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ERRATUM: CONFORMAL MODULES*

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In this erratum we make corrections (a),(b) and (c) to our paper [3].

(a) Here is a correct statement and a proof of Lemma 3.2 of [3].

LEMMA 1. *Let $\mathfrak{g}$ be a finite-dimensional Lie superalgebra and let $\mathfrak{n}$ be a solvable ideal of $\mathfrak{g}$. Let $\mathfrak{a}$ be an even subalgebra of $\mathfrak{g}$ such that $\mathfrak{n}$ is a completely reducible* $\mathrm{ad}\mathfrak{a}$*-module with no trivial summand. Then $\mathfrak{n}$ acts trivially in any irreducible finite-dimensional $\mathfrak{g}$-module V.*

Proof. First note that if $[a, b] = b$, then b is nilpotent on any finite-dimenisonal representation V. To see this, note that if $v \in V$ is an eigenvector of a with eigenvalue λ, then the condition $[a, b] = b$ implies that bv is an eigenvector of a with eigenvalue $\lambda + 1$. Thus there exists a non-zero $w \in V$ such that $bw = 0$. Let W be the space annihilated by b. Then W is a-invariant. Since $W \neq 0$, it follows by induction on the dimension of V that b is nilpotent on V/W. Thus b is nilpotent on V.

Let $\mathfrak{s}$ be an irreducible $\mathrm{ad}\mathfrak{a}$-submodule of $\mathfrak{n}$. By the assumption, it is a module with a non-zero highest weight. Hence there exists $a \in \mathfrak{a}$ and $b \in \mathfrak{s}$ such that $[a, b] = b$, therefore b is nilpotent on V. Moreover all elements from the orbit $\mathrm{Ad}A \cdot b$, where A is the connected Lie group with Lie algebra $\mathfrak{a}$, are nilpotent on V. Since $\mathfrak{s}$ is $\mathrm{Ad}A$-irreducible, this orbit spans $\mathfrak{s}$, hence $\mathfrak{s}$ is spanned by elements that are nilpotent on V. Thus any $\mathfrak{a}$-submodule of $\mathfrak{n}$ is spanned by elements that are nilpotent on V.

To prove the lemma, we may assume that V is a faithful $\mathfrak{g}$-module. Suppose the lemma is not true, i.e. $\mathfrak{n}$ is non-zero. Let $\mathfrak{n}^{(i)}$ be the last non-zero member of its derived series. By the above, $\mathfrak{n}^{(i)}$ is spanned by mutually commuting elements that are nilpotent on V, and hence $\mathfrak{n}^{(i)}$ annihilates a non-zero vector in V. But $\mathfrak{n}^{(i)}$ is an ideal of $\mathfrak{g}$ and hence the subspace of V, annihilated by $\mathfrak{n}^{(i)}$, is a $\mathfrak{g}$-submodule of V. Thus $\mathfrak{n}^{(i)}$ annihilates V and so V is not faithful, which is a contradiction. □

(b) Lemma 1, however, is not applicable to the current algebra of three series of simple Lie superalgebras, namely to the Lie superalgebras $A(m|n)$, for $m \neq n$, $C(n)$ and $W(n)$ (see [4]), for which $\mathfrak{n}$ will contain trivial summands. For these three series the statements of Corollary 3.1 and Theorem 3.2 turn out to be incorrect. Below we will classify irreducible conformal modules over their current algebras. Due to Lemma 3.1 of [3] it suffices to consider finite-dimensional irreducible representations of the Lie superalgebra $\mathfrak{g} \otimes \mathbb{C}[t]/(t^{n+1})$, where $n \geq 0$ and $\mathfrak{g}$ is a member of one of the three series of simple Lie superalgebras above. As the case of $n = 0$ is trivial, we may assume from now on that $n \geq 1$. In fact we will study more general Lie superalgebras $\mathfrak{g}$ satisfying properties we now describe.

