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RETICULAR LAGRANGIAN SINGULARITIES*

TAKAHARU TSUKADA†

1. Introduction. Lagrangian singularity can be found in many problems of differential geometry, calculus of variations and mathematical physics. One of the most successful their applications is the study of singularity of caustics. For example, the light rays incident along geodesics from a smooth hypersurface in a Riemannian manifold to conormal directions define a lagrangian submanifold at a point in the cotangent bundle. The caustic generated by the hypersurface is regarded as the caustic of the lagrangian map defined by the restriction of the cotangent bundle projection to the lagrangian submanifold. Therefore the study of the caustic generated by the smooth hypersurface is reduced to the study of Lagrangian singularity. In [13], I.G.Scherbak studied the case when the hypersurface has a boundary and she explained the caustic generated by the hypersurface with a boundary corresponds to a generalized notion of caustic (i.e., the *boundary caustics*).

In this paper we investigate the more general situation when the hypersurface has an *r-corner*. In this case the incident rays from each edges of the hypersurface to conormal directions gives a *regular r-cubic configuration* (cf., Section 3) at a point of the cotangent bundle which is a generalized notion of Lagrangian submanifolds. The caustic generated by the hypersurface with an r-corner is given by the caustic of the regular r-cubic configurations (cf., Section 3) which is a generalization of the notion of *quasicaustics* given by S.Janeszko (cf., [7]). In complex analytic category, the theory of regular r-cubic configurations has been developed by Nguyen Huu Duc, Nguyen Tien Dai and F.Pham (cf., [3], [6]). But their method does not work well for C^∞-category.

The main purpose of this paper is the investigation of the stability of smooth regular r-cubic configurations and the classification of *stable caustics* given by stable regular r-cubic configurations in C^∞-category. In order to realize this purpose we shall define the notion of *reticular lagrangian maps* in Section 3 which is a generalization of the notion of lagrangian maps for our situations. We shall also prove that the equivalence relation among reticular lagrangian maps is equivalent to a certain equivalence relation of corresponding generating families. In Section 5 we shall define the notion of *stability, homotopically stability, infinitesimal stability* of reticular lagrangian maps and prove that these and the stability of corresponding generating families are all equivalent.

By the above results the classification of stable caustics is reduced to the classifications of function germs. In section 7 we classify unimodal function germs with respect to *reticular R-equivalence.* This gives the classification of stable caustics in manifolds of dimension$\leq$ 6. In [14], D.Siersma classified singularities with *bundle codimension*(=R-codimension−modality) $\leq$ 4 under the same equivalence relation. Hence a part of his classification list is the same as the part of our list. We shall draw the pictures of stable caustics in manifolds of dimension$\leq$ 4 at the last part of this paper.

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†Department of Mathematics, Faculty of Science, Hokkaido University, Kita 8 Nishi 5, Kitaku, Sapporo 060, Japan (ttsukada@math.sci.hokudai.ac.jp).

2. Preliminaries. The propagation mechanism of light rays incident from a hypersurface germ with an r-corner in a smooth manifold is described as follows (Cf., [8]): Let M be an $n(=r+k+1)$-dimensional differentiable manifold and $H: T^*M\backslash 0 \to \mathbf{R}$ be a C^∞-function, called a *Hamiltonian function*, which we suppose to be everywhere positive and positively homogeneous of degree one, that is $H(\lambda\xi) = \lambda H(\xi)$ for all $\lambda > 0$ and $\xi \in T^*M\backslash 0$, where $\pi : T^*M \to M$ is the cotangent bundle. Let X_H denote the corresponding Hamiltonian vector field on $T^*M\backslash 0$, given locally by the Hamiltonian equations:

$$\dot{q}_i = \frac{\partial H}{\partial p_i},\ \dot{p}_i = -\frac{\partial H}{\partial q_i},$$

where (q,p) are local canonical coordinates of T^*M.

We set $E = H^{-1}(1)$ and consider the following canonical projections $\pi : T^*M \to M$, $\pi_E : \mathbf{R}\times E \to E$, $\pi_{\mathbf{R}} : \mathbf{R}\times E \to \mathbf{R}$. We denote E_q the fiber of the spherical cotangent bundle $\pi|_E$ at $q \in M$.

Let $q_0 \in M$, $t_0 > 0$, $\xi_0 \in E_{q_0}$ and η_0 be the image of the phase flow of X_H at (t_0, ξ_0). Since the phase flow of X_H preserves values of H, the local phase flow $\Psi : (\mathbf{R}\times T^*M\backslash 0, (t_0,\xi_0)) \to (T^*M\backslash 0, \eta_0)$ of X_H induces the map $\Phi : (\mathbf{R}\times E, (t_0,\xi_0)) \longrightarrow (\mathbf{R}\times E, (t_0,\eta_0))$ given by $\Phi(t,\xi) = (t, \Psi(t,\xi))$.

We set $exp = \pi_M \circ \Phi : (\mathbf{R}\times E, (t_0,\xi_0)) \to (M, u_0)$, $exp_{q_0} = exp|_{\mathbf{R}\times E_{q_0}}$, $exp^- = \pi_M \circ \Phi^{-1} : (\mathbf{R}\times E, (t_0,\eta_0)) \to (M, q_0)$, $exp^-_{u_0} = exp^-|_{\mathbf{R}\times E_{u_0}}$, $\phi_1 = (\pi_E, exp) : (\mathbf{R}\times E, (t_0,\xi_0)) \to (M\times M, (q_0,u_0))$, $\phi_2 = (exp^-, \pi_M) : (\mathbf{R}\times E, (t_0,\eta_0)) \to (M\times M, (q_0,u_0))$, where $u_0 = \pi(\eta_0)$. Then the following diagram is commutative:

$$\begin{array}{ccccccc}
 & (\mathbf{R}\times E, (t_0,\xi_0)) & & \xrightarrow{\Phi} & & (\mathbf{R}\times E, (t_0,\eta_0)) & \\
 & \swarrow exp & & \phi_1 \searrow \ \swarrow \phi_2 & & exp^- \searrow & \\
(M, u_0) & & \xleftarrow{\pi_2} & (M\times M, (q_0,u_0)) & \xrightarrow{\pi_1} & & (M, q_0)
\end{array}$$

By [8, 2.2] we have the following proposition

PROPOSITION 2.1. *If exp_{q_0} is regular then ϕ_1 and ϕ_2 are diffeomorphisms.*

Let exp_{q_0} be regular, we can define the function germ

$$\tau = \pi_{\mathbf{R}} \circ \phi_1^{-1} = \pi_{\mathbf{R}} \circ \phi_2^{-1} : (M\times M, (q_0,u_0)) \to (\mathbf{R}, t_0).$$

We call τ the *ray length function* associated with the regular point (t_0,ξ_0) of exp_{q_0}. Set $\xi = \pi_E \circ \phi_1^{-1} : (M\times M, (q_0,u_0)) \to (E, \xi_0)$, $\eta = \pi_E \circ \phi_2^{-1} : (M\times M, (q_0,u_0)) \to (E, \eta_0)$. By [8, Lemma 2] we have

$$d_q\tau(q,u) = -\xi(q,u),\ d_u\tau(q,u) = \eta(q,u) \quad \text{for } (q,u) \in (M\times M, (q_0,u_0)).$$

EXAMPLE. Let M be a Riemannian manifold and H be the length of covectors. Then Φ maps each covector in time t a distance t along the geodesic and hence $\tau(q,u)$, $(q,u) \in (M\times M, (q_0,u_0))$, is the length of geodesic which connects q and u. In particular if M be a Euclidean space $\mathbf{R}^n$, then $\Phi(t,q,p) = (q + \frac{p}{|p|}t, p)$ and $\tau(q,u) = |q-u|$, where (q,p) are canonical coordinate of $T^*\mathbf{R}^n$ and $q,u \in \mathbf{R}$.

Let $\mathbf{H}^r = \{(x_1,\cdots,x_r) \in \mathbf{R}^r | x_1 \geq 0, \cdots, x_r \geq 0\}$ be an r-corner. Let V^0 be the hypersurface germ in (M, q_0) satisfying $\xi_0|_{T_{q_0}V^0} = 0$ with an r-corner defined as the image of an immersion $\iota : (\mathbf{H}^r \times \mathbf{R}^k, 0) \to (M, q_0)$. We parameterize V^0 by ι. For each $\sigma \subset I_r = \{1,\cdots,r\}$ we define Λ^0_σ by the set of conormal vectors of $V^0_\sigma := V^0 \cap \{x_\sigma = 0\}$

in (E,ξ_0) as the initial rays incident from V_σ^0. Then we regard the set L_σ the image of covectors in Λ_σ^0 by $\pi_E\circ\Phi$ around time t_0, that is

$$L_\sigma=\{\pi_E\circ\Phi(t,\xi)\in(E,\eta_0)|(t,\xi)\in(\mathbf{R},t_0)\times\Lambda_\sigma^0\},$$

as the set of rays incident from V_σ^0 at time t_0. We also regard the union of L_σ for all $\sigma\subset I_r$ as the set of rays incident from the hypersurface V^0 at time t_0.

Let C_σ be the critical values of $\pi|_{L_\sigma}$ for $\sigma\subset I_r$ and $Q_{\sigma,\tau}=\pi(L_\sigma\cap L_\tau)$ for $\sigma\neq\tau\subset I_r$. We define the *caustic* of the light rays incident from V^0 around q_0 by

$$\bigcup_{\sigma\subset I_r}C_\sigma\cup\bigcup_{\sigma\neq\tau}Q_{\sigma,\tau}.$$

The meaning of the caustic is the following: For example, consider the case $r=2,k=0$. Then $V_0=\{(q_1,q_2,q_3)|q_1\geq 0,q_2\geq 0,q_3=0\}$ for coordinates (q_1,q_2,q_3) of (M,q_0). By the remark of the definition of the caustic in Section 3 we have $\bigcup_{\sigma\neq\tau}Q_{\sigma,\tau}=Q_{\emptyset,1}\cup Q_{\emptyset,2}\cup Q_{1,\{1,2\}}\cup Q_{2,\{1,2\}}$. The bright points generated by incident rays from V^0, $V^0\cap\{q_1=0\}$, $V^0\cap\{q_2=0\}$ and $V^0\cap\{q_1=q_2=0\}$ are $C_\emptyset$, C_1, C_2 and $C_{1,2}$ respectively. On the other hand the light rays incident from the boundary of V^0, $V^0\cap\{q_1=0\}$ and $V^0\cap\{q_2=0\}$ are $Q_{\emptyset,1}\cup Q_{\emptyset,2}$, $Q_{1,\{1,2\}}$ and $Q_{2,\{1,2\}}$ respectively. They appear as the boundary of the shadow defined by the boundary of light rays incident from V^0, $V^0\cap\{q_1=0\}$ and $V^0\cap\{q_2=0\}$ respectively. This definition is a natural extension of *quasicaustic* defined in [7].

The family of lagrangian submanifolds $\{L_\sigma\}_{\sigma\subset I_r}$ is 'generated' by the ray length function τ as the following:

PROPOSITION 2.2. *Let V^0 be the hypersurface germ in (M,q_0) satisfying $\xi_0|_{T_{q_0}V^0}=0$ defined as the image of an immersion $\iota:(\mathbf{H}^r\times\mathbf{R}^k,0)\to(M,q_0)$. Let L_σ be the set of rays incident from $V_\sigma^0:=V^0\cap\{x_\sigma=0\}$ at time t_0 for $\sigma\subset I_r$. Define $F:=\tau\circ(\iota\times id_u)-t_0\in\mathrm{m}(r;k+m)$. Then the following hold:*
(1)

$$\operatorname{rank}\begin{pmatrix}\dfrac{\partial^2F}{\partial x\partial u}\\[2ex]\dfrac{\partial^2F}{\partial y\partial u}\end{pmatrix}_0=r+k.$$

(2)

$$L_\sigma=\{d_uF(x,y,u)\in(T^*M\backslash 0,\eta_0)|x_\sigma=d_{x_{I_r-\sigma}}F(x,y,u)=d_yF(x,y,u)=0\}$$

for $\sigma\subset I_r$, where we identify (M,u_0) and $(\mathbf{R}^n,0)$ by coordinates $(u_1,\cdots,u_n)$ of (M,u_0).

