

GLUING FORMULAE FOR DONALDSON INVARIANTS FOR CONNECTED SUMS ALONG SURFACES*

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Abstract. Following our work in [18], we prove a gluing formula for the Donaldson invariants of the connected sum of two four-manifolds along surfaces of the same genus g , self-intersection zero and representing odd homology classes, solving a conjecture of Morgan and Szabó [14].

1. Introduction. This paper tries to answer the question of the behaviour of the Donaldson invariants under connected sums along surfaces of arbitrary genus $g \geq 2$ and self-intersection zero. The problem was first motivated by the computation of the basic classes of elliptic surfaces. The general elliptic surface with $b^+ > 1$ and simply connected, is constructed by the process of connected sum along embedded tori of self-intersection zero from elementary pieces (rational elliptic surfaces and homotopy $K3$ -surfaces). After establishing the appropriate gluing formula for the behaviour of Donaldson invariants under the operation of connected sum along embedded tori, the information of the basic classes for some particular examples gives the basic classes of any such elliptic surface. This has been carried out by many authors [13] [3] [17] [15] [19].

The following natural case is the connected sum along embedded surfaces of genus $g = 2$. Morgan and Szabó [20] [14] treated the case when the self-intersection of one of the surfaces is 1 (and the other -1). The author [18] [17] has solved the case in which the surfaces have self-intersection zero and are odd in homology, giving a number of nice applications. For this he used the Fukaya-Floer homology as developed in [1].

The case we are going to deal with in this paper is the connected sum along embedded surfaces of higher (arbitrary) genus g , self-intersection zero and odd, giving a positive answer to the conjectures in the literature [14, conjecture 7.2] [17, conjecture 5.27].

Let X be a smooth, compact, oriented four-manifold with $b^+ > 1$ and $b^+ - b_1$ odd. For any $w \in H^2(X; \mathbb{Z})$, D_X^w will denote the corresponding Donaldson invariant [4] [12], which is defined as a linear functional on $\mathbb{A}(X) = \text{Sym}^*(H_0(X) \oplus H_2(X)) \otimes \bigwedge^* H_1(X)$ ($H_*(X)$ will always denote homology with rational coefficients, and similarly for $H^*(X)$). Let $x \in H_0(X)$ be the class of a point. Then Kronheimer and Mrowka [12] define X to be of simple type (with respect to w) when $D_X^w((x^2 - 4)z) = 0$ for all $z \in \mathbb{A}(X)$, and in that case define

$$\mathbb{D}_X^w(z) = D_X^w\left(\left(1 + \frac{x}{2}\right)z\right),$$

for all $z \in \text{Sym}^* H_2(X)$. The series $\mathbb{D}_X^w(e^{t\alpha})$, $\alpha \in H_2(X)$, is even or odd depending on whether

$$d_0 = d_0(X, w) = -w^2 - \frac{3}{2}(1 - b_1 + b^+)$$

is even or odd. When $b_1 = 0$ and $b^+ > 1$, X is of simple type with respect to some w if and only if it is so with respect to any w . In such case, X is just called of simple type.

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PROPOSITION 1 ([12]). *Let X be a manifold of simple type with $b_1 = 0$ and $b^+ > 1$ and odd. Then we have*

$$\mathbb{D}_X^w(e^\alpha) = e^{Q(\alpha)/2} \sum a_{j,w} e^{K_j \cdot \alpha}$$

for finitely many $K_j \in H^2(X; \mathbb{Z})$ (called basic classes) and $a_{j,w} = (-1)^{\frac{K_j \cdot w + w^2}{2}} a_j$, where a_j are rational numbers (the collection is empty when the invariants all vanish). These classes are lifts to integral cohomology of $w_2(X)$. Moreover, for any embedded surface $\Sigma \hookrightarrow X$ of genus g and with $\Sigma^2 \geq 0$, one has $2g - 2 \geq \Sigma^2 + |K_j \cdot \Sigma|$.

DEFINITION 2 ([18]). (w, Σ) is an **allowable pair** if $w, \Sigma \in H^2(X; \mathbb{Z})$, $w \cdot \Sigma \equiv 1 \pmod{2}$ and $\Sigma^2 = 0$. Then we define

$$D_X^{(w, \Sigma)} = D_X^w + D_X^{w+\Sigma}.$$

When $b^+ = 1$, the invariants depend on the metric through a structure of walls and chambers [10] and therefore we have to specify the metric. As $w \cdot \Sigma \equiv 1 \pmod{2}$, we have that Σ is contained in the closure of a unique chamber of type (w, p_1) for every $p_1 < 0$, $p_1 \equiv w^2 \pmod{4}$. So, in the case that the invariants only depend on the metric through the period point (for instance, simply-connected manifolds [11] and $\Sigma \times \mathbb{C}P^1$, with Σ a Riemann surface, which are all the cases we need for our arguments), we shall consider the invariants referring to the chambers defined by Σ .

The series $D_X^{(w, \Sigma)}(e^{t\alpha})$, $\alpha \in H_2(X)$, is even or odd according to whether d_0 is even or odd. Since $(w + \Sigma)^2 \equiv w^2 + 2 \pmod{4}$, we can recover D_X^w and $D_X^{w+\Sigma}$ from $D_X^{(w, \Sigma)}$.

PROPOSITION 3 ([18]). *Suppose X is a manifold of simple type with $b_1 = 0$ and $b^+ > 1$ and odd. Write the Donaldson series as $\mathbb{D}_X^w(e^\alpha) = e^{Q(\alpha)/2} \sum a_{j,w} e^{K_j \cdot \alpha}$. Then setting $d_0 = d_0(X, w) = -w^2 - \frac{3}{2}(1 + b^+)$ we have*

$$D_X^{(w, \Sigma)}(e^\alpha) = e^{Q(\alpha)/2} \sum_{K_j \cdot \Sigma \equiv 2 \pmod{4}} a_{j,w} e^{K_j \cdot \alpha} + e^{-Q(\alpha)/2} \sum_{K_j \cdot \Sigma \equiv 0 \pmod{4}} i^{-d_0} a_{j,w} e^{iK_j \cdot \alpha}$$

So giving \mathbb{D}_X^w is equivalent to giving $D_X^{(w, \Sigma)}$.

REMARK 4. Under the conditions of proposition 3, we can prove in a similar fashion that

$$D_X^{(w, \Sigma)}(x e^\alpha) = 2 e^{Q(\alpha)/2} \sum_{K_j \cdot \Sigma \equiv 2 \pmod{4}} a_{j,w} e^{K_j \cdot \alpha} - 2 e^{-Q(\alpha)/2} \sum_{K_j \cdot \Sigma \equiv 0 \pmod{4}} i^{-d_0} a_{j,w} e^{iK_j \cdot \alpha}$$

and using that $D_X^{(w, \Sigma)}(x^a e^{\alpha+\Sigma}) = \sum \frac{1}{b!} D_X^{(w, \Sigma)}(\Sigma^b x^a e^\alpha)$, one has

$$\begin{aligned} D_X^{(w, \Sigma)}(\Sigma^b x^a e^\alpha) &= e^{Q(\alpha)/2} \sum_{K_j \cdot \Sigma \equiv 2 \pmod{4}} a_{j,w} 2^a ((\alpha + K_j) \cdot \Sigma)^b e^{K_j \cdot \alpha} + \\ &+ e^{-Q(\alpha)/2} \sum_{K_j \cdot \Sigma \equiv 0 \pmod{4}} i^{-d_0} a_{j,w} (-2)^a ((-\alpha + i K_j) \cdot \Sigma)^b e^{iK_j \cdot \alpha} \end{aligned}$$

DEFINITION 5 (compare [18]). We say that (X, Σ) is **permissible** if X is a smooth compact oriented four-manifold and $\Sigma \hookrightarrow X$ is an embedded Riemann surface of genus $g \geq 2$ and self-intersection zero such that $[\Sigma] \in H_2(X; \mathbb{Z})$ is odd (its reduction modulo 2 is non-zero, or equivalently, it is an odd multiple of a primitive homology

class). So we can consider $w \in H^2(X; \mathbb{Z})$ with $w \cdot \Sigma \equiv 1 \pmod{2}$. Then (w, Σ) is an allowable pair (we consider $\Sigma \in H^2(X; \mathbb{Z})$ through Poincaré duality). This implies that $b^+ > 0$. Let $N_\Sigma \cong A = \Sigma \times D^2$ be a tubular neighbourhood of Σ and set $X^o = X - \text{int}(N_\Sigma)$. Then $\partial X^o = Y \cong \Sigma \times \mathbb{S}^1$ (but the isomorphism is not canonical). We consider one such isomorphism fixed and (when necessary) we furnish X^o with a cylindrical end, i.e. we consider $X_{ce}^o = X^o \cup (Y \times [0, \infty))$.

We call **identification** for $Y = \Sigma \times \mathbb{S}^1$ any (orientation preserving) bundle automorphism $\phi : Y \xrightarrow{\sim} Y$. Up to isotopy, ϕ depends only on the isotopy class of the induced diffeomorphism on Σ and on an element of $H^1(\Sigma; \mathbb{Z})$.

DEFINITION 6 ([18]). Let (X_1, Σ_1) and (X_2, Σ_2) be permissible, with Σ_1 and Σ_2 of the same genus g . We pick orientations so that $\partial X_1^o = -\partial X_2^o = Y$ (minus means reversed orientation). Then $X = X(\phi) = X_1^o \cup_\phi X_2^o = X_1 \#_\Sigma X_2$ is called the **connected sum along Σ** of (X_1, Σ_1) and (X_2, Σ_2) (with identification ϕ). It is a compact, naturally oriented, smooth four-manifold with an embedding $Y \hookrightarrow X$ such that $X - Y = (X_1^o)_{ce} \sqcup (X_2^o)_{ce} \subset X_1 \sqcup X_2$.

