized.

As a corollary we obtain that if $X$ is a complex or finite dimensional real symmetric sequence space then $X(X)$ is symmetric if and only if $X = \ell_p$ for some $p$, $1 \leq p \leq \infty$.

We also present characterizations of surjective isometries in more complicated spaces with direct sum structure.

**1. Introduction.** Let $X_0$, $\{X_i\}_{i \in I}$ with $I \subseteq \mathbb{N}$ be sequence spaces (finite or infinite dimensional) over $\mathbb{C}$ or $\mathbb{R}$ with absolute norms $N_i$ for $i \in \{0\} \cup I$, (i.e., with 1-unconditional bases) such that $\dim X_0 = \text{card}(I)$. Define an absolute norm on the cartesian product of $\{X_i\}_{i \in I}$ by

$$N((x_i)_{i \in I}) = N_0((N_i(x_i))_{i \in I}) \quad \text{for all} \quad (x_i)_{i \in I} \in \Pi_{i \in I} X_i.$$

The space of sequences $(x_i)_{i \in I} \in \Pi_{i \in I} X_i$ such that $N((x_i)_{i \in I}) < \infty$ is denoted by $X_0((X_i)_{i \in I})$ (or, with the slight abuse of notation, $X_0(X_1, \ldots, X_k)$) and is called the $X_0$ direct sum ($X_0$ 1-unconditional sum) of spaces $(X_i)_{i \in I}$. If $X_i = Y$ for all $i \in I$, the notation $X_0(Y)$ is used. Since $N_0, \{N_i\}_{i \in I}$ are absolute norms, the norm $N$ on $X_0((X_i)_{i \in I})$ is also absolute. The purpose of this paper is to study the geometry and isometries of $X_0((X_i)_{i \in I})$.

The study of direct sums of normed spaces arises naturally in many areas of mathematics. In particular, they have been a source of examples and counter-examples in geometric theory of Banach spaces (see e.g. [Day, DuV, LT]).

To understand the geometry of a normed vector space, it is useful to know the structure of its isometries. In fact, many authors have studied the isometries of direct sum of Banach spaces. For example, Fleming, Goldstein, Jamison [FGJ] studied isometries of 1-unconditional sums of Euclidean spaces (see also Fleming and Jamison [FJ1, FJ2]) in the complex case and Rosenthal [Ros] obtained the result for the real case, Greim [Gr] studied surjective isometries of $\ell_p$ sums of Banach spaces (see also [KL]), Fleming and Jamison [FJ3] studied isometries of complex $c_0-$sums and $E-$sums, where $E$ is "sufficiently $\ell_p$ like", say, $E$ is a "nice" Orlicz space (see [FJ3] for precise definitions). It turns out that all the results in these papers show that a surjective isometry always preserves the direct sum structure of the space. There is

*Received February 17, 1997; accepted for publication March 23, 1998.

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also a number of papers that address this problem in non-atomic function spaces. For the detailed discussion of the literature we refer the readers to the survey [FJ4].

In the very interesting paper of Schneider and Turner [ST], the authors determine the structure of isometries for an absolute norm $N$ on $\mathbb{C}^n$, which is the space of complex column vectors with $n$ entries and will be viewed as an $n$-dimensional sequence space in our discussion. In particular, it was shown (cf. [ST, (2.3) and (7.7)]) that if the absolute norm is *normalized* so that $N(e_i) = 1$ for all standard unit vectors for $1 \le i \le n$, then $\mathbb{C}^n$ can be decomposed into a direct sum of $Y_i = \operatorname{span}\{v : v \in E_i\}$ for $i = 1, \dots, k$, where $E_1 \cup \dots \cup E_k = \{e_1, \dots, e_n\}$, the standard basis of $\mathbb{C}^n$, and there exists an absolute norm $N_0$ on $\mathbb{C}^k$ such that

(a) each $(Y_i, N)$ is just an $\ell_2$ space, i.e., the Euclidean space, and
(b) $N(x_1, \dots, x_k) = N_0(N(x_1), \dots, N(x_k))$ for each $x = (x_1, \dots, x_k) \in Y_1 \times \dots \times Y_k \cong \mathbb{C}^n$.

Furthermore, an isometry for $N$ must be of the form

$$(x_1, \dots, x_k) \mapsto (U_1 x_{\pi(1)}, \dots, U_k x_{\pi(k)}) \tag{1}$$

for some unitary $U_i$, $1 \le i \le k$, and a permutation $\pi$ of the set $\{1, \dots, k\}$ such that $N_0(z_1, \dots, z_k) = N_0(z_{\pi(1)}, \dots, z_{\pi(k)})$.

This result was later extended to complex infinite dimensional spaces by Kalton and Wood [KaW, Theorem 6.1].

By the above result of [ST, KaW], one sees that there is an intrinsic cross product structure on every complex sequence space with an absolute norm, and such a structure is useful in characterizing isometries. However, the direct sum decomposition in [ST, KaW] can only identify $\ell_2$ components. If such components do not exist, then every *summand* (or *summand space*) $X_i$ will be one dimensional, and the decomposition will not be very interesting. Of course, one can still get the very useful conclusion that every isometry for the norm must be a signed permutation operator, i.e., an operator of the form (1) with all $Y_i$ being 1-dimensional vector spaces (scalars). Nevertheless, the theorem in [ST] and [KaW] seems inadequate to explain the various isometry results on direct sums of Banach spaces. For example, the results in [ST] and [KaW] cannot even describe the isometries of $\ell_p(\ell_q)$ (e.g., see [KL]).

It is also worth mentioning that the results of [ST] and [KaW] do not extend to real sequence spaces and the description of isometries of real spaces is much more difficult. The known results include: Gordon and Loewy [GL] – isometries in spaces with $\Delta$-bases, Greim [Gr] – isometries in $\ell_p$ sums of absolute spaces, Rosenthal [Ros] – isometries in absolute sums of Euclidean spaces. All these results are highly nontrivial and technical.

In this paper, we propose a new way to decompose a complex or real sequence space with an absolute norm into a direct sum of simpler spaces, which are not necessarily Euclidean. Using this decomposition, we obtain a characterization of the isometries of complex sequence spaces that covers all the known isometry results on direct sum spaces (Corollary 3.4 and 3.5) – in particular we describe the isometries of $X(Y)$, where $X, Y$ are arbitrary complex symmetric spaces. Compared to results in [ST, KaW] our characterization gives more detailed information on which permutations $\pi$ of $\{1, \dots, k\}$ are admissible in (1).

We also apply this decomposition in the real spaces and we obtain a unified characterization of isometries of a wide class of real spaces. In particular this class includes all spaces with direct sum structure whose isometries have been described in the literature (as mentioned above).

Our paper is organized as follows. In Section 2, we show a way to decompose (complex or real) sequence spaces with an absolute norm into a direct sum of simpler spaces, which could possibly be further presented as direct sum of subsequent simpler spaces. Thus we obtain a good technical method of describing a "reduced" direct sum structure.

In Section 3, we prove that in a complex sequence space with an absolute norm, every surjective isometry necessarily preserves the intrinsic direct sum structure described in Theorem 2.4. A number of corollaries covering various existing isometry results on complex direct sum spaces (including $X(Y)$, where $X, Y$ are symmetric) are also presented.

In section 4, we study isometries of real spaces with direct sum structure. In particular, we show that our characterization can be applied to all real spaces whose isometry group is contained in the group of signed permutations. This includes for example spaces with $\Delta$-bases [GL] and spaces which are $p$-convex with constant 1 for $2 < p < \infty$ [R1].

However the situation in real spaces is more complicated since there are many natural spaces with direct sum structure which have isometries other than the signed permutation operators i.e., isometries do not always preserve disjointedness of vectors (see the examples in Section 4). Moreover there exist real spaces with explicit direct sum structure which is not preserved by some isometries (see Examples 4 and 5 in Section 4). We feel that such pathology should be rare, but since every finite group of linear operators on $\mathbb{R}^n$ which contains $-I$ can be realized as the group of isometries of some sequence space (see [GL]), we will not attempt here to characterize them completely.

We prove that if $X$, $Y$ are symmetric finite dimensional sequence spaces, i.e., spaces with symmetric norms, then all isometries of $X(Y)$ preserve the direct sum structure except when $X = \ell_p$ and $Y$ can be decomposed as an $\ell_p$-direct sum of two nonzero subspaces. All other possible isometries in the exceptional case are also characterized. It is worth noting that even in this special type of direct sum spaces, the results in the complex case and the real case are quite different when $\dim Y = 2$ or 4 (cf. Corollary 3.5 and Theorem 4.1).

As a corollary we obtain that if $X$ is a real or complex symmetric sequence space then $X(X)$ is symmetric if and only if $X = \ell_p$ for some $p$, $1 \leq p \leq \infty$ (Corollary 3.6 and 4.2).

For simplicity of notation, we shall always assume that we have a normalized absolute norm, i.e., all standard unit vectors have norm 1.

Throughout we follow standard notations as can be found for example in [LT], except that we use symbol $\ell_\infty$ instead of $c_0$ to denote the space of sequences which converge to zero (with the usual sup norm). In the finite dimensional case $\ell_\infty^k = c_0^k$ and we will not study the infinite dimensional space of bounded sequences.

## 2. Intrinsic direct sum structure.

We begin with the definition of fibers which is modelled on the structure of the space $X((Y_i)_{i\in I})$, where each of the $Y_i$ is a "fiber space".

DEFINITION 2.1. *Let $X$ be a sequence space with a normalized absolute norm $N$, and let $\{e_j\}_{j\in J}$, $J \subseteq \mathbb{N}$ be the corresponding 1-unconditional basis. A nonempty proper subset $S$ of $J$ is called a fiber if for all finitely nonzero sequences*

$\{a_s\}_{s\in S}, \{a'_s\}_{s\in S} \subset \mathbb{F}$,

$$N\Big(\sum_{s\in S} a_s e_s\Big) = N\Big(\sum_{s\in S} a'_s e_s\Big)$$

*implies that for all finitely nonzero sequences* $\{b_i\}_{i\in J\setminus S} \subset \mathbb{F}$,

$$N\Big(\sum_{s\in S} a_s e_s + \sum_{i\notin S} b_i e_i\Big) = N\Big(\sum_{s\in S} a'_s e_s + \sum_{i\notin S} b_i e_i\Big) .$$

*Moreover, the corresponding fiber space is defined by*

$$X_S = \text{span}\,\{e_s : s \in S\}.$$

Here we mention a few examples of fibers.

1. Clearly, in any $X$, if $S$ is a singleton then $S$ is a fiber.

It follows from the classical Bohnenblust characterization of $L_p$–spaces that if $X$ is a symmetric space not equal to $\ell_p$ for some $1 \le p \le \infty$ then singletons are maximal fibers in $X$.

2. In $\ell_p$, $1 \le p \le \infty$, every non-empty proper subset $S$ of $\mathbb{N}$ is a fiber.

3. Let $1 \le p \le \infty$. Let $\ell_p^k$ be the $k$-dimensional $\ell_p$ space, $1 < k \le \infty$, and let $\ell_p^k(Y)$, where $Y$ has no nontrivial $\ell_p$–summands (i.e. $Y$ cannot be written as a direct sum of $\{Y_r\}_r$ such that $Y = \ell_p((Y_r)_r)$). Then fiber spaces of $\ell_p^k(Y)$ are of the form $W_1 \times \cdots \times W_k$ where $W_i$ equal $\{0\}$ or $Y$ for all $i \le k$, or $W_i = \{0\}$ for all $i$ except exactly one, say $i_0$, and $W_{i_0}$ is a fiber space in $Y$.

4. In the above example, if the space $\ell_p^k$ is replaced by a different symmetric space $X$, as shown below, we do not need any assumptions on $Y$ (we even allow $Y = X$). We will see that $W_1 \times \cdots \times W_k$ is a fiber space in $X(Y)$ if and only if $W_i = \{0\}$ for all $i$ except exactly one, say $i_0$, and $W_{i_0}$ is a fiber space in $Y$.

We want to analyze fibers on $X$ which are maximal with respect to inclusion. Notice that maximal fibers do not always exist. For example we consider a space $X_{p,q}$ with the norm of finitely supported elements defined inductively by:

$$N(x_1 e_1) = |x_1|$$

$$N\left(\sum_{i=1}^{n} x_i e_i\right) = \left(\left(N\left(\sum_{i=1}^{n-1} x_i e_i\right)\right)^q + |x_n|^q\right)^{1/q} \qquad \text{if } n \ge 2 \text{ is odd}$$

$$= \left(\left(N\left(\sum_{i=1}^{n-1} x_i e_i\right)\right)^p + |x_n|^p\right)^{1/p} \qquad \text{if } n \ge 2 \text{ is even.}$$

where $1 \le p, q < \infty$ and $p \ne q$. Then it is easy to see that all fibers in $X_{p,q}$ are of the form $\{1, \dots, k\}$ for some $k \in \mathbb{N}$ and thus there are no maximal fibers.

