ASIAN J. MATH.
Vol. 1, No. 4, pp. 696–715, December 1997

003

MORSE THEORY FOR MIN-TYPE FUNCTIONS*

V. GERSHKOVICH† AND H. RUBINSTEIN†

At this time the Cheshire Cat vanished quite slowly,
beginning with the end of the tail,
ending with a grin, which remained
some time after the rest of it had gone, [Ca].

0. Introduction. Morse theory for distance functions was initiated by Grove-Shiohama [GS] and Gromov's [G1] paper where basic notions of the theory were formulated; even this very initial level of the theory leads to important geometric applications. Grove and Shiohama [GS] established a generalized sphere theorem by constructing a vector field on a Riemannian manifold, with the property that the distance function had no stationary points along the integral curves at non-singular points of the vector field. They showed that this vector field had exactly two singular points and hence the manifold was homeomorphic to a sphere. Later Gromov [G1] was able to bound the sum of the Betti numbers of a positively curved Riemannian manifold. He controlled the location of critical points of a Riemannian distance function on a positively curved manifold using Toponogov's theorem and then was able to bound the number of critical points of this function and hence the homology of the manifold, using a spectral sequence argument.

Morse theory for Riemannian distance functions was discussed in [Gr], [L] and other papers, which explained its importance for geometric applications rather than developed the theory itself. Even a suitable concept of the index of a critical point has not been developed. As a result, the most powerful tools of the classical Morse theory such as Morse inequalities and the correspondence between critical points and the handle decomposition of the manifold cannot be used. The relationship with the classical Morse theory has not been investigated either; in particular, the connection between notions of the critical points in both theories is not obvious.

In a different direction, the structure of Alexandrov spaces with curvatures bounded below was investigated using distance functions in several papers starting with [BGP] and continuing with [P1], [P2]. A key result obtained is a canonical stratification of such Alexandrov spaces into topological manifolds and again the technique is a type of Morse theory, using the distance function. As Alexandrov spaces do not have as much structure as Riemannian manifolds, our theory gives more detailed information on the nature of critical points and index.

M. Gromov pointed out in [G1] that the Morse theory for Riemannian distance functions can be developed by analogy with the classical Morse theory. The aim of this paper is to construct a Morse theory for functions which are minima of finite families of smooth functions and clarify the connection with Riemannian distance functions for non-positively curved manifolds. We develop the theory in the classical style, including the notion of the index for critical points, and clarify relations with the Grove-Shiohama-Gromov approach.

*Received April 9, 1997; accepted for publication (in revised form) January 6, 1998.

†Department of Mathematics, The University of Melbourne, Parkville, Victoria 3052 AUSTRALIA (rubin@maths.mu.oz.au).

Manifolds of non-positive curvature are the simplest class for such an investigation. A distance function d_x is smooth and has no critical points (but x) on a simply-connected non-positively curved manifold. All the singularities of distance functions for a non-simply-connected manifold M arise from the universal covering $p : \tilde{M} \to M$. Let ρ be a Riemannian metric on M and $\tilde{\rho}$ be the corresponding Riemannian metric on $\tilde{M}$. Fix a preimage $\tilde{x} \in p^{-1}(x)$ for $x \in M$. A point $y \in M$ may have several preimages $\tilde{y}_1, ..., \tilde{y}_m$ such that $\tilde{\rho}(\tilde{x}, \tilde{y}_i) = \rho(x, y)$, for $i = 1, ..., m$. Then we have $d_y(z) = \min_{i=1,...,m} \tilde{d}_{y_i}(z)$ in a neighborhood of x. When we consider a germ of a Riemannian distance function on a non-positively curved manifold, only a finite number of preimages are really involved in (1). This leads to an investigation of the space of Min-type functions f which are minima of a finite number of smooth functions $f = \min\{\alpha_1, ..., \alpha_m\}$. The topology of the space of germs of Min-type functions is described by the condition that small perturbations of germs of f are results of independent small perturbations of germs of α_i.

A similar class of functions which are minima of continuous families of smooth functions was investigated by Arnold's school in a different context, see Bryzgalova [B1], [B2], Matov [M1], [M2], Arnol'd [A]. Morse theory for Minimum-type functions, constructed here, is in a more "polyhedral" style.

The main idea of this paper is: when we extend the class of function outside C^2-functions, the Hessian disappears but its index (suitably defined) which is the only thing necessary to construct Morse theory, still remains, compare the epigraph.

Morse theory for Min-type functions was used to construct Morse theory for distance functions on negatively curved manifolds and for Riemannian metrics on negatively curved surfaces [Ge]. It appears that this theory can be extended to arbitrary Riemannian manifolds [GR].

Here is the plan of the paper.

Section I. We demonstrate that each germ of a distance function for a non-positively curved manifold is a germ of a Min-type function. We define global Min-type functions on compact manifolds, consider minimal representations of germs of Min-type functions and prove that such a represention is, in fact, unique; we discuss a local geometry connected with minimal representations. We define the C^k-topology in the space of germs of Min-type functions.

Section II. We define non-degenerate regular points of Min-type functions and special non-degenerate critical points. For special regular points, gradients of smooth functions in the minimal representations are linearly dependent, and this property is stable. We demonstrate that a Min-type function on a compact manifold can have only a finite number of such points. We show that germs of Min-type functions at non-degenerate regular points have the same normal form as in the smooth case when we extend the transformation group from the group of local diffeomorphisms to the group of almost smooth local homeomorphisms. Finally, we clarify relations between the concept of regular points in the classical Morse theory and Gromov's definition.

Section III. We define non-degenerate critical points for Min-type functions, obtain normal forms for germs of Min-type functions at such points, and define the index of Min-type functions at non-degenerate critical points. We obtain a theorem on approximation of a Morse Min-type function by a smooth Morse function with the same number of critical points which has the same vector of indices. This extends the principal results of the classical Morse theory, including Morse inequalities, decomposition of the manifold in a union of handles corresponding to non-degenerate critical points, to the class of Morse Min-type functions.

Section IV. We define Morse Min-type functions and Morse distance functions. We show that the set of Morse Min-type functions is open and everywhere dense in the set of Min-type functions on M in the C^2-topology.

1. Distance functions and Min-type functions. The aim of this Chapter is to explain the connection between Riemannian distance functions on non-positively curved manifolds and Min-type functions, which are minima of finite families of smooth functions.

1.1. Distance functions as Min-type functions. Let (M_n, ρ) be a Riemannian manifold, $x \in M_n$. Denote by d_x the Riemannian distance function, $d_x(y) = \rho(x,y)$. The aim of this section is to prove the following theorem.

THEOREM 1. *Let (M, ρ) be a compact manifold of non-positive curvature, $x, y \in M$. Then there exists a finite number of smooth functions $\alpha_1, \ldots, \alpha_m$ such that $d_x = \min_{i=1,\ldots,m} \alpha_i$ around y.*

The theorem follows from several statements below, which give also some additional information. Let (M, ρ) be a compact non-positively curved Riemannian manifold and $\tilde{M}$ be the universal covering of M. There exists a unique Riemannian metric $\tilde{\rho}$ on $\tilde{M}$ such that the projection $p : \tilde{M} \to M$ is a local isometry. Let γ be a geodesic on M starting at x and let $\tilde{x} \in \tilde{M}$ with $p(\tilde{x}) = x$; then there exists a unique geodesic $\tilde{\gamma}$ on $\tilde{M}$, starting at $\tilde{x}$ and such that $p(\tilde{\gamma}) = \gamma$. The universal covering preserves lengths of geodesics: *length* $\tilde{\gamma}$ = *length* γ where $\gamma = p(\tilde{\gamma})$.

PROPOSITION 1. *Let $p : (\tilde{M}, \tilde{\rho}) \to (M, \rho)$ be a universal covering, $x, y \in M$. Let $\tilde{x} \in p^{-1}(x)$ and $p^{-1}(y) = \{\tilde{y}_1, ..., \tilde{y}_m, ...\}$. Then*

$$d_y\ (x) = \rho(x,y) = \min_{i=1,\ldots,m,\ldots} \tilde{\rho}\ (\tilde{x}, \tilde{y}_i) = \min_{i=1,\ldots,m,\ldots} \tilde{d}_{\tilde{y}_i}\ (\tilde{x}).$$

Proof. We have $\rho(x,y) \leq \tilde{\rho}\ (\tilde{x}, \tilde{y}_i)$ for any i, and then

$$\rho(x,y) \leq \min_{i=1,\ldots,m,\ldots} \tilde{\rho}\ (\tilde{x}, \tilde{y}_i).$$

Let γ be a shortest geodesic joining x and y. Let $\tilde{\gamma}$ be the geodesic lift to $\tilde{M}$ of γ, which starts at $\tilde{x}$ and ends at $\tilde{y}_{i_0}$. Then $\rho\ (x,y) = \textit{length}\ \gamma = \textit{length}\ (\tilde{\gamma}) = \tilde{\rho}(\tilde{x}, \tilde{y}_{i_0}) \geq \min_{i=1,\ldots,m,\ldots} \rho\ (\tilde{x}, \tilde{y}_i)$.

LEMMA 1. *Let $p : (\tilde{M}, \tilde{\rho}) \to (M, \rho)$ be a universal covering of Riemannian manifolds. Let $x, y \in M$, $\tilde{x} \in p^{-1}\ (x), p^{-1}\ (y) = \{\tilde{y}_1, ..., \tilde{y}_m, ...\}$ and $\tilde{\rho}(\tilde{x}, \tilde{y}_i) \neq \tilde{\rho}(\tilde{x}, \tilde{y}_j)$ for $i \neq j$. Suppose $\tilde{\rho}(\tilde{x}, \tilde{y}_{i_0}) = \min_{j=1,\ldots,m} \tilde{\rho}(\tilde{x}, \tilde{y}_j)$. Then there exists $\epsilon > 0$ such that $\rho(\tilde{x}, \tilde{y}_{i_0}) + \epsilon < \tilde{\rho}(\tilde{x}, \tilde{y}_j)$ for all $j \neq i_0$.*

Proof. If the statement is wrong there exists a sequence $\tilde{y}_m \in p^{-1}(y)$ such that $\rho(\tilde{x}, \tilde{y}_m) < \rho(\tilde{x}, \tilde{y}_{i_0}) + \frac{1}{m}$. Then all $\tilde{y}_m$ are inside the ball of radius $\rho(x,y) + 1$ centered at $\tilde{x}$. This ball is compact, then there exists a subsequence $\tilde{y}_m \to \tilde{y}_\infty$ and we have $p(\tilde{y}_\infty) = y$. This gives a contradiction, since p is a local isometry.

We obtain the following Corollary.