Proof. By [8, Sublemma] we have

$$\begin{pmatrix}d_ud_xF\\ d_uF\end{pmatrix}:T_{u_0}M\to T_{q_0}^*V^0\oplus\mathbf{R}$$

is an isomorphism. This means (1).

Let $\sigma\subset I_r$ and $\eta_u\in(E,\eta_0)$. Then $\eta_u\in L_\sigma$ if and only if $\eta_u=\pi_E\circ\Phi(t,\xi_q)$ for some $\xi_q\in E_q$ and $t\in(\mathbf{R},t_0)$ satisfying $q\in V_\sigma^0$ and $\xi_q|_{T_qV_\sigma^0}=0$ if and only if

$\eta_u = d_u\tau(q,u)$ for some $q \in V^0_\sigma$ and $u \in (M, u_0)$ satisfying $d_q\tau(q,u)|_{T_qV^0_\sigma} = 0$ and this holds if and only if $\eta_u = d_uF(x,y,u)$ for some $(x,y,u) \in (\mathbf{H}^r \times \mathbf{R}^{k+m}, 0)$ satisfying $x_\sigma = 0$ and $d_{x_{I_r-\sigma}}F(x,y,u) = d_yF(x,y,u) = 0$. $\square$

The stability of the caustic of V_0 under perturbations of ι with respect to the fixed Hamiltonian function is studied in [15]. Otherwise let $H = \{(q_1,\cdots,q_n) \in (\mathbf{R}^n,0)|q_1 \geq 0,\cdots,q_r \geq 0, q_{r+1} = \cdots = q_n = 0\}$ be an r-corner and L^0_σ be the conormal bundle of $H \cap \{x_\sigma = 0\}$ for $\sigma \subset I_r$. By theorem 3.2(2), proposition 2.2 implies that the family of lagrangian submanifold $\{L_\sigma\}_{\sigma\subset I_r}$ is a *regular r-cubic configuration*, that is there exists a symplectomorphism $S : (T^*\mathbf{R}^n,0) \to (T^*M\backslash 0, \eta_0)$ such that

$$L_\sigma = S(L^0_\sigma) \quad \text{for} \quad \sigma \subset I_r.$$

Hence in this paper we investigate the stability of the caustic under perturbations of the corresponding symplectomorphism. Generally the stability of the caustic under perturbations of the symplectomorphism is more stronger one of the perturbations of the immersion. Because a small perturbation of the immersion implies a small perturbation of the symplectomorphism, but for any perturbation of the immersion the corresponding lagrangian submanifold L_σ is included in E for all $\sigma \subset I_r$. The stability of the caustic under perturbations of the immersion is studied in [15].

In order to realize our investigation we shall define *reticular lagrangian maps* in more general situation.

3. Reticular lagrangian maps. Here we shall define reticular lagrangian maps, their caustics and equivalence relations.

Let (q,p) be canonical coordinates of $(T^*\mathbf{R}^n,0)$ and $\pi : (T^*\mathbf{R}^n,0) \to (\mathbf{R}^n,0)$ be the cotangent bundle. Let $H = \{(q_1,\cdots,q_n) \in (\mathbf{R}^n,0)|q_1 \geq 0,\cdots,q_r \geq 0, q_{r+1} = \cdots = q_n = 0\}$ be an r-corner and $H_\sigma = \{(q_1,\cdots,q_n) \in H|q_\sigma = 0\}$ be an edge of H for $\sigma \subset I_r$. We define L^0_σ the conormal bundle of H_σ, that is

$$L^0_\sigma = \{(q,p) \in (T^*\mathbf{R}^n,0)|q_\sigma = p_{I_r-\sigma} = q_{r+1} = \cdots = q_n = 0, q_{I_r-\sigma} \geq 0\}.$$

We define a representative of the union of L^0_σ for all $\sigma \subset I_r$ by

$$\mathbf{L}^0 = \{(q,p) \in T^*\mathbf{R}^n|q_1p_1 = \cdots = q_rp_r = q_{r+1} = \cdots = q_n = 0, q_{I_r} \geq 0\}.$$

DEFINITION 3.1. We call the map germ

$$(\mathbf{L}^0,0) \xrightarrow{i} (T^*\mathbf{R}^n,0) \xrightarrow{\pi} (\mathbf{R}^n,0)$$

a *reticular lagrangian map* if there exists a symplectomorphism S on $(T^*\mathbf{R}^n,0)$ such that $i = S|_{\mathbf{L}^0}$. We call S an extension of i and call $\{i(L^0_\sigma)\}_{\sigma\subset I_r}$ the (symplectic) *regular r-cubic configuration* associated with $\pi \circ i$.

REMARK. The definition of symplectic regular r-cubic configurations in the complex analytic category by N.H.Duc [6, p. 631] is as follows: If there exists a symplectomorphism S such that $L_\sigma = S(\{q_\sigma = p_{I_r-\sigma} = q_{r+1} = \cdots = q_n = 0\})$ for $\sigma \subset I_r$ then $\{L_\sigma\}_{\sigma\subset I_r}$ is called a symplectic regular r-cubic configuration.

Caustics: Let $\pi \circ i$ be a reticular lagrangian map. Let C_σ be the caustic of the lagrangian map $\pi \circ i|_{L^0_\sigma}$ for $\sigma \subset I_r$ (i.e., the critical value set of $\pi \circ i|_{L^0_\sigma}$) and let $Q_{\sigma,\tau} = \pi \circ i(L^0_\sigma \cap L^0_\tau)$ for $\sigma \neq \tau \subset I_r$. We define the *caustic* of $\pi \circ i$ by

$$\bigcup_{\sigma\subset I_r} C_\sigma \cup \bigcup_{\sigma\neq\tau} Q_{\sigma,\tau}.$$

We remark that for $\tau_1, \tau_2 \subset I_r$ $(\tau_1 \neq \tau_2)$ we have $Q_{\tau_1,\tau_2} \subset Q_{\sigma,\sigma\cup\{i\}}$, where $\sigma = \tau_1 \cap \tau_2$ and i be any element of $(\tau_1 - \sigma) \cup (\tau_2 - \sigma)$. This means that $\bigcup_{\sigma\neq\tau} Q_{\sigma,\tau}$ is equal to the union of $Q_{\sigma,\tau}$ for $\sigma \subset \tau \subset I_r$, $\#(\tau - \sigma) = 1$. For example, in the case $r = 2$ we have

$$\bigcup_{\sigma\neq\tau} Q_{\sigma,\tau} = Q_{\emptyset,1} \cup Q_{\emptyset,2} \cup Q_{1,\{1,2\}} \cup Q_{2,\{1,2\}}.$$

__Equivalence relations__: We call a homeomorphism germ $\phi : (\mathbf{L}^0, 0) \longrightarrow (\underline{f}L^0, 0)$ a *reticular diffeomorphism* if there exists a diffeomorphism Φ on $(T^*\mathbf{R}^n, 0)$ such that $\phi = \Phi|_{\mathbf{L}^0}$ and $\phi(L^0_\sigma) = L^0_\sigma$ for $\sigma \subset I_r$. We say that reticular lagrangian maps $\pi \circ i_1, \pi \circ i_2 : (\mathbf{L}^0, 0) \to (T^*\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ are *lagrangian equivalent* if there exist a reticular diffeomorphism ϕ and a lagrangian equivalence Θ of π such that the following diagram is commutative:

$$\begin{array}{ccccc}
(\mathbf{L}^0, 0) & \xrightarrow{i_1} & (T^*\mathbf{R}^n, 0) & \xrightarrow{\pi} & (\mathbf{R}^n, 0) \\
\phi \downarrow & & \Theta \downarrow & & g \downarrow \\
(\mathbf{L}^0, 0) & \xrightarrow{i_2} & (T^*\mathbf{R}^n, 0) & \xrightarrow{\pi} & (\mathbf{R}^n, 0),
\end{array}$$

where g is the diffeomorphism of the base space of π induced from Θ.

We remark that if reticular lagrangian maps $\pi \circ i_1, \pi \circ i_2$ are lagrangian equivalent then all lagrangian maps $\pi \circ i_1|_{L^0_\sigma}, \pi \circ i_2|_{L^0_\sigma}$ are lagrangian equivalent.

Here we shall define generating families of reticular lagrangian maps and study the relations between reticular lagrangian maps and their generating families. At first, we define several notations of function germs which are used as generating families of reticular lagrangian maps.

Let $\mathbf{H}^r = \{(x_1, \cdots, x_r) \in \mathbf{R}^r | x_1 \geq 0, \cdots, x_r \geq 0\}$ be an r-corner. Let $\mathcal{E}(r; l)$ be the set of smooth function germs on $(\mathbf{H}^r \times \mathbf{R}^l, 0)$ and $\mathrm{m}(r; l) = \{f \in \mathcal{E}(r; l) | f(0) = 0\}$ be its maximal ideal. We denote simply $\mathcal{E}(l)$ for $\mathcal{E}(0; l)$ and $\mathrm{m}(l)$ for $\mathrm{m}(0; l)$ and denote $\mathcal{B}(r; l)$ the set of diffeomorphism germs on $(\mathbf{H}^r \times \mathbf{R}^l, 0)$ which preserve $(\mathbf{H}^r \cap \{x_\sigma = 0\}) \times \mathbf{R}^l$ for all $\sigma \subset I_r$. We remark that a diffeomorphism germ ϕ on $(\mathbf{H}^r \times \mathbf{R}^l, 0)$ is an element of $\mathcal{B}(r; l)$ if and only if ϕ is written in the form:

$$\phi(x, y) = (x_1 a_1(x, y), \cdots, x_r a_r(x, y), b_1(x, y), \cdots, b_l(x, y)) \text{ for } (x, y) \in (\mathbf{H}^r \times \mathbf{R}^l, 0),$$

where $a_1, \cdots, a_r, b_1, \cdots, b_r \in \mathcal{E}(r; l)$ and $a_1(0) > 0, \cdots, a_r(0) > 0$.

We say that function germs $f, g \in \mathrm{m}(r; l)$ are *reticular R-equivalent* if there exists $\phi \in \mathcal{B}(r; l)$ such that $g = f \circ \phi$.

We say that function germs $F(x, y, u), G(x, y, u) \in \mathrm{m}(r; k+n)$, where $x \in \mathbf{H}^r$, $y \in \mathbf{R}^k$ and $v \in \mathbf{R}^n$, are *reticular R^+-equivalent* (as n-dimensional unfoldings) if there exist $\Phi \in \mathcal{B}(r; k+n)$ and $\alpha \in \mathrm{m}(n)$ satisfying the following:

(1) $\Phi = (\phi, \psi)$, where $\phi : (\mathbf{H}^r \times \mathbf{R}^{k+n}, 0) \to (\mathbf{H}^r \times \mathbf{R}^k, 0)$ and $\psi : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$.

(2) $G(x, y, u) = F(\phi(x, y, u), \psi(u)) + \alpha(u)$ for $(x, y, u) \in (\mathbf{H}^r \times \mathbf{R}^{k+n}, 0)$.

We say (Φ, α) a reticular R^+-isomorphism from G to F and if $\alpha = 0$ we say that F and G are reticular R-equivalent.

We say that function germs $F(x, y_1, \cdots, y_{k_1}, u) \in \mathrm{m}(r; k_1+n)$ and $F(x, y_1, \cdots, y_{k_2}, u) \in \mathrm{m}(r; k_2 + n)$ are *stably reticular R^+-equivalent* if F and G are reticular R^+-equivalent after additions of non-degenerate quadratic forms in the variables y.

A function germ $F(x, y, u) \in \mathrm{m}^2(r; k + n)$ is called *non-degenerate* if

$$x_1, \cdots, x_r, \frac{\partial F}{\partial x_1}, \cdots, \frac{\partial F}{\partial x_r}, \frac{\partial F}{\partial y_1}, \cdots, \frac{\partial F}{\partial y_k}$$

are independent on $(\mathbf{H}^k \times \mathbf{R}^{k+n}, 0)$, that is

$$\operatorname{rank}\begin{pmatrix} \dfrac{\partial^2 F}{\partial x \partial y} & \dfrac{\partial^2 F}{\partial x \partial u} \\ \dfrac{\partial^2 F}{\partial y \partial y} & \dfrac{\partial^2 F}{\partial y \partial u} \end{pmatrix}_0 = r + k.$$

We remark that $F(x, y, u) \in \mathrm{m}^2(r; k+n)$ is non-degenerate only if $r \leq n$.