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a finite-dimensional Lie superalgebra and suppose that $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_k$ is $\mathbb{Z}$-graded such that $\mathfrak{g}_i \subseteq \mathfrak{g}_{\bar{i}}$. Assume that $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathbb{C}c$ is a reductive Lie algebra such that $\mathfrak{a}$ is a semisimple subalgebra and c is a central element. Furthermore suppose that $\mathfrak{g}_i$ as an $\mathfrak{a}$-module has no trivial summand for $i \neq 0$, and there exists an $\mathfrak{a}$-submodule $\mathfrak{g}^*_{-1} \subseteq \mathfrak{g}_1$ contragredient to $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1 = \mathfrak{g}^*_{-1} \oplus \mathfrak{g}'_1$ as $\mathfrak{a}$-modules with $[\mathfrak{g}'_1, \mathfrak{g}_{-1}] \subseteq \mathfrak{a}$. Finally suppose that given any non-zero $a \in \mathfrak{g}_{-1}$,

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$[a, \mathfrak{g}^*_{-1}]$ has non-trivial projection onto $\mathbb{C}c$ and given any non-zero $b \in \mathfrak{g}^*_{-1}$, $[\mathfrak{g}_{-1}, b]$ has non-trivial projection onto $\mathbb{C}c$. Note that it follows from the descriptions of the three series of simple Lie superalgebras $A(m|n)$, for $m \neq n$, $C(n)$ and $W(n)$ in [4] that they satisfy the assumptions of $\mathfrak{g}$ above.

Set $\mathfrak{L} = \mathfrak{g} \otimes \mathbb{C}[t]/(t^{n+1})$. We want to determine finite-dimensional irreducible $\mathfrak{L}$-modules, on which $\mathfrak{g} \otimes t^n$ acts non-trivially. We let

$$G_0 = \mathfrak{g}_0 + \mathfrak{g}_1' + \sum_{i \geq 2} \mathfrak{g}_i.$$

Consider the subalgebra $L \subseteq \mathfrak{L}$, which is defined as follows: For $n = 2k$, an even integer, we let

$$L := G_0 + G_0 \otimes t + \cdots + G_0 \otimes t^{k-1} + (G_0 + \mathfrak{g}_1) \otimes t^k + \mathfrak{g} \otimes t^{k+1} + \cdots + \mathfrak{g} \otimes t^{2k}.$$

If $n = 2k + 1$ is odd, then we let

$$L := G_0 + G_0 \otimes t + \cdots + G_0 \otimes t^k + \mathfrak{g} \otimes t^{k+1} + \cdots + \mathfrak{g} \otimes t^{2k+1}.$$

We first determine finite-dimensional irreducible L-modules. It will turn out that every irreducible $\mathfrak{L}$-module on which $\mathfrak{g} \otimes t^n$ acts non-trivially is obtained from inducing from a suitable irreducible L-module. The main tool we use to prove this assertion is the Lie algebraic analogue of Mackey's irreducibility criterion [1]. Before recalling it, we need some terminology. Let $\mathfrak{k}$ be a finite-dimensional Lie superalgebra and $I \subseteq \mathfrak{k}$ be an ideal of $\mathfrak{k}$. Let (π, V_I) be an irreducible I-module. Define the stabilizer associated to the pair (π, I) to be $K_\pi = \{k \in \mathfrak{k} | \exists A_k \in \mathrm{End}(V_I) \text{ with } \pi([k, i]) = [A_k, \pi(i)], \forall i \in I\}$. K_π is a subalgebra containing I. One can prove using analogous arguments as in [1]

THEOREM 1. *[2] Let V_K be an irreducible representation of K_π such that as an I-module, V_K is a direct sum of copies of π. If the $\mathbb{Z}_2$-graded space $\mathfrak{k}/K_\pi$ is spanned by elements of the same parity, then $\mathrm{Ind}_{K_\pi}^{\mathfrak{k}} V_K$ is an irreducible $\mathfrak{k}$-module.*