COROLLARY 1. *Let $p : (\tilde{M}, \tilde{\rho}) \to (M, \rho)$ be a universal covering of non-positively curved Riemannian manifolds. Let $\tilde{x} \in p^{-1}\ (x)$ and $p^{-1}\ (y) = \{\tilde{y}_1, ..., \tilde{y}_m, ...\}$. If $\tilde{\rho}\ (\tilde{x},\ \tilde{y}_i) \neq \tilde{\rho}(\tilde{x},\ \tilde{y}_j)$ for $i \neq j$ then d_x is smooth at y.*

Similarly to lemma 1 we obtain the following result.

LEMMA 2. *Let (M, ρ) be a non-positively curved Riemannian manifold, $p : (\tilde{M}, \tilde{\rho}) \to (M, \rho)$ be the universal covering and $x, y \in M$. Let $\tilde{x} \in p^{-1}\ (x)$, $p^{-1}\ (y) = Y =$*

$\{\tilde{y}_1, \tilde{y}_2, ..., \tilde{y}_m, ...\}$. *Let* $r = \min_{i=1,...,m,...} \tilde{\rho}\ (\tilde{x}, \tilde{y}_i)$. *Then only a finite number of points* $\tilde{y}_{i_1}, ..., \tilde{y}_{i_k} \in Y$ *satisfy the condition* $r = \tilde{\rho}\ (\tilde{x}, \tilde{y}_{i_j})$.

Let us estimate the set of points where d_x is not smooth.

PROPOSITION 2. *Let* (M, ρ) *be a non-positively curved manifold,* $x \in M$. *The set of points* $y \in M$ *where* d_x *is not smooth is of the first category (that is the union of a countable number of nowhere dense sets).*

The result follows from the next lemma.

LEMMA 3. *Let* $(\tilde{M}_n, \tilde{\rho})$ *be a simply-connected non-positively curved manifold,* $\tilde{x}, \tilde{y} \in \tilde{M}_n$. *Then the set of points* $\tilde{z}$ *satisfying the equation* $\tilde{\rho}(\tilde{x}, \tilde{z}) = \tilde{\rho}(\tilde{y}, \tilde{z})$ *is a hypersurface in* $\tilde{M}_n$ *diffeomorphic to* $\Re^{n-1}$.

Let us start with clear infinitesimal analogy of this statement. Let $x, y \in M$, denote by $B_{\tilde{x},\tilde{y}} \subset \tilde{M}$ the set of points $\tilde{z} \in \tilde{M}$ such that $\tilde{p}(\tilde{x}, \tilde{z}) = \tilde{(}\tilde{y}, \tilde{z})$.

LEMMA 4. *Let* $(\tilde{M}, \tilde{\rho})$ *be a simply-connected non-positively curved manifold,* $\tilde{x}, \tilde{y} \in \tilde{M}_n$, $\tilde{z} \in B_{\tilde{x},\tilde{y}}$. *Let* $B_{\tilde{z}} = \{\xi \in T_{\tilde{z}}\tilde{M} \mid \tilde{\rho}\ (\tilde{x}, \exp_{\tilde{z}}\ \epsilon\xi) < \tilde{\rho}(\tilde{y}, \exp_{\tilde{z}} \epsilon\xi)\}$, *for small enough* ϵ. *Then* B_z *is diffeomorphic to* $\Re^n$ *and its boundary is diffeomorphic to* $\Re^{n-1}$.

The exponential map gives the necessary global result.

REMARK. The representation of d_x as a Min-type function is closely related to the structure of the Dirichlet domain for $\tilde{x}_0 \in \tilde{M}$. This is defined as $\{\tilde{y} \in \tilde{M} \mid \rho\ (\tilde{x}, \tilde{y}) \leq \rho\ (\tilde{x}_i, \tilde{y})$ for all i, where $\rho^{-1}\ (x) = \{\tilde{x}_0, \tilde{x}_1, ...\}$. The Dirichlet domain has been studied for Riemannian manifolds of non-positive curvature, see for example [EI].

1.2. Min-type functions.

DEFINITION 1. *Let* $f : \Re^n \to \Re^1$ *be a germ of a continuous function at* $x \in \Re^n$; f *is said to be a* **germ of a Minimum type (Min-type) function** *at* x *iff there exist germs of smooth functions* $\alpha_1, ..., \alpha_m$ *at* x *such that* $f = \min_{i=1,...,m} \alpha_i$ *around* x. *We shall say that* f *is a* C^k - **Min-type function** *if there exists a representation* $f = \min\{\alpha_i\}$ *with* C^k*-smooth germs* α_i.

DEFINITION 2. *The minimal number of smooth functions in a representation for the germ of* f *at* x *is called the* **Rank** *of* f *at* x, *and is denoted as* $R_x\ f$.

DEFINITION 3. *A representation* $f = \min\{\alpha_1 \dots \alpha_m\}$ *is said to be* **minimal** *at* x *iff* $m = R_x f$.

REMARK. For a minimal representation $f = \min_{1 \leq i \leq m} \alpha_i$ we have $\alpha_1\ (x) = \alpha_2\ (x) = ... = \alpha_m(x) = f(x)$.

Consider two definitions of Min-type functions.

DEFINITION 4. $f : M_n \to \Re^1$ *is said to be a* **Min-type function** *iff each of its germs is a Min-type function in the sense of the previous definitions.*

DEFINITION 5. $f : M_n \to \Re^1$ *is a* **Min-type function** *iff there exist smooth functions* $\alpha_1, ..., \alpha_Q$ *on* M_n *such that* $f = \min_{i=1,...,Q}\ \alpha_i$.

THEOREM 1. *Let* M_n *be a compact manifold. Then the definitions above are equivalent.*

Proof. Evidently any function satisfying Definition 5 satisfies also Definition 4.

Suppose $f : M_n \to \Re^1$ is locally a Min-type function around each point $x \in M_n$; then there exists m_x, an open ball $U(x)$, and smooth functions $\alpha_1, ..., \alpha_Q$ such that $f_{|U_x} = \min\ \{\alpha_1^x, ..., \alpha_{m_x}^x\}$. Since M_n is compact, there exist $x_1, ..., x_r$ such that $M_n = \bigcup_{i=1}^r\ U^0(x_i)$ for smaller balls $U^0(x_i) \subset U(x_i)$. We have $f_{|U(x_i)} = \min\ \{\alpha_1^{x_i}, ..., \alpha_{m_{x_i}}^{x_i}\}$, $i = 1, ..., r$. Extend the functions $\alpha_j^{x_i}$ to the whole of M_n such that they increase faster than f outside of $U_{x_i}^0$. Then we obtain

$$f = \min\ \{\alpha_1^{x_1}, ..., \alpha_{m_{x_1}}^{x_1}; \alpha_1^{x_2}, ..., \alpha_{m_{x_2}}^{x_2}; ...\alpha_1^{x_r}, ..., \alpha_{m_{x_r}}^{x_r}\}.$$

1.3. Minimal representations. We investigate further minimal representations for germs of Min-type functions. In this section we demonstrate that such a representation is, in fact, unique.

DEFINITION 1. *Let $f : M_n \to \Re^1$, a Min-type function at the origin, and $f = \min\ \{\alpha_1, ..., \alpha_m\}$ be a minimal representation in a neighbourhood of the origin; a function α_i is said to be* **active** *at a point y iff $\alpha_i\ (y) = f\ (y)$.*

Denote by ω_i^α the germ of the set of active points for α_i in a minimal representation $f = \min\ \{\alpha_1, ..., \alpha_m\}$.

DEFINITION 2. *Let $f = \min\{\alpha_i\}_{i=1}^m = \min\{\beta\}_{i=1}^k$ be minimal representations. Then $\alpha_i \approx \beta_j$ iff $\omega_i^\alpha = \omega_j^\beta$ and $\alpha_{i|\omega_i^\alpha} = \beta_{j|\omega_j^\beta}$.*

DEFINITION 3. *A minimal representation $f = \min\{\alpha\}_{i=1}^k$ is said to be a* **LIG-representations** *at x if $\{grad\ (\alpha_{i+1} - \alpha_i)\}_{i=1}^{n-1}$ are linearly independent.*

THEOREM 1. *Let $f : \Re^n \to \Re^1$ be a germ of a Min-type function at the origin O which is a non-degenerate regular point for f. Let $f = \min_{i=1,...,m}\ \alpha_i = \min_{j=1,...,m}\ \beta_j$ be two LIG-representations. Then there exists a permutation $\sigma : \{1, ..., m\} \to \{1, ..., m\}$ such that $\alpha_i \approx \beta_{\sigma\ (i)}$, $i = 1, ..., m$.*

To prove the theorem we have to clarify the geometry connected with minimal representations. Let $f : \Re^n \to \Re^1$, $f = \min_{i=1,...,m}\ \alpha_i$ be a germ of a Min-type function at $x \in \Re^n$. Then 1-forms $\{d\alpha_i\}_{i=1}^m$ are linear functionals on $T_x\Re^n$, the LIG-condition means that $d\alpha_i - d\alpha_{i-1}$, $i = 1, ..., m-1$ are linearly independent. Denote $\bar\Omega_i = \{\xi \in T_xM \mid d\alpha_i\ (\xi) = \min_{i=1,...,m}\ d\alpha_j\ (\xi)\}$ and $\Omega_i^0 = \{\xi \in T_xM \mid d\alpha_i\ (\xi) < d\alpha_j(\xi)$ for any $j \neq i\}$.

LEMMA 1. (ON A PARTITION INTO CONES OF THE TANGENT SPACE). *Let $f : \Re^n \to \Re^1$ be a Min-type function, $f = \min\ \{\alpha_1, ..., \alpha_m\}$ be a LIG-representation for f. Then $\Omega_i^0, \bar\Omega_i$ are convex polyhedral cones in $T_xM_n \approx \Re^n$; Ω_i^0 is open, and $\bar\Omega_i$ is the closure of Ω_i^0.*

Proof. $\bar\Omega_i = \cap_{i\neq j}\ \bar\Omega_i^j$, where $\bar\Omega_i^j$ is a closed half-space defined by the linear inequality $d\alpha_i\ (\xi) \leq d\alpha_j\ (\xi)$ then $\bar\Omega_i = \cap_{i\neq j}\ \bar\Omega_i^j$ is a closed convex cone. Similarly, $\Omega_i^0 = \cap_{j\neq i}\ (\Omega_i^j)^0$, where $(\Omega_i^j)^0$ is the open half space, defined by the strict inequality $d\alpha_i\ (\xi) < d\alpha_j(\xi)$. The cones $\bar\Omega_i^0$ are convex and of dimension n, since $m < n$ and the representation is minimal.

