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CONNECTING ORBITS FOR NONLINEAR PARABOLIC EQUATIONS*

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1. Introduction. We consider the scalar reaction diffusion equation

$$u_t = f(x, u, u_x, u_{xx}) \tag{1}$$

with $x \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$, $f \in C^\infty$ and

$$C^{-1} \leq f_{u_{xx}} \leq C\ , \tag{2}$$

where C is a positive constant. Let u be a bounded solution of (1) in $C^1(S^1 \times (0, \infty))$. Then as it follows from [1] and [2], the ω-limit set $\omega(u)$ of the trajectory u contains a solution of (1) which is periodic in t. If all periodic solutions of (1) are hyperbolic, then any bounded solution of equation (1) tends to a periodic solution (which can be in particular a steady state), as $t \to +\infty$. So it is interesting to study the connecting orbits between periodic solutions of (1). Let L be a periodic hyperbolic orbit of (1). We denote by $W^s(L)$ and $W^u(L)$ the stable and unstable manifolds of L respectively. For details of the definition and for general properties of stable and unstable manifolds of periodic solutions, refer to [3], [4], [5]. We denote by $M(L) = \dim W^u(L)$ the Morse index of L.

Let L, L' be periodic hyperbolic orbits of (1). If there exists a solution u of (1) which is defined for all $t \in \mathbb{R}$ and such that $\alpha(u) = L$ and $\omega(u) = L'$, then we say that u connects L and L', and we write in this case $L \to L'$.

THEOREM 1. *Let L, L' be periodic hyperbolic orbits of equation (1) and $L \to L'$. Then $N(L) \geq M(L')$. If L distinguishes from an equilibrium point then $M(L) > M(L')$.*

COROLLARY. *Let L be a periodic hyperbolic orbit of (1) different from an equilibrium point. Then L has no homoclinical trajectory.*

REMARK. The detailed analysis of the connecting orbits for equation (1) with the Dirichlet boundary conditions was done in [6], [7], [8].

The main difference between the Dirichlet boundary condition and the periodic boundary condition for equations (1) and (2) is that in the case of the Dirichlet boundary condition equations (1) and (2) admit no limit cycles.

2. Some properties of linear parabolic equations and proof of the main result.

Let us consider a linear parabolic equation

$$Pu = \frac{\partial u}{\partial t} - a(t,x)\frac{\partial^2 u}{\partial x^2} + b(t,x)\frac{\partial u}{\partial x} + c(t,x)u = 0 \tag{3}$$

with $t \in \mathbb{R}$, $x \in S^1$, $a, b, c \in C^\infty$, $||a||_{C^k}, ||b||_{C^k}, ||c||_{C^k} \leq M(k)$, $C^{-1} \leq a \leq C$, $C > 0$. For $f \in C(S^1)$, we denote by Nf the number of changes in sign of the function f on

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S^1. So, if f is a smooth function with simple zeros then $\# f^{(-1)}(0) = Nf$, and Nf is an even number.

Let $v(t,x) \in C(\mathcal{D})$, $\mathcal{D} = \mathbb{R} \times S^1$, denote

$$NV(t_0) = NV(t_0,x), \|V\|(t_0) = \|V(t_0,x)\|_{C(S^1)} ,$$
$$U_i = \{u \in C^\infty(\mathcal{D}), Pu = 0 \text{ in } \mathcal{D}, Nu(t) \equiv 2i\} \cup \{0\} ,$$
$$i = 0,1,2,\dots .$$

THEOREM 2. *For any i the set U_i is a linear space,* $\dim U_0 = 1$, $\dim U_i = 2$, $i = 1,2,\dots$. *If $i < j$, $u_i \in U_i$, $u_j \in U_j$, $\|u_i\|(0) = \|u_j\|(0)$ then for $t > 0$*

$$\|u_j\|(t) \le c_j' e^{-c_j t} \|u_i\|(t) , \tag{4}$$

for $t < 0$,

$$\|u_i\|(t) \le c_j' e^{-c_j t} \|u_j\|(t) . \tag{5}$$

Here c_j, c_j' are positive constants. If u,v is a basis in U_i, $i = 1,2,\dots$, then the level curves $u = 0$ separates the level curves $v = 0$. Let u be a solution of the problem (3), then $Nu(t) \le i$ for all $t \in \mathbb{R}$ if and only if $u \in U_0 \oplus \cdots \oplus U_i$.