Let $\pi \circ i$ be a reticular lagrangian map and $F(x, y, q) \in \mathrm{m}(r; k+n)^2$ be a non-degenerate function germ. We call F a *generating family* of $\pi \circ i$ if $F|_{x_\sigma = 0}$ is a generating family of $i(L^0_\sigma)$ for $\sigma \subset I_r$, that is

$$i(L^0_\sigma) = \{(q, \frac{\partial F}{\partial q}(x, y, q)) \in (T^*\mathbf{R}^n, 0) | x_\sigma = \frac{\partial F}{\partial x_{I_r - \sigma}} = \frac{\partial F}{\partial y} = 0\} \text{ for } \sigma \subset I_r.$$

We also call F a generating family of $\{i(L^0_\sigma)\}_{\sigma \subset I_r}$

In the case $r = 0$, this definition is the same as that of the generating family of a lagrangian map(cf., [1]).

THEOREM 3.2. (1) *For any reticular lagrangian map $\pi \circ i$, there exists a function germ $F \in \mathrm{m}(r; k+n)^2$ which is a generating family of $\pi \circ i$.*
(2) *For any non-degenerate function germ $F \in \mathrm{m}(r; k+n)^2$, there exists a reticular lagrangian map of which F is a generating family.*
(3) *Two reticular lagrangian maps are lagrangian equivalent if and only if their generating families are stably reticular R^+-equivalent.*

We remark that there exists an analogous result of this theorem for complex analytic categoly in [3, P. 13 Théoréme]. But its proof does not work well for C^∞-categoly because F_t in 'Preuve du lemme i' may be degenerate for some $t \in [0, 1]$. Our proof is available for complex analytic and real analytic categoly.

Proof. (1) Let $\pi \circ i$ be a reticular lagrangian map and S an extension of i. Let P_S be the canonical relation associated with S, that is

$$P_S = \{(Q, P; q, p) \in (T^*\mathbf{R}^n \times T^*\mathbf{R}^n, 0) | (q, p) = S(Q, P)\},$$

where (Q, P) is canonical coordinates of the domain. By considering a lagrangian equivalence of $\pi \circ i$, we may assume that there exists a generating function $T(Q, p)$ of P_S, that is

$$P_S = \{\ (Q, -\frac{\partial T}{\partial Q}(Q, p); -\frac{\partial T}{\partial p}(Q, p), p)\ \}.$$

Define $F \in \mathrm{m}(r; n+n)^2$ by

$$F(x, y, q) = T(x_1, \cdots, x_r, 0, \cdots, 0; y_1, \cdots, y_n) + \sum_{i=1}^{n} y_i q_i.$$

Since T is a generating function of P_S, rank $\frac{\partial^2 T}{\partial x \partial y}(0) = r$. Hence

$$\begin{pmatrix} \dfrac{\partial^2 F}{\partial x \partial y} & \dfrac{\partial^2 F}{\partial x \partial q} \\ \dfrac{\partial^2 F}{\partial y^2} & \dfrac{\partial^2 F}{\partial y \partial q} \end{pmatrix}_0 = \begin{pmatrix} \dfrac{\partial^2 T}{\partial x \partial y} & 0 \\ \dfrac{\partial^2 T}{\partial y^2} & E_n \end{pmatrix}_0$$

has rank $r+n$. This means that F is non-degenerate.

Otherwise, we have

$$\begin{aligned}
&\{(q, \frac{\partial F}{\partial q}(x,y,q))|x_\sigma = \frac{\partial F}{\partial x_{I_r-\sigma}} = \frac{\partial F}{\partial y} = 0\} \\
&= \{(q,y)|x_\sigma = \frac{\partial T}{\partial x_{I_r-\sigma}}(x,0,y) = \frac{\partial T}{\partial y}(x,0,y) + q = 0\} \\
&= \{(q,p)|Q_\sigma = \frac{\partial T}{\partial Q_{I_r-\sigma}}(Q,p) = Q_{r+1} = \cdots = Q_n = \frac{\partial T}{\partial p}(Q,p) + q = 0, Q_{I_r-\sigma} \geq 0\} \\
&= \{(-\frac{\partial T}{\partial p}(Q,p),p)|Q_\sigma = \frac{\partial T}{\partial Q_{I_r-\sigma}}(Q,p) = Q_{r+1} = \cdots = Q_n = 0, Q_{I_r-\sigma} \geq 0\} \\
&= S(L_\sigma^0) = i(L_\sigma^0)
\end{aligned}$$

for $\sigma \subset I_r$. Hence F is a generating family of $\pi \circ i$.

(2) Let $F \in \mathrm{m}(r; k+n)^2$ be an non-degenerate function germ. Choose an $(n-r)\times k$ matrix A and an $(n-r)\times n$ matrix B such that

$$\begin{pmatrix} \dfrac{\partial^2 F}{\partial x \partial y} & \dfrac{\partial^2 F}{\partial x \partial q} \\ \dfrac{\partial^2 F}{\partial y^2} & \dfrac{\partial^2 F}{\partial y \partial q} \\ A & B \end{pmatrix}_0$$

is invertible. Let $F'(x,y,q) \in \mathrm{m}(r+k+n)^2$ be an extension of F and define $G \in \mathrm{m}(k+n+n)^2$ by $G(y,x,x',q) = F'(x,y,q) + x'Ay^t + x'Bq^t$, where $y \in \mathbf{R}^k$, $(x_1,\cdots,x_r,x_1',\cdots,x_{n-r}') \in \mathbf{R}^n$ and $q \in \mathbf{R}^n$. Since $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial x'}, \frac{\partial F}{\partial y}$ are independent,

$$P = \{\ (x,x',-\frac{\partial G}{\partial x},-\frac{\partial G}{\partial x'};q,\frac{\partial G}{\partial q})\ |\ \frac{\partial G}{\partial y} = 0\}$$

is the canonical relation associated with a symplectomorphism S. Hence F is a generating family of the reticular lagrangian map $\pi \circ S|_{\mathbf{L}^0}$.

(3) By using analogous methods of the proof of [1, p.304 Theorem], it is enough to prove the following assertion:

Let $F_0(x,y,q)$, $F_1(x,y,q) \in \mathrm{m}(r;k+n)^2$ be non-degenerate function germs. If F_0 and F_1 are generating families of the same reticular lagrangian map, then F_0 and F_1 are reticular R-equivalent.

We suppose Lemma 3.3 and Lemma 3.4 and begin to prove this assertion. By using analogous methods of the proof of D.(a) $\sim$ (d) in [1, p.304 Theorem], we may assume that

$$F(y,q) := F_0(0,y,q) = F_1(0,y,q), \tag{1}$$

$$\frac{\partial^2 F}{\partial y^2}(0) = 0, \frac{\partial^2 F}{\partial y_J q}(0) = E_k, J \subset \{1,2,\cdots,n\}, |J| = k. \tag{2}$$

We may assume by (1), (2) and Lemma 3.3 that

$$j^2 F_0(0) = j^2 F_1(0). \tag{3}$$

We may assume by (2), (3) and Lemma 3.4 that

$$\Sigma^\sigma := \Sigma^\sigma_{F_0} = \Sigma^\sigma_{F_1}, \frac{\partial F_0}{\partial q} - \frac{\partial F_1}{\partial q}|_{\Sigma^\sigma} \equiv 0 \text{ for all } \sigma \subset I_r, \tag{4}$$

where $\Sigma^\sigma_{F_i} = \{ (x,y,q) \in (\mathbf{H}^r \times \mathbf{R}^{k+n}, 0) \mid x_\sigma = \frac{\partial F_i}{\partial x_{I_r - \sigma}} = \frac{\partial F_i}{\partial y} = 0\}$, $i = 1,2$.

Define the function germ $\bar{F}$ on $(\mathbf{H}^r \times \mathbf{R}^{k+n+1}, 0 \times [0,1])$ by $\bar{F}(x,y,q,t) = (1-t)F_0(x,y,q) + tF_1(x,y,q)$, $t \in [0,1]$ and set $F_t \in \mathfrak{m}(r;k+n)^2$ by $F_t(x,y,q) = \bar{F}(x,y,q,t)$ for each $t \in [0,1]$. Since $j^2F_0(0) = j^2F_1(0)$, F_t is non-degenerate and hence $\Sigma^\sigma_{F_t} = \Sigma^\sigma$ for all $t \in [0,1]$ and $\sigma \subset I_r$ because $\Sigma^\sigma_{F_0} = \Sigma^\sigma_{F_1}$. Hence we have by hypothesis that

$$(x,y,q) \in \Sigma^\sigma \Rightarrow \frac{\partial F_t}{\partial q}(x,y,q) = \frac{\partial F_0}{\partial q}(x,y,q) \ (\ \forall t \in [0,1] \ , \forall \sigma \subset I_r \ ).$$

We now claim that $\frac{\partial \bar{F}}{\partial t}$ is written in the form:

$$\frac{\partial \bar{F}}{\partial t} = \sum_{i=1}^r \xi_i x_i \frac{\partial \bar{F}}{\partial x_i} + \sum_{j=1}^k \eta_j \frac{\partial \bar{F}}{\partial y_j}$$

for some smooth function germs $\xi_1, \cdots, \xi_r, \eta_1, \cdots, \eta_k$ on $(\mathbf{H}^r \times \mathbf{R}^{k+n+1}, 0 \times [0,1])$.

Fix $\sigma \subset I_r$, $(x,y,q) \in \Sigma^\sigma$ and let $c : [0,1] \to \Sigma^\sigma$, $t \mapsto (x(t), y(t), q(t))$, be a smooth path connects 0 and (x,y,q). Then

$$(F_1 - F_0)(x,y,q) = \int_0^1 \frac{d}{dt}(F_1 - F_0)(c(t))dt$$

$$= \int_0^1 (\sum_{j \in I_r - \sigma} \frac{\partial(F_1 - F_0)}{\partial x_j}\frac{dx_j}{dt}(t) + \sum_{j=1}^k \frac{\partial(F_1 - F_0)}{\partial y_j}\frac{dy_j}{dt}(t) + \sum_{i=1}^n \frac{\partial(F_1 - F_0)}{\partial q_i}\frac{dq_i}{dt}(t))dt$$

Since $\frac{\partial F_i}{\partial x_{I_r - \sigma}} = \frac{\partial F_i}{\partial y} = 0$ $(i = 1,2)$, $\frac{\partial(F_1 - F_0)}{\partial q} = 0$ on Σ^σ by (4), we have $(F_1 - F_0)(x,y,q) = 0$. Therefore $(F_1 - F_0)|_{\bigcup_{\sigma \subset I_r} \Sigma^\sigma} = 0$. This means that $\frac{\partial \bar{F}}{\partial t} = 0$ on the set

$$\{(x,y,q,t) \mid x_1 \frac{\partial \bar{F}}{\partial x_1} = \cdots = x_r \frac{\partial \bar{F}}{\partial x_r} = \frac{\partial \bar{F}}{\partial y} = 0 \ \}.$$

Since $x_1, \cdots, x_r, \frac{\partial \bar{F}}{\partial x_1}, \cdots, \frac{\partial \bar{F}}{\partial x_r}, \frac{\partial \bar{F}}{\partial y_1}, \cdots, \frac{\partial \bar{F}}{\partial y_k}$ are independent on $(\mathbf{H}^r \times \mathbf{R}^{k+n+1}, \{0\} \times [0,1])$, we obtain the claim. Moreover since $j^2 \frac{\partial F_t}{\partial t}(0) = 0$, we have $\eta(0,t) = 0$ for $t \in [0,1]$.