LEMMA 2. (ON A PARTITION OF THE GERM OF THE MANIFOLD ON ACTIVE SETS). *Let $f : M \to \Re^1$ be a germ of a regular Min-type function at x and $m = R_x\ f$. Then there exist m germs of maximal open connected sets $U_1, ..., U_m$ such that all the restrictions $f_{|U_i}$ are smooth; the germs of $\{U_i\}$ are uniquely determined up to a permutation. For any minimal representation $f = \min\{\alpha_i\}$ there exists a permutation of $\{1, ..., m\}$ such that $\alpha_{\sigma(i)}{}_{|U_i} = f_{|U_i}$*

Proof. Let $f = \min_{i=1,...,m}\ \alpha_i$ be a minimal representation for f at x and $D_\epsilon(x)$ be a ball centered at x. Define $U_i = \{y \in D_\epsilon\ (x) \mid \alpha_i\ (x) \leq \alpha_j\ (x)$ for all $j \neq i\}$. Then the sets U_i^0 are the maximal connected open sets where f is smooth. Then they are determined canonically (up to a permutation).

COROLLARY 1. *Let $f : M \to \Re^1$ be a germ of a Min-type function at $x \in M$. Suppose we have two LIG-representations $f = \min_{i=1,...,m}\ \alpha_i = \min_{j=1,...,m}\ \beta_j$. Then there exists a permutation $\sigma : \{1, ..., m\} \to \{1, ..., m\}$ such that $U_i^\alpha = U_{\sigma(i)}^\beta$ and $\alpha_i \approx \beta_{\sigma(i)}$.*

This finishes the proof of the theorem.

1.4. Topology in the space of Min-type function. We have to define a suitable topology on the space of Min-type functions. We begin with a definition of C^k-small perturbations of germs of C^k Min-type functions.

DEFINITION 1. *Let $f : \Re^n \to \Re^1$ be a germ of a Min-type function at the origin, $f = \min\{\alpha_1, ..., \alpha_k\}$ be its minimal representation, and $\alpha_i \in C^k(\Re^n)$. We define the class of C^k-small perturbations f^δ of f as $f^\delta = \min\{\alpha_1^\delta, ..., \alpha_k^\delta\}$, where α_i^δ, $i = 1, ..., k$ are C^k-small independent perturbations of α_i. This defines a C^k-topology on the space of Min-type functions.*

We do not demand that the representation is minimal in the definition. However we point out the following statement.

PROPOSITION 1. *Let $f : \Re^n \to \Re^1$ be a germ of a Min-type function at the origin, $f = \min\{\alpha_1, ..., \alpha_k\}$ be its LIG-representation, and $\alpha_i \in C^2$. Then for any sufficiently small C^2-perturbations α_i^δ of α_i, $i = 1, ..., k$, $f^\delta = \min\{\alpha_1^\delta, ..., \alpha_k^\delta\}$ is a LIG-representation in a neighbourhood of a point x^δ close enough to x.*

We present one more statement to motivate the following definition of C^k-distance between Min-type functions. Denote U_i the active set for α_i and U_i^δ the active set for α_i^δ.

LEMMA 1. *Let $f : \Re^n \to \Re^1$ be a germ of a Min-type function at the origin, and $f = \min\{\alpha_1, ..., \alpha_k\}$ is a LIG-representation. Suppose $f^\delta = \min\{\alpha_1^\delta, ..., \alpha_k^\delta\}$ is a δ-perturbation of f. Denote U_i (U_i^δ) the active set of α_i (α_i^δ). Then the Hausdorff distance $dist_H$ between U_i and U_i^δ satisfies the following estimate $dist_H(U_i, U_i^\delta) = O(\delta)$ for $\delta \to 0$.*

This lemma allows us to introduce C^k-distance between C^k-smooth Min-type functions that defines the same C^k-topology. We give first a definition for germs. Let f be a germ of a C^k Min-type function at the origin in $\Re^n$; denote by $N_f \subset \Re^n$ the germ of the set of points where f is not smooth. This germ does not depend on the minimal representation, see the previous section. Denote by N_f^ϵ the ϵ-neighbourhood of N_f, and $\tilde{N}_f^\epsilon = \Re^n \setminus N_f^\epsilon$.

DEFINITION 2. *Let f, g be germs of C^k-smooth Min-type functions at the origin in $\Re^n$. We define the C^k-distance between f and g as*

$$\min_{\epsilon > 0} \left\{ \epsilon + dist_{C^k} \left(f_{|\tilde{N}_f^\epsilon}, g_{|\tilde{N}_f^\epsilon} \right) \right\}.$$

The following proposition is clear.

PROPOSITION 2. *The C^k-distance defines the same C^k-topology on the space of C^k-smooth Min-type functions.*

One more definition of the C^k-distance between Min-type functions is based on the following lemma.

LEMMA 2. *Let $f : \Re^n \to \Re^1$, be a germ of a Min-type function at the origin and $f = \min\{\alpha_1, ..., \alpha_k\}$ be a LIG-representation. Then for a small enough perturbation $f^\delta = \min\{\alpha_1^\delta, ..., \alpha_k^\delta\}$ there exists a germ of a diffeomorphism ϕ^ϵ of $\Re^n$, ϵ-close to the identity and such that $\phi^\epsilon(U_i^\delta) = U_i$, $i = 1, ..., k$.*

2. Regular points of Min-type functions. We introduce and discuss a concept of non-degenerate regular points. The germ of a smooth function can be transformed into the coordinate x_1 using the group $Diff_n$ of local diffeomorphisms of $\Re^n$. For a Min-type function such a transformation is possible when one uses a bigger transformation group Hom_n^s of local homeomorphisms of $\Re^n$ which are smooth almost everywhere, that is outside a set of positive codimension. This bigger group is natural in this theory; the group $Diff_n^{loc}$ is too small to obtain "good" normal forms;

the space of orbits is parametrised by functions even for Min-type functions of one variable. At the same time all Min-type functions at a non-degenerate regular point are on the same orbit of the action of Hom_n^s. (This group arose already in Matov's works, see [Ma1], [Ma2]).

We consider the simplest degenerate regular points, which are codimension-one singularities. Finally we clarify relations between the classical and Gromov's definitions of regular points.

2.1. Non-degenerate regular points.

DEFINITION 1. *Let $f : \Re^n \to \Re^1$ be a germ of a Min-type function; a point x is said to be a* **non-degenerate regular point (NDR-point)** *iff the germ of f at x admits a minimal representation $f = \min_{i=1,\dots,m} \alpha_i$, with smooth functions α_i, $i = 1, \dots, m$ which satisfy the following properties:*

1) The gradients $grad\,(\alpha_1 - \alpha_2), \dots, grad\,(\alpha_{m-1} - \alpha_m)$ are linearly independent at x;

2) $0 \notin Conv\,\{ grad\,\alpha_i\,(x)\}_{i=1}^m$, ($Conv$ means the convex hull).

3) $f_{|G_f}$ is a germ of a Morse function at x, where $G_f = \{y \mid \alpha_1(y) = \dots = \alpha_m(y)\}$;

4) Any $m-1$ gradients among $grad\,\alpha_1, \dots, grad\,\alpha_m$ are linearly independent at x.

REMARK 1. The first condition (it is the LIG-condition) shows that G_f is a smooth submanifold of dimension $n - m + 1$, (G_f is determined by $(m-1)$ equations $(\alpha_i - \alpha_{i+1}) = 0$, with linearly independent gradients). The following two statements clarify properties of G_f.

PROPOSITION 1. *The germ of G_f does not depend on a choice of a minimal representation.*

Proof. A Min-type function f defines uniquely the sets $\bar{U}_i = \{y \mid f = \alpha_i\}$, as the maximal connected sets, where f is smooth, see section 1.3; then $G_f = \cap_{i=1}^m \bar{U}_i$ is also canonically defined.

PROPOSITION 2. *The restriction of f on G_f is a smooth function.*

Proof. G_f is a smooth sub-manifold, and $f_{|G_f}$ coincides with the restriction of a smooth function $\alpha_{i|G_f}$ for each i.

REMARK 2. G_f is the germ of the maximal submanifold at x such that $f_{|G_f}$ is smooth.

COROLLARY 1. *Let $f : \Re^n \to \Re^1$ be a germ of a Min-type function at x, and x be a NDR-point for f. Then for* **any** *minimal representation $f = \min\,\{\beta_1, \dots, \beta_m\}$ the functions β_i, $i = 1, \dots, m$, satisfy all the properties of Definition 1.*

Proof. This follows from Proposition 1 of this section and Theorem 1 of section 1.3.

Let us discuss some corollaries of the conditions above, related to linear independence.

PROPOSITION 3. *For any j, the gradient $grad\,\alpha_j$ is not a convex combination of all other gradients.*

Proof. Suppose there exists j such that $grad\,\alpha_j = \sum_{i\neq j} c_i\,grad\,\alpha_i$, where $\sum_{i\neq j} c_i = 1$ and $c_i > 0$ for all i. Then $\sum_{i\neq j} c_i\,grad(\alpha_i - \alpha_j) = 0$ and then $\{grad\,(\alpha_i - \alpha_{i+1})\}_{i=1}^{m-1}$ are linearly dependent.

All conditions of Definition 1 are "open" and we obtain the next statement.

PROPOSITION 4. *Let $f : M_n \to \Re^1$ be a Min-type function, $f = \min\,\{\alpha_1, \dots, \alpha_m\}$. Then the set of NDR-points is open.*

PROPOSITION 5. *Let $f : \Re^n \to \Re^1$ be a Min-type function, $f = \min\,\{\alpha_1, \dots, \alpha_Q\}$; let U be the set of NDR-points and U_i be the interior of the set of points $\bar{U}_i$, where α_i*

is active. Then the restrictions $\alpha_{i|U\cap U_i}$ *are Morse functions.*

Proof. Let $x \in U \cap U_i$ and $f = \min\{\alpha_{i_1}, ..., \alpha_{i_q}\}$ be a minimal representation around x. When $q = 1$ the germ of G_f coincides with the germ of M_n and then $f = \alpha_{i_1}$ is a germ of a Morse function, in accordance with condition 3 of the Definition. When $q > 1$, for any j the gradient *grad* α_{i_j} does not vanish in accordance with condition 4 of the definition.

Similarly to this statement we obtain the following result.

PROPOSITION 6. *Let* $f : \Re^n \to \Re^1$ *be a Min-type function,* $f = \min\{\alpha_1, ..., \alpha_Q\}$. *Then the restrictions* $(\alpha_i - \alpha_j)_{|U\cap U_i \cap U_j}$ *are Morse functions.*

COMMENTS. Our definition is based on the following idea. We would like to construct a Morse theory for Min-type functions and, in particular, to define Morse Min-type functions by analogy with the classical Morse theory, as functions which have only (non-degenerate) regular and non-degenerate critical points. To obtain an open and everywhere dense subset of Morse Min-type functions we have to admit all stable singularities, but exclude all non-stable critical points which can be eliminated (decomposed) by small perturbations.