The proof of Theorem 2 is based on the following lemmas.

LEMMA 1. *([9]). Let f_n be a sequence of bounded functions on [0,1), continuous from the right, $f_n \not\equiv 0$, and $Nf_n \le k$. Then exists a real valued sequence α_m and subsequence f_{n_m} such that, the sequence*

$$\alpha_m f_{n_m}$$

is convergent, in the sense of distributions, to distribution $\bar{f} \not\equiv 0$ of order $\le k$.

As a consequence of Lemma 1 we have

LEMMA 2. *Let $f_n \in C(S^1)$, $Nf_n \le k$ for all $n = 1,2,\dots$. Then there exists a subsequence n_i and real numbers α_i such that as $i \to \infty$, $\alpha_i f_{n_i}$ converges in the topology of the space of distributions $\mathcal{D}'$ to a non-zero distribution of order less than k.*

LEMMA 3. *([10]). Let u be a solution of (3), $t \in [0,1]$. Then $Nu(t)$ is a non-increasing function on [0,1].*

LEMMA 4. *([11]). Let u be a solution of (3), $t \in (0,1)$. Then for each $t \in (0,1)$, $Nu(t) < \infty$.*

LEMMA 5. *([11]). Let u be a solution of (3), $t \in (0,1)$, $Nu(t) \equiv i$. Then for each $t_0 \in (0,1)$ the function $u(t_0,x)$ has exactly i zeros on S^1, and each zero of $u(t_0,x)$ on S^1 is simple.*

LEMMA 6. *Let u be a solution of (3), $t > 0$, $Nu(t) \equiv k$, $0 < \tau < T$. Then*

$$\|u\|(\tau)/\|u\|(T) < C ,$$

where $C = C(k,\tau,T,M(k)) > 0$.

Proof. We assume the contrary, namely, that there exists a sequence of parabolic equations of the type (3)

$$P_i u_i = 0 \text{ in } S^1 \times (0,\infty) ,$$

$Nu_i(t) \le k$, and

$$\|u_i\|(\tau)/\|u_i\|(T) \to \infty$$

as $i \to \infty$. From Lemma 2 it follows that choosing convergent subsequences P_{i_m}, u_{i_m} we obtain an equation

$$Pu = 0 \text{ in } S^1 \times (0, \infty)$$

such that $u \not\equiv 0$ on $S^1 \times (0, T)$, $u(T, \cdot) \equiv 0$ on S^1. This contradicts the theorem on the uniqueness of the solution of the inverse problem for parabolic equations [12].

LEMMA 7. *Let u be a solution of (3), $t > 0$, $Nu(t) \equiv k$. Denote by $\alpha_1(t), \ldots, \alpha_k(t)$ the zeros of the function $u(t, \cdot)$ on S^1. Then there exists a constant $\beta = \beta(k, M(k)) > 0$ such that*

$$\inf_{\substack{t>1 \\ 1 \le i \le k-1}} \left(\alpha_{i+1}(t) - \alpha_i(t)\right) > \beta \,.$$

Proof. We assume the contrary, namely, that there exists a sequence of parabolic equations of the type (3),

$$P_j u_j = 0 \text{ in } S^1 \times (0, \infty) \,,$$

$Nu_j(t) \equiv k$, $j = 1, 2, \ldots$, and real numbers $t_j > 1$ such that for some i

$$\alpha_{i+1}^j(t_j) - \alpha_i^j(t_j) \to 0$$

as $j \to \infty$, where $\alpha_1^j, \ldots, \alpha_k^j$ are zeros of the function u_j. Without loss of generality, we may assume that all $t_j = 1$, and the coefficients of P_j converge in C^k to the coefficients of P, as $j \to \infty$; $\|u_j\|(1) = 1$, $j = 1, 2, \ldots$.