In our considerations we will restrict ourselves to spaces $X$ which do contain maximal fibers.

We start with the following observation:

PROPOSITION 2.2. *Let $X$ be a $k$-dimensional sequence space with a normalized absolute norm $N$. Suppose there exist two maximal fibers $S, T$ such that $S \cap T \ne \emptyset$. Then $S \cup T = \{1, \dots, k\}$ and $X = \ell_p(X_{T\setminus S}, X_{T\cap S}, X_{S\setminus T})$.*

*Proof.* Suppose $S$ and $T$ are maximal fibers such that $i_0 \in S \cap T \neq \emptyset$. Since $S$ is a fiber for every finitely nonzero sequence $\{a_s\}_{s \in S \cup T} \subset \mathbb{F}$,

$$N\Big(\sum_{s \in S \cup T} a_s e_s\Big) = N\left(N\Big(\sum_{s \in S} a_s e_s\Big) e_{i_0} + \sum_{s \in T \setminus S} a_s e_s\right). \tag{2}$$

Moreover,

$$\begin{aligned}
&N\Big(\sum_{s \in S \cup T} a_s e_s + \sum_{i \notin S \cup T} b_i e_i\Big)\\
&= N\left(N\Big(\sum_{s \in S} a_s e_s\Big) e_{i_0} + \sum_{s \in T \setminus S} a_s e_s + \sum_{i \notin S \cup T} b_i e_i\right) && \text{since } S \text{ is a fiber}\\
&= N\left(N\left(N\Big(\sum_{s \in S} a_s e_s\Big) e_{i_0} + \sum_{s \in T \setminus S} a_s e_s\right) e_{i_0} + \sum_{i \notin S \cup T} b_i e_i\right) && \text{since } T \text{ is a fiber}\\
&= N\left(N\Big(\sum_{s \in S \cup T} a_s e_s\Big) e_{i_0} + \sum_{i \notin S \cup T} b_i e_i\right). && \text{by (2)}
\end{aligned}$$

Therefore $S \cup T$ is a fiber, and by maximality of $S$, $S \cup T = J$. Since $S$ and $T$ are maximal, $S \setminus T \neq \emptyset$ and $T \setminus S \neq \emptyset$. Let $s_0 \in S \setminus T$ and $t_0 \in T \setminus S$. Using consecutively the fact that $T$ and $S$ are fibers, we get for all scalars $x_1, x_2$:

$$N(x_1 e_{s_0} + x_2 e_{i_0}) = N(x_1 e_{s_0} + x_2 e_{t_0}) = N(x_1 e_{i_0} + x_2 e_{t_0}) \tag{3}$$

Next, since $S$ is a fiber and using (3) we get

$$\begin{aligned}
N(x_1 e_{s_0} + x_2 e_{i_0} + x_3 e_{t_0}) &= N(N(x_1 e_{s_0} + x_2 e_{i_0}) e_{s_0} + x_3 e_{t_0})\\
&= N(N(x_1 e_{s_0} + x_2 e_{i_0}) e_{s_0} + x_3 e_{i_0})
\end{aligned} \tag{4}$$

Similarly, since $T$ is a fiber and by (3)

$$\begin{aligned}
N(x_1 e_{s_0} + x_2 e_{i_0} + x_3 e_{t_0}) &= N(x_1 e_{s_0} + N(x_2 e_{i_0} + x_3 e_{t_0}) e_{i_0})\\
&= N(x_1 e_{s_0} + N(x_2 e_{s_0} + x_3 e_{i_0}) e_{i_0})
\end{aligned} \tag{5}$$

Now let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by $f(x_1, x_2) = N(x_1 e_{s_0} + x_2 e_{i_0})$. Then (4) and (5) will take the form:

$$f(f(x_1, x_2), x_3) = f(x_1, f(x_2, x_3)).$$

By a theorem of Bohnenblust [Bo], there exists $p$ with $1 \leq p \leq \infty$ such that

$$f(x_1, x_2) = \begin{cases} (|x_1|^p + |x_2|^p)^{1/p} & \text{if } p < \infty \\ \max(|x_1|, |x_2|) & \text{if } p = \infty . \end{cases}$$

Therefore for all scalars $x_1, x_2, x_3$

$$N(x_1 e_{s_0} + x_2 e_{i_0} + x_3 e_{t_0}) = \ell_p(x_1, x_2, x_3). \tag{6}$$

By the fact that $S$ and $T$ are fibers and by (6), for any finitely nonzero sequence $\{x_s\}_{s=1}^k$, we have:

$$N\Big(\sum_{s \in T} x_s e_s\Big) = N\left(N\Big(\sum_{s \in S \cap T} x_s e_s\Big) e_{s_0} + \sum_{s \in T \setminus S} x_s e_s\right)$$

$$= N\left(N\Big(\sum_{s\in S\cap T} x_s e_s\Big)e_{s_0} + N\Big(\sum_{s\in T\setminus S} x_s e_s\Big)e_{t_0}\right)$$

$$= \ell_p\left(N\Big(\sum_{s\in S\cap T} x_s e_s\Big), N\Big(\sum_{s\in T\setminus S} x_s e_s\Big)\right) .$$

Similarly, since $S\cup T = J$:

$$N\Big(\sum_{s=1}^{k} x_s e_s\Big) = N\left(N\Big(\sum_{s\in T} x_s e_s\Big)e_{t_0} + \sum_{s\in S\setminus T} e_s x_s\right)$$

$$= N\left(N\Big(\sum_{s\in T} x_s e_s\Big)e_{t_0} + N\Big(\sum_{s\in S\setminus T} e_s x_s\Big)e_{s_0}\right)$$

$$= \ell_p\left(N\Big(\sum_{s\in T} x_s e_s\Big), N\Big(\sum_{s\in S\setminus T} e_s x_s\Big)\right)$$

$$= \ell_p\left(N\Big(\sum_{s\in T\setminus S} x_s e_s\Big), N\Big(\sum_{s\in T\cap S} x_s e_s\Big), N\Big(\sum_{s\in S\setminus T} e_s x_s\Big)\right) .$$

Since finitely supported elements are dense in $X$, thus $X = \ell_p(X_{T\setminus S}, X_{T\cap S}, X_{S\setminus T})$. □

The main theorem of this section says that maximal fibers determine the direct sum structure of a sequence space $X$. First we need a definition of a special 2-dimensional real space different form $\ell_p^2$, which can be decomposed into $\ell_p$ sum of its nonzero subspaces (see [LaW]):

DEFINITION 2.3. *Let $1 \le p < \infty$, $p \neq 2$. Let $E_p(2)$ denote the space $\mathbb{R}^2$ with the following norm:*

$$\|(x,y)\|_{E_p} = \left(\frac{|x+y|^p}{2} + \frac{|x-y|^p}{2}\right)^{1/p} .$$

*If $p = \infty$ define:*

$$\|(x,y)\|_{E_\infty} = \max(|x+y|, |x-y|) = \|(x,y)\|_{\ell_1}.$$

Observe that $E_p(2)$ is isometric to $\ell_p^2$ through the isometry $T : E_p(2) \longrightarrow \ell_p^2$ defined by $T(x,y) = 2^{-1/p}(x+y, x-y)$.

THEOREM 2.4. *Let $X$ be a sequence space over a scalar field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ with a normalized absolute norm $N$, and let $\{e_j\}_{j\in J}$, $2 \le k = \operatorname{card}(J) \le \infty$ be the corresponding 1-unconditional basis. Suppose that $X \neq \ell_p^k$ for any $1 \le p \le \infty$ and that $X$ has maximal fibers. Then there exist $m$, $2 \le m \le \infty$, a set $\emptyset \subseteq S_1 \subset J$ and a partition $S_2, \ldots, S_m$ of $J \setminus S_1$ such that $X$ is a direct sum of $X_i = \operatorname{span}\{e_s : s \in S_i\}$, $1 \le i \le m$, and exactly one of the following holds:*

(i) *For each $1 \le i \le m$, $S_i$ is a maximal fiber and $X = X_0((X_i)_{i\le m})$ where the norm $N_0$ on $X_0$ is defined by*

$$N_0((a_i)_{i\le m}) = N\left(\sum_{i=1}^{m} a_i e_{s_i}\right)$$

*for some $s_i \in S_i$. In this case maximal fibers of $X_0$ exist and are singletons.*

(ii) *For each $2 \leq i \leq m$, $J \setminus S_i$ is a maximal fiber and there exists $p$ with $1 \leq p \leq \infty$ such that $X = \ell_p(X_1, \dots, X_m)$, where*

(a) $X_1 = \ell_p^{\operatorname{card}(S_1)}$ *(possibly $\dim X_1 = 0$),*

(b) *some of the $X_i$'s equal to $E_p(2)$ if $\mathbb{F} = \mathbb{R}$ and $p \neq 2$,*

(c) *and the rest of the $X_i$'s are such that $\dim X_i \geq 2$ and $X_i$ is not an $\ell_p$ sum of two nonzero subspaces.*

*Proof.* Note first that if $\{S_i\}_{1 \leq i \leq m}$ are maximal fibers in $X$ and $F \subset \{i : 1 \leq i \leq m\}$ is a fiber in $X_0$ then $S_F = \bigcup_{i \in F} S_i$ is a fiber in $X$.

Indeed, select for each $i \leq m$, $s_i \in S_i$, $s_F \in \bigcup_{i=1}^m \{s_i\}$ and let $\{a_s\}_{s \in S_F}, \{a'_s\}_{s \in S_F} \subset \mathbb{F}$ be finitely nonzero sequences with

$$N\Big(\sum_{s \in S_F} a_s e_s\Big) = N\Big(\sum_{s \in S_F} a'_s e_s\Big).$$

Then for all finitely nonzero sequences $\{b_i\}_{i \in J \setminus S_F} \subset \mathbb{F}$,

$$\begin{aligned}
&N\Big(\sum_{s \in S_F} a_s e_s + \sum_{i \in J \setminus S_F} b_i e_i\Big) \\
&\underset{(1)}{=} N\Big(\sum_{i \in F} N\Big(\sum_{s \in S_F \cap S_i} a_s e_s\Big) e_{s_i} + \sum_{i \notin F} N\Big(\sum_{i \in (J \setminus S_F) \cap S_i} b_s e_s\Big) e_{s_i}\Big) \\
&\underset{(2)}{=} N\Big(N\Big(\sum_{i \in F} N\Big(\sum_{s \in S_F \cap S_i} a_s e_s\Big) e_{s_i}\Big) e_{s_F} + \sum_{i \notin F} N\Big(\sum_{i \in (J \setminus S_F) \cap S_i} b_s e_s\Big) e_{s_i}\Big) \\
&\underset{(3)}{=} N\Big(N\Big(\sum_{s \in S_F} a_s e_s\Big) e_{s_F} + \sum_{i \notin F} N\Big(\sum_{i \in (J \setminus S_F) \cap S_i} b_s e_s\Big) e_{s_i}\Big) \\
&\underset{(4)}{=} N\Big(N\Big(\sum_{s \in S_F} a'_s e_s\Big) e_{s_F} + \sum_{i \notin F} N\Big(\sum_{i \in (J \setminus S_F) \cap S_i} b_s e_s\Big) e_{s_i}\Big) \\
&\underset{(5)}{=} N\Big(\sum_{s \in S_F} a'_s e_s + \sum_{i \in J \setminus S_F} b_i e_i\Big),
\end{aligned}$$

where equality (1) holds because all $S_i$ are fibers and our elements are finitely supported, (2) holds since $F$ is a fiber in $X_0$, (3) uses again the fact that $S_i$'s are fibers in $X$, (4) was our assumption and (5) is the final effect of applying again (4), (3), (2) and (1).

Thus, by maximality of $S_i$'s, only sigletons can be fibers in $X_0$.

Suppose next that (i) does not hold, i.e. that it is not possible to form a partition of $J$ consisting of maximal fibers. Then there must exist two nondisjoint maximal fibers. Thus, by Proposition 2.2, there exists $p$, $1 \leq p \leq \infty$ and spaces $Y_1, Y_2, Y_3$ so that $X = \ell_p(Y_1, Y_2, Y_3)$ and $Y_i = \operatorname{span}\{e_s : s \in A_i\}$, for some partition $\{A_i\}_{i=1}^3$ of $J$. Among all the decompositions of the space $X$ into $\ell_p$ sum, let $R_1 \cup \dots \cup R_s$ $(s \leq \infty)$ be a maximal partition of $J$ so that $X = \ell_p((Z_i)_{i \leq s})$ with $Z_i = \operatorname{span}\{e_r : r \in R_i\}$. If $X$ is real and $p \neq 2$, then for each $1 \leq i \leq s$ we have one of the three possibilities (cf. [LaW]):

(a) $R_i$ is a singleton.