REMARK 3. Principles, listed above do not fix the definition uniquely. One possible technically useful (but not principal) modification is to add conditions that all α_i and all their differences $(\alpha_i - \alpha_j)$, $i \neq j$ are Morse functions (everywhere on the manifold M_n, not only on the set of active points of these functions, see Propositions 5 and 6).

2.2. Linearly dependent gradients. The equations $\alpha_1 = \alpha_2 = ... = \alpha_m$ define an $(n - m + 1)$ dimensional submanifold. The linear dependence of *grad* $\alpha_1, ...,$ *grad* α_m means that one of them, say *grad* α_m, is a linear combination of the others. This means that *grad* $\alpha_m(x)$ belongs to the $(m - 1)$ dimensional linear subspace of T_xM_n, generated by *grad* $\alpha_i(x)$, $i = 1, \dots, m-1$; this is a condition of "codimension" $n - (m - 1) = n - m + 1$ on the $n - m$ dimensional submanifold G_f; hence there exists, generically, only a finite number of points in M_n, satisfying this condition. This motivates the following definition.

DEFINITION 1. *We shall call a NDR-point with linearly dependent gradients* **special** *(and* **non-special** *when the gradients are linearly independent). To be shorter we denote such points as* **SR-points (NSR-points)**.

The following two examples present linearly independent and linearly dependent gradients. In both examples functions are stable.

1. NSR-point. Let $f : \Re^2 \to \Re^1$, $f = \min\{\alpha_1, \alpha_2\}$, where $\alpha_1 = x + y, \alpha_2 = x - y$. Then *grad* $\alpha_1 = (1, 1)$, *grad* $\alpha_2 = (1, -1)$ are linearly independent and then α_1, α_2 satisfy also condition 2, we have *Convex* (*grad* α_1, *grad* α_2) $= 1 \times [-1, 1] \not\ni 0$. The origin is an NSR-point.

2. SR-point. Let $f : \Re^2 \to \Re^1, f = \min(\alpha_1, \alpha_2), \alpha_1 = x + y^2, \alpha_2 = 2x + y^2$. Then *grad* $\alpha_1(0) = (1, 0)$, *grad* $\alpha_2(0) = (2, 0)$ are linearly dependent at the origin. One can destroy the linear dependence at the origin, however a point with linearly dependent gradients arises near the origin.

We present one more example to comment on the third condition in the Definition.

3. The restriction on G_f. Let $g = \min\{x + y^3, 2x + y^3\}$; then G_g is the y-axis, and $g|_{G_g} = y^3$ has a birth-death singularity at the origin.

THEOREM 1. (ON SR-POINTS). *Let* $f : \Re^n \to \Re^1$ *be a germ of a Min-type function,* $f = \min\{\alpha_1, ..., \alpha_m\}$ *be a minimal representation, and* p *be an SR-point. Then all the other points of a small* ϵ*-ball around* p *are NSR-points.*

Proof. Let p be an SR-point of $f : \Re^n \to \Re^1, f = \min\{\alpha_1, ..., \alpha_m\}$. Any $(m-1)$ gradients among $grad\ \alpha_1, ..., grad\ \alpha_m$ are linearly independent at p and thus in a small ϵ-ball $D_\epsilon\ (p)$ around p. Let $q \in D_\epsilon\ (p) \setminus G_f$, then f is a minimum of a smaller number of smooth functions and hence is an NSR-point.

Consider a point $q \in D_\epsilon(p) \cap G_f$. One can choose a local coordinate system around p in such a way that $\alpha_1 = x_1, ..., \alpha_{m-1} = x_{m-1}$, then we have $f = \min\{x_1, ..., x_{m-1}, \alpha_m\}$. Since any $(m-1)$ gradients among $grad\ \alpha_i$ are linearly independent and $0 \notin Convex\ \{grad\ \alpha_i\}_{i=1}^m$ we have $\alpha_m = \sum_{i=1}^{m-1} c_i x_i + \beta$, where $grad\ \beta = 0$ and $c_i > 0$. We have $\sum_{i=1}^{m-1} c_i \neq 1$, since $grad\ \alpha_m$ is not a convex combination of all the other gradients due to Proposition 3.

Let $\mathcal{M}$ be the maximal ideal in the local ring of germs of smooth functions at the origin. We have $G_f = \{(x_1, ..., x_n) \mid x_1 = ... = x_{m-1} = \sum_{i=1}^{m-1} c_i x_i + \sum_{i,j=1}^n a_{i,j} x_i x_j + O\ (\mathcal{M}^3)\}$. The equations for G_f can be transformed into $x_1 = ... = x_{m-1} = \sum_{i,j=1}^n a_{i,j} x_i x_j + O\ (\mathcal{M}^3)$ and then $x_1 = ... = x_{m-1} = \sum_{i,j=m}^n \tilde{a}_{i,j} x_i x_j + O\ (\mathcal{M}^3)$.

The restriction of α_m on G_f is a Morse function, then there exists coordinates $x_m, ..., x_n$ such that $\alpha_{m|G_f} = \sum_{i=1}^{m-1} c_i x_i + \sum_{i=m}^n \pm x_i^2$, compare [Mi1]. In this coordinate system one has $\alpha_m = \sum_{i=1}^{m-1} c_i x_i + \sum_{i,j \le m-1} a_{i,j} x_i x_j + \sum_{i=m}^n \pm x_i^2 + O(\mathcal{M}^3)$.

For a point of $D_\epsilon(x) \cap G_f$, where at least one of $x_i, i = m, ..., n$ does not vanish $\{grad\ \alpha_i\}_{i=1}^m$ are linearly independent. The origin is the only point on G_f, defined by the equations $x_m = ... = x_n = 0$, and then the only SR-point inside $D_\epsilon(x)$.

COROLLARY 2. *SR-points are isolated.*

The next statement clarifies the connection between conditions 3 and 4 in the Definition of NDR-points.

PROPOSITION 7. *Let $f : \Re^n \to \Re^1$, $f = \min\{\alpha_1, ..., \alpha_m\}$ and the origin O be a NDR-point. Then $\{grad\ f_{|G_f}(x) = 0\} \iff \{O$ is special $\}$.*

Proof. $\left(grad\ f_{|G_f}\ (x) = 0\right) \iff \Big($for each i, $grad\ \alpha_i\ (x)$ is a linear combination of $grad\ (\alpha_j - \alpha_{j+1})(x), j = 1, ...m\Big) \iff \left(rk\ \{grad\ \alpha_i\}_{i=1}^m\ (x) = m - 1\right)$.

REMARK. This statement means that linear independence of the gradients provides all the other conditions of the Definition of NDR-points.

2.3. The simplest example and motivation. We consider Min-type functions of one variable to motivate our approach to normal forms of Min-type functions based on Hom_n^s- equivalence. Consider a regular germ of a Min-type function $f : \Re^1 \to \Re^1$ at 0. When smooth, f is smoothly equivalent to x. A non-smooth germ is a minimum of two smooth functions and we obtain the following elementary proposition. Denote by $\mathcal{M}$ the maximal ideal of functions vanishing at O.

PROPOSITION 1.

1) Let $f : \Re^1 \to \Re^1$ be the germ of a non-smooth Min-type function at an NDR-point q. Then one can choose a (smooth) coordinate around q such that $f = Const +$ $\min\ (x,\ x\ \alpha(x))$, where $\alpha(q) > 0$ and $\alpha(q) \neq 1$.

2) Any germ $\alpha : \Re^1 \to \Re^1$ of a smooth function at q, satisfying the condition $\alpha(q) > 0$ and $\alpha(q) \neq 1$, defines a regular germ of a smooth Min-type function $f = Const + \min\ (x, x\alpha(x))$.

3) For any two such germs α_1, α_2 the corresponding functions f_1, f_2 are C^∞-equivalent iff $\alpha_1 - \alpha_2 \in \mathcal{M}^\infty$ (in particular there exists an infinite set of orbits for the *action of* $Diff_1^{loc}$ *).*

4) All the germs of Min-type functions are on the same orbit of the action of Hom_1^s.

2.4. Functions of n-variables. We consider SR-points and NSR-points separately.

a) NSR-points ($rk\ \{grad\ \alpha_i\}_{i=1}^m = m$). Let $f : \Re^n \to \Re^1$ be a Min-type function q is an NSR-point, $f = \min_{i=1,...,m}\ \alpha_i$ be a minimal representation at q. Then there exists a local coordinate system $\{x_i\}$ around q such that $\alpha_i = x_i,\ i = 1, ..., m$ and $f = \min\ \{x_1, ..., x_m\}$.

We obtain the following theorem.

THEOREM 1. *The set of germs of Min-type functions of n variables at an NSR-point consists of n orbits of the action of $Diff_n^{loc}$ numbered by the number of functions in a minimal representation.*

b) SR-points: ($rank\ \{grad\ \alpha_i\}_{i=1}^m = (m-1)$). One can choose a coordinate system $\{x_i\}$ in such a way that $\alpha_i - \alpha_1 = x_{i-1},\ i = 2, ..., m-1$. Then $f = \min\ \{\alpha_1, ..., \alpha_m\} = \min\ \{\alpha_1,\ \alpha_1 + x_1, ...,\ \alpha_1 + x_{m-1}\} = \alpha_1 + \min\ \{0, x_1, ..., x_{m-1}\}$ and G_f is the coordinate plane defined by the equations $x_1 = ... = x_{m-1} = 0$. Denote $\tilde{\alpha}_1 = \alpha_{1|G_f}$; this is a Morse function and as we know $grad\ \tilde{\alpha}_1\ (x) = 0$; One can choose a coordinate system $x_m, ..., x_n$ on G_f such that $\tilde{\alpha}_1 = -\sum_{i=m}^k\ x_i^2 + \sum_{i=k+1}^n\ x_i^2$. Define $\tilde{f} = f - \tilde{\alpha}_1$, then $\tilde{f}_{|G_f} \equiv 0$. We obtain the following lemma.