From Lemma 5 it follows that there is a subsequence u_{j_m} which is convergent to the solution of the equation

$$Pu = 0 \text{ in } S^1 \times (0, 1) \,.$$

From Lemmas 3 and 5, it follows that $Nu(t) \le k - 1$ for $t > 1$, and there is a $t_0 > 1$ such that all zeros of the function $u(t_0, \cdot)$ are simple. As $u_{j_m}(t_0, \cdot) \to u(t_0, \cdot)$ in C^1 then there is such u_n that $Nu_n(t) \le k - 1$.

LEMMA 8. *Let u, v be solutions of (3), $t > 0$, $Nu(t) \equiv i$, $Nu(t) \equiv j$, $i < j$, $\|u\|(1) = \|v\|(1)$.*

Proof. Let us assume that for some $T > 1$ $\|v\|(T)/\|u\|(T)$ is sufficiently large. Then by Lemmas 5, 6, 7 for the sufficiently small $\varepsilon > 0$, $N(u + \varepsilon v)(1) = i$, $N(u + \varepsilon v)(T) = j$. As $i < j$ then our assumption contradicts Lemma 2.

LEMMA 9. *Let u, v be solutions of (3), $t > 0$, $Nu(t) \equiv i$, $Nv(t) \equiv j$, $i < j$, $\|u\|(1) = \|v\|(1)$. Then there exists $T > 1$, $T = T(j, M(j))$, such that*

$$\|u\|(T)/\|v\|(T) > 2.$$

Proof. Let us assume the opposite. Then from Lemma 8 it follows that there is a sequence $t_k \to \infty$, $\|u\|(t_k)/\|v\|(t_k) \to s > 0$ as $k \to \infty$. Further, we may assume that the following sequences are convergent in $C^1(S^1)$:

$$\frac{u(t_k, x)}{\|u\|(t_k)} \to \varphi \,,$$

$$\frac{v(t_k, x)}{\|v\|(t_k)} \to \psi \,,$$

as $k \to \infty$. Since $N\varphi = i$, $N\psi = j$, then for any $\varepsilon > 0$ there is an $\alpha \in \mathbb{R}$ such that the distance between a pair of zeros of the function $\varphi + \alpha\psi$ is less than ε. So the same is true for the function $u(t_k, x) + \alpha s v(t_k, x)$ for sufficiently large k. The last statement contradicts Lemma 7.

LEMMA 10. *For all $T > 0$, $i = 1, 2, \dots$ there exists a two-dimensional space $U(T, i)$ of solutions of the problem (2) in $S^1 \times (-T, T)$ such that if $u \in U(T, i)$, $u \not\equiv 0$, then $Nu(t) = 2i$, $t \in (0, T)$. If u, v is a basis in $U(T, i)$, then the level curves $u = 0$ separates the level curves $v = 0$.*

Proof. **1.** We define parabolic operators P_θ, $\theta \in [0, 1]$ on $S^1 \times (-T, 3T)$ by

$$P_1 = \frac{\partial}{\partial t} - a_1(t, x)\frac{\partial^2}{\partial x^2} + b_1(t, x)\frac{\partial}{\partial x} + c_1(t, x) \ ,$$

$$a_1(t, x) = \begin{cases} a(t, x) & \text{for } x \in (-T, T] \ , \\ -a(3T - t, x) & \text{for } x \in (T, 3T) \ , \end{cases}$$

$$b_1(t, x) = \begin{cases} b(t, x) & \text{for } x \in (-T, T] \ , \\ -b(3T - t, x) - \frac{\partial}{\partial x} a(3T - tx) & \text{for } x \in (T, 3T) \ , \end{cases}$$

$$c_1(t, x) = \begin{cases} c(t, x) & \text{for } x \in (-T, T] \ , \\ c(3T - t, x) - \frac{\partial}{\partial x} b(3T - t, x) - \frac{\partial^2}{\partial x^2} a(3T - t, x) & \text{for } x \in (T, 3T) \ , \end{cases}$$

$$P_0 = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \ ,$$

$$P_\theta = (1 - \theta)P_0 + \theta P_1 \ .$$

Let us consider the problem

$$\begin{cases} P_\theta u = 0 & \text{in } S^1 \times (-T, 3T) \ , \\ u(0, x) = g(x). \end{cases} \tag{6}$$