(b) $R_i$ has two elements and $Z_i = E_p(2)$.

(c) $R_i$ has at least two elements and $Z_i$ cannot be decomposed as an $\ell_p$-direct sum of two nonzero subspaces.

If $X$ is complex or if $p = 2$ only (a) and (c) can happen (cf. [BL]).

Let $S_1$ be the union of the $R_i$ which are singletons if they exist, and rename the other $R_i$ as $S_j$ if necessary. We see that condition (ii) holds. □

Some remarks are in order in connection with Theorem 2.4.

1. In both cases (i) and (ii) we present $X$ as a direct sum of summand subspaces $X_1, \dots, X_m$ ($m \leq \infty$). Notice that $(X_i)_{i \leq m}$ are uniquely determined by $X$, up to a permutation, and that each of the spaces $X_i$ may be further decomposable into summands (we do not consider summands of $X_i$'s as summands of $X$, sometimes we will call them second generation summands of $X$).

2. Evidently, if $X$ has an explicit direct sum structure i.e. if $X = Y((Y_i)_{i \leq m})$, where $X$ (and, equivalently, $Y$) has maximal fibers, then Theorem 2.4 can be used to regroup the summands of $X$ so that condition (i) or (ii) of Theorem 2.4 holds. If no regrouping is necessary, we say that $X$ has *reduced direct sum structure.* Some examples of spaces with the explicit reduced direct sum structure include

type (i): $X(\ell_p)$, $X(X)$, $X(Y)$, $X(Y, \ell_p)$, $X(Y(\ell_p))$, $X(Y(Z))$, where $X, Y, Z$ are symmetric spaces not equal to any $\ell_r$,

type (ii): $\ell_p(\ell_q)$, $\ell_p^2(\ell_p, \ell_q)$, $\ell_p(E_p(2))$, $\ell_p^4(\ell_p, \ell_q, E_p(2), \ell_q(\ell_r))$, where $p \neq q \neq r$, etc.

**3. Isometries of complex sequence spaces.** Before further analysis of isometries of $X$ we need to introduce another definition. As before, we denote by $\{e_j\}_{j \in J}$ the 1-unconditional basis of $X$. After [ST] (cf. also [KaW]) we define an equivalence relation $\sim$ on the indices $J$. We say that $s \sim t$ if $N(\sum_{j \in J} a_i e_i) = N(\sum_{j \in J} b_i e_i)$ whenever $\ell_2(a_s, a_t) = \ell_2(b_s, b_t)$ and $a_i = b_i$ for all $i \neq s, t$.

Schneider and Turner showed that $\sim$ is indeed an equivalence relation and that equivalence classes of $\sim$ are isometrically isomorphic to $\ell_2$ (with appropriate dimension) [ST, Lemma 2.3]. If $X$ is isometric to $\ell_2$ then relation $\sim$ has only one equivalence class equal to the whole set $J$. Otherwise equivalence classes of $\sim$ are fibers in $X$, and hence they are contained in maximal fibers of $X$. We will call equivalence classes of $\sim$ maximal $\ell_2$-fibers. Notice that every subset of a maximal $\ell_2$-fiber is also a fiber; we will call it a (non-maximal) $\ell_2$-fiber.

The results in [ST] and [KaW] state that every isometry of $X$ preserves maximal $\ell_2$-fibers. This fact has very important consequences for us. Namely we have:

THEOREM 3.1. *Let $X$ be a complex sequence space with 1-unconditional basis $\{e_j\}_{j \in J}$, $J \subseteq \mathbb{N}$, $X \neq \ell_2$, and let $X = X_0(X_1, \dots, X_m)$, $m \leq \infty$, where $X_1, \dots, X_m$ are summands as described in Theorem 2.4. Then $T$ is a surjective isometry of $X$ if and only if there exists a permutation $\pi$ of $\{1, \dots, m\}$ such that the norm $N_0$ on $X_0$ satisfies $N_0(z_1, \dots, z_m) = N_0(z_{\pi(1)}, \dots, z_{\pi(m)})$ and there exists a family of surjective isometries $S_j : X_{\pi(j)} \longrightarrow X_j$ such that*

$$T(x_1, \dots, x_m) = (S_1 x_{\pi(1)}, \dots, S_m x_{\pi(m)}) \tag{7}$$

*for all $(x_1, \dots, x_m) \in X_0(X_1, \dots, X_m) = X$.*

*Proof.* For the proof of Theorem 3.1 we will need two lemmas. We start with introducing some notation.

We will use $\mathcal{M}$ to denote the collection of all maximal fibers in $X$ and $\mathcal{M}_2 = \{J_\lambda\}_{\lambda \in \Lambda}$, $\Lambda \subseteq \mathbb{N}$ the collection of all maximal $\ell_2$-fibers in $X$. By [KaW, Theorem 6.1] there exists a permutation $\sigma$ of $\Lambda$ such that for all $\lambda \in \Lambda$

$$\operatorname{supp}(T(\operatorname{span}\{e_s : s \in J_\lambda\})) = J_{\sigma(\lambda)}. \tag{8}$$

Let $\mathcal{U}$ be the class of subsets of $J$ which can be presented as unions of maximal $\ell_2$-fibers. Define a map $\widetilde{T} : \mathcal{U} \longrightarrow \mathcal{U}$ by

$$\widetilde{T}(A) = \operatorname{supp}\left(T(\operatorname{span}\{e_s : s \in A\})\right) = \bigcup_{\lambda \in \Lambda_A} J_{\sigma(\lambda)}$$

where $A = \bigcup_{\lambda \in \Lambda_A} J_\lambda \in \mathcal{U}$, $\Lambda_A \subset \Lambda$. In this notation we have:

LEMMA 3.2. $\widetilde{T}(\mathcal{M} \cap \mathcal{U}) = \mathcal{M} \cap \mathcal{U}$.

LEMMA 3.3. *$\mathcal{M} \subset \mathcal{U}$, unless $X$ has the form described in Theorem 2.4(ii) with $p = 2$ and $S_1 \neq \emptyset$; in this case $\mathcal{M} \cap \mathcal{U} = \{J \setminus S_i\}_{i \geq 2}$.*

Before proving the lemmas we show that they indeed imply the conclusion of Theorem 3.1. We have the following cases:

*Case 1.* If $X$ has the form described in Theorem 2.4(i) then $\mathcal{M} = \{S_i\}_{i \leq m}$, and by Lemma 3.3 $\mathcal{M} \subset \mathcal{U}$. By Lemma 3.2, $\widetilde{T}(\mathcal{M}) = \mathcal{M}$. Thus there exists a permutation $\pi$ of $\{1, \ldots, m\}$ so that $\widetilde{T}(S_{\pi(i)}) = S_i$. Hence $T\big|_{X_{\pi(i)}}$ is an isometry of $X_{\pi(i)}$ onto $X_i$ and $T$ has form (7).

*Case 2.* If $X$ has the form described in Theorem 2.4(ii) then $\mathcal{M} = \{J \setminus S_i\}_{2 \leq i \leq m} \cup \{J \setminus \{s\}\}_{s \in S_1}$. If $S_1 = \emptyset$ then, by Lemma 3.3, $\mathcal{M} \subset \mathcal{U}$ and the proof is the same as in Case 1. Thus we will assume that $S_1 \neq \emptyset$.

*Case 2(a).* If $p \neq 2$ then, by Lemma 3.3, $\mathcal{M} \subset \mathcal{U}$ and by Lemma 3.2, $\widetilde{T}(\mathcal{M}) = \mathcal{M}$.

Since $p \neq 2$, $\{s\} \in \mathcal{M}_2$ for every $s \in S_1$. Thus, by [KaW, Theorem 6.1], $\widetilde{T}(\{s\}) \in \mathcal{M}_2$ and $\operatorname{card}(\widetilde{T}(\{s\})) = 1$. By Lemma 3.2, $\widetilde{T}(J \setminus \{s\}) = J \setminus \widetilde{T}(\{s\}) \in \mathcal{M}$, so, since for all $i \geq 2$ $\operatorname{card}(S_i) \geq 2$, there exists $s' \in S_1$ so that $\widetilde{T}(J \setminus \{s\}) = J \setminus \{s'\}$. Hence

$$\widetilde{T}(\{J \setminus S_i\}_{2 \leq i \leq m}) = \{J \setminus S_i\}_{2 \leq i \leq m}.$$

and there exists a permutation $\pi$ of $\{1, \ldots, m\}$ so that $\pi(1) = 1$ and $\widetilde{T}(J \setminus S_{\pi(i)}) = J \setminus S_i$ for $i \geq 2$. Thus for all $i$ we have $\widetilde{T}(S_{\pi(i)}) = S_i$ and $T\big|_{X_{\pi(i)}}$ is an isometry of $X_{\pi(i)}$ onto $X_i$ i.e. $T$ has form (7).

*Case 2(b).* If $p = 2$ then $S_1 \in \mathcal{M}_2$ and, by Lemma 3.3, $\mathcal{M} \cap \mathcal{U} = \{J \setminus S_i\}_{i \geq 2}$. By Lemma 3.2, $\widetilde{T}(\mathcal{M} \cap \mathcal{U}) = \mathcal{M} \cap \mathcal{U}$ so there exists a permutation $\pi$ of $\{1, \ldots, m\}$ so that $\pi(1) = 1$ and $\widetilde{T}(J \setminus S_{\pi(i)}) = J \setminus S_i$ for $i \geq 2$.

On the other hand $S_1 \in \mathcal{M}_2$, so by [KaW, Theorem 6.1], $\widetilde{T}(S_1) \in \mathcal{M}_2$, and since $S_1 = \bigcap_{2 \leq i \leq m} (J \setminus S_i)$, we have

$$\widetilde{T}(S_1) = \bigcap_{2 \leq i \leq m} \widetilde{T}(J \setminus S_i) = \bigcap_{2 \leq i \leq m} (J \setminus S_i) = S_1.$$

Thus for all $i \geq 1$ we have $\widetilde{T}(S_{\pi(i)}) = S_i$ and the theorem follows same as in the previous cases. □

*Proof of Lemma 3.2.* Let $S \in \mathcal{M} \cap \mathcal{U}$, then $S^c \in \mathcal{U}$ and $\widetilde{T}(S)$, $\widetilde{T}(S^c)$ are disjoint.

Now let $a, a' \in X$ be such that $\operatorname{supp} a \cup \operatorname{supp} a' \subset \widetilde{T}(S)$ and $N(a) = N(a')$, and let $b \in X$ with $\operatorname{supp} b \subset \widetilde{T}(S^c)$. Then $\operatorname{supp} T^{-1}(a) \cup \operatorname{supp} T^{-1}(a') \subset S$ and $\operatorname{supp} T^{-1}(b) \subset S^c$. Thus, since $S$ is a fiber, we have

$$N(a + b) = N(T^{-1}(a) + T^{-1}(b)) = N(T^{-1}(a') + T^{-1}(b)) = N(a' + b)\ .$$

Therefore $\widetilde{T}(S)$ is a fiber in $X$ and $(\widetilde{T^{-1}})(\widetilde{T}(S)) = S$.

Assume now that $\widetilde{T}(S) \notin \mathcal{M}$, say $\widetilde{T}(S)$ is a subfiber of a proper fiber $S_1$ then $(\widetilde{T^{-1}})(S_1)$ is a proper fiber in $X$ which contains $(\widetilde{T^{-1}})(\widetilde{T}(S)) = S$ which contradicts the maximality of $S$. Thus $\widetilde{T}(\mathcal{M} \cap \mathcal{U}) \subset \mathcal{M} \cap \mathcal{U}$. Also $(\widetilde{T^{-1}})(\mathcal{M} \cap \mathcal{U}) \subset \mathcal{M} \cap \mathcal{U}$. Hence $\widetilde{T}(\mathcal{M} \cap \mathcal{U}) = \mathcal{M} \cap \mathcal{U}$. □

*Proof of Lemma 3.3.* Assume that $\mathcal{M} \not\subset \mathcal{U}$. Then there exists $S \in \mathcal{M}$ and $F \in \mathcal{M}_2$ so that $F \cap S \neq \emptyset$ and $F \cap S^c \neq \emptyset$. Let $V$ be a maximal fiber such that $F \subset V$. Then $V \cap S \neq \emptyset$ and by Proposition 2.2 there exists $p$, $1 \leq p \leq \infty$ so that $X = \ell_p(X_{V\setminus S}, X_{V\cap S}, X_{S\setminus V})$. But $F \cap (V \setminus S) \neq \emptyset$ and $F \cap (V \cap S) \neq \emptyset$. Let $i \in F \cap (V \setminus S)$ and $k \in F \cap (V \cap S)$. Then for all $a_i, a_k \in \mathbb{C}$:

$$N(a_i e_i + a_k e_k) = (|a_i|^p + |a_k|^p)^{1/p},$$

but $i, k \in F$ and $F \in \mathcal{M}_2$ so:

$$N(a_i e_i + a_k e_k) = (|a_i|^2 + |a_k|^2)^{1/2}.$$

Thus $p = 2$.