The induced homology classes $[\Sigma_1]$ and $[\Sigma_2]$ coincide and are induced by an embedded $\Sigma \hookrightarrow X$. Then (X, Σ) is permissible. Choose $w_i \in H^2(X_i; \mathbb{Z})$, $i = 1, 2$, and $w \in H^2(X; \mathbb{Z})$ such that $w_i \cdot \Sigma_i \equiv 1 \pmod{2}$, $w \cdot \Sigma \equiv 1 \pmod{2}$, in a compatible way (i.e. the restriction of w to $X_i^o \subset X$ coincides with the restriction of w_i to $X_i^o \subset X_i$). We have $w^2 \equiv w_1^2 + w_2^2 \pmod{2}$. Changing w by $w + \Sigma$ if necessary, we can always suppose $w^2 \equiv w_1^2 + w_2^2 \pmod{4}$. In general, we shall call w all of them, not making explicit to which manifold they refer. Note also that if $b_1(X_1) = b_1(X_2) = 0$ then $b_1(X) = 0$ and $b^+(X) > 1$.

REMARK 7. There is a case when there is a preferred identification. Suppose X_1 and X_2 are complex surfaces, Σ is a complex curve of genus g and there are holomorphic embeddings $\Sigma \hookrightarrow X_i$, with image Σ_i , $i = 1, 2$, such that $[\Sigma_i]$ is odd and has self-intersection zero. Then the holomorphic normal bundle to Σ_i gives a preferred isomorphism $\partial N_{\Sigma_i} \xrightarrow{\sim} \Sigma_i \times \mathbb{S}^1 \cong \Sigma \times \mathbb{S}^1$, and hence a preferred identification $\partial X_1^o \xrightarrow{\sim} -\partial X_2^o$.

REMARK 8. Let X_1 and X_2 be as in definition 6 and $X = X_1 \#_\Sigma X_2$. Then homology orientations for X_1 and X_2 induce canonically a homology orientation for X . To see this, define $H_+^2(X_i^o)$ to be a maximal definite positive subspace for the intersection pairing of X_i^o restricted to the image of $H_2(X_i^o) \rightarrow H_2(X_i^o, \partial X_i^o) \cong H^2(X_i^o)$. Then it is easy to see that the exact sequence for the pair (X_i, X_i^o) and the Mayer-Vietoris sequence for $X = X_1^o \cup X_2^o$ yield the following exact sequences

$$0 \rightarrow H^1(X_i) \rightarrow H^1(X_i^o) \rightarrow H^2(A, \partial A) \rightarrow H_+^2(X_i) \rightarrow H_+^2(X_i^o) \rightarrow 0$$

$$0 \rightarrow H^1(X) \rightarrow H^1(X_1^o) \oplus H^1(X_2^o) \rightarrow H^1(Y) \rightarrow H_+^2(X) \rightarrow H_+^2(X_1^o) \oplus H_+^2(X_2^o) \rightarrow 0,$$

where $A = \Sigma \times D^2$ as before. We fix an orientation of $H^1(Y)$ (it does not change when we reverse the orientation of Y). So homology orientations for X_1 and X_2 induce a homology orientation for X .

Let $\mathcal{H} = \{D \in H_2(X)/D|_Y = k[\mathbb{S}^1] \in H_1(Y), \text{ some } k \in \mathbb{Q}\}$. This subspace of $H_2(X)$ contains the image of $H_2(X_1^o) \oplus H_2(X_2^o)$. For every $D \in \mathcal{H}$, choose $D_i \in H_2(X_i)$ agreeing with D (i.e. $D_i|_{X_i^o} = D|_{X_i^o}$, $i = 1, 2$) and with $D^2 = D_1^2 + D_2^2$. Moreover, we can suppose that the map $D \mapsto (D_1, D_2)$ is linear. Actually, once chosen one of such maps, any other is of the form $D \mapsto (D_1 + r\Sigma, D_2 - r\Sigma)$, for a (rational) number r . Let us state now our main result.

THEOREM 9. *Suppose (X_1, Σ_1) and (X_2, Σ_2) are permissible with Σ_1 and Σ_2 of the same genus $g \geq 2$. Suppose also that X_1, X_2 have both $b_1 = 0$ and $b^+ > 1$ and are of simple type. Let*

$$\mathbb{D}_{X_1}^w(e^\alpha) = e^{Q(\alpha)/2} \sum a_{j,w} e^{K_j \cdot \alpha}$$

and

$$\mathbb{D}_{X_2}^w(e^\alpha) = e^{Q(\alpha)/2} \sum b_{k,w} e^{L_k \cdot \alpha}.$$

Choose any identification ϕ and let $X = X(\phi) = X_1 \#_\Sigma X_2$ be the connected sum along Σ , with the induced homology orientation. Suppose finally that X is of simple type. Let $w \in H^2(X; \mathbb{Z}), w_i \in H^2(X_i; \mathbb{Z}), i = 1, 2$, in a compatible way, such that $w_i \cdot \Sigma_i \equiv 1 \pmod{2}$, $w \cdot \Sigma \equiv 1 \pmod{2}$ and $w^2 \equiv w_1^2 + w_2^2 \pmod{4}$. For every $D \in \mathcal{H}$, we choose $D_i \in H_2(X_i)$ agreeing with D and satisfying $D^2 = D_1^2 + D_2^2$ in such a way that the map $D \mapsto (D_1, D_2)$ is linear. Then

$$\begin{aligned} \mathbb{D}_X^w(e^{tD}) &= e^{Q(tD)/2} \left(\sum_{K_j \cdot \Sigma = L_k \cdot \Sigma = 2g-2} -2^{7g-9} a_{j,w} b_{k,w} e^{(K_j \cdot D_1 + L_k \cdot D_2 + 2\Sigma \cdot D)t} + \right. \\ (1) \quad &+ \left. \sum_{K_j \cdot \Sigma = L_k \cdot \Sigma = -(2g-2)} (-1)^g 2^{7g-9} a_{j,w} b_{k,w} e^{(K_j \cdot D_1 + L_k \cdot D_2 - 2\Sigma \cdot D)t} \right). \end{aligned}$$

REMARK 10. If we do not assume $w^2 \equiv w_1^2 + w_2^2 \pmod{4}$, we get an extra factor $\epsilon = (-1)^{(g-1)(w^2 - w_1^2 - w_2^2)/2}$ in front of formula (1).

REMARK 11. The reason for the sign is easy to work out. First, $w^2 \equiv w_1^2 + w_2^2 \pmod{2}$. Also,

$$-\frac{3}{2}(1 - b_1(X) + b^+(X)) = -\frac{3}{2}(1 - b_1(X_1) + b^+(X_1)) - \frac{3}{2}(1 - b_1(X_2) + b^+(X_2)) - 3(g - 1).$$

Therefore, $d_0(X, w) \equiv d_0(X_1, w) + d_0(X_2, w) + g - 1 \pmod{2}$. Now the sign comes from the fact that the coefficient for the basic class $-\kappa$ is $(-1)^{d_0} c_\kappa$, being c_κ the coefficient for the basic class κ .

REMARK 12. If we are in the conditions of theorem 9, but $g = 1$, we get a slightly different answer [17, chapter 4] [14]. For all basic classes, it is $K_j \cdot \Sigma = L_k \cdot \Sigma = 0$, and

$$\begin{aligned} \mathbb{D}_X^w(e^{tD}) &= e^{Q(tD)/2} \left(\sum_{K_j, L_k} -\frac{1}{4} a_{j,w} b_{k,w} e^{(K_j \cdot D_1 + L_k \cdot D_2 + 2\Sigma \cdot D)t} + \right. \\ &+ \left. \sum_{K_j, L_k} -\frac{1}{4} a_{j,w} b_{k,w} e^{(K_j \cdot D_1 + L_k \cdot D_2 - 2\Sigma \cdot D)t} + \sum_{K_j, L_k} -\frac{1}{2} a_{j,w} b_{k,w} e^{(K_j \cdot D_1 + L_k \cdot D_2)t} \right). \end{aligned}$$

COROLLARY 13. *Suppose we are in the conditions of theorem 9. Write $\mathbb{D}_X(e^\alpha) = e^{Q(\alpha)/2} \sum c_\kappa e^{\kappa \cdot \alpha}$, $\mathbb{D}_{X_1}(e^\alpha) = e^{Q(\alpha)/2} \sum a_j e^{K_j \cdot \alpha}$ and $\mathbb{D}_{X_2}(e^\alpha) = e^{Q(\alpha)/2} \sum b_k e^{L_k \cdot \alpha}$ for the Donaldson series for X, X_1 and X_2 , respectively. Then given any pair $(K, L) \in H^2(X_1^o; \mathbb{Z}) \oplus H^2(X_2^o; \mathbb{Z})$, we have*

$$\sum_{\{\kappa/\kappa|_{X_1^o}=K, \kappa|_{X_2^o}=L\}} c_\kappa = (\pm 1)^{g-1} 2^{7g-9} \left(\sum_{K_j|_{X_1^o}=K} a_j \right) \cdot \left(\sum_{L_k|_{X_2^o}=L} b_k \right)$$

whenever $K|_Y = L|_Y = \pm(2g - 2)\text{P.D.}[\mathbb{S}^1]$. Otherwise, the left hand side is zero.