Since the time dependent vector field

$$X = \sum_{i=1}^r \xi_i x_i \frac{\partial}{\partial x_i} + \sum_{j=1}^k \eta_j \frac{\partial}{\partial y_j}$$

vanishes on $\{x = y = q = 0\}$, the flow $\Phi_t(x,y,q)$ of X with the initial condition $\Phi_0(x,y,q) = (x,y,q)$ exists for all $t \in [0,1]$. By the uniqueness of the flow, Φ_t is written in the following form:

$$\Phi_t(x,y,q) = (x_1 a^1_t(x,y,q), \cdots, x_r a^r_t(x,y,q), h_t(x,y,q), q),$$

for each $t \in [0,1]$. Then Φ_1 defines a reticular right equivalence from F_0 to F_1, that is $F_1 \circ \Phi_1 = F_0$. □

LEMMA 3.3. *Let $F_1, F_2 \in \mathrm{m}(r; k+n)^2$ be non-degenerate function germs. Suppose that the following conditions hold:*

$$L_\sigma := L^\sigma_{F_1} = L^\sigma_{F_2} \text{ for } \sigma = I_r - \{1\}, \cdots, I_r - \{r\}, \emptyset,$$

$$F := F_1|_{x=0} = F_2|_{x=0},$$

$$\frac{\partial^2 F}{\partial y^2}(0) = 0, \frac{\partial^2 F}{\partial y \partial q_J}(0) = E_k \quad (J \subset I_r, |J| = k),$$

where $\{L^\sigma_{F_i}\}_{\sigma \subset I_r}$ be the symplectic regular r-cubic configuration defined by F_i. Then there exist positive numbers $a_1, \cdots, a_r$ and an $r \times k$-matrix B such that $F_3(x,y,q) = F_2(a_1x_1, \cdots, a_rx_r, y + xB, q)$ satisfies $j^2F_1(0) = j^2F_3(0)$.

As a result, F_1 and F_3 are generating families of the same reticular lagrangian map and the conditions (1) and (2) in the proof of Theorem 3.2 (3) hold for F_1 and F_3.

Proof. Let $I = I_n - J$. We denote $\frac{\partial^2 F_i}{\partial x \partial y}(0)$ by F^i_{xy} and denote other notations analogously. By hypothesis we have

$$L_\sigma = \{ \ (q, \frac{\partial F_i}{\partial q}(x,y,q)) \mid x_\sigma = \frac{\partial F_i}{\partial x_{I_r - \sigma}}(x,y,q) = \frac{\partial F_i}{\partial y}(x,y,q) = 0\},$$

for $\sigma \subset I_r$, $i = 1,2$. Therefore for any vector v in T_0L_σ, there exists $(a^i_\tau, b^i, c^i) \in \mathbf{R}^{|\tau|+k+n}$ (i=1,2) such that $a_\tau \geq 0$,

$$\begin{pmatrix} F^i_{x_\tau^2} & F^i_{x_\tau y} & F^i_{x_\tau q} \\ F^i_{yx_\tau} & F_{y^2}(=0) & F_{yq} \end{pmatrix} \begin{pmatrix} a^i_\tau \\ b^i \\ c^i \end{pmatrix} = 0 \tag{5}$$

and

$$v = c^i \frac{\partial}{\partial q}|_0 + \left(F^i_{qx_\tau} a^i_\tau + F_{qy} b^i + F_{qq} c^i\right) \frac{\partial}{\partial \kappa}|_0, \tag{6}$$

where $\tau = I_r - \sigma$ and (q, κ) are the canonical coordinates of $T^*\mathbf{R}^l$. Since

$$\mathrm{rank} \begin{pmatrix} F^i_{x_\tau y} & F^i_{x_\tau q} \\ F_{y^2} & F_{yq} \end{pmatrix} = |\tau| + k,$$

we can arbitrarily choose a^i_τ.

Fix $(a^1_\tau, b^1, c^1), (a^2_\tau, b^2, c^2)$ which define the same vector in $T^*\mathbf{R}^n$. By comparing the coefficients of $\frac{\partial}{\partial q}|_0$, $\frac{\partial}{\partial \kappa_I}|_0$, $\frac{\partial}{\partial \kappa_J}|_0$ of (6), we have

$$c^1 = c^2 \ (= c)$$

$$F^1_{q_I x_\tau} a^1_\tau + F_{q_I y} b^1 = F^2_{q_I x_\tau} a^2_\tau + F_{q_I y} b^2 \tag{7}$$

$$F^1_{q_J x_\tau} a^1_\tau + b^1 = F^2_{q_J x_\tau} a^2_\tau + b^2. \tag{8}$$

By (5), we have $F^1_{yx_\tau} a^1_\tau + F_{yq} c = F^2_{yx_\tau} a^2_\tau + F_{yq} c$. Hence

$$F^1_{yx_\tau} a^1_\tau = F^2_{yx_\tau} a^2_\tau. \tag{9}$$

By (7)$-F_{q_I y}$(8), we have

$$(F^1_{q_I x_\tau} - F_{q_I y} F^1_{q_J x_\tau}) a^1_\tau = (F^2_{q_I x_\tau} - F_{q_I y} F^2_{q_J x_\tau}) a^2_\tau. \tag{10}$$

By ($(9)^t$,$(10)^t$), we have

$$a^{1\,t}_\tau (F^1_{x_\tau y}, F^1_{x_\tau q_I} - F^1_{x_\tau q_J} F_{y q_I}) = a^{2\,t}_\tau (F^2_{x_\tau y}, F^2_{x_\tau q_I} - F^2_{x_\tau q_J} F_{y q_I}). \tag{11}$$

Otherwise, since F_i is non-degenerate, we have

$$\operatorname{rank}\begin{pmatrix} F^i_{x_\tau y} & F^i_{x_\tau q} \\ F^i_{y^2} & F^i_{yq} \end{pmatrix} = \operatorname{rank}\begin{pmatrix} F^i_{x_\tau y} & F^i_{x_\tau q_I} & F^i_{x_\tau q_J} \\ 0 & F_{y q_I} & E_k \end{pmatrix} = |\tau| + k.$$

By multiplying the invertible matrix $\begin{pmatrix} E_{|\tau|} & -F^i_{x_\tau q_J} \\ 0 & E_k \end{pmatrix}$ on the left hand side of the above, we have

$$\operatorname{rank}\begin{pmatrix} F^i_{x_\tau y} & F^i_{x_\tau q_I} - F^i_{x_\tau q_J} F_{y q_I} & 0 \\ 0 & F_{y q_I} & E_k \end{pmatrix} = |\tau| + k.$$

Hence

$$\operatorname{rank}(F^i_{x_\tau y}, F^i_{x_\tau q_I} - F^i_{x_\tau q_J} F_{y q_I}) = |\tau|. \tag{12}$$

Consider the case $\tau = \{s\}$ and $a^1_s = 1$. By (11) and (12) we have $a^2_s > 0$. Therefore if we denote $F_2(a^2_1 x_1, \cdots, a^2_r x_r, y, q)$ instead of F_2, then we have

$$(F^1_{xy}, F^1_{x q_I} - F^1_{x q_J} F_{y q_I}) = (F^2_{xy}, F^2_{x q_I} - F^2_{x q_J} F_{y q_I}). \tag{13}$$

Hence $a^1 = a^2 (= a)$. Set $B = F^1_{x q_J} - F^2_{x q_J}$ and define $F_3(x, y, q) = F_2(x, y + xB, q)$. Then we need only to check that $F^1_{xx} = F^3_{xx}, F^1_{xq} = F^3_{xq}$ in order to complete the proof. We have

$$F^3_{x q_J} = F^2_{x q_J} + B F_{y q_J} = F^2_{x q_J} + B = F^2_{x q_J} + F^1_{x q_J} - F^2_{x q_J} = F^1_{x q_J},$$

$$\begin{aligned} F^3_{x q_I} &= F^2_{x q_I} + B F_{y q_I} = F^2_{x q_I} + F^1_{x q_J} F_{y q_I} - F^2_{x q_J} F_{y q_I}, \\ &= (F^2_{x q_I} - F^2_{x q_J} F_{y q_I}) + F^1_{x q_J} F_{y q_I} = (F^1_{x q_I} - F^1_{x q_J} F_{y q_I}) + F^1_{x q_J} F_{y q_I} = F^1_{x q_I}. \end{aligned}$$

Therefore $F^1_{xq} = F^3_{xq}$.

Finally repeat this proof between F_1 and F_3. In the case $\sigma = \emptyset$ let (a^1, b^1, c^1), $(a^3, b^3, c^3) \in \mathbf{R}^{r+k+n}$ define the same vector. Then we have $a^1 = a^3 (= a)$ by (11) and have $b^1 = b^3$ by (8) and have $(F^1_{xx} - F^3_{xx})a = 0$ by (5). Since a is an arbitrary real number, we have $F^1_{xx} = F^3_{xx}$. □

LEMMA 3.4. *Let $F_1, F_2 \in \mathrm{m}(r; k+n)^2$ be non-degenerate function germs. Suppose that the following conditions hold:*

$$L_\sigma := L^\sigma_{F_1} = L^\sigma_{F_2} \ (\ \forall \sigma \subset I_r \ )$$

$$j^2 F_1(0) = j^2 F_2(0), \frac{\partial^2 F}{\partial y^2}(0) = 0, \frac{\partial^2 F}{\partial y \partial q_J}(0) = E_k \quad (\ J \subset I_r, |J| = k \ ).$$

Set

$$\Sigma^{\sigma}_{F_i} = \{\ (x,y,q) \in (\mathbf{H}^r \times \mathbf{R}^{k+n},0) \mid x_\sigma = \frac{\partial F_i}{\partial x_{I_r-\sigma}} = \frac{\partial F_i}{\partial y} = 0\ \},$$

$$p^{\sigma}_i : \Sigma^{\sigma}_{F_i} \xrightarrow{\sim} L_\sigma\ (\ (x,y,q) \mapsto (q, \frac{\partial F_i}{\partial q})\ )$$

for each $\sigma \subset I_r$. Then there exists $G \in \mathcal{B}(r;k+n)$ such that G preserves q and $G_{*0} = id|_{T_0\mathbf{H}^r\times\mathbf{R}^{k+n}}$ and for each $\sigma \subset I_r$ the following diagram is commutative:

$$\begin{array}{ccc} \Sigma^{\sigma}_{F_1} & \xrightarrow{G|_{\Sigma^{\sigma}_{F_1}}} & \Sigma^{\sigma}_{F_2} \\ & p^{\sigma}_1 \searrow & \downarrow p^{\sigma}_2 \\ & & L_\sigma \end{array}$$

As a result $F_3 = F_2 \circ G$ is reticular R-equivalent to F_2 and $\Sigma^{\sigma}_{F_1} = \Sigma^{\sigma}_{F_3} (= \Sigma^\sigma)$, $\frac{\partial F_1}{\partial q} - \frac{\partial F_3}{\partial q}|_{\Sigma^\sigma} \equiv 0$ for each $\sigma \subset I_r$.

Proof. For each $\sigma \subset I_r$, we set

$$G_\sigma = (p^{\sigma}_2)^{-1} \circ p^{\sigma}_1 : \Sigma^{\sigma}_{F_1} \xrightarrow{\sim} \Sigma^{\sigma}_{F_2}.$$

Since $L^{\sigma}_{F_1} = L^{\sigma}_{F_2}$ for each $\sigma \subset I_r$, we have

$$G_{\sigma\cap\sigma'} := G_\sigma|_{\Sigma^{\sigma}_{F_1}\cap\Sigma^{\sigma'}_{F_1}} = G_{\sigma'}|_{\Sigma^{\sigma}_{F_1}\cap\Sigma^{\sigma'}_{F_1}}\ (\forall \sigma,\sigma' \subset I_r\ ).$$

Since $j^2F_1(0) = j^2F_2(0)$ and F_1, F_2 are non-degenerate, there exist function germs $w_1, \cdots, w_{n-r}$ on $(\mathbf{H}^r \times \mathbf{R}^{k+n},0)$ such that

$$x_1,\cdots,x_r,\frac{\partial F_i}{\partial x_1},\cdots,\frac{\partial F_i}{\partial x_r},\frac{\partial F_i}{\partial y_1},\cdots,\frac{\partial F_i}{\partial y_k},w_1,\cdots,w_{n-r}\quad (\ i=1,2\ )$$

define coordinates of $(\mathbf{H}^r \times \mathbf{R}^{k+n},0)$. By using analogous methods of [3, lemme i], there exists a diffeomorphism G on $(\mathbf{H}^r \times \mathbf{R}^{k+n},0)$ for which the diagrams are commutative. By $j^2F_1(0) = j^2F_2(0)$, we have $G_{*0} = id|_{T_0\mathbf{H}^r\times\mathbf{R}^{k+n}}$. We have to modify G such that $G \in \mathcal{B}(r;k+n)$ and G preserves q.