Let $X = \ell_2(X_1, \ldots, X_m)$ $(m \leq \infty)$ be the decomposition of $X$ described in Theorem 2.4(ii). Assume that there exist $F \in \mathcal{M}_2$ and $2 \leq i, k \leq m$, $i \neq k$, with $F \cap S_i \neq \emptyset$ and $F \cap S_k \neq \emptyset$, say $j_k \in F \cap S_k$.

Note that if $F \cap S_i = S_i$ then $X_i = \ell_2^{\operatorname{card}(S_i)}$ $(\operatorname{card}(S_i) \geq 2)$ contrary to the assumption that $X_i$ cannot be decomposed into $\ell_2$ sum of nonzero subspaces. Thus $F \cap S_i^c \neq \emptyset$. Let $x \in X_i$, then

$$\begin{aligned} N(x) &= N\left(\sum_{j \in F\cap S_i} x_j e_j + \sum_{j \in F^c \cap S_i} x_j e_j\right) \\ &\underset{(1)}{=} N\left(N\left(\sum_{j\in F\cap S_i} x_j e_j\right) e_{j_k} + \sum_{j \in F^c\cap S_i} x_j e_j\right) \\ &\underset{(2)}{=} \left(N\left(\sum_{j\in F\cap S_i} x_j e_j\right)^2 + N\left(\sum_{j\in F^c\cap S_i} x_j e_j\right)^2\right)^{\frac{1}{2}}, \end{aligned}$$

where (1) holds since $F$ is a fiber, and (2) holds since $j_k \notin S_i$ and $X = \ell_2(X_1, \ldots, X_m)$.

Thus $X_i$ is the $\ell_2$ sum of $X_{F\cap S_i}$ and $X_{F\cap S_i^c}$ again contradicting our assumption of nondecomposability of $X_i$.

Thus $\{J \setminus S_i\}_{i\geq 2} \subset \mathcal{U}$ and $\mathcal{M} \neq \{J \setminus S_i\}_{i \geq 2}$ only if $S_1 \neq \emptyset$. □

We would like to make some remarks:

1. Notice that (7) is very similar to (1). Theorem 3.1 refines the results of [ST, KaW] by determining precisely which permutations of standard basis vectors generate isometries and which do not (see also Corollary 3.5).

2. Lemma 3.3 is also valid in real sequence spaces with maximal fibers. The proof does not change.

3. If the isometry group of $X$ is contained in the group of signed permutations then $X$ has no nontrivial $\ell_2$–fibers and (8) is trivially satisfied. Thus Lemma 3.2 and Theorem 3.1 will follow.

4. Theorem 3.1 is valid in those real spaces with maximal fibers for which Lemmas 3.2, 3.3 and formula (8) hold. In particular, by the discussion in the preceding paragraphs, Theorem 3.1 is valid for any real sequence space with maximal fibers, whose isometry group is contained in the group of signed permutations.

Theorem 3.1 provides a complete description of surjective isometries of complex sequence spaces with maximal fibers. Below we present some immediate corollaries about the form of isometries of spaces with explicit cross product structure (cf. Remark 3 after Theorem 2.4).

COROLLARY 3.4. *Let $X = Z(X_1, \dots, X_m)$ be the space with explicit reduced direct sum structure. Suppose that $X$ or, equivalently, $Z$ has maximal fibers. Then every surjective isometry of $X$ onto itself has form (7).*

COROLLARY 3.5. *Suppose $X, Y$ are complex symmetric sequence spaces not both equal to $\ell_p$ with the same $p$. Let $\{e_{ij}\}_{i=1,j=1}^{\dim X, \dim Y}$ be the standard basis vectors of $X(Y)$ (i.e. $N_{X(Y)}(\sum_{i,j} a_{ij}e_{ij}) = N_X(\sum_i (N_Y(\sum_j a_{ij}e_j^Y)e_i^X))$. Then $T : X(Y) \longrightarrow X(Y)$ is a surjective isometry if and only if there exist a family of numbers $\lambda_{ij} \in \mathbb{C}$ with $|\lambda_{ij}| = 1$ and permutations $\pi_1$ of $\{1, \dots, \dim X\}$ and $\pi_2$ of $\{1, \dots, \dim Y\}$ such that*

$$Te_{ij} = \lambda_{ij} e_{\pi_1(i)\pi_2(j)}$$

*for all $i, j$.*

COROLLARY 3.6. *(cf.* [BVG]*) If $X$ is a complex symmetric sequence space then $X(X)$ is symmetric if and only if $X = \ell_p$ for some $p$, $1 \leq p \leq \infty$.*

Notice that if summands $X_1, X_2, \dots, X_m$ have maximal fibers and can be further decomposed into simpler second generation summands (as mentioned in Remark 1 after Theorem 2.4), then one can again use Theorem 3.1 to conclude that the isometries $S_1, \dots, S_m$ have form (7). In particular, one can inductively describe isometries of spaces of the form $X_1(X_2(\dots(X_m)\dots))$, where $X_1, X_2, \dots, X_m$ are complex symmetric sequence spaces such that for any $i = 1, \dots, m-1$, $X_i$ and $X_{i+1}$ are not simultaneously equal to $\ell_p$ with the same $p$. We leave the exact statement to the interested reader.

It is interesting to note that the group of isometries of $X_1(X_2(\dots(X_m)\dots))$ does not depend on entire isometry groups of $X_1, \dots, X_{m-1}$, but only on intersection of these groups with the group of signed permutation operators and the isometry group of $X_m$.

**4. Isometries of real sequence spaces.** The description of isometries of real sequence spaces is more complicated than in the complex case. The main difference is in classification of spaces whose group of isometries is contained in the group of signed permutations. In the complex case Schneider, Turner [ST] and Kalton, Wood [KaW] showed that the group of isometries is contained in the group of signed permutations if and only if the space does not have nontrivial $\ell_2$–fibers. In the real case similar classification is not valid. In fact, we have the following examples of spaces which do not contain any copies of $\ell_2$ and which allow non-disjointedness preserving isometries, i.e., isometries that are not signed permutation operators.

1. Let $p \neq 2$ and let $E_p(2)$ be the space defined in Definition 2.3. Then $E_p(2)$ is isometric to $\ell_p^2$ through the isometry $T : E_p(2) \longrightarrow \ell_p^2$, $T(x, y) = 2^{-1/p}(x + y, x - y)$. Let $X = Y(\ell_p^2, E_p)$ for any 2-dimensional real symmetric space $Y$, and let $S : X \longrightarrow X$ be an isometry defined by $S(v_1, v_2) = (Tv_2, T^{-1}v_1)$ where $v_1 \in \ell_p$, $v_2 \in E_p$. Then $S$ is a non-disjointedness preserving isometry of $X$.

2. Let $X$ be any 2-dimensional real symmetric space, $X \neq \ell_2$. Put

$$\|(x,y)\|_{E_X} = \frac{1}{\|(1,1)\|_X}\|(x+y, x-y)\|_X \ .$$

Consider $Z = Y(X, E_X)$, where $Y$ is any 2-dimensional real symmetric space. Then, similarly to Example 1, there exists a non-disjointedness preserving isometry of $Z$.

3. Spaces in Example 2 can be generalized to higher dimensions by taking any spaces $X_1, X_2$ which are isometric through a non-disjointedness preserving isometry (spaces like that can be constructed e.g. by taking direct products of $X$ and $E_X$, cf. also [R1, Theorem 4]). Then let $Z = Y(X_1, X_2)$ for any symmetric space $Y$.

The above examples show isometries which are not signed permutations but which nevertheless "preserve the direct sum structure", i.e., have canonical form (7). One would hope that this is always true, however the following examples show the contrary.

4. Consider $\ell_p(\ell_p^3, E_p(2)) = (\mathbb{R}^3 \times \mathbb{R}^2, N)$ with

$$N(x_1, x_2, x_3, y_1, y_2) = \ell_p(\ell_p(x_1, x_2, x_3), E_p(y_1, y_2))\}.$$

As described in Example 1, $\ell_p^2$ is isometric with $E_p(2)$ via the isometry $T$. Thus one can define isometry $S$ on $\ell_p(\ell_p^3, E_p(2))$ by

$$S(x_1, x_2, x_3, y_1, y_2) = \big(T(y_1, y_2), x_3, T^{-1}(x_1, x_2)\big)\ .$$

Clearly $S$ does not have form (7).

Surprisingly, a similar pathology is possible even in spaces of the form $X(Y)$, where $X$, $Y$ are symmetric.

5. Consider the space $\ell_p^2(E_p(2)) = \ell_p(E_p(2), E_p(2))$. Then we have

$$\begin{aligned}
&N(x_1, x_2, y_1, y_2)\\
&= \ell_p(E_p(x_1, x_2), E_p(y_1, y_2))\\
&= \ell_p(2^{-1/p}\ell_p(x_1 + x_2, x_1 - x_2), 2^{-1/p}\ell_p(y_1 + y_2, y_1 - y_2))\\
&= 2^{-1/p}\ell_p(x_1 + x_2, x_1 - x_2, y_1 + y_2, y_1 - y_2)\\
&= 2^{-1/p}\ell_p(\ell_p(x_1 + x_2, y_1 + y_2), \ell_p(x_1 - x_2, y_1 - y_2))\\
&= \ell_p\bigg(E_p\Big(2^{-1}(x_1 + x_2 + y_1 + y_2, x_1 + x_2 - y_1 - y_2)\Big),\\
&\qquad E_p\Big(2^{-1}(x_1 - x_2 + y_1 - y_2, x_1 - x_2 - y_1 + y_2)\Big)\bigg)\\
&= N\Big(2^{-1}(x_1 + x_2 + y_1 + y_2, x_1 + x_2 - y_1 - y_2,\\
&\qquad x_1 - x_2 + y_1 - y_2, x_1 - x_2 - y_1 + y_2)\Big)
\end{aligned}$$

Thus the linear map defined by

$$\begin{array}{ll}
S(1,0,0,0) = 2^{-1}(1,1,1,1) & S(0,1,0,0) = 2^{-1}(1,1,-1,-1)\\
S(0,0,1,0) = 2^{-1}(1,-1,1,-1) & S(0,0,0,1) = 2^{-1}(1,-1,-1,1)
\end{array}$$

is an isometry, and clearly $S$ does not preserve disjointedness of vectors. This isometry in the case when $p = \infty$, i.e. $X = \ell^2_\infty(\ell^2_1)$, and $p = 1$, i.e. $X = \ell^2_1(\ell^2_\infty)$ is described in [KL, Theorem 3.1(b)]. Notice that $E_p(2)$ can be decomposed as an $\ell_p$-direct sum of two nonzero subspaces. Thus this example is consistent with [Gr, Proposition 2], which says that if $E$ is an $\ell_p$−sum of two nonzero subspaces then there exists an isometry of $\ell_p(E)$ which is not of the form (9). It is interesting that this is, in fact, the only possible example as shown in Theorem 4.1 below.

It becomes of interest to characterize spaces with direct sum structure, which is preserved under action of all isometries.

First we list classes of spaces whose isometry group is contained in the group of signed permutations:

(1) spaces with $\Delta$-bases ([GL]).

(1a) In particular for spaces of the form $Z(X_1, X_2, \dots, X_k)$ where $k \le \infty$, $\dim X_i \in \mathbb{N} \setminus \{2, 4\}$ and $X_i$ is a symmetric space not equal to $\ell_2$, $i = 1, \dots, k$, $Z$ is arbitrary.

(2) spaces which are $p$-convex with constant 1 for $2 < p < \infty$ ([R1]).

(2a) spaces which are strictly monotone, smooth at every basis vector and $q$-concave with constant 1 for $1 < q < 2$ ([R1]).