Proof. Allowing D to vary in \mathcal{H} , formula (1) gives

$$\sum_{\{\kappa/\kappa|_{X_1^o}=K, \kappa|_{X_2^o}=L\}} c_{\kappa,w} e^{(\kappa \cdot D)t} = \sum_{\substack{\kappa_j|_{X_1^o}=K \\ L_k|_{X_2^o}=L}} -(\pm 1)^{g-1} 2^{7g-9} a_{j,w} b_{k,w} e^{(K_j \cdot D_1 + L_k \cdot D_2 \pm 2\Sigma \cdot D)t}.$$

(2)

We cannot have more precise information on κ as we cannot evaluate \mathbb{D}_X^w on all $D \in H_2(X)$. Now take $w = \text{P.D.}[D] \in H^2(X; \mathbb{Z})$, $w_1 = \text{P.D.}[D_1] \in H^2(X_1; \mathbb{Z})$ and $w_2 = \text{P.D.}[D_2] \in H^2(X_2; \mathbb{Z})$ with $w^2 = w_1^2 + w_2^2$ and $w \cdot \Sigma = 1$. Substitute $t = \pi i/2$ in (2) and multiply by $(-1)^{w^2/2} = (-1)^{w_1^2/2} (-1)^{w_2^2/2}$, to get the sought expression. \square

REMARK 14. In theorem 9 we cannot hope for having a similar formula for classes D such that $D|_Y$ is not a multiple of $[\mathbb{S}^1]$ in $H_1(Y)$. This is due to the fact that we cannot find $D_i \in H_2(X_i)$ agreeing with D (i.e. $D_i|_{X_i^o} = D|_{X_i^o}$, for $i = 1, 2$). We would need to relate the invariant $\mathbb{D}_X^w(e^{tD})$ with invariants of the form $\mathbb{D}_{\tilde{X}_i}^w(e^{tD_i})$, for suitable manifolds \tilde{X}_i containing X_i^o , $D_i \in H_2(\tilde{X}_i)$, such that $D_i|_{X_i^o} = D|_{X_i^o}$. This was done in [18, theorem 10] for the case $g = 2$ (see also conjecture in section 5).

This limitation prevents us from having more general results. For example, we do not know whether (under the conditions of theorem 9) there are basic classes κ for $X = X_1 \#_{\Sigma} X_2$ such that $|\kappa \cdot \Sigma| < 2g - 2$ or not (compare [18, corollary 11]).

In some cases, theorem 9 is all that we need to find explicitly the basic classes for X . This is due to the fact that there is a subspace $V \subset H_2(X)$ where all the basic classes vanish, such that $H_2(X) = \mathcal{H} \oplus V$. In [14, definition 4.1], Morgan and Szabó define admissible identification, which is a condition which implies that $X = X_1 \#_{\Sigma} X_2$ is of simple type and such V exists.

COROLLARY 15. *Suppose there exists a subspace $V \subset H_2(X)$ where all the basic classes vanish such that $H_2(X) = \mathcal{H} \oplus V$. Then there are no basic classes κ for X such that $|\kappa \cdot \Sigma| < 2g - 2$. The basic classes for X are indexed by pairs of basic classes (K_i, L_j) for X_1 and X_2 respectively, such that $K_i \cdot \Sigma = L_j \cdot \Sigma = \pm(2g - 2)$.*

REMARK 16. Corollary 13 agrees with the results of the kind for the Seiberg-Witten invariants [17, section 7.3]. Morgan, Szabó and Taubes [16] have proved the analogous result to corollary 13 for the Seiberg-Witten basic classes (not the part corresponding to basic classes κ for X with $|\kappa \cdot \Sigma| < 2g - 2$). Both results are equivalent supposing true the conjecture of Witten [22] about the relationship of Donaldson and Seiberg-Witten invariants.

Our last result is

THEOREM 17. *Let $S = \Sigma \times \mathbb{C}\mathbb{P}^1$, $w = \text{P.D.}[\mathbb{C}\mathbb{P}^1] \in H^2(S; \mathbb{Z})$. Suppose that S is of finite type of order $n \geq 1$ with respect to w and $w + \Sigma$ for the metrics defined by Σ (i.e. $D_S^{(w, \Sigma)}((x^2 - 4)^n z) = 0$, for all $z \in \mathbb{A}(S)$). Then for any (X, Σ) permissible, X is of finite type of order at most n with respect to any $w \in H^2(X; \mathbb{Z})$ with $w \cdot \Sigma \equiv 1 \pmod{2}$ (and for metrics defined by Σ , in the case $b^+ = 1$).*

2. Gluing theory. Now we are going to develop the gluing theory necessary to prove theorem 9. The set up is as follows, $X = X_1^o \cup_Y X_2^o$, where $\partial X_1^o = -\partial X_2^o = Y$, an oriented three-manifold (later $Y = \Sigma \times \mathbb{S}^1$), $w \in H^2(X; \mathbb{Z})$, $w_i = w|_{X_i} \in H^2(X_i^o; \mathbb{Z})$, and $D \in H_2(X)$ a homology class. So $D|_Y \in H_1(Y)$. We want to evaluate $\mathbb{D}_X^w(D^d)$, the invariant being linear, we may multiply D by any non-zero rational number, and hence suppose that $D|_Y$ is either zero or primitive in $H_1(Y; \mathbb{Z})$. Now we represent D by a cycle $D \subset X$ and put $D_i^o = D \cap X_i^o$, which we shall write formally as $D = D_1^o + D_2^o$, $D_i^o \subset X_i^o$. We can suppose $\partial D_1^o = -\partial D_2^o = \gamma$, with $\gamma \subset Y$ an embedded curve in Y ,

so when we give X_1^o a cylindrical end, we have $D_1^o \cap (Y \times [0, \infty)) = \gamma \times [0, \infty)$ (and analogously for X_2^o).

PROPOSITION 18 ([2] [1] [17]). *Suppose $w|_Y$ odd. Then we have one of the following cases:*

- $D|_Y = 0$ in $H_1(Y; \mathbb{Z})$. Represent D by a cycle so $D = D_1^o + D_2^o$, $D_i^o \subset X_i^o$, $\partial D_1^o = \partial D_2^o = \emptyset$. Consider the Floer homology groups [2] $HF_*(Y)$ (graded mod 4). Then (X_i^o, D_i^o) define relative invariants $\phi^{w_1}(X_1^o, e^{tD_1^o}) \in HF_*(Y) \otimes \mathbb{Q}[[t]]$, $\phi^{w_2}(X_2^o, e^{tD_2^o}) \in HF_*(-Y) \otimes \mathbb{Q}[[t]]$. There is a natural pairing $HF_*(Y) \otimes HF_*(-Y) \rightarrow \mathbb{Q}$, such that

$$(3) \quad D_X^{(w, \Sigma)}(e^{tD}) = \langle \phi^{w_1}(X_1^o, e^{tD_1^o}), \phi^{w_2}(X_2^o, e^{tD_2^o}) \rangle .$$

- $D|_Y \neq 0$. Substitute D by a rational multiple if necessary so that $D|_Y \in H_1(Y; \mathbb{Z})$ is a primitive element in $H_1(Y; \mathbb{Z})$. Represent D by a cycle so $D = D_1^o + D_2^o$, $D_i^o \subset X_i^o$, $\partial D_1^o = -\partial D_2^o = \gamma$, with $\gamma \subset Y$ an embedded curve in Y . Consider the Fukaya-Floer homology groups [1] $HFF_*(Y, \gamma)$ (graded mod 4). Then (X_i^o, D_i^o) define relative invariants $\phi^{w_1}(X_1^o, e^{tD_1^o}) \in HFF_*(Y, \gamma)$, $\phi^{w_2}(X_2^o, e^{tD_2^o}) \in HFF_*(-Y, -\gamma)$. There is a natural pairing $HFF_*(Y, \gamma) \otimes HFF_*(-Y, -\gamma) \rightarrow \mathbb{Q}[[t]]$, such that

$$(4) \quad D_X^{(w, \Sigma)}(e^{tD}) = \langle \phi^{w_1}(X_1^o, e^{tD_1^o}), \phi^{w_2}(X_2^o, e^{tD_2^o}) \rangle .$$

When $b^+ = 1$, the invariants are calculated for a long neck, i.e. we refer to the invariants defined by Σ .

Proof.

- As explained in [2], the Floer homology groups $HF_*(Y)$ are well-defined since $w|_Y$ is odd (this rules out problems with flat reducible connections on Y). Also, there are invariants $\phi^{w_1}(X_1^o, (D_1^o)^n) \in HF_*(Y)$, $\phi^{w_2}(X_2^o, (D_2^o)^m) \in HF_*(-Y)$ such that

$$D_X^{(w, \Sigma)}((D_1^o)^n (D_2^o)^m) = \langle \phi^{w_1}(X_1^o, (D_1^o)^n), \phi^{w_2}(X_2^o, (D_2^o)^m) \rangle .$$

Now we write

$$\phi^{w_1}(X_1^o, e^{tD_1^o}) = \sum \frac{t^n}{n!} \phi^{w_1}(X_1^o, (D_1^o)^n),$$

from where the statement of the theorem.