Since

$$x_1 \circ G|_{x_1=x_2\frac{\partial F_1}{\partial x_2}=\cdots=x_r\frac{\partial F_1}{\partial x_r}=\frac{\partial F_1}{\partial y}=0} = x_1 \circ G|_{\bigcup_{1\in\sigma\subset I_r}\Sigma^{\sigma}_{F_1}} = 0,$$

$x_1 \circ G$ can be written in the form:

$$x_1 \circ G = x_1 \cdot a^1_1 + \sum_{i=2}^{r} x_i\frac{\partial F_1}{\partial x_i}\cdot a^1_i + \sum_{j=1}^{k}\frac{\partial F_1}{\partial y_j}\cdot b^1_j,$$

where $b^1_1,\cdots,b^1_k$ are independent on x_1. By $G_{*0} = id$, we have $a^1_1(0) = 1$. For each $i = 2,\cdots,r$, take $a^i_1,\cdots,a^i_r,b^i_1,\cdots,b^i_k$ which have the similar properties. Otherwise since

$$q_i \circ G|_{x_1\frac{\partial F_1}{\partial x_1}=\cdots=x_r\frac{\partial F_1}{\partial x_r}=\frac{\partial F_1}{\partial y}=0} = q_i \circ G|_{\sum_{\sigma\subset I_r}\Sigma^{\sigma}_{F_1}} = q_i \text{ for } i=1,\cdots,n\ ,$$

each $q_i \circ G$ can be written in the following form:

$$q_i \circ G = q_i + \sum_{j=1}^{r} x_i\frac{\partial F_1}{\partial x_j}\cdot c^i_j + \sum_{j=1}^{k}\frac{\partial F_1}{\partial y_j}\cdot d^i_j.$$

Define $G'(x,y,q) = (x_1a^1_1,\cdots,x_ra^r_r,y\circ G',q)$, then the diagrams are also commutative for G' and $G'_{*0} = id$, so that $G' \in \mathcal{B}(r;k+n)$ and G' preserves q. □

4. Stability of unfoldings. In order to study the stabilities of reticular lagrangian maps, we shall prepare the results of the singularity theory of function germs with respect to *reticular R^+-equivalence.* Basic techniques for the characterization of the stabilities we use in this paper depend heavily on the results in this section, however the all arguments are the almost parallel along the ordinary theory of the right-equivalence (cf., [18]), so that we omit the detail.

We denote $J^l(r+k,1)$ the set of l-jets at 0 of germs in $\mathrm{m}(r;k)$ and let $\pi_l : \mathrm{m}(r;k) \to J^l(r+k,1)$ be the natural projection. We denote $j^l f(0)$ the l-jet of $f \in \mathrm{m}(r;k)$.

LEMMA 4.1. *Let $f \in \mathrm{m}(r;k)$ and $O^l_{rR}(j^l f(0))$ be the submanifold of $J^l(r+k,1)$ consist of the image by π_l of the orbit of reticular R-equivalence of f. Put $z = j^l f(0)$. Then*

$$T_z(O^l_{rR}(z)) = \pi_l(\langle x_1\frac{\partial f}{\partial x_1}, \cdots, x_r\frac{\partial f}{\partial x_r}\rangle_{\mathcal{E}(r;k)} + \mathrm{m}(r;k)\langle\frac{\partial f}{\partial y_1}, \cdots, \frac{\partial f}{\partial y_k}\rangle).$$

We say that a function germ $f \in \mathrm{m}(r;k)$ is *reticular R-l-determined* if all function germ which has same l-jet of f is reticular R-equivalent to f.

LEMMA 4.2. *Let $f \in \mathrm{m}(r;k)$ and let*

$$\mathrm{m}(r;k)^{l+1} \subset \mathrm{m}(r;k)(\langle x_1\frac{\partial f}{\partial x_1}, \cdots, x_r\frac{\partial f}{\partial x_r}\rangle + \mathrm{m}(r;k)\langle\frac{\partial f}{\partial y_1}, \cdots, \frac{\partial f}{\partial y_k}\rangle) + \mathrm{m}(r;k)^{l+2},$$

then f is reticular R-l-determined. Conversely let $f \in \mathrm{m}(r;k)$ be reticular R-l-determined, then

$$\mathrm{m}(r;k)^{l+1} \subset \langle x_1\frac{\partial f}{\partial x_1}, \cdots, x_r\frac{\partial f}{\partial x_r}\rangle_{\mathcal{E}(r;k)} + \mathrm{m}(r;k)\langle\frac{\partial f}{\partial y_1}, \cdots, \frac{\partial f}{\partial y_k}\rangle.$$

Let $F \in \mathrm{m}(r;k+n_1)$, $G \in \mathrm{m}(r;k+n_2)$ be unfoldings of $f \in \mathrm{m}(r;k)$. We say that F *is reticular R^+-f-induced from G* if there exist smooth map germs $\phi : (\mathbf{H}^r \times \mathbf{R}^{k+n_2}, 0) \to (\mathbf{H}^r \times \mathbf{R}^k, 0)$, $\psi : (\mathbf{R}^{n_2}, 0) \to (\mathbf{R}^{n_1}, 0)$ and $\alpha \in \mathrm{m}(0;n_2)$ satisfying the following conditions:

(1) $\phi((\mathbf{H}^r \cap \{x_\sigma = 0\}) \times \mathbf{R}^{k+n_2}) \subset (\mathbf{H}^r \cap \{x_\sigma = 0\}) \times \mathbf{R}^k$ for $\sigma \subset I_r$.

(2) $G(x,y,v) = F(\phi(x,y,v), \psi(v)) + \alpha(v)$ for $x \in \mathbf{H}^r$, $y \in \mathbf{R}^k$ and $v \in \mathbf{R}^{n_2}$.

DEFINITION 4.3. Here we define several stabilities of unfoldings. Let $f \in \mathrm{m}(r;k)$ and $F \in \mathrm{m}(r;k+n)$ be an unfolding of f.

We define a smooth map germ

$$j^l_1 F : (\mathbf{R}^{r+k+n}, 0) \longrightarrow (J^l(r+k,1), j^l f(0))$$

as follow: Let $\tilde{F} : U \to \mathbf{R}$ be a representative of F. For each $(x,y,u) \in U$, We define $F_{(x,y,u)} \in \mathrm{m}(r;k)$ by $F_{(x,y,u)}(x',y') = F(x+x', y+y', u) - F(x,y,u)$. Now define $j^l_1 F(x,y,u)$ =the l-jet of $F_{(x,y,u)}$. $j^l_1 F$ depends only on the germ at 0 of F. We say that F is *reticular R^+-l-transversal* if $j^l_1 F|_{x=0}$ is transversal to $O^l_{rR}(j^l f(0))$. It is easy to check that F is reticular R$^+$-l-transversal if and only if

$$\mathcal{E}(r;k) = \langle x_1\frac{\partial f}{\partial x_1}, \cdots, x_r\frac{\partial f}{\partial x_r}, \frac{\partial f}{\partial y_1}, \cdots, \frac{\partial f}{\partial y_k}\rangle_{\mathcal{E}(r;k)} + V_F + \mathrm{m}(r;k)^{l+1},$$

where $V_F = L_{\mathbf{R}}\langle 1, \frac{\partial F}{\partial u_1}|_{u=0}, \cdots, \frac{\partial F}{\partial u_n}|_{u=0}\rangle$.

We say that F is *reticular R^+-stable* if the following condition holds: For any neighborhood U of 0 in $\mathbf{R}^{r+k+n}$ and any representative $\tilde{F} \in C^\infty(U, \mathbf{R})$ of F, there exists a neighborhood $N_{\tilde{F}}$ of $\tilde{F}$ such that for any element $\tilde{G} \in N_{\tilde{F}}$ the germ $\tilde{G}|_{\mathbf{H}^r \times \mathbf{R}^{k+n}}$ at $(0, y_0, u_0')$ is reticular R^+-equivalent to F for some $(0, y_0, u_0') \in U$.

We say that F is *reticular R^+-versal* if F is reticular R^+-f-induced from all unfolding of f.

We say that F is *reticular R^+-infinitesimal versal* if

$$\mathcal{E}(r;k) = \langle x_1 \frac{\partial f}{\partial x_1}, \cdots, x_r \frac{\partial f}{\partial x_r}, \frac{\partial f}{\partial y_1}, \cdots, \frac{\partial f}{\partial y_k} \rangle_{\mathcal{E}(r;k)} + V_F.$$

We say that F is *reticular R^+-infinitesimal stable* if

$$\mathcal{E}(r;k+n)$$
$$= \langle x_1 \frac{\partial F}{\partial x_1}, \cdots, x_r \frac{\partial F}{\partial x_r}, \frac{\partial F}{\partial y_1}, \cdots, \frac{\partial F}{\partial y_k} \rangle_{\mathcal{E}(r;k+n)} + \langle 1, \frac{\partial F}{\partial u_1}, \cdots, \frac{\partial F}{\partial u_n} \rangle_{\mathcal{E}(n)}.$$

We say that F is *reticular R^+-homotopically stable* if for any smooth path-germ $(\mathbf{R}, 0) \to \mathcal{E}(r; k+n), t \mapsto F_t$ with $F_0 = F$, there exists a smooth path-germ $(\mathbf{R}, 0) \to \mathcal{B}(r; k+n) \times \mathcal{E}(n), t \mapsto (\Phi_t, \alpha_t)$ with $(\Phi_0, \alpha_0) = (id, 0)$ such that each (Φ_t, α_t) is a reticular R^+-isomorphism and $F_0 = F_t \circ \Phi_t + \alpha_t$.

THEOREM 4.4 (Transversality lemma). *Let U be a neighborhood of 0 in $0 \in \mathbf{R}^{r+k+n}$ with the coordinates $(x_1, \cdots, x_r, y_1, \cdots, y_k, u_1, \cdots, u_n)$ and A be a submanifold of $J^l(r+k, 1)$. Then the set*

$$T_A = \{F \in C^\infty(U, \mathbf{R}) \mid j_1^l F|_{x=0} \textit{ is transversal to } A\}$$

is dense in $C^\infty(U, \mathbf{R})$ with respect to C^∞-topology, where $j_1^l F(x, y, u)$ is the l-jet of the map $(x', y') \mapsto F(x + x', y + y', u)$ at 0. The transversality we used is a slightly different for the ordinary one [18], however we can also prove this theorem by the method which is the same as the ordinary method.

THEOREM 4.5. *Let $F \in \mathrm{m}(r; k+n)$ be an unfolding of $f \in \mathrm{m}(r;k)$. Then the following are equivalent.*
(1) *F is reticular R^+-stable.*
(2) *F is reticular R^+-versal.*
(3) *F is reticular R^+-infinitesimal versal.*
(4) *F is reticular R^+-infinitesimal stable.*
(5) *F is reticular R^+-homotopically stable.* For $f \in \mathrm{m}(r;k)$ we define the *reticular R-codimension of f* by the $\mathbf{R}$-dimension of the vector space

$$\mathcal{E}(r;k)/\langle x_1 \frac{\partial f}{\partial x_1}, \cdots, x_r \frac{\partial f}{\partial x_r}, \frac{\partial f}{\partial y_1}, \cdots, \frac{\partial f}{\partial y_k} \rangle_{\mathcal{E}(r;k)}.$$

By the above theorem if $1, a_1, \cdots, a_n \in \mathcal{E}(r;k)$ is a representative of a basis of the vector space, then $f + a_1 v_1 + \cdots a_n v_n \in \mathrm{m}(r; k+n)$ is a reticular R^+-stable unfolding of f.

5. Stability of reticular lagrangian maps. In this section we shall define several notions of stabilities for reticular lagrangian maps and prove that they and the notion of stabilities for corresponding generating families are all equivalent.

In order to consider symplectomorphisms and symplectomorphism germs on $T^*\mathbf{R}^n$, we introduce canonical coordinates (Q, P) and (q, p) of $T^*\mathbf{R}^n$, where (Q, P) are the coordinates of the source and (q, p) are the coordinates of the target.