Thus if a space $X$ belongs to one of the above classes and has maximal fibers then Theorem 3.1 can be applied to conclude that indeed the direct sum structure is preserved by all isometries. Further Rosenthal [Ros] (cf. also [R2]) showed that Theorem 3.1 holds in real spaces of the form:

(3) $Z(X_1, \dots, X_k)$ where $k \le \infty$, $\dim X_i \ge 2$ and $X_i = \ell_2$ for all $i = 1, \dots, k$, $Z \ne \ell_2$.

Below we study the group of isometries of spaces of the form $X(Y)$ where $X, Y$ are finite-dimensional symmetric spaces with $\dim Y \ge 2$. This class of spaces has a sizable intersection with classes (1a) and (3), but we do allow $\dim Y$ to be 2 or 4, which are excluded by (1a). Thus we allow the situation when the isometry group is not contained in the group of signed permutations. Our proof is somewhat technical but it is more elementary than the one in [GL]. It is possible that our different approach may lead to some insight to the general problem. So we present the entire proof including the previously proven cases.

THEOREM 4.1. *Suppose $X$ and $Y$ are finite dimensional symmetric spaces such that* $\dim X = n$ *and* $\dim Y = m$, *and not both $N_1$ and $N_2$ are $\ell_p$ (with the same $p$). Then $\Psi$ is an isometry for $N$ on $X(Y)$ if and only if*

(i) *$\Psi$ is of the form*

$$(y_1, \dots, y_n) \mapsto (S_1 y_{\pi(1)}, \dots, S_k y_{\pi(n)}) \tag{9}$$

*for some isometries $S_i$ of $Y$ and some permutation $\pi$ of $\{1, \dots, n\}$, or*

(ii) *$X = \ell_p^n$, $Y = E_p(2)$ for some $p$, $1 \le p \le \infty$, $p \ne 2$, and*

(10) $\Psi$ permutes the matrices in the set $\{\pm(e_1^Y \pm e_2^Y)(e_j^X)^t : 1 \le j \le n\}$,

*where $e_i^Y$, $e_i^X$ denote column basis vectors in $Y$ and $X$, respectively.*

Notice that, clearly, if $N_1 = N_2 = \ell_p$ then $X(Y) = \ell_p^{mn}$ and the isometry group is well known.

COROLLARY 4.2. *(cf. [BVG]) If $X$ is a real symmetric sequence space with* $\dim X = n$ *then $X(X)$ is symmetric if and only if $X = \ell_p$ for some $p$, $1 \le p \le \infty$.*

The choices of $S_i$ in Theorem 4.1 (i) are very restrictive. If $N_2 = \ell_2$, the $S_i$ is orthogonal (on $Y$); otherwise, $S_i$ a signed permutation operator on $Y$ unless for $m = 4$ or $m = 2$. When $m = 4$, there are a few more possibilities for $S_i$. One may, for example, see [Ro] (see also [DLR, Br]) for more details. This exceptional case will be treated separately in our proof. When $m = 2$, $S_i$ must be chosen from a dihedral group.

In the following discussion, we shall identify $X(Y)$ with the space $\mathbb{R}^{m\times n}$ of $m \times n$ real matrices, and identify the linear operator $\Psi$ on $X(Y)$ with its matrix representation relative to the standard basis

$$\{e_{11}, e_{21}, \ldots, e_{m1}, \ldots, e_{1n}, e_{2n}, \ldots, e_{mn}\}.$$

We shall also use the following notations in our discussion.

$O(m)$: the orthogonal group on $\mathbb{R}^m$.

$O(mn)$: the orthogonal group on $\mathbb{R}^{m\times n}$.

$P(m)$: the group of $m \times m$ permutation matrices.

$GP(m)$: the group of $m \times m$ signed permutation matrices.

$GP(mn)$: the group of linear operators on $\mathbb{R}^{m\times n}$ that permute and change the signs of the entries of $A \in \mathbb{R}^{m\times n}$.

$P \otimes Q$: the tensor product of the matrices $P$ and $Q$ given by $(P_{ij}Q)$.

$e_{ij}^{(n)}$: the $n \times n$ matrix with one at the $(i, j)$ position and zero elsewhere.

$E_{pq}^{(ij)}$: the $mn \times mn$ matrix $e_{ij}^{(n)} \otimes e_{pq}^{(m)}$.

We shall also use the concept of the *Wreath product* of two groups of linear operators. For simplicity, we consider the special case when $G$ is a group of linear operators (identified as matrices) acting on $\mathbb{R}^m$. The Wreath product of $G$ and $P(n)$, denoted by $G * P(n)$, is the group of linear operators on $\mathbb{R}^{m\times n}$ of the form

$$[y_1|\cdots|y_n] \mapsto [U_1(y_1)|\cdots|U_n(y_n)]V$$

for some $U_1, \ldots, U_n \in G$ and $V \in P(n)$.

With this definition, Theorem 4.1 implies that the isometry group of $N$ is the Wreath product of the isometry group of $N_2$ and $P(n)$ if $m \neq 2$. In particular, isometries will always preserve the direct sum structure of $X(Y)$.

It is also interesting to note that in our proof, we actually determine all possible closed overgroups $G$ of $H = GP(m) * GP(n)$ in $O(mn)$. In particular, if $G$ is infinite then $G = O(m) * GP(n)$ or $O(mn)$; if $G$ is finite then $G$ is one of the following:

(a) $GP(m) * GP(n)$: the Wreath product of $GP(m)$ and $GP(n)$,

(b) $GP(mn)$: the group of signed permutations on $\mathbb{R}^{m\times n}$,

(c) $A_{2n}$: the Weyl group of $A_{2n}$ type realized as the group of orthogonal operators on $\mathbb{R}^{2\times m}$ that permute the set $\{\pm(e_1^Y \pm e_2^Y)(e_j^X)^t : 1 \leq j \leq n\}$ if $m = 2$,

(d) $D_h * GP(n)$: the Wreath product of the dihedral group $D_h$ and $GP(n)$ if $m = 2$,

(e) $F_4$ and the normalizer of $F_4$: the Weyl group of $F_4$, if $m = n = 2$.

In (e), there are two possible realizations of $F_4$, namely, an overgroup of $GP(4)$ (e.g., see [DLR]) or the group generated by $H$ and $L_1$ mentioned in the proof of Lemma 4.8. However, only the first realization can be an isometry group of X(Y).

Our proof of Theorem 4.1 uses the basic ideas in [DLR] (cf. also [Br]) and some intricate arguments. It would be nice to have a shorter conceptual proof. We begin our proof with the following corollary of Auerbach's Theorem (see e.g. [Ro, Theorem IX.5.1], [KL, Theorem 2.3]).

LEMMA 4.3. *Let $G$ be the isometry group of $N$. Then $G < O(mn)$, i.e., $G$ is a subgroup of $O(mn)$.*

We first deal with the case when the isometry group of $N$ is infinite.

LEMMA 4.4. *If the isometry group $G$ of $N$ is infinite, then either $G = O(mn)$ or $G$ is the Wreath product $O(m) * P(n)$.*

*Proof.* By Lemma 4.3, $G$ is a subgroup of $O(mn)$. Since $G$ is closed and $O(mn)$ is a compact Lie group, $G$ is also a compact Lie group. It is well-known that the Lie algebra of $O(mn)$ is $o(mn)$, the algebra of all skew-symmetric $mn \times mn$ matrices over real under the Lie product $[A, B] = AB - BA$. Suppose $\mathbf{g}$ is the Lie algebra of $G$. Then $\mathbf{g}$ is a subalgebra of $o(mn)$. Furthermore, by definition of $N$, we have $H := GP(m) * GP(n) < G$ and $H$ acts on $G$ by conjugation, and so $\mathbf{g}$ is a $H$-module under the action $(P, A) \mapsto P^tAP$ for any $P \in H$ and $A \in \mathbf{g}$. We shall show that there are only two subalgebras of $o(mn)$ which are $H$-modules, and the two Lie groups corresponding to the subalgebras are $O(mn)$ and $O(m) * P(n)$.

For any $A \in \mathbf{g}$ we write $A = (A^{(ij)})$ in $n \times n$ block form such that each block $A^{(ij)} \in \mathbb{R}^{m\times m}$. If there exists $i < j$ such that $A^{(ij)} \neq 0$, we claim that $\mathbf{g} = o(mn)$. First, note that there is $P \in P(n)$ such that the $(1,2)$ entry of $P^tZP$ is the $(i, j)$ entry of $Z$ for any $Z \in \mathbb{R}^{n\times n}$. Then $P \otimes I_m \in H$ and hence $(P \otimes I_m)^t A(P \otimes I_m) \in \mathbf{g}$ will have nonzero $(1,2)$ block. So, we may assume $(i, j) = (1, 2)$. Now suppose the $(p, q)$ entry of $A^{(12)}$ is nonzero. Then $D_1 = I_{mn} - E_{pp}^{(11)} \in H$ and hence $A_1 = A - D_1AD_1 \in \mathbf{g}$. Note that only the $p$th row and $p$th column of $A_1$ are nonzero. Now $D_2 = I_{mn} - E_{pp}^{(22)} \in H$ and hence $A_2 = A_1 - D_2A_1D_2 \in \mathbf{g}$. Since $A_2$ is a nonzero multiple of $E_{pq}^{(12)} - E_{qp}^{(21)}$, it follows that $Q^t(E_{pq}^{(12)} - E_{qp}^{(21)})Q \in \mathbf{g}$ for all $Q \in H$. As a result, $\mathbf{g}$ contains $E_{rs}^{(ij)} - E_{sr}^{(ji)}$ for all $1 \le i < j \le n$ and $1 \le s, r \le m$. In particular, it contains the Lie product $[E_{11}^{(12)} - E_{11}^{(21)}, E_{12}^{(21)} - E_{21}^{(12)}] = E_{12}^{(11)} - E_{21}^{(11)}$. It then follows that $Q^t(E_{12}^{(11)} - E_{21}^{(11)})Q \in \mathbf{g}$ for all $Q \in H$. As a result, $\mathbf{g}$ also contains $E_{rs}^{(ii)} - E_{sr}^{(ii)}$ for all $1 \le i \le n$ and $1 \le r < s \le m$. So, $\mathbf{g}$ contains a basis of $o(mn)$ and we conclude that $G = O(mn)$.

Next, suppose all $A = (A^{(ij)}) \in \mathbf{g}$ satisfy $A^{(ij)} = 0$ if $i \neq j$. Then we can show that $\mathbf{g}$ contains $E_{rs}^{(ii)} - E_{sr}^{(ii)}$ for all $1 \le i \le n$ and $1 \le r < s \le m$, by arguments similar to the preceding case. Taking the exponential map for the elements of $\mathbf{g}$, we see that $G$ contains the Wreath product $O(m) * \{I_n\}$. Since $G$ also contains $GP(m) * GP(n)$, we conclude that $O(m) * P(n) = O(m) * GP(n) < G$. In particular, $N_2 = \ell_2$. By the results in [Ros] (see also [R1]), we conclude that $G = O(m) * GP(n)$. □

Next we consider the case when the isometry group of $N$ is finite. We use the approach in [DLR], namely, determining all the finite overgroups of $GP(m) * GP(n)$ in $O(mn)$. We begin with the following lemma, which explains why one needs to exclude the cases when $n = 2, 4$ in [GL].

LEMMA 4.5. *Suppose* $\dim Y \neq 4, 2$. *If $G$ is the isometry group of $N$ and $G$ is finite, then $G < GP(mn)$.*

*Proof.* For each $\Psi \in G$, we write $\Psi = (\Psi^{(ij)})$ in $n \times n$ block form such that $\Psi^{(ij)} \in \mathbb{R}^{m\times m}$ for all $(i, j)$. Since $G < O(mn)$, $\Psi \in G$ implies $\Psi^t \in G$ as well.

We use the technique in [DLR] to show that for any $\Psi \in G$ (in its matrix representation) the entries of $\Psi$ can only be $0, 1$ or $-1$. Suppose this is not true. Let

$$\mu = \min\{a > 0 : a \text{ is an entry of } \Psi \text{ for some } \Psi \in G\}.$$

Since $G < O(mn)$ by Lemma 4.3, we have $0 < \mu < 1$. Denote by $H = GP(m) * GP(n)$ as in the proof of Lemma 4.4. Let $\Phi \in G$ have one of its entries equal to $\mu$. Then there exist $P, Q \in H$ such that the $(1,1)$ entry of $P\Phi Q$ equal $\mu$. Thus we may assume that the $(1,1)$ entry of $\Phi = (\Phi^{(ij)})$ equals $\mu$. We consider several cases.

First, suppose $n$ is odd. By the arguments in the proof of Theorem 2.4 in [DLR], there exists $S \in GP(m)$ such that $(\Phi^{(11)})^t S \Phi^{(11)}$ has a positive $(1,1)$ entry equal to $\eta < \mu$.