- Analogously, in [1] the Fukaya-Floer homology groups $HFF_*(Y, \gamma)$ are defined when $w|_Y$ is odd. Associated to (X_i^o, D_i^o) , there are invariants $\phi^{w_1}(X_1^o, D_1^o) \in HFF_*(Y, \gamma)$ and $\phi^{w_2}(X_2^o, D_2^o) \in HFF_*(-Y, -\gamma)$, where $\phi^{w_1}(X_1^o, D_1^o)$ is represented by a Fukaya-Floer chain $(\phi^{w_1}(X_1^o, (D_1^o)^n))_{n \geq 0}$, where $\phi^{w_1}(X_1^o, (D_1^o)^n) \in HF_*(Y)$ (and analogously for (X_2^o, D_2^o)), such that

$$D_X^{(w, \Sigma)}(D^r) = \sum_{0 \leq n \leq r} \binom{r}{n} \langle \phi^{w_1}(X_1^o, (D_1^o)^n), \phi^{w_2}(X_2^o, (D_2^o)^{r-n}) \rangle .$$

So we write formally

$$\phi^{w_1}(X_1^o, e^{tD_1^o}) = \sum \frac{t^n}{n!} \phi^{w_1}(X_1^o, (D_1^o)^n)$$

for the given Fukaya-Floer homology class. \square

Now we particularize to the case which concerns us, $Y = \Sigma \times \mathbb{S}^1$. Conjugation in the second factor produces an isomorphism $Y \cong (-Y)$ (also $(Y, \gamma) \cong (-Y, -\gamma)$). As explained in [1] [17], $HFF_*(Y, \gamma)$ is the limit of a spectral sequence whose E_3 -term is $HF_*(Y) \otimes \hat{H}_*(\mathbb{C}\mathbb{P}^\infty)$ (here $\hat{H}_*(\mathbb{C}\mathbb{P}^\infty)$ means the natural completion of $H_*(\mathbb{C}\mathbb{P}^\infty)$, i.e. $\mathbb{Q}[[t]]$, t of degree 2), with differential d_3 given by

$$\mu(\gamma) : HF_i(Y) \otimes H_j(\mathbb{C}\mathbb{P}^\infty) \rightarrow HF_{i-3}(Y) \otimes H_{j+2}(\mathbb{C}\mathbb{P}^\infty).$$

Let $\gamma = \text{pt} \times \mathbb{S}^1 \subset \Sigma \times \mathbb{S}^1$. Now all the differentials in the E_3 term of the spectral sequence are of the form (see Proposition 20) $H_{\text{odd}}(M_\Sigma^{\text{odd}}) \rightarrow H_{\text{even}}(M_\Sigma^{\text{odd}})$ and $H_{\text{even}}(M_\Sigma^{\text{odd}}) \rightarrow H_{\text{odd}}(M_\Sigma^{\text{odd}})$. The boundary cycle $\gamma = \mathbb{S}^1$ is invariant under the action of the group $\text{Diff}(\Sigma)$ on $Y = \Sigma \times \mathbb{S}^1$, so the differentials commute with the action of $\text{Diff}(\Sigma)$. As there are elements $\rho \in \text{Diff}(\Sigma)$ acting as -1 on $H^1(\Sigma)$, we have that ρ acts as -1 on $H_{\text{odd}}(M_\Sigma^{\text{odd}})$ and as 1 on $H_{\text{even}}(M_\Sigma^{\text{odd}})$. Therefore the differentials are zero and the spectral sequence degenerates in the third term. This implies that $HFF_*(Y, \gamma) = HF_*(Y) \otimes \hat{H}^*(\mathbb{C}\mathbb{P}^\infty) = V[[t]]$, where $V = HF_*(Y)$. Now the relative invariants for (X_1°, D_1°) can be written as

$$\phi^{w_1}(X_1^\circ, e^{tD_1^\circ}) = \sum \frac{t^n}{n!} \phi^{w_1}(X_1^\circ, (D_1^\circ)^n) \in V[[t]]$$

where $\phi^{w_1}(X_1^\circ, (D_1^\circ)^n) \in HF_*(Y)$ has perfect meaning. Under the isomorphism $HFF_* = HFF_*(Y, \gamma) = V[[t]]$, HFF_* becomes a $\mathbb{Q}[[t]]$ -module and $HFF_* \otimes HFF_* \rightarrow \mathbb{Q}[[t]]$ is $\mathbb{Q}[[t]]$ -bilinear.

COROLLARY 19.

1. *There is a (rational) vector space $V = HF_*(Y)$ endowed with a bilinear form such that for every permissible (X, Σ) , $w \in H^2(X^\circ; \mathbb{Z})$ with $w|_Y$ an odd multiple of P.D. $[\mathbb{S}^1]$, and cycle $D^\circ \subset X^\circ$ with $\partial D^\circ = \emptyset$, we have $\phi^w(X^\circ, e^{tD^\circ}) \in V[[t]]$. For $X = X_1^\circ \cup_Y X_2^\circ$, $D = D_1^\circ + D_2^\circ$, $\partial D_1^\circ = \partial D_2^\circ = \emptyset$, we have*

$$D_X^{(w, \Sigma)}(e^{tD}) = \langle \phi^{w_1}(X_1^\circ, e^{tD_1^\circ}), \phi^{w_2}(X_2^\circ, e^{tD_2^\circ}) \rangle .$$

2. *There is a canonical isomorphism $HFF_*(Y, \mathbb{S}^1) \cong V[[t]]$, such that for every permissible (X, Σ) , $w \in H^2(X^\circ; \mathbb{Z})$ with $w|_Y$ an odd multiple of P.D. $[\mathbb{S}^1]$, and cycle $D^\circ \subset X^\circ$ with $\partial D^\circ = \mathbb{S}^1$, we have $\phi^w(X^\circ, e^{tD^\circ}) \in V[[t]]$. For $X = X_1^\circ \cup_Y X_2^\circ$, $D = D_1^\circ + D_2^\circ$, $\partial D_1^\circ = -\partial D_2^\circ = \mathbb{S}^1$, we have*

$$D_X^{(w, \Sigma)}(e^{tD}) = \langle \phi^{w_1}(X_1^\circ, e^{tD_1^\circ}), \phi^{w_2}(X_2^\circ, e^{tD_2^\circ}) \rangle .$$

PROPOSITION 20. *Let M_Σ^{odd} be the moduli space of odd degree rank two stable vector bundles on Σ , which is a smooth variety [9]. Then there is an isomorphism*

$$HF_*(Y) \cong H_*(M_\Sigma^{\text{odd}})$$

as vector spaces (we are using rational coefficients), where we reduce the grading of $H_(M_\Sigma^{\text{odd}})$ modulo 4.*

Proof. Dostoglou and Salamon [5] prove $HF_*(\Sigma \times \mathbb{S}^1) \cong HF_*^{\text{symp}}(M_\Sigma^{\text{odd}})$. It is the particular case where we consider $\phi = \text{id} : \Sigma \rightarrow \Sigma$, in which the mapping torus of ϕ is $\Sigma \times \mathbb{S}^1$. As explained in the introduction of [5], the symplectic manifold M_Σ^{odd} is connected, simply connected and $\pi_2(M_\Sigma^{\text{odd}}) = \mathbb{Z}$, so the groups $HF_*^{\text{symp}}(M_\Sigma^{\text{odd}})$ are well-defined. Now $HF_*^{\text{symp}}(M_\Sigma^{\text{odd}}) \cong H_*(M_\Sigma^{\text{odd}})$ is a standard result obtained by Floer himself [7] for proving the Arnold conjecture. \square

There is a map

$$\mu : H_*(\Sigma) \rightarrow H^{4-*}(M_{\Sigma}^{\text{odd}})$$

given by $\mu(\alpha) = -\frac{1}{4}p_1(\mathcal{V})/4$, where $\mathcal{V} \rightarrow \Sigma \times M_{\Sigma}^{\text{odd}}$ is the associated universal $SO(3)$ -bundle, $p_1(\mathcal{V}) \in H^4(\Sigma \times M_{\Sigma}^{\text{odd}})$ its first Pontrjagin class. Fix a basis $\{\gamma_i\}$ of $H_1(\Sigma)$. Let $a = \mu(\Sigma)$, $b = \mu(x)$, $c_i = \mu(\gamma_i)$. These elements generate $H^*(M_{\Sigma}^{\text{odd}})$ as a ring [9] [21]. So there is a basis for $V = H^*(M_{\Sigma}^{\text{odd}})$ with elements of the form

$$f_{\alpha} = a^n b^m c_{i_1} \cdots c_{i_r} \in V,$$

for a finite set of indices of the form $\alpha = (n, m; i_1, \dots, i_r)$, $n, m \geq 0$, $r \geq 0$, $1 \leq i_1 < \dots < i_r \leq 2g$. Let $N = \dim V$. We order the set of indices $\{\alpha\}$ so we identify such set with $\{1, \dots, N\}$ and write $1 \leq \alpha \leq N$ in general.

Let I be the ideal of $H^*(M_{\Sigma}^{\text{odd}})$ generated by c_1, \dots, c_{2g} . Then the elements $a^n b^m$, $0 \leq n, m < g$, generate the quotient $H^*(M_{\Sigma}^{\text{odd}})/I$ (see [9]). So we can suppose these elements are the first g^2 elements in the basis $\{f_{\alpha}\}$, i.e. for $1 \leq \alpha \leq g^2$.

The intersection pairing in $H^*(M_{\Sigma}^{\text{odd}})$ is given by

$$(5) \quad \langle f_{\alpha}, f_{\beta} \rangle = \langle f_{\alpha} \cup f_{\beta}, [M_{\Sigma}^{\text{odd}}] \rangle .$$

Therefore the intersection matrix $(\langle f_{\alpha}, f_{\beta} \rangle)$ is invertible.