Denote by L^θ the linear operator $L^\theta : g(x) \to u(3T, x)$. For any $\theta \in [0, 1]$, L^θ is a selfadjoint operator in $L_2(S^1)$.

2. Let g be an eigenfunction of L^θ, u be a solution of (6). Then $Nu(-T) = Nu(3T)$, and hence by Lemma 5 the zeros of the function $u(t_0, x)$, $t_0 \in [-T, 3T]$ are simple.

3. Denote by $\lambda_1^\theta > \lambda_2^\theta \geq \lambda_3^\theta \geq \cdots$ the eigenvalues of L^θ, $\theta \in [0, 1]$. Then $\lambda_1^0 = 1$, $\lambda_2^0 = \lambda_3^0 = e^{-4T}$, $\lambda_4^0 = \lambda_5^0 = e^{-16T}, \dots$. The eigenfunctions of L^0 are: $1, \sin nx, \cos nx$, $n = 1, 2, \dots$. Since the eigenvalues λ_n^θ and their eigenvector subspaces are continuously dependent on the parameter θ then from 2 it follows that for any $i = 1, 2, \dots$ and $\theta \in [0, 1]$ the operator L^θ has exactly two linear independent eigenfunctions φ^θ, ψ^θ (or two-dimensional subspace if $\lambda_{2i}^\theta = \lambda_{2i+1}^\theta$) such that $N\varphi^\theta = N\psi^\theta = i$.

4. Let u, v, w be solutions of (6) with $g = \varphi^\theta, \psi^\theta$, $h = a\varphi^\theta + b\psi^\theta \not\equiv 0$, $a, b \in \mathbb{R}$. We prove that the function w has simple zeros. Denote $L\varphi^\theta = \alpha\varphi^\theta$, $L\psi^\theta = \beta\psi^\theta$. If $\alpha = \beta$, the statement follows from Assertion 2.

If $\alpha \neq \beta$, we assume the contrary. Then by Lemma 5 $N\omega(-T) > N\omega(3T)$. Since

$$(L^\theta)^k(h) = \alpha^k a\varphi^\theta + \beta^k b\psi^\theta \ ,$$

$k \in \mathbb{Z}$, then for sufficiently large k

$$N\big[(L^\theta)^k(h)\big] = N\big[(L^\theta)^{-k}(h)\big] = i \ .$$

This contradicts the inequality $Nw(-T) > Nw(3T)$.

5. From 4 it follows that $Nw(t) \equiv i$, $t \in [-T, 3T]$ and that $(u^\theta)^2 + (v^\theta)^2 > 0$ on $S^1 \times [-T, 3T]$. Hence we may consider u^1, v^1 as a basis in $U(T,i)$.

Proof of Theorem 2. Let $U(T,i)$ be defined as in Lemma 10. By Lemma 6 we can choose sequences α_k, $T_k \in \mathbb{R}$, $T_k \to +\infty$ as $k \to \infty$ such that $U(T_k, i) \to U_i \not\equiv 0$ in $\mathcal{D}$ as $k \to \infty$. If $u \in U_i$ then by Lemmas 5 and 6, $Nu(t) \equiv 2i$, $t \in \mathbb{R}$, $i = 1, 2, \ldots$. The inequalities (4) and (5) follow from Lemma 9.