Indeed, let $P = (P^{(ij)}) \in H$ be such that $P^{(11)} = S$, $P^{(2k,2k+1)} = I_m$ and $P^{(2k+1,2k)} = -I_m$ for $1 \le k \le (n-1)/2$, and $P^{(ij)} = 0$ for all other $(i,j)$. Then $\Phi^t P \Phi \in G$ has $(1,1)$ entry equal to $\eta = \mu^2 < \mu$, which contradicts the definition of $\mu$.

Next, suppose $n$ is even. If $m$ is even, we can again obtain $S \in GP(m)$ such that $(\Phi^{(11)})^t S \Phi^{(11)}$ has a positive $(1,1)$ entry equal to $\eta < \mu$. Let $P = (P^{(ij)}) \in H$ be such that $P^{(11)} = S$, $P^{(ii)} = \sum_{j=1}^{m/2} (e_{2j-1,2j}^{(m)} - e_{2j,2j-1}^{(m)})$ for $i = 2, \ldots, n$, and $P^{(rs)} = 0$ for other $(r,s)$. Then $\Phi^t P \Phi \in G$ has $(1,1)$ entry equal to $\eta < \mu$, which contradicts the definition of $\mu$.

Finally, suppose $n$ is even and $m$ is odd. Note that there exists $P \in H$ such that the first column of $P\Phi$ is nonnegative. So, we may assume that $\Phi$ has nonnegative column. Furthermore, we may assume that $\Phi_{11}^{(21)}$ attains the minimum among all entries in the first columns of $\Phi^{(j1)}$ for $j = 2, \ldots, n$. Then either

(i) $\Phi_{11}^{(21)} = 0$, or

(ii) $0 < (\Phi_{11}^{(21)})^2 \le \left\{\sum_{j=2}^{n} \sum_{k=1}^{m} (\Phi_{k1}^{(j1)})^2\right\} / \{(n-1)m\} \le 1/\{(n-1)m\}$.

If (i) holds, let $S \in GP(m)$ such that $(\Phi^{(11)})^t S \Phi^{(11)}$ has a positive $(1,1)$ entry equal to $\eta < \mu$. Let $P = (P^{(ij)}) \in H$ be such that $P^{(11)} = S$, $P^{(22)} = \sum_{j=1}^{m/2} (e_{2j+1,2j}^{(m)} - e_{2j,2j+1}^{(m)})$, $P^{(2k-1,2k)} = I_m$ and $P^{(2k,2k-1)} = -I_m$ for $k = 2, \ldots, n/2$. Then $\Phi^t P \Phi \in G$ has $(1,1)$ entry equal to $\eta < \mu$, which contradicts the definition of $\mu$.

If (ii) holds and $(m,n) \ne (3,2)$, then $0 < \Phi_{11}^{(21)} < 1/2$. Let $P = (P^{(ij)}) \in H$ be such that $P^{(12)} = I_m$, $P^{(21)} = 2e_{11}^{(m)} - I_m$, $P^{(2k-1,2k)} = I_m$ and $P^{(2k,2k-1)} = -I_m$ for $k = 2, \ldots, n/2$. Then $\Phi^t P \Phi \in G$ has $(1,1)$ entry equal to $2\Phi_{11}^{(11)} \Phi_{11}^{(21)} < \Phi_{11}^{(11)} = \mu$, which contradicts the definition of $\mu$.

If (ii) holds and $(m,n) = (3,2)$, let the first column of $\Phi$ be $v = (a_1, a_2, a_3, b_1, b_2, b_3)^t$. By our assumption, $\mu = a_1 \le b_1 \le b_2 \le b_3$. Since $v$ is a unit vector, we have $a_1 \le 1/2$. We claim that $a_1 = 1/2$ and hence $b_1 = b_2 = b_3 = 1/2$. Assume that $a_1 < 1/2$. We consider two cases.

Case 1. Suppose $a_1 > 1/4$. Note that $b_1 \ge 1/2$. Otherwise, one can find $R \in H$ such that the $(1,1)$ entry of $\Phi^t R \Phi$ equals $2a_1 b_1 < a_1$, which is a contradiction. Let $P = P_1 \oplus P_2 \in H$ with $P_1 = -e_{11}^{(3)} - e_{23}^{(3)} + e_{32}^{(3)}$ and $P_2 = e_{11}^{(3)} - e_{23}^{(3)} + e_{32}^{(3)}$. Then $\Phi^t P \Phi$ has $(1,1)$ entry equal to the positive number $b_1^2 - a_1^2 \le (b_1^2 + b_2^2 + b_3^2)/3 - a_1^2 \le (v^t v - 4a_1^2)/3 = (1 - 4a_1^2)/3 < a_1 = \mu$, because the roots of the equation $4t^2 + 3t - 1 = 0$ are $-1$ and $1/4$.

Case 2. Suppose $a_1 \le 1/4$. Let $P = I_6 - 2e_{11}^{(6)}$. Then $\Psi = \Phi^t P \Phi$ has $(1,1)$ entry equal to $1 - 2a_1^2$. Let $u = (c_1, c_2, c_3, d_1, d_2, d_3)$ be the first column of $\Psi$. We may assume that $u$ is a nonnegative vector. Since $u$ is a unit vector, there are other nonzero entries besides $c_1$. In particular, we may assume that $c_3^2 + d_3^2 > 0$, otherwise multiply $\Psi$ by a suitable $P \in H$. Let $Q = e_{12}^{(3)} - e_{21}^{(3)} + e_{33}^{(3)}$. Then $Q \oplus Q \in H$ and the $(1,1)$ entry of $\Psi^t (Q \oplus Q) \Psi$ equals $c_3^2 + d_3^2 \le u^t u - c_1^2 = 4a_1^2(1 - a_1^2) < 4a_1^2 < a_1 = \mu$.

In both cases, we get the desired contradiction. So, we have $v = (1,0,0,1,1,1)^t/2$. But then if we define $Q$ as in Case 2, the $(1,1)$ entry of $\Phi^t (Q \oplus Q) \Phi$ will be $1/4 <$

$1/2 = a_1 = \mu$, which contradicts the definition of $\mu$.

Combining the about analysis, we get the conclusion. □

LEMMA 4.6. *Let $G$ be the isometry group of $N$ and suppose that $G$ is infinite or $G < GP(mn)$. Then every isometry in $G$ has form (9), i.e. Theorem 4.1(i) holds.*

*Proof.* If $G$ is infinite, then by Lemma 4.4 either $G = O(mn)$ or $G = O(m) * P(n)$. In the former case, clearly, we have $N_1 = N_2 = \ell_2$. In the latter case, we have $N_2 = \ell_2 \neq N_1$. The conclusion of Theorem 4.1$(i)$ holds.

Suppose $G < GP(mn)$. The result follows from Remark 4 before Corollary 3.4. □

Next we turn to the exceptional case when $\dim Y = m = 4$.

LEMMA 4.7. *Suppose* $\dim Y = 4$. *The conclusion of Theorem 4.1(i) holds (i.e. isometries have form* (9)*).*

*Proof.* Let $G$ be the isometry group of $N$. By Lemma 4.6 it remains to consider the case when $G$ is finite and is not a subgroup of $GP(mn)$.

For that, define $\mu$ as in the proof of Lemma 4.5, and let $\Phi \in G$ have $(1,1)$ entry equal to $\mu$ and a nonnegative first column. We divide the proof into three assertions. The matrix

$$Q = e_{12}^{(4)} + e_{34}^{(4)} - e_{21}^{(4)} - e_{43}^{(4)}$$

will be used frequently in our arguments.

*Assertion* 1. The first column of $\Phi$ equals $v = (1,1,1,1,0,0,\ldots,)^t/2$. In particular, $\mu = 1/2$.

Suppose the first column of $\Phi$ equals $v = (v_1, v_2, \ldots)^t$ with $0 \leq v_2 \leq v_3 \leq v_4$. Then $v_2 \neq 0$. Otherwise, we may let $P = (e_{34}^{(4)} - e_{43}^{(4)}) \oplus (I_{n-1} \otimes Q) \in H$, where $Q$ is defined as above, so that the $(1,1)$ entry of $\Phi^t P \Phi$ equals $v_1^2 < v_1 = \mu$, which contradicts minimality of $\mu$.

Thus, we may assume that $v_1 \leq v_2 \leq v_3 \leq v_4$.

To prove that $v$ is of the asserted form it is enough to show that $v_1 = 1/2$, since $v^t v = 1$.

Since $v_1^2 \leq (v_1^2 + v_2^2 + v_3^2 + v_4^2)/4 = v^t v/4 = 1/4$, we see that $v_1 \leq 1/2$. One can show that $v_1 < 1/2$ is impossible as in the last part (Case 2) of the proof of Lemma 4.5 (see also the first proof of Theorem 3.2 in [DLR]).

*Assertion* 2. A column of $\Psi \in G$ must be of the form $e_i^X \otimes (Pu)$ for some $1 \leq i \leq n$, $P \in GP(4)$ and $u = (1,0,0,0)^t, (1,1,1,1)^t/2$, or $(1,1,0,0)^t/\sqrt{2}$.

Let $v$ be the $k$th column of $\Psi \in G$. Multiplying $\Psi$ by a suitable $R \in H$, we may assume that $k = 1$ and the $(1,1)$ entry of $\Psi$ is nonzero. We need to show that $v = e_1^X \otimes (Px)$. If $v$ is not of this form, then (see the first proof of Theorem 3.2 in [DLR]) there exists $S \in GP(4)$ such that the $(1,1)$ entry of $(\Psi^{(11)})^t S \Psi^{(11)}$ equals $\eta < 1/2$. Let $R = S \oplus (I_{n-1} \otimes Q) \in H$, where $Q$ is defined as above. Then the $(1,1)$ entry of $\Psi^t R \Psi \in G$ equals $\eta < 1/2$, which is a contradiction.

*Assertion* 3. Suppose $\Psi = (\Psi^{(ij)}) \in G$ is in $n \times n$ block form with $\Psi^{(ij)} \in \mathbb{R}^{4\times 4}$ for each $(i,j)$. Then each nonzero $\Psi^{(ij)}$ must be of the form $P, PAR$ or $PBR$, with $P, R \in GP(4)$, where

$$A = (I_4 - (1,1,1,1)^t(1,1,1,1)/2) \quad \text{and} \quad B = \frac{1}{2}\left\{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right\}.$$

Consider a nonzero $\Psi^{(ij)}$. Multiplying $\Psi$ by a suitable $R \in H$, we may assume that $(i,j) = (1,1)$, the first column of $\Psi$ is nonnegative, and the $(1,1)$ entry $\alpha$ of $\Psi$ has the smallest magnitude among all nonzero entries in $\Psi^{(11)}$. We consider 3 cases.
Case 1. Suppose $\alpha = 1/2$. Then the first column of $\Psi$ equals $(1,1,1,1,0,\ldots)^t/2$. Since $\Psi^t \in G$ as well, we see that the first columns of $\Psi^t$ are of the form $e_1^X \otimes P(1,1,1,1)^t/2$ for some $P \in GP(4)$ by Assertion 2. Thus $\Psi^{(11)} = PAR$ for some $P, R \in GP(4)$.

Note that it follows from Assertion 1 and the above arguments that $\Phi^{(11)}$ is of the form $PAR$ for some $P, R \in GP(4)$.
Case 2. Suppose $\alpha = 1/\sqrt{2}$. By Assertion 2, we may assume that the first column of $\Psi^{(11)} = (1,1,0,0)^t/\sqrt{2}$. Since $\Psi^t \in G$ as well, we may assume the first row of $\Psi^{(11)}$ equals $(1,1,0,0)/\sqrt{2}$. Since the first two column of $\Psi$ are orthogonal, the second column of $\Psi^{(11)}$ equals $(1,-1,0,0)^t/\sqrt{2}$. By Assertion 2 and the knowledge about the first two columns of $\Psi$, one easily sees that the $(1,1)$ and $(2,1)$ entries of $\Psi^t\Phi \in G$ are $1/\sqrt{2}$ and 0. By Assertion 2 again, one of the $(3,1)$ and $(4,1)$ entries is 0, and the other has magnitude $1/\sqrt{2}$. It follows that the third and the four columns of $\Psi$ have the same form. Thus $\Psi^{(11)} = PBR$ for some $P, R \in GP(4)$.
Case 3. Suppose $\alpha = 1$. Then the first column of $\Phi\Psi \in G$ equals $(1,1,1,1,0,\ldots)^t/2$. By the result in Case 1, the first four columns of $\Phi\Psi$ are of the form $e_1^X \otimes P(1,1,1,1)/2$ for some $P \in GP(4)$. Thus $\Psi_{(11)} \in GP(4)$.