Here we recall that we have defined the manifold $A = \Sigma \times D^2$, with boundary $Y = \Sigma \times S^1$, and let $\Delta = \text{pt} \times D^2 \subset A$ be the horizontal slice with $\partial\Delta = S^1$. Put $w = \text{P.D.}[\Delta] \in H^2(A; \mathbb{Z})$. Put $\mathbb{A}(\Sigma) = \text{Sym}^*(H_0(\Sigma) \oplus H_2(\Sigma)) \otimes \wedge^* H_1(\Sigma)$, so there is a natural map $\mathbb{A}(\Sigma) \rightarrow \mathbb{A}(X)$, whenever $\Sigma \hookrightarrow X$. Then we define

$$\begin{aligned} z_{\alpha} &= \Sigma^n x^m \gamma_{i_1} \cdots \gamma_{i_r} \in \mathbb{A}(\Sigma), \\ e_{\alpha} &= \phi^w(A, z_{\alpha} e^{t\Delta}) \in \text{HFF}_*(Y, S^1) = V[[t]]. \end{aligned}$$

LEMMA 21. *The intersection matrix $(\langle e_{\alpha}, e_{\beta} \rangle)$ (with coefficients in $\mathbb{Q}[[t]]$) is invertible. Therefore, $\{e_{\alpha}\}$ is a basis for $\text{HFF}_*(Y) = V[[t]]$.*

Proof. As the elements $f_{\alpha} \in H^*(M_{\Sigma}^{\text{odd}})$ have an integer degree between 0 and $6g - 6 = \dim M_{\Sigma}^{\text{odd}}$, we can reorder the basis $\{f_{\alpha}\}$ such that the degree goes increasing (we use this special ordering only in this lemma). Now $z_{\alpha} = \Sigma^n x^m \gamma_{i_1} \cdots \gamma_{i_r}$, $z_{\beta} = \Sigma^{n'} x^{m'} \gamma_{i'_1} \cdots \gamma_{i'_r}$, and

$$\langle e_{\alpha}, e_{\beta} \rangle = \langle \phi^w(A, z_{\alpha} e^{t\Delta}), \phi^w(A, z_{\beta} e^{t\Delta}) \rangle = D_{\Sigma \times \mathbb{C}P^1}^{(w, \Sigma)}(z_{\alpha} z_{\beta} e^{t\mathbb{C}P^1}),$$

where $w = \text{P.D.}[\mathbb{C}P^1] \in H^2(\Sigma \times \mathbb{C}P^1; \mathbb{Z})$. The matrix $(\langle e_{\alpha}, e_{\beta} \rangle)$ has coefficients in $\mathbb{Q}[[t]]$, so it is invertible if and only if its determinant is a unit in $\mathbb{Q}[[t]]$, i.e. when we put $t = 0$ we obtain an invertible matrix with rational coefficients. Now

$$\langle e_{\alpha}, e_{\beta} \rangle |_{t=0} = D_{\Sigma \times \mathbb{C}P^1}^{(w, \Sigma)}(z_{\alpha} z_{\beta}).$$

The lowest dimension of the moduli spaces of anti-self-dual connections for $\Sigma \times \mathbb{C}P^1$ is $6g - 6$, so if $\deg z_{\alpha} + \deg z_{\beta} < 6g - 6$ then $\langle e_{\alpha}, e_{\beta} \rangle |_{t=0} = 0$. The moduli space of dimension $6g - 6$ corresponds to $p_1 = 0$ and $w = \text{P.D.}[\mathbb{C}P^1]$. All of these connections are flat and irreducible, actually pull-back of flat connections on Σ with $w|_{\Sigma}$ odd, so

the corresponding moduli space is isomorphic to M_Σ^{odd} . Thus if $\deg z_\alpha + \deg z_\beta = 6g - 6$ then $\langle e_\alpha, e_\beta \rangle|_{t=0} = \langle f_\alpha, f_\beta \rangle$. The matrix $(\langle f_\alpha, f_\beta \rangle)$ is of the form

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & A_0 \\ 0 & 0 & \cdots & A_1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{6g-6} & 0 & \cdots & 0 & 0 \end{pmatrix}$$

where A_i are submatrices corresponding to the intersection product

$$H^i(M_\Sigma^{\text{odd}}) \otimes H^{6g-6-i}(M_\Sigma^{\text{odd}}) \rightarrow \mathbb{Q}.$$

So all A_i have non-vanishing determinant $\det A_i \in \mathbb{Q}$. Finally, we have that the matrix $(\langle e_\alpha, e_\beta \rangle|_{t=0})$ is

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & A_0 \\ 0 & 0 & \cdots & A_1 & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{6g-6} & * & \cdots & * & * \end{pmatrix}$$

and it is invertible. \square

3. Proof of Theorem 9. By the above lemma, $\{e_\alpha\}$ is a basis of $V[[t]]$, so there is an isomorphism

$$\begin{aligned} V[[t]] &\rightarrow \mathbb{Q}^N[[t]] \\ \phi &\mapsto (\langle \phi, e_\alpha \rangle)_\alpha. \end{aligned}$$

The important feature is that if (X_1, Σ_1) is permissible, $w \in H^2(X_1; \mathbb{Z})$ with $w \cdot \Sigma \equiv 1 \pmod{2}$, $D_1^\circ \subset X_1^\circ$ with $\partial D_1^\circ = \mathbb{S}^1$, $D_1 = D_1^\circ + \Delta$, then $\phi = \phi^w(X_1^\circ, e^{tD_1^\circ})$ goes to $(c_{X_1, \alpha}(t))_{1 \leq \alpha \leq N} \in \mathbb{Q}^N[[t]]$, where

$$c_{X_1, \alpha}(t) = D_{X_1}^{(w, \Sigma)}(z_\alpha e^{tD_1}).$$

The pairing in $V[[t]]$ corresponds through the isomorphism to a pairing in $\mathbb{Q}^N[[t]]$, which is $\mathbb{Q}[[t]]$ -bilinear, hence given by a matrix of $\mathbb{Q}[[t]]$ -coefficients $(M_{\alpha\beta}(t))_{1 \leq \alpha, \beta \leq N}$. This matrix is universal (only dependent on the data necessary for the construction of the Fukaya-Floer groups, i.e. (Y, γ) , and on the chosen basis).

Now if (X_1, Σ_1) and (X_2, Σ_2) are permissible, let $X = X(\phi) = X_1 \#_\Sigma X_2$ (with an identification ϕ) be the connected sum along Σ , with the induced homology orientation. Let $D \in H_2(X)$ with $D = D_1^\circ + D_2^\circ$, $\partial D_1^\circ = -\partial D_2^\circ = \mathbb{S}^1$. Put $D_i = D_i^\circ + \Delta$. Then (here $c_{X_1, \alpha}(t) = D_{X_1}^{(w, \Sigma)}(z_\alpha e^{tD_1})$, $c_{X_2, \beta}(t) = D_{X_2}^{(w, \Sigma)}(z_\beta e^{tD_2})$),

$$(6) D_X^{(w, \Sigma)}(e^{tD}) = \langle \phi^w(X_1^\circ, e^{tD_1^\circ}), \phi^w(X_2^\circ, e^{tD_2^\circ}) \rangle = \sum_{1 \leq \alpha, \beta \leq N} c_{X_1, \alpha}(t) M_{\alpha\beta}(t) c_{X_2, \beta}(t)$$

Now suppose that X_1 has $b_1 = 0$. Then $c_{X_1, \alpha}(t) = 0$ whenever $r > 0$ (recall $\alpha = (n, m; i_1, \dots, i_r)$). So the only non-zero coordinates correspond to $z_\alpha = \Sigma^n x^m$, $0 \leq n, m \leq g - 1$. Suppose furthermore X_1 of simple type with $b_1 = 0$ and $b^+ > 1$, so for $z = (x^2 - 4)\Sigma^n x^{m-2}$, $0 \leq n \leq g - 1$, $2 \leq m \leq g - 1$, $D_{X_1}^{(w, \Sigma)}(ze^{tD_1}) = 0$, hence

changing the basis $\{z_\alpha\}$, all coordinates $c_{X_1,\alpha}(t)$ are zero except for the first $2g$ of them, corresponding to $z_\alpha = \Sigma^n$ and $z_\alpha = \Sigma^n x$, $0 \leq n \leq g - 1$.