LEMMA 1. *Let L be a plane defined by the equations $x_1 = ... = x_{m-1} = 0$. Suppose $\beta : \Re^n \to \Re^1$ is a smooth function such that $\beta_{|L} \equiv 0$. Then there exist smooth functions $\beta_1, ..., \beta_{m-1}$ such that $\beta = \sum_{i=1}^{m-1}\ x_i \beta_i$.*

This lemma gives

$$f = -\sum_{i=m}^{k}\ x_i^2 + \sum_{i=k+1}^{n}\ x_i^2 + \sum_{i=1}^{m-1}\ x_i\beta_i + \min\ \{0, x_1, ..., x_{m-1}\}. \tag{2}$$

The functions β_i satisfy the following conditions (*): $\beta_i\ (O) \neq 0$ - this is equivalent to the condition of linear independence of any $m-1$ gradients among $grad\ \alpha_1, ..., grad\ \alpha_m$; $\{-\beta_1\ (O), ..., -\beta_{m-1}\ (O)\}$ does not belong to the standard simplex $S = \{t_1, ..., t_{m-1} \mid t_i \geq 0$ and $\sum\ t_i = 1\}$, this is equivalent to the condition $0 \notin Convex\ \{grad\ \alpha_1, ..., grad\ \alpha_m\}$. (Both equivalences can be immediately obtained from the following formulas: $grad\ \alpha_1\ (0) = \sum_{i=1}^{m-1}\ \beta_i\ (O)\partial_i$, $grad\ \alpha_i\ (O) = \partial_{i-1} + \sum_{i=1}^{m-1}\ \beta_i\ (O)\partial_i$, for $i = 2, ..., m$ we denote $\partial_i = \frac{\partial}{\partial x_i}$). On the other hand any smooth functions $\beta_i, i = 1, ..., m-1$ satisfying the two conditions above define a Min-type function with linearly dependent gradients.

2.5. Normal forms (group of local diffeomorphisms). Let us describe the orbit structure of germs of Min-type functions $\Re^n \to \Re^1$ for the group $Diff_{loc}^n$.

PROPOSITION 1. *Let $f, g : \Re^n \to \Re^1$ be Min-type functions such that $f = \min_{i=1,...,l}\ \alpha_i$ and $g = \min_{j=1,...,m}\ \beta_j$ are minimal representations and $l \neq m$. Then f and g are not smoothly equivalent.*

Proof. Smooth equivalence would give the same decomposition of the tangent space $T_x\Re^n$ on maximal open sets where a Min-type function is smooth, and then gives $l = m$.

There is one more (smooth) invariant of a non-degenerate regular point.

PROPOSITION 2. *Let $f, g : \Re^n \to \Re^1$ be germs of Min-type functions at q. Let the origin q be an SR-point for f and an NSR-point for g. Then the f and g are not smoothly equivalent.*

Proof. A smooth equivalence conserves linear dependence of gradients.

We have to obtain normal forms for an SR-point. Formula (2) gives a representation for any germ of a Min-type function of n variables at an SR-point for Min-type functions with rank m. The parameters in (2) are the natural number $k_f \leq n-m+1$ of negative squares and $m-1$ smooth functions $\{\beta_i\}_{i=1}^{m-1}$ satisfying conditions (*). Denote by $f_{k,\{\beta_i\}_{i=1}^{m-1}}$ the Min-type function f determined by k_f and $\beta_i, i = 1, ..., m-1$, satisfying (*). We have to determine which k and $\{\beta_i\}$ define smoothly equivalent Min-type functions.

PROPOSITION 3. *k is a smooth invariant.*

Proof. Let there exist a local diffeomorphism ϕ of $\Re^n$ such that $\phi(f) = \tilde{f}$. Then $\phi(G_f) = G_{\tilde{f}}$ and $\phi(f_{|G_f}) = \tilde{f}_{|G_{\tilde{f}}}$ and these (Morse) functions has equal indices, that is $k_f = k_{\tilde{f}}$.

REMARK. It is natural to call k the **(smooth) index** at an SR-point.

PROPOSITION 4. *Let $\{\beta_i, \tilde{\beta}_i\}_{i=1}^{m-1} : \Re^1 \to \Re^1$ be germs of smooth functions at O satisfying α and β. Then Min-type functions defined by (2) (with the same smooth index) are smoothly equivalent iff there exists a permutation $\sigma : \{1, ..., m-1\} \to \{1, ..., m-1\}$ such that $\tilde{\beta}_i - \sigma(\beta_i) \in \mathcal{M}_1^\infty$, $i = 1, ..., m-1$.*

COROLLARY 1. *For SR-points we have a continuum of $Diff_n^{loc}$-orbits, parametrized by ∞-jets of $m-1$ smooth functions of n variables.*

2.6. Group of almost smooth local homeomorphisms.

THEOREM 1. *Let $f : \Re^n \to \Re^1$ be a germ of a Min-type function regular at the O. Then there exists $\phi \in Hom_n^s$ which transforms f into $Const + x_1$.*

Proof. We have to consider separately SR-points and NSR-points.

1) NSR-points. There exists a local coordinate system $\{x_i\}$ such that $f = Const + \min_{i \leq k} x_i$, where $k \leq n$.

We prove that the Min-type function $f : \Re^n \to \Re^1$, $f = \min(x_1, x_2, \ldots, x_m)$ is Hom_n^s-equivalent to x_1.

We define $2u = x_1 + x_2$ and $2v = x_1 - x_2$. Then we have $\min\{x_1, x_2\} = \min\{u + v, u - v\} = u - |v| \sim u - v^2 \approx u \approx x_1$. (The equivalence $\sim$ is for the action of Hom_n^s, $\approx$ is for $Diff_n^{loc}$). Then we have $\min\{x_1, ..., x_m\} = \min\{\min\{x_1, ..., x_{m-1}\}, x_m\} \sim \min\{x_1, \ldots, x_{m-1}\} \sim \ldots \sim x_1$.

2. SR-points. Let $f : \Re^n \to \Re^1, f = \min\{\alpha_1, ..., \alpha_m\}$ be a Min-type function, and O be an SR-point, Then $\alpha_1, ..., \alpha_{m-1}$ can be chosen as local coordinates in a neighbourhood of O and we have $f = \min\{x_1, ..., x_{m-1}, \alpha_m\}$; $\min\{x_1, ..., x_{m-1}, \alpha_m\} = \min\{\min\{x_1, ..., \tilde{x}_{m-1}\}, \alpha_m\} \sim \min\{\tilde{x}_1, \alpha_m\}$. Hom_n^s does not conserve linear dependence of gradients, so we have to consider two cases.

a) $\frac{\partial}{\partial \tilde{x}_1}$ and $grad\ \alpha_m$ are linearly independent. Then there exists a local coordinate system such that $\alpha_m = x_2$ and we obtain $\min\{x_1, \alpha_m\} = \min\{x_1, x_2\} \sim x_1$.

b) $\frac{\partial}{\partial \tilde{x}_1}$ and $grad\ \alpha_m$ are linearly dependent. Then $\alpha_m = c\tilde{x}_1 + \beta$, where $grad\ \beta = 0$. The gradient $grad\ \tilde{x}_1$ is a linear combination of $\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_{m-1}}$ with positive coefficients, then we have $c > 0$. The almost smooth local homeomorphism Φ can be defined as $x_1 \to x_1$ when $x_1 > 0$, $x_1 \to cx_1 + \beta$ when $x_1 < 0$; ϕ is non-smooth only on G_f and transforms $\min\{x_1, \alpha_m\}$ into x_1.

2.7. Gradients of Min-type functions. The convex hull $Conv\ \{grad\ \alpha_i\}$

plays an important role in the theory. This motivates the next definition of the Gradient of Min-type functions (compare with the definition of sub-gradients in non-smooth analysis).

DEFINITION 1. *Let M_n be a Riemannian manifold, $f : M_n \to \Re^1$ be a Min-type function, $x \in M_n$. We define* **gradient** *Grad $f(x)$ as Conv $\{$ grad $\{\alpha_i(x)\}_{i=1}^m\} \in T_xM_n$, where $f(x) = \min_{i=1,\dots,m} \{\alpha_i\ (x)\}$ is a minimal representation for the germ of f at x.*

REMARK. *Grad* (x) is a simplex in T_xM of dimension $(m-1)$ at a NDR-point. One of its vertices is *grad* α_1 and $\{grad\ (\alpha_i - \alpha_1)\}_{i=1}^m$ are linearly independent edges starting at this vertex.

PROPOSITION 1. *Grad f (x) does not depend on a choice of a minimal representation.*

Proof. *Grad f* is defined by values of functions α_i at their active points, which are defined uniquely (up to a permutation of the $\{\alpha_i\}$). The definition of the *Grad* is invariant under permutations of the functions α_i.

DEFINITION 2. *Let M_n be a Riemannian manifold, $f : M_n \to \Re^1$ be a Min-type function. Then we define* **Gradient** *as the field of simplexes Grad $\subset TM$, such that Grad $\cap\, T_xM =$ Grad (x).*

The dimension of simplexes *Grad f* (x) depends on the point x, however we have the following result.

PROPOSITION 2. *Let M_n be a Riemannian manifold, $f : M_n \to \Re^1$ be a Min-type function. Then the germ of the simplex field Grad is lower semi-continuous at any non-degenerate regular point.*

REMARK. (DEFINITION OF "SEMI-CONTINUITY"). The germ of the tangent bundle is a direct product $TU = U \times \Re^n$ for a neighbourhood $U = U(x)$; a Riemannian metric allows us to identify $T_xM = \Re^n$ over different points as Euclidean spaces, using an orthonormal frame of vector fields on U. Semi-continuity of the family of subsets *Grad* $(x) \subset T_x\ M_n$ is defined for Hausdorff distance in $\Re^n$.

2.8. Simplest degenerate regular points. We demonstrate in this paper that Morse theory for Min-type functions is in a sense equivalent to Morse theory of smooth functions (there is a one-to-one correspondence between stable singularities). At the same time singularity theory for Min-type functions is more complicated than singularity theory for smooth functions. The only codimension one singularities for smooth functions are so called birth-death singularities. (The normal form of a smooth function at a point with a birth-death singularity is $x_1^3 + \sum_{i=2}^n \pm x_i^2$). At the same time, Min-type functions admit several types of codimension one degenerations.

An investigation of singularities is beyond our scope. We only point out several types of codimension one singularities.

1) $f : \Re^n \to \Re^1$, $f = \min(\alpha_1, \dots, \alpha_m)$ $m \leq n$, q is a regular point, $\{grad\ (\alpha_i - \alpha_{i-1})\}$ are linearly independent, and $f|_{G_f}$ has a birth-death singularity (an example is $\alpha_1 = x + y^3$, $\alpha_2 = 2x + y^3$).

A small perturbation decomposes a degenerate regular point at the origin into a union of two NDR-points.

2) $f : \Re^n \to \Re^1$, $f = \min(\alpha_1, \dots, \alpha_{n+2})$

1) for any $n+1$ indices $1 \leq i_1 \dots < i_{n+1} \leq m$ the gradients $grad(\alpha_{i_1} - \alpha_{i_2}), \dots, grad(\alpha_{i_n} - \alpha_{i_{n+1}+1})$ are linearly independent;

2) $0 \notin$ *Convex* $\{grad\ \alpha_i\}$.