Stability: For any open set U in $T^*\mathbf{R}^n$ we denote $S(U, T^*\mathbf{R}^n)$ the space of symplectic embeddings from U to $T^*\mathbf{R}^n$ with C^∞-topology. We say that a reticular lagrangian map $\pi \circ i : (\mathbf{L}^0, 0) \to (T^*\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ is *stable* if the following holds: For any extension S of i and any representative $\tilde{S} \in S(U, T^*\mathbf{R}^n)$ of S, there exists a neighborhood $N_{\tilde{S}}$ of $\tilde{S}$ such that for any $\tilde{T} \in N_{\tilde{S}}$ the reticular lagrangian maps $\pi \circ (\tilde{T}|_{\mathbf{L}^0}$ at $x_0)$ and $\pi \circ i$ are lagrangian equivalent for some $x_0 = (0, \cdots, 0; 0, \cdots, 0, P^0_{r+1}, \cdots, P^0_n) \in U$.

Homotopical Stability: Let $\pi \circ i : (\mathbf{L}^0, 0) \to (T^*\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ be a reticular lagrangian map. A map germ $\bar{i} : (\mathbf{L}^0 \times \mathbf{R}, (0,0)) \to (T^*\mathbf{R}^n, 0)((Q,P,t) \mapsto i_t(Q,P))$ is called a *reticular lagrangian deformation* of i if $i_0 = i$ and there exists a one-parameter family of symplectomorphisms $\bar{S} : (T^*\mathbf{R}^n \times \mathbf{R}, (0,0)) \to (T^*\mathbf{R}^n, 0)$ $((Q,P,t) \mapsto S_t(Q,P))$ such that $i_t = S_t|_{\mathbf{L}^0}$ for t around 0. We call $\bar{S}$ an *extension* of $\bar{i}$. Let $\phi : (\mathbf{L}^0, 0) \to (\mathbf{L}^0, 0)$ be a reticular diffeomorphism. A map germ $\bar{\phi} : (\mathbf{L}^0 \times \mathbf{R}, (0,0)) \to (\mathbf{L}^0, 0)((Q,P,t) \mapsto \phi_t(Q,P))$ is called a *one-parameter deformation of reticular diffeomorphisms* of ϕ if $\phi_0 = \phi$ and there exists a one-parameter family of diffeomorphisms $\bar{\Phi} : (T^*\mathbf{R}^n \times \mathbf{R}, (0,0)) \to (T^*\mathbf{R}^n, 0)((Q,P,t) \mapsto \Phi_t(Q,P))$ such that ϕ_t is a reticular diffeomorphism defined by $\phi_t = \Phi_t|_{\mathbf{L}^0}$ for t around 0. We call $\bar{\Phi}$ an *extension* of $\bar{\phi}$. We say that a reticular lagrangian map $\pi \circ i : (\mathbf{L}^0, 0) \to (T^*\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ is *homotopically stable* if for any reticular lagrangian deformation $\bar{i} = \{i_t\}$ of i there exists a one-parameter deformation of reticular diffeomorphisms $\bar{\phi} = \{\phi_t\}$ of $id_{\mathbf{L}^0}$ and a one-parameter family of lagrangian equivalences $\bar{\Theta} = \{\Theta_t\}$ with $\Theta_0 = id_{T^*\mathbf{R}^n}$ such that $i_t = \Theta_t \circ i \circ \phi_t$ for t around 0.

Infinitesimal Stability: A vector field v on $(T^*\mathbf{R}^n, 0)$ is said to be *tangent* to $\mathbf{L}^0$ if $v|_{L^0_\sigma}$ is tangent to L^0_σ for all $\sigma \subset I_r$. A function germ H on $(T^*\mathbf{R}^n, 0)$ is said to be *fiber preserving* if there exist function germs $h_0, \cdots, h_n$ on the base of π such that $H(q,p) = \sum_{i=1}^n h_i(q)p_i + h_0(q)$ for $(q,p) \in (T^*\mathbf{R}^n, 0)$. We say that a reticular lagrangian map $\pi \circ i : (\mathbf{L}^0, 0) \to (T^*\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ is *infinitesimal stable* if for any function germ f on $(T^*\mathbf{R}^n, 0)$ there exists a fiber preserving function germ H on $(T^*\mathbf{R}^n, 0)$ and a vector field v on $(T^*\mathbf{R}^n, 0)$ such that v is tangent to $\mathbf{L}^0$ and $X_f \circ i = X_H \circ i + i_* v$, where X_f and X_H are the Hamiltonian vector fields of f and H respectively and $i_* v$ is defined by $i_* v = (S_* v) \circ i$ for an extension S of i.

At first we prepare some lemmas to construct continuous maps between mapping spaces. Let U, V be open sets in $\mathbf{R}^n, \mathbf{R}^m$ respectively. We define

$$N_f(l, \varepsilon, K) = \{\ g \in C^\infty(U, V) \mid |D^\alpha(g - f)_x| < \varepsilon\ \forall x \in K, |\alpha| < l\ \}$$

for each $f \in C^\infty(U, V), l \in \mathbf{N}, \varepsilon > 0$ and compact set K in U. Then the family of sets $N_f(l, \varepsilon, K)$ forms a basis for the C^∞-topology on $C^\infty(U, V)$.

LEMMA 5.1. *Let U be an open ball around 0 in $\mathbf{R}^n$. Then the map*

$$\int : C^\infty(U, \mathbf{R}) \longrightarrow C^\infty(U, \mathbf{R}) \quad (\ f \mapsto (x \mapsto \int_0^1 f(tx)dt)\ )$$

is continuous.

Proof. Let $f \in C^\infty(U, \mathbf{R})$ and a neighborhood N of $\int f$ be given. We may assume that $N = N_{\int f}(l, \varepsilon, K)$ for some l, ε, K. Choose a closed ball K' around 0 in U such that $K \subset K'$ and set $N' = N_f(l, \varepsilon, K')$. Then for any $g \in N', x \in K$,

$$|D^\alpha(\int g - \int f)_x| = |D^\alpha(\int_0^1 (g(tx) - f(tx))dt)|$$

$$= |\int_0^1 (t^{|\alpha|}D^\alpha(g(tx) - f(tx)))dt|$$
$$\leq \int_0^1 t^{|\alpha|}|D^\alpha(g(tx) - f(tx))|dt$$
$$< \int_0^1 1 \cdot \varepsilon dt = \varepsilon$$

for any $|\alpha| < l$. It follows that $\int(N') \subset N$. Hence $\int$ is continuous. □

PROPOSITION 5.2. *Let U, V be open sets in $\mathbf{R}^n$ satisfying $0 \in U \subset V$ and $i : U \to V$ be the inclusion map. Choose $\varepsilon > 0$ such that $\overline{U_{3\varepsilon}(0)} \subset U$ and set $U_1 = U_{3\varepsilon}(0), V_1 = U_\varepsilon(0)$. Then there exists a neighborhood N_0 of i in $C^\infty(U, V)$ such that $g|_{U_1}$ is embedding and $V_1 \subset g(U_1)$ for $g \in N_0$. Moreover*

$$N_0 \longrightarrow C^\infty(V_1, U) \quad (f \mapsto (g|_{U_1})^{-1}|_{V_1})$$

is continuous.

Proof. We define the neighborhood N_0 of i by

$$g \in N_0 \Leftrightarrow \begin{cases} \frac{\partial g_i}{\partial x_i} > \frac{1}{2}, |\frac{\partial g_i}{\partial x_j}| < \frac{1}{2n} \quad (i \neq j) \\ \det \frac{\partial g}{\partial x}(x) \neq 0 \\ |g(x) - x| < \varepsilon \qquad\qquad\qquad \text{for } x \in \overline{U_1}. \end{cases}$$

Let $g \in N_0$ and $a, b \in U_1$ $(a \neq b)$ be given, we may assume that $|a_1 - b_1| \geq |a_i - b_i| (i = 2, \cdots, n)$. Set $c(t) = (1-t)a + tb, t \in [0, 1]$. Since U_1 is convex, we have $c([0, 1]) \subset U_1$.

$$\begin{aligned} |g_1(b) - g_1(a)| &= |\int_0^1 \frac{d}{dt} g_1 \circ c(t) dt| \\ &= |\sum_{i=1}^n \int_0^1 \frac{\partial g_1}{\partial x_i} \circ c(t)(b_i - a_i)dt| \\ &\geq |\int_0^1 \frac{\partial g_1}{\partial x_1} \circ c(t)(b_1 - a_1)dt| - \sum_{i=2}^n |\int_0^1 \frac{\partial g_1}{\partial x_i} \circ c(t)(b_i - a_i)dt| \\ &= |b_1 - a_1||\int_0^1 \frac{\partial g_1}{\partial x_1} \circ c(t)dt| - \sum_{i=2}^n |b_i - a_i||\int_0^1 \frac{\partial g_1}{\partial x_i} \circ c(t)dt| \\ &> |b_1 - a_1|\frac{1}{2} - (n-1)|b_1 - a_1|\frac{1}{2n} = \frac{1}{2n}|b_1 - a_1| > 0. \end{aligned}$$

It follows that $g|_{U_1}$ is an injective. Hence $g|_{U_1}$ is an embedding. It is easy to prove that $V_1 \subset g(U_1)$ because of the definition of U_1, V_1 and the fact that $|g(x) - x| < \varepsilon$.

Let $f_0 \in N_0$ and a neighborhood N of $g_0 = (f_0|_{U_1})^{-1}|_{V_1}$ be given. We may assume that $N = N_{g_0}(l, \varepsilon', K)$ for a l, ε', K. Since the l-jet extension of $(f|_{U_1})^{-1}|_{V_1}$ is written as a continuous map of the l-jet extension of $f|_{\overline{U_1}}$ for each $f \in N_0$, it follows that there exists $\varepsilon'' > 0$ such that $(f|_{U_1})^{-1}|_{V_1} \in N$ for any $f \in N_{f_0}(l, \varepsilon'', \overline{U_1})$. □

We have the following lemma as a corollary of Proposition 5.2.

LEMMA 5.3. *Let U, V be open sets in $\mathbf{R}^n$ such that $0 \in U$ and let $f_0 : U \to V$ be a embedding. Then there exist a neighborhood U_1 of 0 in U and an open ball V around $f_0(0)$ in V and a neighborhood N_1 of f_0 in $C^\infty(U, V)$ such that $f|_{U_1}$ is embedding and $V_1 \subset f(U_1)$ for all $f \in N_1$. Moreover*

$$N_1 \longrightarrow C^\infty(V_1, U) \quad (f \mapsto (f|_{U_1})^{-1}|_{V_1})$$

is continuous.

LEMMA 5.4. *For any one-parameter family of lagrangian equivalences* $\bar{\Theta} : (T^*\mathbf{R}^n \times \mathbf{R}, (0,0)) \to (T^*\mathbf{R}^n, 0)((Q,P,t) \mapsto \Theta_t(Q,P))$ *with* $\Theta_0 = id$, *there exists a fiber preserving function germ* H *on* $(T^*\mathbf{R}^n, 0)$ *such that* $X_H = \frac{d\Theta_t}{dt}|_{t=0}$. *Conversely for any fiber preserving function germ* H *on* $(T^*\mathbf{R}^n, 0)$, *the flow* $\bar{\Theta} = \{\Theta_t\}$ *of* X_H *with the initial condition* $\Theta_0 = id : (T^*\mathbf{R}^n, 0) \to (T^*\mathbf{R}^n, 0)$ *is a one-parameter family of lagrangian equivalences.*

THEOREM 5.5. *Let* $\pi \circ i : (\mathbf{L}^0, 0) \to (T^*\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ *be a reticular lagrangian map with the generating family* $F(x,y,q) \in \mathrm{m}(r+k+n)^2$. *Then the following are equivalent.*

(1) F *is a reticular* R^+*-stable unfolding of* $F|_{q=0}$.
(2) $\pi \circ i$ *is homotopically stable.*
(3) $\pi \circ i$ *is infinitesimal stable.*
(4) *For any function germ* f *on* $(T^*\mathbf{R}^n, 0)$, *there exists a fiber preserving function germ* H *on* $(T^*\mathbf{R}^n, 0)$ *such that* $f \circ i = H \circ i$.
(5) $\pi \circ i$ *is stable.*

Proof. We shall prove (1)⇔(5), (1)⇒(2)⇒(3)⇒(4)⇒(1).