LEMMA 22. *Let (X_1, Σ_1) be permissible, with Σ_1 of genus g , X_1 of simple type with $b_1 = 0$ and $b^+ > 1$, (w, Σ) allowable and $D_1 \in H_2(X_1)$ with $D_1 \cdot \Sigma = 1$. Then $D_{X_1}^{(w,\Sigma)}(e^{tD_1} z) = 0$ for*

$$z = \begin{cases} (1 - \frac{x}{2})(\Sigma + t)((\Sigma + t)^2 + 4^2)((\Sigma + t)^2 + 8^2) \cdots ((\Sigma + t)^2 + (2g - 4)^2) & g \text{ even} \\ (1 + \frac{x}{2})(\Sigma - t)^2 - 2^2)((\Sigma - t)^2 - 6^2) \cdots ((\Sigma - t)^2 - (2g - 4)^2) & g \text{ odd} \end{cases}$$

Proof. Suppose, for instance, g even. By remark 4, for any polynomial $p(\Sigma)$ in Σ (with coefficients in $\mathbb{Q}[[t]]$),

$$D_{X_1}^{(w,\Sigma)}((1 - \frac{x}{2})p(\Sigma)e^{tD_1}) = 2e^{-Q(tD_1)/2} \sum_{K_j \cdot \Sigma \equiv 0 \pmod{4}} i^{-d_0} p((-tD_1 + iK_j) \cdot \Sigma) a_{j,w} e^{itK_j \cdot D_1}.$$

Now $D_1 \cdot \Sigma = 1$, so the expression above vanishes when $p(\Sigma)$ has roots $-t, -t \pm 4i, \dots -t \pm (2g - 4)i$. \square

Note that $z = (1 \pm \frac{x}{2})p(\Sigma)$ with p of degree $g - 1$. Let us choose a basis $\{z_\alpha\}$ with z_1, \dots, z_{2g-1} being the elements

$$1, \Sigma, \Sigma^2, \dots, \Sigma^{g-1}, x, \Sigma x, \dots, \Sigma^{g-2} x,$$

$z_{2g} = z$, also z_{2g+1}, \dots, z_{g^2} being the elements $(x^2 - 4)\Sigma^n x^{m-2}$, $0 \leq n \leq g - 1$, $2 \leq m \leq g - 1$ and e_{g^2+1}, \dots, e_N having all $r > 0$. So, when X_1 is of simple type with $b_1 = 0$ and $b^+ > 1$,

$$\phi^w(X_1^o, e^{tD_1^o}) \in \mathbb{Q}^{2g-1}[[t]] \subset V[[t]],$$

where $\mathbb{Q}^{2g-1}[[t]]$ is the orthogonal complement to $\langle e_{2g}, \dots, e_N \rangle$ in $V[[t]]$.

Formula (6) reduces to (when both X_i are of simple type with $b_1 = 0$ and $b^+ > 1$)

$$(7) \quad D_X^{(w,\Sigma)}(e^{tD}) = \sum_{1 \leq \alpha, \beta \leq 2g-1} c_{X_1,\alpha}(t) M_{\alpha\beta}(t) c_{X_2,\beta}(t),$$

where

$$c_{X_1,\alpha}(t) = \begin{cases} D_{X_1}^{(w,\Sigma)}(\Sigma^n e^{tD_1}) & \text{if } 0 \leq n \leq g - 1, \alpha = (n, 0;) = n + 1 \\ D_{X_1}^{(w,\Sigma)}(\Sigma^n x e^{tD_1}) & \text{if } 0 \leq n \leq g - 2, \alpha = (n, 1;) = n + g + 1 \end{cases}$$

$$= \begin{cases} e^{Q(tD_1)/2} \sum_{K_j \cdot \Sigma \equiv 2 \pmod{4}} a_{j,w} (1 + K_j \cdot \Sigma)^n e^{tK_j \cdot D_1} + \\ \quad + e^{-Q(tD_1)/2} \sum_{K_j \cdot \Sigma \equiv 0 \pmod{4}} i^{-d_0} a_{j,w} (-1 + iK_j \cdot \Sigma)^n e^{tiK_j \cdot D_1} \\ 2e^{Q(tD_1)/2} \sum_{K_j \cdot \Sigma \equiv 2 \pmod{4}} a_{j,w} (1 + K_j \cdot \Sigma)^n e^{tK_j \cdot D_1} - \\ \quad - 2e^{-Q(tD_1)/2} \sum_{K_j \cdot \Sigma \equiv 0 \pmod{4}} i^{-d_0} a_{j,w} (-1 + iK_j \cdot \Sigma)^n e^{tiK_j \cdot D_1} \end{cases}$$

and analogously for $c_{X_2,\beta}(t)$.

So it is easy to find another basis $\{z_1, \dots, z_{2g-1}\}$ (which we do not write explicitly) spanning \mathbb{Q}^{2g-1} such that

$$c_{X_1, \alpha}(t) = \begin{cases} e^{Q(tD_1)/2} \sum_{K_j \cdot \Sigma = 2p} a_{j,w} e^{tK_j \cdot D_1} & \text{if } p \text{ is odd} \\ e^{-Q(tD_1)/2} \sum_{K_j \cdot \Sigma = 2p} i^{-d_0} a_{j,w} e^{ti K_j \cdot D_1} & \text{if } p \text{ is even} \end{cases}$$

where $\alpha = 1, \dots, 2g - 1$ corresponds to $p = g - 1, -(g - 1), g - 2, -(g - 2), \dots, 0$. Formula (7) yields

$$(8) \quad D_X^{(w, \Sigma)}(e^{tD}) = \sum_{1 \leq \alpha, \beta \leq 2g-1} c_{X_1, \alpha}(t) M_{\alpha\beta}(t) c_{X_2, \beta}(t)$$

with $c_{X_2, \beta}(t)$ defined analogously to $c_{X_1, \alpha}(t)$, with the letter q in the place of p .

LEMMA 23. *The matrix $(M_{\alpha\beta}(t))$ is diagonal.*

Proof. First we note that

$$D_X^{(w, \Sigma)}(e^{tD+s\Sigma}) = \sum_{p \text{ odd}} c_{X, \alpha}(t) e^{ts+2ps} + \sum_{p \text{ even}} c_{X, \alpha}(t) e^{-ts+2pis}.$$

We use equation (8) for $D + \frac{s}{t}\Sigma$, $D_1 + \frac{s}{t}\Sigma$ and D_2 , so

$$D_X^{(w, \Sigma)}(e^{tD+s\Sigma}) = \sum_{\substack{1 \leq \alpha, \beta \leq 2g-1 \\ p \text{ odd}}} c_{X_1, \alpha}(t) M_{\alpha\beta}(t) c_{X_2, \beta}(t) e^{ts+2ps} + \sum_{\substack{1 \leq \alpha, \beta \leq 2g-1 \\ p \text{ odd}}} c_{X_1, \alpha}(t) M_{\alpha\beta}(t) c_{X_2, \beta}(t) e^{-ts+2pis}.$$

Then

$$(9) \quad c_{X, \alpha}(t) = \sum_{\beta} c_{X_1, \alpha}(t) M_{\alpha\beta}(t) c_{X_2, \beta}(t).$$

Let us see that $M_{\alpha\beta}(t) = 0$ unless $\beta = \alpha$. Suppose, for instance, that p is odd. If we write now $D = (D_1^o + r\Sigma) + (D_2^o - r\Sigma)$, the left hand side of (9) remains unchanged, but the right hand side is a sum of exponentials $e^{(2p-2q)r}$, q odd, and $e^{(2p-2qi)r}$, q even. So when $q \neq p$, it is $M_{\alpha\beta}(t) = 0$. \square

Formula (8) gives

$$D_X^{(w, \Sigma)}(e^{tD}) = \sum_{K_j \cdot \Sigma = L_k \cdot \Sigma = 2p, p \text{ odd}} e^{Q(tD_1)/2 + Q(tD_2)/2} M_{\alpha\alpha}(t) a_{j,w} b_{k,w} e^{tK_j \cdot D_1 + tL_k \cdot D_2} + \sum_{K_j \cdot \Sigma = L_k \cdot \Sigma = 2p, p \text{ even}} \left\{ e^{-Q(tD_1)/2 - Q(tD_2)/2} M_{\alpha\alpha}(t) i^{-d_0(X_1, w_1) - d_0(X_2, w_2)} a_{j,w} b_{k,w} e^{ti K_j \cdot D_1 + ti L_k \cdot D_2} \right\}.$$

Obviously $D^2 = D_1^2 + D_2^2$. We are assuming that X is of simple type and recall that $b_1(X) = 0$ and $b^+(X) > 1$. By remark 11, $d_0(X, w) - d_0(X_1, w_1) - d_0(X_2, w_2) = w^2 - w_1^2 - w_2^2 - 3(g - 1) \equiv g - 1 \pmod{4}$ (since we are assuming $w^2 \equiv w_1^2 + w_2^2 \pmod{4}$), so

$$\mathbb{D}_X^w(e^{tD}) = e^{Q(tD)/2} \sum_{\substack{K_j \cdot \Sigma = L_k \cdot \Sigma = 2p \\ -(g-1) \leq p \leq g-1}} i^s M_{\alpha\alpha}(t) a_{j,w} b_{k,w} e^{tK_j \cdot D_1 + tL_k \cdot D_2}$$

where $s = 0$ when p is odd and $s = g - 1$ when p is even. As g is not dependent on the particular manifolds, we absorb this factor into the matrix without affecting its universality.

This expression is valid for any $D \in H_2(X)$ such that $D|_Y = \mathbb{S}^1$. We note that it does not change if we change (D_1, D_2) for $(D_1 + r\Sigma, D_2 - r\Sigma)$, as expected. This means that we only need to assume the conditions: D and D_i coincide in X_i^o , $i = 1, 2$, and $D^2 = D_1^2 + D_2^2$. Now take a linear map $D \mapsto (D_1, D_2)$ from the subspace $\mathcal{H} = \{D \in H_2(X)/D|_Y = k[\mathbb{S}^1], \text{ some } k\}$, satisfying the former conditions. As the set of $D \in H_2(X)$ with $D|_Y$ a non-zero multiple of \mathbb{S}^1 is dense in \mathcal{H} , we have that for any $D \in \mathcal{H}$,

$$(10) \quad \mathbb{D}_X^w(e^{tD}) = e^{Q(tD)/2} \sum_{K_j \cdot \Sigma = L_k \cdot \Sigma = 2p} M_{\alpha\alpha}(t(D \cdot \Sigma)) a_{j,w} b_{k,w} e^{tK_j \cdot D_1 + tL_k \cdot D_2}$$

LEMMA 24. $M_{\alpha\alpha}(t) = 0$, except for $\alpha = 1, 2$.