Let u be a solution of (2), $t \in \mathbb{R}$, and $Nu(t) \equiv i$. We prove that $u \in U_i$. Assume the contrary, namely, that u, u_i, v_i are linear independent, where $u_i, v_i \in U_i$. Then there is a linear combination $w = au + bu_i + cv_i$ such that $w(0,0) = \frac{\partial}{\partial x} w(0,0) = 0$. By Lemma 5 $Nw(-1) > Nw(1)$. If $Nw(1) < i$ then by Lemma 9, $\|u\|(t)$, $\|u_i\|(t)$, $\|v_i\|(t) = o(\|w\|(t))$ as $t \to +\infty$, which is impossible. If $Nw(1) \geq i$ then $Nw(-1) > i$ and $\|u\|(t)$, $\|u_i\|(t)$, $\|v_i\|(t) = o(w(t))$ as $t \to -\infty$, which is also impossible. So, $u \in U_i$. Equality $\dim U_0 = 1$ is evident. If $u \in U_0 \oplus \cdots \oplus U_i$, the inequality $Nu(t) \leq i$ follows from (5).

Let u be a solution of (3), $t \in \mathbb{R}$, $Nu(t) \leq i$. First we prove that $u \in U_0 \oplus \cdots \oplus U_{i+1}$. Let us assume the contrary. Then there is $v \in U_0 \oplus \cdots \oplus U_{i+1}$ such that $N(u+v)(0) > i+1$. By Lemma 9 $\|u\|(t)$, $\|v\|(t) = o(\|u+v\|(t))$ as $t \to -\infty$ which is impossible. So, $u = \alpha v + \beta w$, with $v \in U_0 \oplus \cdots \oplus U_i$, $w \in U_{i+1}$. If $\beta \neq 0$, then from inequality (5) it follows that for sufficiently large $T > 0$, $Nu(-T) = i+1$. Hence $u \in U_0 \oplus \cdots \oplus U_i$, and so Theorem 2 is proved.

If in Theorem 2 we consider Dirichlet or Neumann boundary conditions the situation becomes more simple. Let us consider the problem

$$\begin{cases} Pu = 0 & \text{in } \mathbb{R} \times [0,1]\,, \\ u(t,0) = u(t,1) = 0. \end{cases} \tag{7}$$

Let us denote by V_i the set of solutions of (7) such that for $v \in V_i$, $Nv(t) \equiv i$.

THEOREM 3. *For any i the set V_i is a one-dimensional linear space. If $i < j$, $v_i \in V_i$, $v_j \in V_j$, $\|v_i\|(0) = \|v_j\|(0)$, then for $t > 0$*

$$\|v_j\|(t) \leq c_j' e^{-c_j t} \|v_i\|(t)\,,$$

and for $t < 0$

$$\|v_i\|(t) \leq c_j' e^{-c_j t} \|v_j\|(t)\,,$$

where $c_j, c_j' > 0$.

Evidently it is possible to prove Theorem 3 by the same reasoning as in the proof of Theorem 2. We don't need Theorem 3 for the following, but it is interesting to compare both cases.

Proof of Theorem 1. Let u be a connecting orbit between L and L'. Let us consider the function

$$v_T(t,x) = \frac{\partial u/\partial t}{\|\partial u/\partial t\|(T)}(t+T, x)\,.$$

Then $v_T(t,x)$ converges to the exponential growing solution of the variational equation on L as $T \to -\infty$ and $v_T(t,x)$ converges to the exponentially decreasing solution of the variational equation on L' as $T \to +\infty$. By Lemma 3, $k = N\frac{\partial u}{\partial t}(-\infty) \geq N\frac{\partial u}{\partial t}(+\infty) = m$. So by Theorem 2

$$\dim W^n(L) \geq 2k\,, \dim W^u(L') \leq 2m\,. \tag{8}$$

Let us assume now that L is the orbit of the periodic solution w of (1) and $\partial w/\partial t \not\equiv 0$. Then by Theorem 2, $N\frac{\partial w}{\partial t} \geq k$. Thus $\dim W^u(L) \geq 2k+1$ and hence $M(L) > M(L')$.

Theorem 1 is proved.

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