We now consider the subalgebra

$$B = \mathfrak{b} + \mathfrak{g}_1' + \sum_{i \geq 2} \mathfrak{g}_i + (L \cap \mathfrak{g} \otimes \mathbb{C}[t]t),$$

where $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}_0$. Then B is a solvable subalgebra of L and $[B_{\bar{1}}, B_{\bar{1}}]$ has trivial projection onto $\mathbb{C}c$. However, since $\mathfrak{g}$ as an $\mathfrak{a}$-module contains only one trivial summand, namely $\mathbb{C}c$, it is clear that $[\mathfrak{b}, \mathfrak{g}_i] = \mathfrak{g}_i$, for all $i \neq 0$, and $[\mathfrak{b}, \mathfrak{g}_0] = \mathfrak{a}$. In fact $[\mathfrak{b}, \mathfrak{g}^*_{-1}] = \mathfrak{g}^*_{-1}$ and $[\mathfrak{b}, \mathfrak{g}_1'] = \mathfrak{g}_1'$. This discussion implies that $[B_{\bar{1}}, B_{\bar{1}}] \subseteq [B_{\bar{0}}, B_{\bar{0}}]$. Therefore by [4] every finite-dimensional irreducible representation of B is one-dimensional, say $\mathbb{C}v_\lambda$, where $\lambda : \mathfrak{h} + \mathbb{C}c \otimes \mathbb{C}[t] \to \mathbb{C}$ is a linear form extended trivially to B and $\mathfrak{h}$ is a Cartan subalgebra in $\mathfrak{b}$. However $\mathrm{Ind}_B^L \mathbb{C}v_\lambda$ is spanned by elements of the form $e_{-\alpha_1}^{k_{\alpha_1}} \cdots e_{-\alpha_m}^{k_{\alpha_m}} v_\lambda$, where $e_{-\alpha_j}$ are negative root vectors of $\mathfrak{a}$. It follows that $c \otimes t^j$ acts as a scalar on $\mathrm{Ind}_B^L \mathbb{C}v_\lambda$. Set $\mathfrak{g}^c = \mathfrak{g}_{-1} + \mathfrak{a} + \sum_{i \geq 1} \mathfrak{g}_i$. Clearly $[\mathfrak{a}, \mathfrak{g}^c] = \mathfrak{g}^c$. From this it follows by induction on $k_{\alpha_1} + \cdots + k_{\alpha_m}$ that $\mathfrak{g}_1' + \sum_{i \geq 2} \mathfrak{g}_i + (\mathfrak{g}^c \otimes \mathbb{C}[t]t) \cap L$ acts trivially on this induced module. Thus we obtain

PROPOSITION 1. *Every irreducible L-module is an irreducible $\mathfrak{g}_0 \oplus (c \otimes \mathbb{C}[t]t)$-module, on which $\mathfrak{g}_1' + \sum_{i \geq 2} \mathfrak{g}_i + (\mathfrak{g}^c \otimes \mathbb{C}[t]t) \cap L$ acts trivially.*

Next we introduce the auxilary subalgebra

$$\bar{\mathfrak{L}} := G_0 + \mathfrak{g} \otimes \mathbb{C}[t]t.$$

From Proposition 1 we obtain the classification of finite-dimensional irreducible $\bar{\mathfrak{L}}$-modules.

PROPOSITION 2. *Let V_L be an irreducible L-module such that $c \otimes t^n$ acts as a non-zero scalar. Then $\mathrm{Ind}_L^{\bar{\mathfrak{L}}} V_L$ is irreducible. Furthermore suppose that W is an irreducible $\bar{\mathfrak{L}}$-module on which $\mathfrak{g} \otimes t^n$ acts non-trivially, then $W \cong \mathrm{Ind}_L^{\bar{\mathfrak{L}}} V_L$, where V_L is as above.*

Proof. Let $I = (G_0 + \mathfrak{g}_1) \otimes t^k + \mathfrak{g} \otimes \mathbb{C}[t]t^{k+1}$, if $n = 2k$ and $I = \mathfrak{g} \otimes \mathbb{C}[t]t^{k+1}$, if $n = 2k+1$. Note that I is an ideal of $\bar{\mathfrak{L}}$. By Proposition 1 V_L is a direct sum of 1-dimensional mutually isomorphic I-modules $\mathbb{C}v_\lambda$, where $(c \otimes t^j)v_\lambda = \lambda(c \otimes t^j)v_\lambda$. Consider the stabilizer of the I-module $\mathbb{C}v_\lambda$ in $\bar{\mathfrak{L}}$. Since for every vector $g \in \mathfrak{g}_{-1} + \mathfrak{g}^*_{-1}$ there exists a vector $g' \in \mathfrak{g}_{-1} + \mathfrak{g}^*_{-1}$ such that $[g, g'] = \mu c + a$, where $\mu \neq 0$ and $a \in \mathfrak{a} + \mathfrak{g}_2$, it follows that the stabilizer is precisely L, if $\lambda(c \otimes t^n) \neq 0$. Since also $\bar{\mathfrak{L}}/L$ is a completely odd vector space, we can employ Theorem 1 and irreducibility follows. On the other hand suppose that W is an irreducible $\bar{\mathfrak{L}}$-module on which $\mathfrak{g} \otimes t^n$ acts non-trivially. Let V_L be an irreducible L-submodule inside W. If $c \otimes t^n$ acts trivially on V_L, then $\mathfrak{g} \otimes t^n$ acts trivially on V_L by Proposition 1. But $\mathfrak{g} \otimes t^n$ is an ideal of $\bar{\mathfrak{L}}$ and hence acts trivially on W, which is a contradiction. □