Proof. Let Y be the $K3$ surface, which is a manifold of simple type with $b_1 = 0$ and $b^+ > 1$ (see [12]). Consider a tight embedded Riemann surface $\Sigma' \hookrightarrow Y$ of genus $g' < g$ and self-intersection zero. By definition of tightness (see [12]), $(\Sigma')^2 = 2g' - 2$. To construct it, we consider an elliptic fibration for the $K3$ surface. Let T be a generic fibre (which is a torus of self-intersection zero) and let S be a section represented by a sphere of self-intersection -2 . Then consider S together with g' generic fibres and smooth out the intersection points. Call the resulting Riemann surface Σ' . It is homologous to $S + g'T$, it has genus g' and self-intersection $2g' - 2$, as required.

We blow-up Y at $2g' - 2$ points in Σ' , to get $X_1 = Y \# (2g' - 2)\overline{\mathbb{C}\mathbb{P}^2}$. The proper transform of Σ' is a Riemann surface of genus g' and self-intersection zero. Perform an internal connected sum with $g - g'$ homologically trivial tori to obtain a Riemann surface Σ_1 of genus g and self-intersection zero. If $E_1, \dots, E_{2g'-2}$ are the exceptional divisors, Σ_1 is homologous to $\Sigma' + E_1 + \dots + E_{2g'-2}$.

Moreover, the basic classes of X_1 are $\pm E_1 + \dots \pm E_{2g'-2}$. They all satisfy $\kappa \cdot \Sigma_1 \leq 2g' - 2$ and there is exactly one, $K = E_1 + \dots + E_{2g'-2}$, satisfying the equality.

Let $(X_2, \Sigma_2) = (X_1, \Sigma_1)$, and consider $X = X_1 \#_{\Sigma} X_2$ with the preferred identification of remark 7 (double of X_1 along Σ_1). Then X splits off a $\mathbb{S}^2 \times \mathbb{S}^2$, so its invariants are zero (see [16, section 4.3] for this well-known phenomenon). As in the proof of lemma 23, $c_{X,\alpha}(t) = c_{X_1,\alpha}(t)M_{\alpha\alpha}(t)c_{X_2,\alpha}(t)$. We proceed by induction from $g' = 1, 2, \dots, g - 1$ and get $M_{\alpha\alpha}(t) = 0$, for $\alpha \geq 3$. \square

Formula (10) becomes

$$(11) \quad \begin{aligned} \mathbb{D}_X^w(e^{tD}) = & e^{Q(tD)/2} \sum_{K_j \cdot \Sigma = L_k \cdot \Sigma = 2g-2} M_{11}(t(D \cdot \Sigma)) a_{j,w} b_{k,w} e^{tK_j \cdot D_1 + tL_k \cdot D_2} + \\ & + e^{Q(tD)/2} \sum_{K_j \cdot \Sigma = L_k \cdot \Sigma = -(2g-2)} M_{22}(t(D \cdot \Sigma)) a_{j,w} b_{j,w} e^{tK_j \cdot D_1 + tL_k \cdot D_2} \end{aligned}$$

Now let us compute $M_{11}(t)$ and $M_{22}(t)$. We can do that with a particular example of connected sum along a surface of genus g in which the invariants are known for X_1, X_2 and $X = X_1 \#_{\Sigma} X_2$.

DEFINITION 25. We are going to define the following smooth manifolds:

- Let S_n be the minimal elliptic surface with no multiple fibres and geometric genus $p_g = n - 1$. This is unique up to diffeomorphism [8]. S_1 is the rational elliptic surface, i.e. $S_1 = \mathbb{C}\mathbb{P}^2 \# 9\overline{\mathbb{C}\mathbb{P}^2}$.

- Let \tilde{S}_1 be the blow-up of S_1 at one point. Therefore, $\tilde{S}_1 = \mathbb{C}P^2 \# 10\overline{\mathbb{C}P^2}$. Consider a particular elliptic fibration for S_1 with section σ (of self-intersection -1) and fibre F . Call E the exceptional divisor, so \tilde{S}_1 has an elliptic fibration with fibre the torus $T_1 = F$ and there is another embedded torus T_2 homologous to $\sigma + T_1 - E$ with $T_2^2 = 0$ and $T_1 \cdot T_2 = 1$.
- Let $B_g = \underbrace{\tilde{S}_1 \#_{T_1} \tilde{S}_1 \#_{T_1} \cdots \#_{T_1} \tilde{S}_1}_g$ (connected sums along T_1 's with the preferred identification), which is diffeomorphic to $S_g \#_g \overline{\mathbb{C}P^2}$. It contains an embedded torus T_1 of self-intersection zero and a Riemann surface of genus g (and self-intersection zero) made up by gluing smoothly the T_2 's coming from each \tilde{S}_1 . Actually, the elliptic surface S_g has a section σ_g with $\sigma_g^2 = -g$, and Σ_g can be taken to be the proper transform of σ_g . Clearly, $\Sigma_g \cdot T_1 = 1$, so (B_g, Σ_g) is permissible. B_g is of simple type with $b_1 = 0$, $b^+ > 1$ (as $g \geq 2$).

Let $C_g = B_g \#_{\Sigma_g} B_g$ with the preferred identification. It contains a Riemann surface $\hat{\Sigma}_2$ of genus 2 and self-intersection zero made up from gluing smoothly the T_1 's. If we perform instead the connected sum of two \tilde{S}_1 along T_2 , we get $\hat{B}_2 = \tilde{S}_1 \#_{T_2} \tilde{S}_1$ with an embedded Riemann surface $\hat{\Sigma}_2$ of genus 2 and self-intersection zero, coming from smoothly gluing the T_1 's. Clearly $(\hat{B}_2, \hat{\Sigma}_2) \cong (B_2, \Sigma_2)$. Now

$$C_g = B_g \#_{\Sigma_g} B_g = \underbrace{\hat{B}_2 \#_{\hat{\Sigma}_2} \cdots \#_{\hat{\Sigma}_2} \hat{B}_2}_g.$$

By [18, theorem 10], C_g is of simple type with $b_1 = 0$ and $b^+ > 1$. Alternatively, we can use [12], since it contains a torus of self-intersection 0 intersecting an embedded (-2) -sphere transversely in one point (see proposition 27).

PROPOSITION 26. Consider (B_g, Σ_g) . Let K_{B_g} be the canonical class of B_g , and $w = \text{P.D.}[T_1] \in H^2(B_g; \mathbb{Z})$. Then $\mathbb{D}_{B_g}^w(e^\bullet) = \frac{1}{2^{2g-2}} e^{Q/2} e^{K_{B_g}} + \frac{1}{2^{2g-2}} e^{Q/2} e^{-K_{B_g}} + \dots$, where the dots correspond to basic classes κ for B_g with $|\kappa \cdot \Sigma_g| < 2g - 2$.

Proof. Write B_g as $S_g \#_g \overline{\mathbb{C}P^2}$. Let F be the fibre of the natural elliptic fibration (i.e. $F = T_1$). Let E_1, \dots, E_g be the exceptional divisors. Then the basic classes are $kF \pm E_1 \pm E_2 \cdots \pm E_g$, with $-(g - 2) \leq k \leq (g - 2)$ and $k \equiv g - 2 \pmod{2}$ (see [12] [6]). So the only basic class κ with $\kappa \cdot \Sigma_g = 2g - 2$ is the canonical class $K_{B_g} = (g - 2)F + E_1 + E_2 + \cdots + E_g$. Therefore [12] [6] [17]

$$\begin{aligned} \mathbb{D}_{B_g}(e^\bullet) &= e^{Q/2} (\sinh F)^{g-2} \sinh E_1 \cdots \sinh E_g \\ &= \frac{1}{2^{2g-2}} e^{Q/2} e^{K_{B_g}} + \frac{1}{2^{2g-2}} e^{Q/2} e^{-K_{B_g}} + \dots \end{aligned}$$

Now we note from proposition 1 that for $w = T_1$, $\mathbb{D}_{B_g}^w = \mathbb{D}_{B_g}$, since $w \cdot \kappa = 0$ for all basic classes κ , hence the result. \square

PROPOSITION 27. C_g is of simple type. For $w = \text{P.D.}[\hat{\Sigma}_2] \in H^2(C_g; \mathbb{Z})$,

$$\mathbb{D}_{C_g}^w(e^\alpha) = -2^{3g-5} e^{Q(\alpha)/2} e^{K_{C_g} \cdot \alpha} + (-1)^g 2^{3g-5} e^{Q(\alpha)/2} e^{-K_{C_g} \cdot \alpha},$$

for a unique basic class $K = K_{C_g} \in H^2(C_g; \mathbb{Z})$, such that $K \cdot \hat{\Sigma}_2 = 2$, $K \cdot \Sigma_g = 2g - 2$.