We are now ready to complete the proof of the lemma. Let $\mathcal{A}$ be the group generated by $GP(4)$ and $A$, and let $\mathcal{B}$ be the group generated by $GP(4)$ and $B$. It is known (e.g., see [DLR, Theorem 3.2]) that $\mathcal{A}$ is a normal subgroup of $\mathcal{B}$, and they are the only other possible isometry groups of a symmetric norm on $\mathbb{R}^4$ besides $O(4)$ and $GP(4)$. By Assertion 3, one easily concludes that if $G$ is not infinite and $G$ is not a subgroup of $GP(mn)$, then $G$ must be of the form $G_2 * P(n)$, where $G_2 = GP(4), \mathcal{A}$ or $\mathcal{B}$. In each case, $G_2$ is clearly the isometry group of $N_2$. □

Finally, we deal with the exceptional case when $\dim Y = m = 2$ in the next two lemmas.

LEMMA 4.8. *Suppose* $\dim Y = 2$. *Then the isometry group of* $N$ *has form (9) or (10).*

Proof. Let $G$ be the isometry group of $N$. By Lemma 4.6, we only need to consider the case when $G$ is finite and not a subgroup of $GP(mn)$. Define $\mu$ as in the proof of Lemma 4.5. Then $0 < \mu < 1$. Let $\Phi \in G$ have $(1,1)$ entry equal to $\mu$ and a nonnegative first column $v = (v_1, \ldots, v_{2n})^t$. Similarly as in Lemma 4.7, we divide the proof into several assertions.

*Assertion* 1. The vector $v$ cannot have more than four nonzero entries.

If the assertion is not true, then

$$\eta = \min\{v_j : 2 \le j \le 2n, v_j > 0\} \le \left\{ (\sum_{j=2}^{2n} v_j^2)/(k-1) \right\}^{1/2} < 1/2,$$

where $k$ is the number of nonzero entries of $v$. Let $H = GP(2) * GP(n)$. If $v_2 = \eta$, we can find $P \in H$ such that the first column of $P\Phi$ equals $(v_2, v_1, v_4, -v_3, v_6, -v_5, \ldots, v_{2n}, -v_{2n-1})^t$. If $v_2 > \eta = v_j$ for some $j > 2$, we may assume that $j = 3$ after multiplying $\Phi$ by a suitable $Q \in H$ on the left. Then we can find $P \in H$ such that the first column of $P\Phi$ equals $(v_3, -v_4, v_1, v_2, v_6, -v_5, \ldots, v_{2n})^t$. In both cases, $\Phi^t P \Phi$ has $(1,1)$ entry equal to $2\mu\eta < \mu$, which contradicts minimality of $\mu$.

*Assertion* 2. We have $v_2 \ge 1/2$.

If $v_2 = 0$, we can find $P \in H$ such that the first column of $P\Phi$ equals

$$(v_1, v_2, v_4, -v_3, v_6, -v_5, \ldots, v_{2n}, -v_{2n-1})^t$$

so that the $(1,1)$ entry of $\Phi^t P \Phi$ is $\mu^2 < \mu$, which is a contradiction. If $0 < v_2 < 1/2$, we can find $P \in H$ such that the first column of $P\Phi$ equals $(v_2, v_1, v_4, -v_3, v_6, -v_5, \ldots, v_{2n}, -v_{2n-1})^t$ so that $\Phi^t P \Phi$ has $(1,1)$ entry equal to $2v_2\mu < \mu$, which is a contradiction.

*Assertion* 3. The vector $v$ cannot have exactly 3 nonzero entries.

If the assertion is not true, we may assume without loss of generality that $v_j = 0$ for $j = 4, \ldots, 2n$(by replacing $\Phi$ with $P\Phi$ for some $P \in H$, if necessary).

If $n \geq 3$, let $R \in H$ be such that the first column of $R\Phi$ equals $(0, 0, v_1, v_2, v_3, 0, \ldots)^t$, then the $(1,1)$ entry of $\Phi^t R \Phi$ equals $\mu v_3 < \mu$, which is a contradiction.

If $n = 2$, let $S \in H$ be such that the first column of $S\Phi$ equals $(-v_1, v_2, v_3, 0)^t$. Then one can find $R \in H$ such that the first column of $\Psi = R\Phi^t S \Phi \in G$ is nonnegative and equals $(1-2\mu^2, r, s, t)$ with $s \geq t$. Furthermore, suppose $\eta = v_2 r + v_3 s - \mu(1-2\mu^2)$ is the $(1,1)$ entry of $\Psi^t S \Phi \in G$. Since

$$(v_2 r + v_3 s)^2 \leq (v_2^2 + v_3^2)(r^2 + s^2) \quad \text{and} \quad r^2 + s^2 + t^2 + (1-2\mu^2)^2 = 1 = v_1^2 + v_2^2 + v_3^2,$$

we have

$$\begin{aligned} \eta &\leq \{(v_2^2 + v_3^2)(r^2 + s^2)\}^{1/2} - \mu(1 - 2\mu^2) \\ &\leq \{(1 - v_1^2)(1 - (1 - 2\mu^2)^2)\}^{1/2} - \mu(1 - 2\mu^2) = \mu, \end{aligned} \tag{11}$$

and $\eta = \mu$ if and only if $(r, s, t)$ is a multiple of $(v_2, v_3, 0)$.

Note that $v_3 \geq 1/2$. Indeed, otherwise, one can find $P \in H$ such that the first column of $P\Phi$ equals $(v_3, 0, v_1, v_2)^t$ so that $\Phi^t P \Phi$ has $(1,1)$ entry equal to $2v_3\mu < \mu$, which is a contradiction.

If $s = 0$, then $t = 0$ and $r^2 = 1 - (1 - 2\mu^2)^2 = 4\mu^2(1 - \mu^2)$. Then $(r, s, t)$ is not a multiple of $(v_2, v_3, 0)$ and we have $\mu > \eta = v_2 r - \mu(1 - 2\mu^2) \geq r/2 - \mu(1 - 2\mu^2) = \mu\sqrt{1 - \mu^2} - \mu(1 - 2\mu^2) > 0$, since $\mu \leq 1/\sqrt{3}$, which contradicts minimality of $\mu$.

If $r = 0$, then $(r, s, t)$ is not a multiple of $(v_2, v_3, 0)$, and hence $\mu > \eta$. If $\eta = v_3 s - \mu(1 - 2\mu^2) \neq 0$, then we can find $P \in H$ such that the $(1,1)$ entry of $\Psi^t P \Phi$ equals $|\eta| < \mu$, which is a contradiction. Similarly, if $v_3 t - \mu(1 - 2\mu^2) \neq 0$, we have a contradiction. Suppose $t^2 = s^2 = 2\mu^2(1-\mu^2)$. Then $v_3^2 = (1-2\mu^2)^2/(2(1-\mu^2))$. Since $v_3^2 - \mu^2 \geq 0$, we have $6\mu^4 - 6\mu^2 + 1 \geq 0$. Thus $\mu^2 \leq 1/2 - 1/\sqrt{12}$ or $\mu^2 \geq 1/2 + \sqrt{12}$. Since $\mu^2 \leq (v_1^2 + v_2^2 + v_3^2)/3 = 1/3$, we have $\mu^2 \leq 1/2 - 1/\sqrt{12} \leq 1/4$. Now, if $\mu < 1/4$ or $1/4 \leq \mu < 1/2$, one can derive a contradiction as in the last part of the proof of Lemma 4.5 (see also the first proof of Theorem 3.2 in [DLR]).

If $r, s \neq 0$, then $r, s \geq \mu$. Since $v_2, v_3 \geq 1/2$, we have

$$\eta \geq \mu(v_2 + v_3) - \mu(1 - 2\mu^2) \geq \mu - \mu(1 - 2\mu^2) > 0.$$

If $(r, s, t)$ is not a multiple of $(v_2, v_3, 0)$, then, by (11), $\mu > \eta > 0$, which is a contradiction. Thus $(r, s, t)$ is a multiple of $(v_2, v_3, 0)$ and $t = 0$. We can find $P \in H$ such that the $(1,1)$ entry of $\Psi^t P \Phi$ equals $\delta = |v_2 r - \mu(1 - 2\mu^2)| < \mu$, which is a contradiction if $\delta > 0$. If $\delta = 0$, one can let $(r, s) = c(v_2, v_3)$ and solve the equations:

$$v_2^2 + v_3^2 = 1 - \mu^2$$

$$r^2 + s^2 = c^2(v_2^2 + v_3^2) = 1 - (1 - 2\mu^2)^2 = 4\mu^2(1 - \mu^2),$$

$$0 = v_2 r - \mu(1-2\mu^2) = cv_2^2 - \mu(1-2\mu^2)$$

to conclude that $v_2^2 = 1/2 - \mu^2$ and $v_3^2 = 1/2$. Since, by Assertion 2, $v_2 \geq \mu$, we have $\mu \leq 1/2$ and since $2\mu^2 + 1/2 \leq v_1^2 + v_2^2 + v_3^2 = 1$ we conclude that $\mu = 1/2 = v_2$.

Now, we may assume that the first row of $\Phi$ is $v^t$. Otherwise replace $\Phi$ by $\Phi R$ for some $R \in H$. One readily sees that $P\Phi Q$ equals $L_1$ or $L_2$ for some $P, Q \in H$, where

$$L_1 = 2^{-1}\begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ \sqrt{2} & 0 & -1 & -1 \\ 0 & \sqrt{2} & -1 & 1 \end{pmatrix} \quad \text{and} \quad L_2 = 2^{-1}\begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is routine to check that $\langle H, L_1\rangle = \langle H, L_2\rangle$ consists of matrices of the form $P, PL_1Q$ or $PL_2Q$ with $P, Q \in H$. By the previous discussion, this group is contained in $G$. We shall show that this is impossible. Suppose $\langle H, L_1\rangle = \langle H, L_2\rangle < G$. Then

$$\begin{aligned} N_2(e_1^Y + e_2^Y) &= N(e_{11} - e_{21}) = N(L_2(e_{11} - e_{21})) \\ &= N(\sqrt{2}e_{12}) = \sqrt{2}N_2(e_1^X) = \sqrt{2}. \end{aligned}$$

Thus

$$\begin{aligned} N_1(ae_1^X + be_2^X) &= N(ae_{11} + be_{12}) \\ &= N(2^{-1/2}a(e_{11} - e_{21}) + be_{22}) \\ &= N(L_2(2^{-1/2}a(e_{11} - e_{21}) + be_{22})) \\ &= N(ae_{12} + be_{22}) = N_2(ae_1^Y + be_2^Y) \end{aligned}$$

for any $a, b \in \mathbb{R}$. Furthermore,

$$\begin{aligned} N_2(ae_1^Y + be_2^Y) &= N(ae_{11} + be_{21}) \\ &= N(L_2(e_{11} + be_{21})) \\ &= N((a+b)(e_{11} + e_{21})/2 + (a-b)e_{12}/\sqrt{2}) \\ &= N((a+b)e_{11} + (a-b)e_{12})/\sqrt{2} \\ &= N_1((a+b)e_1^X + (a-b)e_2^X)/\sqrt{2} \\ &= N_2((a+b)e_1^X + (a-b)e_2^X)/\sqrt{2} \end{aligned}$$

for any $a, b \in \mathbb{R}$. Thus $A = 2^{-1/2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is an isometry for $N_2$, and $\Gamma = A \oplus I_2$ is an isometry for $N$. Note that the first column of $\Gamma$ is neither of the form $P(1,0,\ldots)^t$ nor $P(1/2, 1/2, 1/\sqrt{2}, 0, \ldots)^t$ with $P \in H$. We can find $R \in H$ such that the $(1,1)$ entry of $\Gamma L_2$ is positive and less than $1/2$, which is a contradiction with the fact that $\mu = 1/2$.

*Assertion* 4. Suppose the first column of $\Phi$ has exactly 4 nonzero entries. Then the first column of $P\Phi$ equals $(1,1,1,1,0,\ldots)^t/2$ for some $P \in H$, and every $\Psi \in G$ has form (10).

Suppose $v$ has exactly 4 nonzero entries. One can show that all of them equal $1/2$ by arguments similar to those in the proof of Lemma 4.7 (cf. Assertion 1). By Assertion 2, we have $v_1 = v_2 = 1/2$. Since $\mu = 1/2$, by Assertion 2, for each $1 \leq j \leq n$, $v_{2j-1}$ and $v_{2j}$ are either both zero or both nonzero. Thus $P\Phi$ has first column equal to $(1,1,1,1,0,\ldots)^t/2$ for some $P \in H$ as asserted.