(1)⇒(5). Let S_0 be an extension of i and $\tilde{S}_0 \in S(U, T^*\mathbf{R}^n)$ be a representative of S_0. We shall construct the map (14) which maps a symplectic embedding around $\tilde{S}_0$ to a function around a representative of F. Define

$$\pi_{\tilde{S}} : U \longrightarrow \mathbf{R}^{2n} \quad ((Q,P) \mapsto (Q, p_{\tilde{S}}(Q,P)))$$

for each $\tilde{S} = (q_{\tilde{S}}, p_{\tilde{S}}) \in S(U, T^*\mathbf{R}^n)$. By taking some lagrangian equivalence of $\pi \circ i$ and shrinking U if necessary, we may assume that $\pi_{\tilde{S}_0}$ is embedding. By Lemma 5.3, there exist a neighborhood $N_{\tilde{S}_0}$ of $\tilde{S}_0$ and a neighborhood U_1 of 0 in U and a convex neighborhood V of 0 in $\mathbf{R}^{2n}$ with the coordinates (Q,p) such that the map

$$N_{\tilde{S}_0} \longrightarrow C^\infty(V,U) \; (\tilde{S} \mapsto (\pi_{\tilde{S}}|_{U_1})^{-1}|_V = (id_Q, P_{\tilde{S}}))$$

is well defined and continuous. Let $\tilde{S} \in N_{\tilde{S}_0}$. Then the set

$$P_{\tilde{S}} = \{ (Q, P_{\tilde{S}}(Q,p); q_{\tilde{S}}(Q,p), p) \in U \times T^*\mathbf{R}^n \mid (Q,p) \in V\},$$

where $q_{\tilde{S}}(Q,p) := q_{\tilde{S}}(Q, P_{\tilde{S}}(Q,p))$ for $(Q,p) \in V$, is a canonical relation associated $\tilde{S}$. Therefore there exists a smooth function $H_{\tilde{S}}$ on V such that $H_{\tilde{S}}(0,0) = 0$ and

$$P_{\tilde{S}} = \{ (Q, -\frac{\partial H_{\tilde{S}}}{\partial Q}(Q,p), -\frac{\partial H_{\tilde{S}}}{\partial p}(Q,p), p) \}.$$

But

$$\begin{aligned} H_{\tilde{S}}(Q,p) &= \int_0^1 \frac{d}{dt} H_{\tilde{S}}(tQ,tp)dt \\ &= \sum_{i=1}^k Q_i \int_0^1 \frac{\partial H_{\tilde{S}}}{\partial Q_i}(tQ,tp)dt + \sum_{i=1}^k p_i \int_0^1 \frac{\partial H_{\tilde{S}}}{\partial p_i}(tQ,tp)dt \\ &= \sum_{i=1}^k Q_i (\int \frac{\partial H_{\tilde{S}}}{\partial Q_i})(Q,p) + \sum_{i=1}^k p_i (\int \frac{\partial H_{\tilde{S}}}{\partial p_i})(Q,p) \end{aligned}$$

and the maps

$$N_{\tilde{S}_0} \longrightarrow C^\infty(V, \mathbf{R}) \quad (\tilde{S} \mapsto \frac{\partial H_{\tilde{S}}}{\partial Q_i}(= -p^i_{\tilde{S}}), \frac{\partial H_{\tilde{S}}}{\partial p_i}(= -q^i_{\tilde{S}})) \ (i = 1, \cdots, n)$$

are continuous, we have by Lemma 5.1 that the map

$$N_{\tilde{S}_0} \longrightarrow C^\infty(V, \mathbf{R}) \ (\tilde{S} \mapsto H_{\tilde{S}})$$

is continuous. Let $V' = V \cap \{ Q_{r+1} = \cdots = Q_n = 0 \}$. Now we define the following continuous map

$$(14) \quad \phi : N_{\tilde{S}_0} \longrightarrow C^\infty(V' \times \mathbf{R}^n, \mathbf{R}) \ (\tilde{S} \mapsto \tilde{F}_{\tilde{S}}(x, y, q) = H_{\tilde{S}}(x, 0; y) + < y, q >).$$

Since $\tilde{F}_{\tilde{S}_0}|_{\mathbf{H}^r \times \mathbf{R}^{2n}}$ at 0 is a generating family of $\pi \circ i$, we may assume that $\tilde{F}_0 = \tilde{F}_{\tilde{S}_0}$ is a representative of F. Since F is a reticular stable unfolding of $F|_{q=0}$, there exists a neighborhood $N_{\tilde{F}_0}$ of $\tilde{F}_0$ such that for any $\tilde{G} \in N_{\tilde{F}_0}$ the function germ $\tilde{G}$ at $(0, y^0, q^0)$ and F are reticular R^+-equivalent for some $(0, y^0, q^0) \in V' \times \mathbf{R}^n$. Set $N'_{\tilde{S}_0} = \phi^{-1}(N_{\tilde{F}_0})$. Let $\tilde{S} \in N'_{\tilde{S}_0}$. Take $(0, y^0, q^0) \in V' \times \mathbf{R}^n$ such that the above condition holds for $\tilde{F}_{\tilde{S}}$. If we denote $\{L^{\tilde{S}_0}_\sigma\}_{\sigma \subset I_r}$ the symplectic regular r-cubic configuration defined by $F_{\tilde{S}} = \tilde{F}_{\tilde{S}}|_{\mathbf{H}^r \times \mathbf{R}^{2n}}$ at $(0, y^0, q^0)$, then for each $\sigma \subset I_r$

$$\begin{aligned}
L^{\tilde{S}_0}_\sigma &= \{ (q_0 + q, \frac{\partial F_{\tilde{S}}}{\partial q}(x, y_0 + y, q_0 + q)) | x_\sigma = \frac{\partial F_{\tilde{S}}}{\partial x_{I_r - \sigma}} = \frac{\partial F_{\tilde{S}}}{\partial y} = 0, x_{I_r - \sigma} \geq 0 \} \\
&= \{ (q_0 + q, y_0 + y) | x_\sigma = \frac{\partial H_{\tilde{S}}}{\partial x_{I_r - \sigma}}(x, 0; y_0 + y) = \\
&\qquad\qquad \frac{\partial H_{\tilde{S}}}{\partial y}(x, 0; y_0 + y) + q_0 + q = 0, x_{I_r - \sigma} \geq 0 \} \\
&= \{ (-\frac{\partial H_{\tilde{S}}}{\partial p}(Q; y_0 + p), y_0 + p) | Q_\sigma = \frac{\partial H_{\tilde{S}}}{\partial Q_{I_r - \sigma}}(Q; y_0 + p) = \\
&\qquad\qquad Q_{r+1} = \cdots = Q_n = 0, Q_{I_r - \sigma} \geq 0 \} \\
&= \tilde{S}((L^0_\sigma + (0; 0, P_0))),
\end{aligned}$$

where $(0; 0, P_0) = \tilde{S}^{-1}(q_0, y_0)$. This implies the reticular lagrangian maps

$$\pi \circ (\tilde{S}|_{\mathbf{L}^0 \text{ at } (0;0,P_0)})$$

and $\pi \circ i$ are lagrangian equivalent.

(5)⇒(1). Let S_0 be an extension of i. By taking some lagrangian equivalence of $\pi \circ i$, we may assume that there exists a generating function $T_0(Q, p)$ of the canonical relation P_{S_0} associated with S_0. Then $F_0(x, y, q) = T_0(x, 0; y) + < y, q > \in \mathfrak{m}^2(r; n+n)$ is a generating family of $\pi \circ i$. We prove that F_0 is reticular R^+-stable unfolding of $F_0|_{q=0}$. Let $\tilde{F}_0 \in C^\infty(U, \mathbf{R})$ be a representative of F_0. We construct the map (15) which maps a function around $\tilde{F}_0$ to a symplectic embedding around a representative of S_0. The following construction is summarized in the diagram after the proof.

By shrinking U if necessary, we may assume that there exist a neighborhood U_1 of 0 in $\mathbf{R}^n$ with the coordinates Q, U_2 of 0 in $\mathbf{R}^n$ with the coordinates y, U_3 of 0 in $\mathbf{R}^n$ with the coordinates q and $\tilde{T}_0(Q, y) \in C^\infty(U_1 \times U_2, \mathbf{R})$ such that the following conditions hold:

(a) $\tilde{T}_0$ is a representative of T_0
(b) The map $U = (U_1 \cap \{Q_{r+1} = \cdots = Q_n = 0\}) \times U_2 \times U_3$
(c) $U_1 \times U_2 \times U_3 \to U_1 \times U_2 \times \mathbf{R}^n$ given by $(Q, y, q) \mapsto (Q, y, \frac{\partial \tilde{T}_0}{\partial y}(Q, y) + q)$ is an embedding.
(d) The map $U_1 \times U_2 \to U_1 \times \mathbf{R}^n$ given by $(Q, y) \mapsto (Q, -\frac{\partial \tilde{T}_0}{\partial Q}(Q, y)$ is an embedding.

Define the representative $\tilde{F}_0 \in C^\infty(U, \mathbf{R})$ of F_0 by $\tilde{F}(x, y, q) = \tilde{T}(x, 0; y) + \langle y, q\rangle$ and define $\bar{F}_0 \in C^\infty(U_1 \times U_2 \times U_3, \mathbf{R})$ by $\bar{F}_0(Q, y, q) = \tilde{T}(Q, y) + \langle y, q\rangle$. Since the map

$$C^\infty(U, \mathbf{R}) \to C^\infty(U_1 \times U_2 \times U_3, \mathbf{R}) \;\; (\, \tilde{F} \mapsto \bar{F}(Q, y, q) = \bar{F}_0(Q, y, q) + (\tilde{F} - \tilde{F}_0)(Q', y, q)\,),$$

where $Q' = (Q_1, \cdots, Q_r)$, is continuous, the map

$$C^\infty(U, \mathbf{R}) \to C^\infty(U_1 \times U_2 \times U_3, U_1 \times U_2 \times \mathbf{R}^n) \;\; (\, \tilde{F} \mapsto \phi_{\tilde{F}}(Q, y, q) = (Q, y, \frac{\partial \bar{F}}{\partial y})\,)$$

is also continuous. Since $\phi_{\tilde{F}_0}$ is embedding by (c), there exist a neighborhood $N^1_{\tilde{F}_0}$ of $\tilde{F}_0$ and a neighborhood U' of 0 in $U_1 \times U_2 \times U_3$ and a open ball V around 0 in $U_1 \times U_2 \times \mathbf{R}^n$ such that

$$N^1_{\tilde{F}_0} \to C^\infty(V, U_1 \times U_2 \times U_3) \;\; (\, \tilde{F} \mapsto (\phi_{\tilde{F}}|_{U'})^{-1}|_V\,)$$

is well defined and continuous. Let $V_1 = V \cap (U_1 \times U_2 \times \{0\})$. Then

$$N^1_{\tilde{F}_0} \to C^\infty(V_1, U_1 \times U_2 \times U_3) \;\; (\, \tilde{F} \mapsto (\phi_{\tilde{F}}|_{U'})^{-1}|_{V_1}\,)$$

is also continuous. We denote $(\phi_{\tilde{F}}|_{U'})^{-1}|_{V_1}(Q, y)$ by $(Q, y, q_{\tilde{F}}(Q, y))$ for $(Q, y) \in V_1$. Then the map

$$N^1_{\tilde{F}_0} \to C^\infty(V_1, U_1 \times \mathbf{R}^n) \;\; (\, \tilde{F} \mapsto \psi_{\tilde{F}}(Q, y) = (Q, -\frac{\partial \bar{F}}{\partial Q}(Q, y, q_{\tilde{F}}(Q, y))\,)\,)$$

is also continuous. Since $\psi_{\tilde{F}_0}$ is embedding by (d), there exists a neighborhood $N^2_{\tilde{F}_0}$ of $\tilde{F}_0$ in $N^1_{\tilde{F}_0}$ and a neighborhood V_2 of 0 in V_1 and a neighborhood W of 0 in $U_1 \times \mathbf{R}^n$ such that the map

$$N^2_{\tilde{F}_0} \to C^\infty(W, V_1) \quad (\, \tilde{F} \mapsto (\psi_{\tilde{F}}|_{V_2})^{-1}|_W)$$

is well defined and continuous. We denote $(\psi_{\tilde{F}}|_{V_2})^{-1}|_W(Q, P)$ by $(Q, y_{\tilde{F}}(Q, P))$. Then the map

$$N^2_{\tilde{F}_0} \to C^\infty(W, U_1 \times U_2 \times U_3) \;\; (\, \tilde{F} \mapsto (\; (Q, P) \mapsto (Q, y_{\tilde{F}}(Q, P), q_{\tilde{F}}(Q, y_{\tilde{F}}(Q, P)))\;)\;)$$

is also continuous. Hence the map

$$N^2_{\tilde{F}_0} \to S(W, T^*\mathbf{R}^n) \;\; (\, \tilde{F} \mapsto \tilde{S}_{\tilde{F}}(Q, P) = (q_{\tilde{F}}, \frac{\partial \tilde{F}}{\partial q}(Q, y_{\tilde{F}}, q_{\tilde{F}})\,)\,) \tag{15}$$

is well defined and continuous. Since $\tilde{S}_{\tilde{F}_0}$ is a representative of S_0, there exists a neighborhood $N^3_{\tilde{F}_0}$ of $\tilde{F}_0$ in $N^2_{\tilde{F}_0}$ such that for any $\tilde{F} \in N^3_{\tilde{F}_0}$ the reticular lagrangian maps $\pi \circ (\tilde{S}_{\tilde{F}}|_{\mathbf{L}^0 \text{ at } (Q^0, P^0)})$ and $\pi \circ i$ are lagrangian equivalent for some $(Q^0, P^0) =$