3) Any n gradients among $\{grad\ \alpha_i\}_{i=1}^m$ are linearly independent.

REMARK 1. In this case G_f is a point, and so we must not demand that $f_{|G_f}$ is Morse.

PROPOSITION 1. *There exists a small perturbation $\alpha_1^\epsilon, ..., \alpha_{n+2}^\epsilon$ such that $f^\epsilon = \min\{\alpha_1^\epsilon, ..., \alpha_{n+2}^\epsilon\}$ has $n+2$ non-degenerate regular points around q, such that minimal representations at each point includes $n+1$ functions.*

Birth-death singularities which lead to creation-annihilations of a pair of critical points with neigbouring indices are far more interesting and important for the theory; the reader can find examples in [Ge].

2.9. Gromov's and the classical definitions of Min-type regular points. We have two definitions of regular points for Riemannian distance functions d_p:

1) Regular point of this function as a Min-type function (we shall call them M-regular);

2) Regular point in the sense of the Gromov definition, see [G1], (we shall call them G-regular); we recall this definition.

DEFINITION 1. (GROMOV [G1]). *A point q is said to be G-critical for a distance function d_p iff for any tangent vector $v \in T_qM$ there exists a shortest geodesic γ connecting p and q such that the angle between $\dot\gamma\,(q)$ and v is at most $\frac{\pi}{2}$. (The point is said to be G-regular otherwise).*

This definition admits several simple reformulations.

DEFINITION 2. *A point q is said to be G-regular for d_p iff there exists an open half-space $T_p^+M \subset T_pM$ such that $\dot\gamma\,(q) \in T_q^+M$ for any shortest geodesic γ connecting p and q.*

DEFINITION 3. *A point q is said to be G-critical for d_p iff the linear envelop of $\{\dot\gamma_i(q)\}$ (for all shortest geodesics γ_i connecting p and q) is a linear subspace in T_qM.*

DEFINITION 4. *A point q is said to be G-critical for d_p iff 0 is a convex combination of $\dot\gamma_i(q)$ (for all shortest geodesics γ_i connecting p and q).*

PROPOSITION 1. *Let M_n be a negatively curved manifold, $p, q \in M_n$ and p, q are connected by $l \le n+1$ shortest geodesics. Then the definitions (1-4) of a G-regular point are equivalent to the definition of an M-regular point for $f = d_p$.*

3. Non-degenerate critical points. We define non-degenerate critical points for Min-type functions, obtain normal forms for germs of Min-type functions at non-degenerate critical points for the action of Hom_n^s and define the index of Min-type functions at non-degenerate critical points.

We obtain a theorem on approximation of a Morse Min-type function by a smooth Morse function. This approximation preserves topological properties: there exists a one-to-one correspondence between critical points of these functions, the corresponding critical points are ϵ-close and have the same index. This theorem extends results of the classical Morse theory: Morse inequalities and the handle decomposition of a manifold, corresponding to a smooth Morse function, to the class of Morse Min-type functions.

3.1. Definition.

DEFINITION 1. *Let $f : \Re^n \to \Re^1$ be a a germ of a Min-type function at a point q, and f admits a minimal representation $f = \min_{i=1,...,k}\ \alpha_i$, at q satisfying the following properties:*

1) *The gradients: $grad\,(\alpha_1 - \alpha_2), ..., grad\,(\alpha_{m-1} - \alpha_m)$ are linearly independent at q;*
2) *$O \in Grad\, f$;*
3) *The restriction $f_{|G_f}$ is a Morse function;*

4) Any $(m-1)$ gradients among grad $\alpha_1, ..., grad\ \alpha_m$ are linearly independent at q;

Then q is said to be a **non-degenerate critical point (an NDC-point)** *for f.*

PROPOSITION 1. *q is a non-degenerate critical point for the restriction $f_{|G_f}$.*

Proof. We have $O \in Grad_q\ f$, then $\sum_{i=1}^{k}\ p_i\ grad\ \alpha_i = 0$ for some non-negative p_i, such that $\sum_{i=1}^{k}\ p_i = 1$. In particular, for any $\xi \in T_qG_f(x)$ we obtain (all α_i have the same restriction on G_f) $\sum_{i=1}^{k}\ p_i\ grad\ \alpha_i\ (\xi) = grad\ (\alpha_{i|_{G_f}})(\xi)\ = 0$. Then $grad\ (f_{|_{G_f}}) = grad\ (\alpha_{i|_{G_f}}) = 0$; hence q is a critical point of the restriction of f on G_f. (The third condition of the definition states that this critical point is non-degenerate).

REMARK. In particular, $grad\ \alpha_1(x), ..., grad\ \alpha_m\ (x)$ are linearly dependent for an NDC-point.

The previous statement can be reformulated as follows.

PROPOSITION 2. *Let $f : \Re^n \to \Re^1$ be a germ of a Min-type function at q, $f = \min_{i=1,\dots,k}\ \alpha_i$, and q is an NDC-point. Then there exists a local coordinate system $\{x_i\}$ such that the germ of G_f is a germ of a coordinate plane of dimension $n-k+1$, and the restriction $f_{|G_f}$ is given by the equation $f_{|G_f} = -\sum_{i=k}^{l}\ x_i^2 + \sum_{i=l+1}^{n}\ x_i^2$.*

Proof. The submanifold G_f is determined by the equations $\beta_1 = ... = \beta_{k-1} = 0$, (where $\beta_i = \alpha_i - \alpha_{i+1}$) with linearly independent gradients. One can choose a local coordinate system $\{x_i\}$ such that $\beta_i = x_i$, $i = 1, \dots, k-1$. Then the germ of G_f is the germ of the coordinate plane $x_1 = x_2 = ... = x_{k-1} = 0$. Denote $\bar{f}\ (x_k, ..., x_n) = f_{|G_f}$; $\bar{f}$ is a Morse function and then $\bar{f} = -\sum_{i=k}^{l} x_i^2 + \sum_{i=l+1}^{n}\ x_i^2$.

DEFINITION 2. (INDEX). *Let $f : \Re^n \to \Re^1$ be a Min-type function,*

$$f = \min_{i=1,\dots,k}\ \alpha_i$$

at the origin, and the origin is an NDC-point. Then we define the **index** *$Ind_x\ f$ of f at x as $Ind_x\ f = (k-1) + Ind_x\ (f_{|G_f})$.*

$Ind_x\ (f_{|G_f})$ is a smooth invariant of f at an SR-point, see section 2.3; however, all functions regular at x are equivalent under the action of Hom_n^s. For NDC-points the index $Ind_x f$ is a Hom_n^s-invariant and has a clear geometric interpretation (similar to the index of the Hessian for smooth functions).

PROPOSITION 3. *Let $f : \Re^n \to \Re^1$ be a germ of a Min-type function at q, and q be an NDC-point for f. Then $Ind_x\ f$ is the maximal dimension of a germ of a submanifold N_q at q such that the restriction $f_{|\mathcal{N}_q}$ has a strict minimum at q.*

In the following sections (ss. 3.2-3.4) we prove that $Ind_x\ f$ is the unique invariant of a germ of a Min-type function at an NDC-point.

3.2. Normal forms.

THEOREM 1. *Let $f : \Re^n \to \Re^1$ be a Min-type function, $f = \min\ \{\alpha_1, ..., \alpha_k\}$, and q be an NDC-point. There exists a local homeomorphism $\phi \in Hom_n^c$ which transforms $f \to -\sum_{i=1}^{m}\ z_i^2 + \sum_{i=m+1}^{n}\ z_i^2$, where $m = Ind_q f$.*

We transform f to its normal form in several steps.

PROPOSITION 1. *Let $f : \Re^n \to \Re^1$ be a germ of a Min-type function, q be an NDC-point and $f = \min\ \{\alpha_1, ..., \alpha_k\}$ be a minimal representation of f at q. Then there exists a smooth local coordinate system $x_1, ..., x_n$ around q such that $f = \min\{x_1, ..., x_{k-1}, -(\sum_{i=1}^{k-1}\ c_ix_i + \beta\ (x_1, ..., x_n))\}$, where $c_i > 0$, for $i = 1, ..., k-1$, and grad $\beta\ (q) = 0$.*

Proof. There exist $k-1$ linearly independent gradients among $grad\ \alpha_1, ...,$ $grad\ \alpha_k$. We can assume the first $(k-1)$ gradients are linearly independent and then $\alpha_1, ..., \alpha_{k-1}$ can be used as the first $(k-1)$ coordinates. The expression for the last function α_k is due to the fact that $0 \in Grad\ f$.

To obtain the normal form we present f as a sum of "smooth" and "non-smooth" parts.

PROPOSITION 2. *Let $f : \Re^n \to \Re^1$ be a germ of a Min-type function, q be an NDC-point and $f = \min \{\alpha_1, ..., \alpha_k\}$ be a minimal representation at q. Then there exists a local smooth coordinate system $\{y_1, ..., y_n\}$ such that $f = \beta(y_1, ..., y_n) + \min \left\{y_1, ..., y_{k-1}, -\sum_{i=1}^{k-1} c_i y_i\right\}$, where $c_i > 0$ for $i = 1, ..., k-1$, and $grad\ \beta\ (q) = 0$.*

Proof. Define $z_i = y_i + \lambda\beta$ for $i = 1, ..., k-1$, $z_i = y_i$ for $i = k, ..., n$. Then $\{z_1, ..., z_n\}$ is a local coordinate system and

$$f = \min \left\{ z_1 - \lambda\beta, ..., z_{k-1} - \lambda\beta,\ -\sum_{i=1}^{k-1} c_i z_i + \beta + \lambda\beta \sum_{i=1}^{k-1} c_i \right)\right\}.$$

We would like to obtain $-\lambda = 1 + \lambda \sum_{i=1}^{k-1} c_i \Leftrightarrow -\lambda(1 + \sum_{i=1}^{k} c_i) = 1$. Recall that all c_i are strictly positive; we obtain $\lambda = -(1 + \sum_{i=1}^{k-1} c_i)^{-1}$. Denote $\tilde{\beta} = -\lambda\beta$. Then we have $f = \tilde{\beta} + \min \{z_1, ..., z_{k-1}, -\sum_{i=1}^{k-1} c_i z_i\}$, and $grad\ \tilde{\beta}\ (0) = 0$.

We recall some standard results related to representations of smooth functions vanishing at the origin.