Since $(P\Phi)^t \in G$ and the first four entries in the first row of $(P\Phi)^t$ equal $1/2 = \mu$, the second column $w$ of $P\Phi$ is of the form $(\pm 1, \pm 1, \pm 1, \pm 1, 0, \ldots)^t/2$ by Assertion 2. We claim that $w = \pm(1, 1, -1, -1, 0, \ldots)^t/2$. If it is not true, we may assume that $w = (1, -1, 1, -1, 0, \ldots)^t/2$. Otherwise interchange the third and fourth rows of $P\Phi$, and multiply the second column of $P\Phi$ by $-1$ if necessary. It follows that

$$\begin{aligned} N_2((a,b)^t) &= N(ae_{11} + be_{21}) = N(P\Phi(ae_{11} + be_{21})) \\ &= N(a+b)(e_{11} + e_{12}) + (a-b)(e_{21} + e_{22}))/2 \\ &= N_2((a+b, a-b)^t)N_1(1,1)/2. \end{aligned}$$

It follows that $A = k\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is an isometry for $N_2$ for $k = N_1((1,1)^t)/\sqrt{2}$. Since an isometry for $N_2$ must be orthogonal (e.g., see [DLR]), $k = 1/\sqrt{2}$. But then $\Psi = A \oplus I_{2n-2} \in G$ and the first column of $\Gamma = \Psi P \Phi$ has 3 nonzero entries, and there exists $Q \in H$ such that $\Gamma^t Q \Phi$ has a positive nonzero entry less than $1/2 = \mu$, which is impossible.

Now for any $\Psi \in G$, the columns of $\Psi$ must be of the form $Q(1,1,1,1,0,\ldots)^t/\sqrt{2}$, $Q(1,1,0,\ldots)^t/\sqrt{2}$, or $Q(1,0,\ldots)^t$ for some $Q \in H$. Otherwise, one can find $R, S \in H$ such that $R\Psi^t S\Phi$ has positive $(1,1)$ entry less than $1/2 = \mu$. Now if $\Psi$ has a column of the form $Q(1,1,0,\ldots)^t/\sqrt{2}$, then one can show that $2^{-1/2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is an isometry for $N_2$, and get a contradiction as in the preceding paragraph. Furthermore, if the $(2j-1)$th (respectively, $(2j)$th) column of $\Psi$ is of the form $Q(1,1,1,1,0,\ldots)^t/2$, then the $(2j)$th (respectively, $(2j-1)$th column must be of the form $\pm Q(1,1,-1,-1,0,\ldots)^t/2$ by arguments similar to those in the analysis of the second column of $\Phi$. It follows that every $\Psi \in G$ permutes the matrices in the set $\{\pm(e_1^Y \pm e_2^Y)(e_j^X)^t : 1 \le j \le n\}$ as asserted.

*Assertion* 5. Suppose $v$ has exactly 2 nonzero entries. The conclusion of Theorem 4.1($i$) holds.

By Assertion 2, if $v$ has exactly two nonzero entries, then we may assume that $(v_1, v_2) = (\sin t, \cos t)$ for some $t \in (0, \pi/4)$. Now, it is easy to see the columns of $\Psi \in G$ must be of the form $P(a, b, 0, \ldots)^t$ with $a^2 + b^2 = 1$. Otherwise, one can find $R, S \in H$ such that the $(1,1)$ entry of $R\Psi^t S\Phi$ is positive and is less than $\sin t$. Moreover, if the $(2j-1)$th (respectively, the $(2j)$th) column of $\Psi$ is of the form $P(a,b,0,\ldots)^t$ with $ab \neq 0$, then by the fact that $\Psi \in G$ one can conclude that the $(2j)$th (respectively, the $(2j-1)$th) column must be of the form $\pm P(b, -a, 0, \ldots)^t$. Thus $P\Psi P^t$ is a direct sum of a signed permutation matrix $A$ and a number of $2 \times 2$ orthogonal matrices $B_i$. Furthermore, we may assume that $A$ is a direct sum of matrices in $GP(2)$. Otherwise, $\Gamma = A \oplus I_k \in G$ will satisfy the hypothesis of Lemma 2.4, and hence both $N_1$ and $N_2$ equal $\ell_p$ for some $p \ge 1$. Thus $P\Psi P^t$ must be a direct sum of isometries for $Y$ for some $P \in H$, and the conclusion follows. □

LEMMA 4.9. *If isometries of $X(Y)$ have form (10), then $X = \ell_p^n$ and $Y = E_p(2)$ for some $p$, $1 \le p < \infty$, $p \neq 2$.*

*Proof.* Suppose $m = 2$. The elements in $Y$ will be written as $(y_1, y_2)$, and the elements in $X(Y)$ will be written as

$$(x_{11}, x_{21}, x_{12}, x_{22}, \ldots, x_{1n}, x_{2n}).$$

If isometries of $X(Y)$ have form (10) but condition (i) of Theorem 4.3 does not hold, then the isometry group $G$ of $X(Y)$ must be $A_{2n}$ as mentioned before Lemma 4.3.

Therefore, the linear map $T : X(Y) \longrightarrow X(Y)$ defined by

$$\begin{aligned} &T(x_{11}, x_{21}, x_{12}, x_{22}, \ldots, x_{1n}, x_{2n}) \\ &= \Big(2^{-1}(x_{11} + x_{21} + x_{12} + x_{22}), 2^{-1}(x_{11} + x_{21} - x_{12} - x_{22}), \\ &\quad 2^{-1}(x_{11} - x_{21} + x_{12} - x_{22}), 2^{-1}(x_{11} - x_{21} - x_{12} + x_{22}), x_{13}, x_{23}, \ldots, x_{1n}, x_{2n}\Big) \end{aligned}$$

is an isometry of $X(Y)$. Assume that $x_{13} = x_{23} = \ldots = x_{1n} = x_{2n} = 0$, then

$$N(x_{11}, x_{21}, x_{12}, x_{22}, 0, 0, \ldots, 0, 0) = N_1(N_2(x_{11}, x_{21})e_1^X + N_2(x_{12}, x_{22})e_2^X).$$

In particular, when $x_{11} = x_{21} = a$ and $x_{12} = x_{22} = b$ we get

$$N(a, a, b, b, 0, \ldots, 0) = N_1(aN_2(1, 1)e_1^X + bN_2(1, 1)e_2^X) = N_2(1, 1)N_1(ae_1^X + be_2^X).$$

Since $T$ is an isometry

$$\begin{aligned} N(a, a, b, b, 0, \ldots, 0) &= N(T(a, a, b, b, 0, \ldots, 0)) \\ &= N(a + b, a - b, 0, \ldots, 0) = N_1(N_2(a + b, a - b)e_1^X) \\ &= N_2(a + b, a - b). \end{aligned}$$

Thus

$$N_1(ae_1^X + be_2^X) = \frac{1}{N_2(1,1)} N_2(a + b, a - b) \tag{12}$$

for all $a, b \in \mathbb{R}$. Also

$$N_2(c, d) = N_2(1, 1)N_1(\frac{1}{2}(c + d)e_1^X + \frac{1}{2}(c - d)e_2^X) \tag{13}$$

for all $c, d \in \mathbb{R}$. Further, since $T$ is an isometry, we get

$$N(x_{11}, x_{21}, x_{12}, x_{22}, 0, 0, \ldots, 0, 0) = N(T(x_{11}, x_{21}, x_{12}, x_{22}, 0, 0, \ldots, 0, 0)).$$

It follows that

$$\begin{aligned} &N_1\Big(N_2(x_{11}, x_{21})e_1^X + N_2(x_{12}, x_{22})e_2^X\Big) \\ &= N_1\Big[N_2\Big(2^{-1}(x_{11} + x_{21} + x_{12} + x_{22}), 2^{-1}(x_{11} + x_{21} - x_{12} - x_{22})\Big)e_1^X \\ &\qquad + N_2\Big(2^{-1}(x_{11} - x_{21} + x_{12} - x_{22}), 2^{-1}(x_{11} - x_{12} - x_{21} + x_{22}\Big)e_2^X\Big] \end{aligned} \tag{14}$$

Put

$$\begin{aligned} x_{11} + x_{21} &= y_1 & x_{11} - x_{21} &= y_2 \\ x_{12} + x_{22} &= y_3 & x_{12} - x_{22} &= y_4. \end{aligned}$$

Then by (13) and (14) we get

$$\begin{aligned} &2^{-1}N_2(1, 1)N_1\Big(N_1(y_1e_1^X + y_2e_2^X)e_1^X + N_1(y_3e_1^X + y_4e_2^X)e_2^X\Big) \\ &= 2^{-1}N_2(1, 1)N_1\Big(N_1(y_1e_1^X + y_3e_2^X)e_1^X + N_1(y_2e_1^X + y_4e_2^X)e_2^X\Big) \end{aligned} \tag{15}$$

Define $f(a,b) = N_1(ae_1^X + be_2^X)$. Then (15) becomes:

$$f(f(y_1,y_2), f(y_3,y_4)) = f(f(y_1,y_3), f(y_2,y_4))$$

for all $y_1, y_2, y_3, y_4 \in \mathbb{R}$. When $y_4 = 0$ we have

$$f(f(y_1,y_2),y_3) = f(f(y_1,y_2), f(0,y_3)) = f(f(y_1,0), f(y_2,y_3)) = f(y_1, f(y_2,y_3)).$$

By a theorem of Bohnenblust [Bo]

$$f(a,b) = \max(|a|,|b|) \quad \text{or} \quad f(a,b) = (|a|^p + |b|^p)^{1/p}$$

for some $p$, $1 \le p < \infty$. Hence

$$N_1(ae_1^X + be_2^X) = \ell_p(a,b) \tag{16}$$

for some $p$, $1 \le p \le \infty$. By (13) we see that $Y = E_p(2)$.

To see that $X = \ell_p^n$, let $k \le n$ be the maximal number such that

$$N_1(a_1e_1^X + \ldots + a_ke_k^X) = \ell_p(a_1, \ldots, a_k) \tag{17}$$

for all $a_1, \ldots, a_k \in \mathbb{R}$. If $k = n$ there is nothing to prove. If $k < n$ let $(a_l)_{l=1}^{k+1} \subset \mathbb{R}^{k+1}$ be arbitrary and set $x_{1l} = x_{2l} = a_l$ for $l = 1, \ldots, k+1$, and $x_{1l} = x_{2l} = 0$ for $l > k+1$. Then

$$\begin{aligned} &N(a_1, a_1, \ldots, a_{k+1}, a_{k+1}, 0, \ldots, 0) \\ &= N_1(a_1N_2(1,1)e_1^X + \cdots + a_{k+1}N_2(1,1)e_{k+1}^X) \\ &= N_2(1,1)N_1(a_1e_1^X + \cdots + a_{k+1}e_{k+1}^X) \end{aligned} \tag{18}$$

Since $T$ is an isometry we get

$$\begin{aligned} &N(a_1, a_1, \ldots, a_{k+1}, a_{k+1}, 0, \ldots, 0) \\ &= N(T(a_1, a_1, \ldots, a_{k+1}, a_{k+1}, 0, \ldots, 0)) \\ &= N((a_1 + a_2, a_1 - a_2, 0, 0, a_3, a_3, \cdots, a_{k+1}, a_{k+1}, 0, \ldots, 0)) \\ &= N_1(N_2(a_1 + a_2, a_1 - a_2)e_1^X + a_3N_2(1,1)e_3^X + \cdots + a_{k+1}N_2(1,1)e_{k+1}^X) \\ &= N_1(N_2(1,1)\ell_p(a_1,a_2)e_1^X + a_3N_2(1,1)e_3^X + \cdots + a_{k+1}N_2(1,1)e_{k+1}^X) \\ &\hspace{20em} \text{by (12) and (16)} \\ &= N_2(1,1)N_1(\ell_p(a_1,a_2)e_1^X + a_3e_3^X + \cdots + a_{k+1}e_{k+1}^X) \\ &= N_2(1,1)\ell_p(\ell_p(a_1,a_2), a_3, \ldots, a_{k+1}) \qquad \text{by (17) and symmetry of } X \\ &= N_2(1,1)\ell_p(a_1, a_2, a_3, \ldots, a_{k+1}). \end{aligned}$$

Hence, by (18)

$$N_1(a_1e_1^X + \ldots + a_{k+1}e_{k+1}^X) = \ell_p(a_1, \ldots, a_{k+1}),$$

which contradicts maximality of $k$. Thus $(X, N_1) = \ell_p^n$.

Finally, we conclude that $p \ne 2$. Indeed, if $p = 2$ then $E_p(2) = \ell_2^2$ and then $X(Y) = \ell_2(\ell_2^2) = \ell_2^{2n}$ whose isometry group is not of the form (10). □

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