Proposition 1 says that when inducing from L to $\bar{\mathfrak{L}}$ irreducibility is preserved. So our final goal is to show that induction from $\bar{\mathfrak{L}}$ to $\mathfrak{L}$ also preserves irreducibility. Note that from Proposition 2 we get

COROLLARY 1. *Every irreducible $\bar{\mathfrak{L}}$-module, on which $\mathfrak{g} \otimes t^n$ acts non-trivially, considered as a $\mathfrak{g} \otimes t^n$-module, is a direct sum of the same 1-dimensional $\mathfrak{g} \otimes t^n$-module $\mathbb{C}v_\lambda$. Furthermore $c \otimes t^n$ acts as a non-zero scalar $\lambda(c \otimes t^n)$, while $\mathfrak{g}^c \otimes t^n$ acts trivially.*

Now we are ready to classify irreducible $\mathfrak{L}$-modules.

PROPOSITION 3. *Let V be an irreducible $\bar{\mathfrak{L}}$-module on which $c \otimes t^n$ acts as a non-zero scalar. Then $\mathrm{Ind}_{\bar{\mathfrak{L}}}^{\mathfrak{L}} V$ is an irreducible $\mathfrak{L}$-module. Furthermore every irreducible $\mathfrak{L}$-module, on which $\mathfrak{g} \otimes t^n$ acts non-trivially, is of this form.*

Proof. We will set up to employ Theorem 1 again. By Corollary 1 V is a direct sum of copies of the $\mathfrak{g} \otimes t^n$-module $\mathbb{C}v_\lambda$ and $\lambda(c \otimes t^n) \neq 0$. We compute the stabilizer of the $\mathfrak{g} \otimes t^n$-module $\mathbb{C}v_\lambda$ now in $\mathfrak{L}$. Again using the same argument as in the proof of Proposition 2, one checks that the stabilizer is precisely $\bar{\mathfrak{L}}$, if $\lambda(c \otimes t^n) \neq 0$. Thus by Theorem 1 again, the induced module is irreducible. The same argument as in the proof of Proposition 2 gives that these are all such irreducibles. □

Combining Proposition 2 and 3 gives

THEOREM 2. *All finite-dimensional irreducible representations of $\mathfrak{L}$, on which $\mathfrak{g} \otimes t^n$ acts non-trivially, are of the form $\mathrm{Ind}_L^{\mathfrak{L}} V_L$, where V_L is an irreducible representation of L, on which $c \otimes t^n$ acts as a non-zero scalar. Furthermore all such representations are irreducible.*

In [2] it was shown that if S is a finite-dimensional Lie superalgebra such that $[S, S] = S$ and $\Lambda(n)$ is the Grassmann superalgebra in n odd variables, then every finite-dimensional irreducible $S \otimes \Lambda(n)$-module is an irreducible S-module, on which the $S \otimes N$ acts trivially, where N is the maximal nilpotent ideal of $\Lambda(n)$. Thus for supercurrents $\mathfrak{g}[t, \theta] = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \otimes \Lambda(1)$, where θ is an odd variables and $\mathfrak{g}$ is one of the three series of simple Lie superalgebras, we may apply this result and Theorem 2 to obtain their irreducible conformal modules.

(c) In the proof of Lemma 3.1 in [3] we use v to denote the vector $\sum_{i=1}^n p_i(\partial)v_i$.

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