LEMMA 1. [M1]. *Let $\phi : \Re^n \to \Re^1$ be a smooth function vanishing at the origin q and $\{x_1, ..., x_n\}$ be a coordinate system around q. Then there exist smooth functions $g_1, ..., g_n$ such that $\phi = \sum_{i=1}^{n} x_i g_i$, where $g_i\ (q) = \frac{\partial \phi}{\partial x_i}(q)$.*

An iteration of this lemma gives the following result.

LEMMA 2. [M1]. *Let $\phi : \Re^n \to \Re^1$ be a smooth function vanishing at q together with all its first derivatives, and $\{x_1, ..., x_n\}$ be a local coordinate system. Then there exist germs of smooth functions $\{g_{i,j}\}_{i,j=1}^{n}$ such that $\phi = \sum_{i,j=1}^{n} x_i x_j g_{i,j}$, where $g_{i,j} = g_{j,i}$ and $g_{i,j}\ (q) = \frac{1}{2} \frac{\partial^2 \phi}{\partial x_i \partial x_j}(q)$.*

We can present the function $\tilde{\beta}$ (in Proposition 2) as $\tilde{\beta}(z_1, ..., z_n) = Const + \sum_{i,j=1}^{n} z_i z_j H_{i,j}$, where $\{H_{i,j}\}$ is symmetric and $H_{i,j}$ is a smooth function. The germ of the submanifold G_f, is the germ of the coordinate plane $z_1 = z_2 = ... = z_{k-1}$. The condition of the non-degenericity of the Hessian for the restriction of f on G_f means that the minor $(H_{i,j}\ (O))_{i,j=k}^{n}$ is non-degenerate. By analogy with the classical Morse theory, see [M1], we obtain the following statement.

PROPOSITION 3. *Let $f : \Re^n \to \Re^1$ be a germ of a Min-type function, q be an NDC-point, and $f = \min \{\alpha_1, ..., \alpha_k\}$ be a minimal representation at q. Then there exists a local coordinate system $x_1, ..., x_n$ such that $f = \min \{x_1, ..., x_{k-1}, -\sum_{i=1}^{k-1} c_i x_i\} - \sum_{i=k}^{m} x_i^2 + \sum_{i=m+1}^{n} x_i^2 + \sum_{i,j=1}^{k-1} x_i x_j H_{i,j}\ (x_1, ..., x_n)$, where $c_i \geq 0$, for $i = 1, ..., k-1,$, the matrix $\{H_{i,j}\}$ is symmetric and $H_{i,j}$ is a smooth function.*

The following clear geometric lemma finishes the proof of the theorem.

LEMMA 3. *The germs of functions $\kappa\ (x_1, ..., x_k) = -\sum_{i=1}^{k} x_i^2$ and $\psi = \min \{x_1, ..., x_k, -\sum_{i=1}^{k} c_i x_i\}$ at the origin of $\Re^k$ are Hom_k^s-equivalent.*

The normal form in a neighbourhood of a non-degenerate critical point gives, in particular, the following statement.

COROLLARY 1. *Non-degenerate critical points are isolated.*

3.3. Approximations by smooth Morse functions.

DEFINITION 1. *A Min-type function is called a* **Morse Min-type function** *if it has only non-degenerate regular and non-degenerate critical points.*

THEOREM 1. *Let f be a Morse Min-type function on a compact manifold M_n and $y_1, ..., y_m$ be its critical points with indices $q_1, ..., q_m$. Then for any $\epsilon > 0$ there exists a smooth Morse function f_ϵ satisfying the following properties:*

1) f_ϵ is an ϵ-approximation of f in the C^0-metric;

2) f_ϵ is an ϵ-approximation of f in the C^2-metric, for any neighbourhood $U(x)$, where f is smooth.

3) f_ϵ has the same number of critical points $y_1^\epsilon, ..., y_m^\epsilon$ with the same indices

$$\left(Ind_{y_i^\epsilon}\, f = Ind_{y_i} f\right).$$

4) $\rho\left(y_i, y_i^\epsilon\right) \leq \epsilon$ for $i = 1, ..., m$.

This theorem is based on results in Chapter 2 and on the next lemma.

LEMMA 1. *1) A germ of a Min-type function $f : \Re^n \to \Re^1$ at an NDC-point q of index m can be ϵ-approximated by a germ of a smooth Morse function f_ϵ such that*

a) f_ϵ is an ϵ-approximation of f in the C^0 - topology;

b) f_ϵ is an ϵ-approximation in the C^2 - topology in a neighbourhood of any point where f is smooth.

c) f_ϵ is a germ of a smooth Morse function at q and q is a critical point of index m.

Proof. Let x be a non-degenerate critical point of f and $f = \min\{\alpha_1, ..., \alpha_k\}$ be a minimal representation at x. One can choose a local coordinate system $(x_1, ..., x_n)$ around x which is a smooth transformation of the initial coordinate system and such that

$$f = \min\left\{x_1, ..., x_{k-1}, -\sum_{i=1}^{k-1} c_i x_i + \sum_{i=1}^{k-1} H_{i,j} x_i x_j\right\} - \sum_{i=k}^{m} x_i^2 + \sum_{i=m+1}^{k} x_i^2,$$

see section 3.2. Clearly the "non-smooth part" of f (in the brackets) can be approximated by $-\sum_{i=1}^{k-1} x_i^2$ which gives the necessary result.

This theorem allows us to extend all results of classical Morse theory to Morse Min-type functions.

4. Morse Min-type functions.

4.1. Morse Min-type functions.

DEFINITION 1. *Let $f : M_n \to \Re^1$, be a Min-type function $f = \min\{\alpha_1, ..., \alpha_Q\}$; f is said to be a* **Morse (Min-type) function** *iff f has only non-degenerate regular and non-degenerate critical points.*

The aim of this section is to prove the following theorem.

THEOREM 1. *The set of Morse Min-type functions is open and everywhere dense in the space of C^2-smooth Min-type functions on any compact manifold M_n.*

Note that only the second conditions are different in the definition of NDR-points and NDC-points and these conditions are complementary. We eliminate these to define **non-degenerate** points. We modify the definition slightly to apply for global functions on a compact manifold.

DEFINITION 2. *Let $f : M_n \to \Re^1, f = \min\{\alpha_1, ..., \alpha_Q\}$, be a Min-type function, $x \in M_n$ is said to be a* **non-degenerate point** *for f iff the functions $\{\alpha_1, ..., \alpha_Q\}$ satisfy the following three conditions:*

1) grad $(\alpha_{i_1} - \alpha_{i_2}), ..., grad\ (\alpha_{i_{l-1}} - \alpha_{i_l})$ *are linearly independent at each point* x *where* $\alpha_{i_1}, ..., \alpha_{i_l}$ *are active;*

2) The restriction $f_{|G_f^x}$ *is a Morse (smooth) function for each* x*, where*

$$G_f^x = \{y \mid \alpha_{i_1}\ (y) = ... = \alpha_{i_l}(y)\ ,\ where\ \alpha_{i_1}, ..., \alpha_{i_l}\ \ are\ active\ at\ x\};$$

3) Let $x \in M_n$ *and* $\alpha_{i_1}, ..., \alpha_{i_l}$ *are active at* x*. Then any* $(l-1)$ *gradients among* $grad\, \alpha_{i_1}, ..., grad\, \alpha_{i_l}$ *are linearly independent at* x*.*

DEFINITION 3. (REFORMULATION). $f : M_n \to \Re^1,\ f = \min\ \{\alpha_1, ..., \alpha_l\}$ *is said to be a* **Morse Min-type function** *iff all the points of* M_n *are non-degenerate for* f*.*

4.2. Minima of two smooth functions. The first series of lemmata relate to two smooth functions; they give the base of the induction for the general case.

LEMMA 1. *Let* M_n *be a compact manifold. Define the subset* $\Lambda_2 \subset C^2\ (M_n) \times C^2\ (M_n)$ *consisting of pairs* (α_1, α_2) *of smooth functions, satisfying the following five conditions,*

a) α_1, α_2 *are Morse functions;*
b) $\beta = \alpha_1 - \alpha_2$ *is a Morse function;*
c) 0 *is a regular level for* β *(denote* G *the hypersurface* $\beta^{-1}\ (0)$*);*
d) $grad\, \alpha_1, grad\, \alpha_2$ *do not vanish at any point of* G*;*
e) The restriction $\alpha_{1|G} = \alpha_{2|G}$ *is a smooth Morse function on* G*.*

Then Λ_2 *is open and everywhere dense in* $C^2\ (M_n) \times C^2\ (M_n)$*.*

Proof. 1) Evidently Λ_2 is open.

2) Λ_2 is everywhere dense. Each of the conditions in the Lemma is "open". We will perturb sequentially the functions α_1, α_2 to satisfy the conditions. After each step, we choose the following perturbations so small that they do not break any of the previous conditions.

a-b) Choose Morse functions $\tilde{\alpha}_1, \tilde{\alpha}_2$ close enough to α_1, α_2 and perturb slightly $\tilde{\alpha}_1$ such that $\tilde{\beta} = \tilde{\alpha}_1 - \tilde{\alpha}_2$ is also Morse.

c) Choose a small ϵ such that ϵ is a regular level for $\tilde{\alpha}_1$ and for $\tilde{\beta}$. Define $\bar{\alpha}_1 = \tilde{\alpha}_1 - \epsilon$ and $\bar{\beta} = \bar{\alpha}_1 - \alpha_2$. Then $\bar{\alpha}_1, \alpha_2$ satisfy the condition c). (To simplify notation, we shall return to the notation α_1, α_2 after each step).

d) α_1, α_2 are Morse functions and then have only a finite number of critical points on M_n. Define G^ϵ by the equation $\beta = \epsilon$. Any small enough ϵ is a regular level of β. Choose such a small ϵ that G_f^ϵ does not contain critical points of α_1 and of α_2 and define $\alpha_1^\epsilon = \alpha_1 - \epsilon$ (note that α_1 and α_1^ϵ have the same set of critical points on M_n). Then the gradients of $\alpha_1^\epsilon, \alpha_2$ do not vanish on $G_f^\epsilon = G_{\bar{f}}$, where $f = \min\ \{\alpha_1^\epsilon, \alpha_2\}$.

e) Smooth Morse functions are everywhere dense in $C^2\ (G)$. Then there exists a smooth function μ small in the $C^2\ (G)$ -topology such that $\alpha_1 + \mu_{|G}$ is Morse; μ can be extended to a Morse function $\bar{\mu}$ which is C^2-small on M_n. Then $\alpha_1 + \bar{\mu}$ and $\alpha_2 + \bar{\mu}$ are Morse functions on M_n.