Proof. By proposition 26 and proposition 3,

$$D_{\hat{B}_2}^{(T_2, \hat{\Sigma}_2)}(e^\alpha) = \frac{1}{4} e^{Q(\alpha)/2} e^{K_{\hat{B}_2} \cdot \alpha} + \frac{1}{4} e^{Q(\alpha)/2} e^{-K_{\hat{B}_2} \cdot \alpha} + \dots$$

where the dots correspond to basic classes κ with $\kappa \cdot \hat{\Sigma}_2 = 0$. Now we express $C_g = \underbrace{\hat{B}_2 \#_{\hat{\Sigma}_2} \cdots \#_{\hat{\Sigma}_2} \hat{B}_2}_g$ and use [18, theorem 6],

$$\begin{aligned} D_{C_g}^{(\Sigma_g, \hat{\Sigma}_2)}(e^\alpha) &= (-32)^{g-1} \left(\frac{1}{4}\right)^g e^{Q(\alpha)/2} e^{K_{C_g} \cdot \alpha} + 32^{g-1} \left(\frac{1}{4}\right)^g e^{Q(\alpha)/2} e^{K_{C_g} \cdot \alpha} = \\ &= (-1)^{g-1} 2^{3g-5} e^{Q(\alpha)/2} e^{K_{C_g} \cdot \alpha} + 2^{3g-5} e^{Q(\alpha)/2} e^{K_{C_g} \cdot \alpha}, \end{aligned}$$

where $K = K_{C_g} \in H^2(C_g; \mathbb{Z})$ is defined as the unique cohomology class such that

- $K \cdot \hat{\Sigma}_2 = 2$
- $K \cdot \Sigma_g = 2g - 2$
- Writing $C_g = C_{g-1}^o \cup_Y \hat{B}_2^o$, one has $K \cdot (\alpha_1 + \alpha_2) = K_{C_{g-1}} \cdot \alpha_1 + K_{\hat{B}_2} \cdot \alpha_2$, where $\alpha_1 \in H_2(C_{g-1}^o)$ and $\alpha_2 \in H_2(\hat{B}_2^o)$.
- Write $\hat{B}_2^o = (\tilde{S}_1 - (N_{T_1} \cup N_{T_2})) \cup_{Y - N_{T_1} \cap \partial N_{T_2}} (\tilde{S}_1 - (N_{T_1} \cup N_{T_2}))$. For any $\gamma \subset \Sigma \subset \Sigma \times \mathbb{S}^1 = \partial N_{T_1}$ disjoint from N_{T_2} , we consider the vanishing discs for \tilde{S}_1 with elliptic fibration with fibre T_1 (see [8, page 167]). These can be considered disjoint from N_{T_2} , so they are embedded (-1) -discs in \hat{B}_2^o with boundary included in $\partial \hat{B}_2^o$. In the same vein we find many embedded (-1) -discs in C_{g-1}^o , so, in the terminology of [14], the preferred identification for $C_g = C_{g-1}^o \cup_Y \hat{B}_2^o$ is admissible. By remark 14, in the first place C_g is of simple type. In the second place, there is a subspace $V \subset H_2(X)$ such that $H_2(X) = \mathcal{H} \oplus V$ and $K \cdot \alpha = 0$, for all $\alpha \in V$.

For $w = \text{P.D.}[\hat{\Sigma}_2]$, using proposition 3 again,

$$\mathbb{D}_{C_g}^{\hat{\Sigma}_2}(e^\alpha) = -2^{3g-5} e^{Q(\alpha)/2} e^{K_{C_g} \cdot \alpha} + (-1)^g 2^{3g-5} e^{Q(\alpha)/2} e^{K_{C_g} \cdot \alpha} \quad \square$$

Finally, we can find $M_{11}(t)$ and $M_{22}(t)$. Let $(X_1, \Sigma_1) = (X_2, \Sigma_2) = (B_g, \Sigma_g)$, $w_1 = w_2 = \text{P.D.}[T_2]$, $D_1 = D_2 = T_2$ and $X = C_g$, $w = \text{P.D.}[\hat{\Sigma}_2]$, $D = \hat{\Sigma}_2$, $\Sigma = \Sigma_g$, so formula (11) and proposition 26 yield

$$\mathbb{D}_X^w(e^{tD+s\Sigma}) = e^{ts} (M_{11}(t) \frac{1}{2^{2g-2}} \frac{1}{2^{2g-2}} e^{(2g-2)s} + M_{22}(t) \frac{1}{2^{2g-2}} \frac{1}{2^{2g-2}} e^{-(2g-2)s})$$

and use proposition 27 to get

$$\mathbb{D}_X^w(e^{tD+s\Sigma}) = e^{ts} (-2^{3g-5} e^{2t+(2g-2)s} + (-1)^g 2^{3g-5} e^{-2t-(2g-2)s}),$$

from where $M_{11}(t) = -2^{7g-9} e^{2t}$, $M_{22}(t) = (-1)^g 2^{7g-9} e^{-2t}$. This finishes the proof of theorem 9.

4. Proof of Theorem 17. Let $S = \Sigma \times \mathbb{C}P^1$, $w = \text{P.D.}[\mathbb{C}P^1] \in H^2(S; \mathbb{Z})$. Suppose that S is of finite type of order $n \geq 1$ with respect to w and $w + \Sigma$ for the metrics defined by Σ . Then

$$0 = D_S^{(w, \Sigma)}((x^2 - 4)^n z_\alpha e^{t\mathbb{C}P^1}) = \langle \phi^w(A, (x^2 - 4)^n e^{t\Delta}), e_\alpha \rangle$$

for all α . From lemma 21, $\phi^w(A, (x^2 - 4)^n e^{t\Delta}) = 0$. Then consider (X, Σ) permissible, $w \in H^2(X; \mathbb{Z})$ with $w \cdot \Sigma \equiv 1 \pmod{2}$, $X = X^o \cup_Y A$. For any $D \in H_2(X)$ with $D \cdot \Sigma = 1$, we can write $D = D^o + \Delta$, so

$$D_X^{(w, \Sigma)}((x^2 - 4)^n e^{tD}) = \langle \phi^w(X^o, e^{tD^o}), \phi^w(A, (x^2 - 4)^n e^{t\Delta}) \rangle = 0.$$

We conclude that $D_X^{(w, \Sigma)}((x^2 - 4)^n e^{tD}) = 0$ for all $D \in H_2(X)$, i.e. X is of finite type of order at most n with respect to w .

5. Conjecture. Following our results in [18] for the case of genus $g = 2$, we propose the following conjecture.

For any (X, Σ) permissible, define $\tilde{X} = X \#_{\Sigma} B_g$ (we need to fix one identification arbitrarily). Then $b^+(\tilde{X}) > 1$ and also $b_1(\tilde{X}) = 0$ whenever $b_1(X) = 0$. For any cycle $D^o \subset X^o$ with $\partial D^o = \gamma \subset Y$ an embedded curve (when X^o is given a cylindrical end, $D^o \cap (Y \times [0, \infty)) = \gamma \times [0, \infty)$), we choose cappings $D = D^o + D_{B_g}^o$, $\partial D_{B_g}^o = -\gamma$. The cappings have to satisfy the following condition. In $B_g = S_g \# \overline{\mathbb{C}P^2}$ (see definition 25), we fix an embedded surface S representing σ_g , intersecting Σ_g transversely in g points. Put $S^o = S \cap B^o$, then we impose that $D_{B_g}^o \cdot S^o = 0$ (the pairing makes sense as long as γ and ∂S^o are disjoint)

CONJECTURE 28. Let (X_1, Σ_1) and (X_2, Σ_2) be permissible with X_i having $b_1 = 0$ (we do not suppose that they are of simple type). Consider \tilde{X}_i . Then \tilde{X}_i are of simple type. Put $\mathbb{D}_{\tilde{X}_1}^w(e^\alpha) = e^{Q(\alpha)/2} \sum \tilde{a}_{j,w} e^{\tilde{K}_j \cdot \alpha}$ and $\mathbb{D}_{\tilde{X}_2}^w(e^\alpha) = e^{Q(\alpha)/2} \sum \tilde{b}_{k,w} e^{\tilde{L}_k \cdot \alpha}$. Choose any identification ϕ and let $X = X(\phi) = X_1 \#_{\Sigma} X_2$ be the connected sum along Σ , with the induced homology orientation. Then X is of simple type. Choose $w \in H^2(X; \mathbb{Z})$, $w_i \in H^2(\tilde{X}_i; \mathbb{Z})$, $i = 1, 2$ in a compatible way such that $w \cdot \Sigma \equiv 1 \pmod{2}$, $w_i \cdot \Sigma \equiv 1 \pmod{2}$ and $w_i|_{B_g} = kP.D.[T_1]|_{B_g}$. For every $D \in H_2(X)$, we write $D = D_1^o + D_2^o$, $D_i^o \subset X_i^o$, with $\partial D_1^o = -\partial D_2^o = \gamma$ disjoint with ∂S^o and consider cappings $D_i \in H_2(\tilde{X}_i)$ of D_i^o as above, in such a way that the map $D \mapsto (D_1, D_2)$ is linear. Then

$$\begin{aligned} \mathbb{D}_X^w(e^{tD}) &= \epsilon e^{Q(tD)/2} \left(\sum_{\tilde{K}_j \cdot \Sigma = \tilde{L}_k \cdot \Sigma = 2g-2} -2^{-3g+5} \tilde{a}_{j,w} \tilde{b}_{k,w} e^{(\tilde{K}_j \cdot D_1 + \tilde{L}_k \cdot D_2)t} + \right. \\ &\quad \left. + \sum_{\tilde{K}_j \cdot \Sigma = \tilde{L}_k \cdot \Sigma = -(2g-2)} (-1)^g 2^{-3g+5} \tilde{a}_{j,w} \tilde{b}_{k,w} e^{(\tilde{K}_j \cdot D_1 + \tilde{L}_k \cdot D_2)t} \right), \end{aligned}$$

for $\epsilon = (-1)^{(g-1)(w^2 - w_1^2 - w_2^2)/2}$.

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