COROLLARY 1. *Let* M_n *be a compact manifold. Then there exists an open and everywhere dense subset* Λ_2 *of pairs* $(\alpha_1, \alpha_2) \in C^2\ (M_n) \times C^2\ (M_n)$ *such that* $f = \min\ \{\alpha_1, \alpha_2\}$ *is a Morse Min-type function.*

4.3. Minima of n **functions.**

THEOREM 2. *Let* Λ_Q *be the set of vector functions* $\{\alpha_1, ..., \alpha_Q\} \in C^2\ (M_n) \times ... \times C^2\ (M_n)$ *satisfying the following five conditions:*

a) α_i *are Morse for all* i*;*

b) $\alpha_i - \alpha_{i+1}$ *are Morse for* $i = 1, ..., m-1$;

c) for any $i_1 < ... < i_l$ *the gradients grad* $(\alpha_{i_j} - \alpha_{i_{j+1}})$, $j = 1, ..., l-1$ *are linearly independent at any point* y *where* $\alpha_{i_1}(y) = ... = \alpha_{i_l}(y)$;

This condition means $\alpha_{i_1}(y) = ... = \alpha_{i_l}(y)$ *defines an* $(n-l+1)$ *dimensional submanifold (which we denote as* G_f^x*).*

d) Any $(l-1)$ *gradients among grad* $\alpha_{i_1}, ..., grad\ \alpha_{i_l}$ *are linearly independent, when* $\alpha_{i_1}, ..., \alpha_{i_l}$ *are active at* x.

e) $f_{|G_f^x} = \alpha_{i_1|G_f^x} = ... = \alpha_{i_l|G_f^x}$ *is a Morse (smooth) function on* G_f^x.

Then Λ_Q *is open and everywhere dense in* $C^2(M_n) \times ...C^2(M_n)$.

Proof. 1) Evidently Λ_Q is open.

2) Λ_Q is everywhere dense. We use the same scheme as in the proof of Lemma 1.

a-b) We can choose Morse functions $(\tilde{\alpha}_1, ..., \tilde{\alpha}_Q)$ close enough to $(\alpha_1, ..., \alpha_Q)$. There exists a small neighbourhood of these functions consisting of Morse functions. We can further perturb these functions $\alpha_i \to \bar{\alpha}_i$, $i = 1, ..., m$ inside these neighbourhoods such that $\bar{\alpha}_i - \bar{\alpha}_j$ are Morse for all pairs $i \neq j$.

c) We shall prove this statement by induction. Lemma 1 gives the necessary result for $Q = 2$. Assume we have the result for minima of any $Q-1$ (or less) smooth functions. All conditions are of a local character, then if $l \leq Q-1$ we obtain the necessary result from the induction assumption (since the minimal (local) representation includes not more than $Q-1$ smooth functions). Let $l = Q$, that is, we consider a point x, where $\alpha_1(x) = \ldots = \alpha_Q(x)$.

By the induction assumption *grad* $(\alpha_1 - \alpha_2), ..., grad\ (\alpha_{Q-2} - \alpha_{Q-1})$ are linearly independent at x (and then $Q \leq n+2$). The equations $\alpha_1 = ... = \alpha_{Q-1}$ define an $(n-Q+2)$ dimensional submanifold N. Denote N_x the connected component of N containing x. The restrictions of all the functions $\bar{\alpha}_1, ..., \bar{\alpha}_{Q-1}$ on N_x coincide. Consider the pair of functions α_{Q-1}, α_Q on N_x. In accordance with Lemma 1, there exists small perturbations $\tilde{\alpha}_{Q-1}, \tilde{\alpha}_Q$ of α_{Q-1}, α_Q such that *grad* $(\tilde{\alpha}_{Q-1} - \tilde{\alpha}_Q)$ does not vanish on the zero level set of $\tilde{\beta} = \tilde{\alpha}_{Q-1} - \tilde{\alpha}_Q$ on N_x. We define $\delta = \tilde{\alpha}_{Q-1} - \alpha_{Q-1}$ and $\tilde{\alpha}_i = \alpha_i + \delta$ for $i = 1, ..., Q-2$.

d) We prove this statement by induction. When $l < Q$ the induction assumption gives the necessary result. Let $l = Q$ and x is a point where $\alpha_1(x) = \ldots = \alpha_Q(x)$. Let us prove that there exist small perturbations $\tilde{\alpha}_1, ..., \tilde{\alpha}_{Q-1}, \tilde{\alpha}_Q$ of $\alpha_1, ..., \alpha_Q$ such that *grad* $\tilde{\alpha}_1, ..., grad\ \tilde{\alpha}_{Q-1}$ are linearly independent. The induction assumption guarantees that there exist small perturbations $\tilde{\alpha}_1, ..., \tilde{\alpha}_Q$ such that *grad* $\tilde{\alpha}_1, ..., grad\ \tilde{\alpha}_{Q-1}$ are linearly independent on N_x. We have to apply Lemma 1 to the functions $\tilde{\alpha}_{Q-1}, \tilde{\alpha}_Q$ on N_x.

e) The proof is the same as in Lemma 1.

Proof of Theorem 1. Clearly, any function $f \in \Lambda_Q$ is Morse and then the set of Morse functions is everywhere dense. The set of Morse functions is evidently open.

4.4. Morse distance functions.

DEFINITION 1. *Let* M_n *be an almost non-positively curved n-manifold,* $p \in M_n$. *The distance function* d_p *is said to be a* **Morse distance function** *iff* d_p *is a Morse Min-type function.*

Results presented in this paper allows us to define index of critical points for distance functions on negatively curved manifolds which have a clear geometric interpretation ($ind_q d_p = m_p(q) - 1$, where $m_p(q)$ is the number of shortest geodesies connecting p and q minus one). Also one can apply all results of the classical Morse theory to Morse distance functions. The result concerning density of Morse distance functions is correct.

THEOREM [GE]. *Let M_n be a compact manifold, $p \in M_n$ and $\rho \in \rho_-(M_n)$. Then there exists an open and everywhere dense subset of metrics in $\rho_-(M_n)$ such that d_p^ρ is a Morse distance function.*

Nevertheless, it must be proved independently in a more "geometric way". The reason is that the location of Riemannian distance functions inside the set of Min-type functions cannot be described in a reasonable way.

4.5. Morse metric on negatively curved manifolds. We point out one more application of the theory constructed.

THEOREM 1. [GE]. *Let M_n be a compact manifold, ρ be a negatively curved Riemannian metric. Then $\rho : M_n \times M_n \to \Re^1$ is a Min-type function.*

DEFINITION 1. *A negatively curved Riemannian matric ρ is said to be Morse if $\rho : M_n \times M_n \to \Re^1$ is a Morse Min-type function.*

THEOREM 2. [GE]. *Almost all negatively curved metrics on M_n are Morse.*

4.6. Positively curved manifolds. The Morse theory can be extended to distance functions on positively curved manifolds (they are not Min-type functions but admit only a finite number of additional types of stable singularities), see [GR].

4.7. Non-degenerate critical submanifolds. We can also extend our theory to the case when the distance function has critical sub-manifolds (we follow Bott's generalization of the classical Morse theory, see [Bo]).

The simplest example is the projective space $P_n\Re$ with the standard metric. The distance function d_p reaches its maximum on $P_{n-1}\Re$ and in a neighbourhood of a point $q \in P_{n-1}\Re$ it looks like $d_p(q) = \frac{\pi}{2} - \mid \phi(q) \mid$ where ϕ is the coordinate along the geodesic connecting p and q. All notions and results of the Bott generalization of the Morse theory (including the index of non-degenerate critical submanifolds and Morse inequalities) can be extended for Min-type functions.

REFERENCES

[AGV] V. ARNOLD, GUSSEIN-ZADE, AND A. VARCHENKO, *Singularities of differentiable maps*, Nauka, Moscow, 1982.

[Bo] R. BOTT, *Lectures on Morse theory Old and New*, Bull. AMS 7, 331-358(1982).

[BGP] YU. BURAGO, M. GROMOV AND G. PERELMAN, *Alexandrov spaces with curvatures bounded below*, Uspehi matem. nauk. 47:2 (1992), 3-51.

[BL] TH. BRÖKER AND L. LANDER, *Differential germs and catastrophes*, Cambridge University Press, 1975.

[B1] L.N. BRYZGALOVA, *Singularities of maximums of parametrically dependent functions*, Functional Analysis and Applications, 11(1977), 49-51.

[B2] L.N. BRYZGALOVA, *On maximum of parametrically dependent functions*, Functional Analysis and Application, 12(1977), 66-67.

[Ca] L. CARROLL, *Alice's Adventures in Wonderland.*

[C] J. CHEEGER, *Critical points of distance functions and application to geometry*, Lecture Notes in Math., 1504(1990), 1-38, Springer-Verlag.

[CGR] B. CRAVEN, V.GERSHKOVICH, AND D. RALPH, *Morse theory and relations between smooth and non-smooth optimization*, Proc. Ballarat Optimization mini-conference, University of Ballarat, July 1994.

[EI] P. EHRLICH AND H. C. IM HOF, *Dirichlet regions in manifolds without conjugate points*, Comm. Math. Helv. 54(1979), 642-658.

[Ge] V. GERSHKOVICH, *Singularity theory for distance functions on non-positively curved surfaces*, in Proc. Pacific Rim Singapore Conference, Walter de Gruyter, 1997.

[GR] V. GERSHKOVICH AND H. RUBINSTEIN, *Riemannian distance functions on surfaces with birth-death singularities*, preprint.

[GG] M. GOLUBITSKY AND V. GUILLEMIN, *Stable mapping and their singularities*, Springer-Verlag, 1973.

[G1] M. GROMOV, *Curvature, diameter, and Betti numbers*, Comm. Math. Helv., 56(1989), 179-195.

[Gr] K. GROVE, *Critical Point Theory for Distance Functions*, Proc. Symp. in Pure Math., Vol 54(1993), p. 3, 357-385.

[GS] K. GROVE AND K. SHIOHAMA, *Generalized sphere theorem* Annals of Math, (2) 106(1977), 201-211.

[Ma1] V. MATOV, *Topological classification of germs of maxima and minima of generic families of functions*, Uspehi Mat. Nauk, 37:4(1982), 129-130.

[Ma2] V. MATOV, *Singularities of functions of maximum of manifolds with boundaries*, Trudi Petrovski Seminar, 6(1981), 195-222.

[M1] J. MILNOR, *Morse theory*, Princeton Univ. Press 1963.

[M2] J. MILNOR, *Lectures on the h-cobordism theorem*, Princeton Univ. Press, 1965.

[P1] G. PERELMAN, *Spaces with curvatures bounded from below*, Proceedings of the International Congress of Mathematicians, Zurich, Switzerland, 1994, 517-525.

[P2] G. PERELMAN, *Elements of Morse theory in Alexandrov spaces*, St Petersburg Math. Journal 5:1 (1994), 205-213.