$(0,\cdots,0;0,\cdots,0,P^0_{r+1},\cdots,P^0_n)\in W$. Let $(0,y^0,q^0)=(0,y_{\bar F}(Q^0,P^0),q_{\bar F}(Q^0,y_{\bar F}(Q^0,P^0)))$. Since $\bar F$ at $(0,y^0,q^0)$ is a generating family of $\pi\circ(\tilde S_{\bar F}|_{\mathbf{L}^0 \text{ at } (Q^0,P^0)})$, $\bar F|_{\mathbf{H}^r\times\mathbf{R}^{2n}}$ at $(0,y^0,q^0)$ and F_0 is reticular R^+-equivalent.

$$\begin{array}{ccccccc}
U_1\times U_2\times U_3 & = & U_1\times U_2\times U_3 & & U_1\times\mathbf{R}^n & \supset & W\\
(Q,y,q) & & (Q,y,q_{\bar F}) & \to & (Q,-\frac{\partial\bar F}{\partial Q}(Q,y,q_{\bar F})) & = & (Q,P)\\
\phi_{\bar F}\downarrow & & \uparrow & \nearrow\psi_{\bar F} & & & \downarrow\tilde S_{\bar F}\\
(Q,y,\frac{\partial\bar F}{\partial y}) & & (Q,y,0) & & & & (q_{\bar F},\frac{\partial\bar F}{\partial q}(Q,y_{\bar F},q_{\bar F}))\\
U_1\times U_2\times\mathbf{R}^n & \supset & V_1 & & & & T^*\mathbf{R}^n
\end{array}$$

(1)⇒(2). Let $\bar i:(\mathbf{L}^0\times\mathbf{R},(0,0))\to(T^*\mathbf{R}^n,0)((Q,P,t)\mapsto i_t(Q,P))$ be a reticular lagrangian deformation of i. Take a one-parameter family of symplectomorphisms $\bar S:(T^*\mathbf{R}^n\times\mathbf{R},(0,0))\to(T^*\mathbf{R}^n,0)((Q,P,t)\mapsto S_t(Q,P)=(q_t(Q,P),p_t(Q,P)))$ such that $i_t=S_t|_{\mathbf{L}^0}$ for t around 0. We may assume that there exists a function germ $\bar T:(\mathbf{R}^{2l}\times\mathbf{R},(0,0))\to(\mathbf{R},0)((Q,p,t)\mapsto T_t(Q,p))$ such that T_t is a generating function of the canonical relation associated with S_t for t around 0. Define $F(x,y,q,t)\in\mathcal{E}(r;n+n+1)$ by $F(x,y,q,t)=F_t(x,y,q)=T_t(x,0;y)+\langle y,q\rangle$, then F_t is a generating family of $\pi\circ i_t$ for all t. By hypothesis, there exists a one-parameter family of reticular R^+-equivalences of the form

$$F_t(x,y,q)=F(x_1a^1_t(x,y,q),\cdots,x_ra^r_t(x,y,q),h_t(x,y,q),g_t(q))+\alpha_t(q).$$

Set a one-parameter family of lagrangian equivalences $\bar\Theta=\{\Theta_t\}$ by $\Theta_t=g^*_t+d\alpha_t|_{\pi\circ g^*_t}$. Then we have $i_t(L^0_\sigma)=\Theta_t\circ i(L^0_\sigma)$ for all $\sigma\subset I_r,t$ around 0. Therefore we may define the one-parameter family of reticular diffeomorphisms $\bar\phi=\{\phi_t\}$ by $\phi_t=(S_0)^{-1}\circ\Theta_t^{-1}\circ S_t|_{\mathbf{L}^0}$. Then we have $i_t=\Theta_t\circ i\circ\phi_t$ for t around 0.

(2)⇒(3). Take an extension S of i. Let a function germ f on $(T^*\mathbf{R}^n,0)$ be given. Let $\bar S=\{\bar S_t\}$ be the flow of X_f with the initial condition $\bar S_0=S$. Because $\bar i=\{\bar i_t=\bar S_t|_{\mathbf{L}^0}\}$ is a reticular lagrangian deformation of i, there exists a one-parameter family of lagrangian equivalences $\bar\Theta=\{\bar\Theta_t\}$ with $\bar\Theta_0=id$ and a one-parameter deformation of reticular diffeomorphisms $\bar\phi=\{\bar\phi_t\}$ of id such that $\bar i_t=\bar\Theta_t\circ i\circ\bar\phi_t$ for t around 0. Let $\bar\Phi=\{\Phi_t\}:(T^*\mathbf{R}^n\times\mathbf{R},(0,0))\to(T^*\mathbf{R}^n,0)$ be an extension of $\bar\phi$. Then we have

$$X_f\circ i=\frac{d\bar S_t}{dt}|_{t=0}|_{\mathbf{L}^0}=\frac{d\bar\Theta_t}{dt}|_{t=0}\circ i+(S_*\frac{d\bar\Phi_t}{dt}|_{t=0})\circ i=X_H\circ i+i_*v.$$

This implies that $\pi\circ i$ is infinitesimal stable.

(3)⇒(4). Let a function germ f on $(T^*\mathbf{R}^n,0)$ be given. By hypothesis, there exists a fiber preserving function germ H on $(T^*\mathbf{R}^n,0)$ and a vector field v on $(T^*\mathbf{R}^n,0)$ such that v is tangent to $\mathbf{L}^0$ and $X_f\circ i=X_H\circ i+i_*v$. Set $i_\sigma=i|_{L^0_\sigma},v_\sigma=v|_{L^0_\sigma}$ for each $\sigma\subset I_r$, then it is easy to prove that $(f-H)\circ i_\sigma=$ constant because $X_f\circ i_\sigma=X_H\circ i_\sigma+(i_\sigma)_*v_\sigma$. Since $\mathbf{L}^0$ is connected, we have that $(f-H)\circ i=$ constant. By replacing $H+$ constant by H if necessary, we have $f\circ i=H\circ i$.

(4)⇒(1). Take an extension $S=(q,p)$ of i. We may assume that there exists a generating function $T=T(Q,p)$ of the canonical relation associated with S. We define a generating family $F(x,y,q)\in\mathrm{m}(r;n+n)^2$ of $\pi\circ i$ by $F(x,y,q)=T(x,0;y)+\langle y,q\rangle$. Since $(Q,P)\mapsto(q(Q,P),P)$ is invertible, there exists $I\subset\{1,\cdots,n\}(|I|=r)$ such that $\phi:(x,y)\mapsto(q_I(x,0;y),y),x=(x_1,\cdots,x_r)$, is also invertible. Otherwise since $(Q,p)\mapsto(Q,P(Q,p))$ is invertible, $\psi:(x,y)\mapsto(x,P(x,0;y))$ is also invertible. We define $S' = \phi \circ \psi^{-1}$.

$$\begin{array}{ccc} (Q,p) & \rightarrow & (q,p) \\ \downarrow & \nearrow S & \\ (Q,P) & & \end{array} \qquad\qquad \begin{array}{ccc} (x,y) & \overset{\phi}{\rightarrow} & (q_I,p) \\ \downarrow \psi & \nearrow S' & \\ (x,P) & & \end{array}$$

Let $f \in \mathcal{E}(r;k)$ be given. Set $g(q,y) = f \circ \phi^{-1}(q_I, y)$. Since

$$S(x,0;P)|_{x_1P_1=\cdots=x_rP_r=0,x\geq 0} = i,$$

there exists a fiber preserving function germ $H(q,p) = \sum_{i=1}^n h_i(q)p_i + h_0(q)$ on $(T^*\mathbf{R}^n, 0)$ such that

$$g \circ S(x,0;P)|_{x_1P_1=\cdots=x_rP_r=0,x\geq 0} = H \circ S(x,0;P).$$

Therefore there exist function germs $a_1, \cdots, a_r \in \mathcal{E}(r;n)$ such that

$$g \circ S(x,0;P) = H \circ S(x,0;P) + \sum_{j=1}^r x_jP_ja_j(x,P) \text{ for } (x,P) \in (\mathbf{H}^r \times \mathbf{R}^n, 0).$$

Hence

$$\begin{aligned}
& f(x,y) \\
&= (f \circ \phi^{-1}) \circ (\phi \circ \psi^{-1}) \circ \psi(x,y) = g \circ S' \circ \psi(x,y) \\
&= g \circ S(x,0;P(x,0;y)) = g(q(x,0,y),y) \\
&= \sum_{i=1}^n h_i(q(x,0;y))y_i + h_0(q(x,0;y)) + \sum_{j=1}^r x_jP_j(x,0;y)a'_j(x,y) \\
&= \sum_{i=1}^n h_i(-\frac{\partial T}{\partial y}(x,0;y))y_i + h_0(-\frac{\partial T}{\partial y}(x,0;y)) + \sum_{j=1}^r x_j(-\frac{\partial T}{\partial x_j}(x,0;y))a'_j(x,y) \\
&\equiv \sum_{i=1}^n h_i(0)y_i + h_0(0) \bmod \langle x_1\frac{\partial T}{\partial x_1}(x,0;y), \cdots, x_r\frac{\partial T}{\partial x_r}(x,0;y), \frac{\partial T}{\partial y}(x,0;y)\rangle_{\mathcal{E}(r;n)},
\end{aligned}$$

where $a'_j(x,y) = a_i(x, P(x,0,y))$ for $j = 1, \cdots, r$. This implies that F is a reticular R^+ infinitesimal versal unfolding of f. □

6. Adjacencies of singularities. We shall study the structure of the caustics of stable regular r-cubic configurations. Firstly we investigate the adjacencies of singularities classified in Section 7 because the investigation of caustics means that of adjacencies of corresponding functions germs. The following list is the classification list of simple or unimodal singularities. This includes the classification list of singularities of R-codimension≤ 7. Therefore the stable caustics in manifolds of dimension≤ 6 are classified.

The classification list of simple or unimodal singularities

r	k	Normal form	rR-codim	Conditions	Notation
1	0	$\pm x^n$	n	$n \geq 2$	B_n
1	1	$xy \pm y^n$	n	$n \geq 3$	C_n
		$\pm x^2 + y^3$	4		F_4
		$\pm x^3 + ax^2y + y^3$	6	$4a^2 + 9 \neq 0$	$F_{1,0}$
		$ax^{n+3} \pm xy^2 + y^3$	$n+6$	$n \geq 1, a \neq 0$	$F_{1,n}$
		$\pm x^4 + y^3 + ax^3y$	8		F_8
		$\pm x^3y + y^3 + ax^2y^2$	9		F_9
		$\pm x^5 + y^3 + ax^4y$	10		F_{10}
		$\pm y^4 + axy^2 \pm x^2$	6	$a^2 \neq \pm\pm 4$	$K_{4,2}$
		$\pm y^4 \pm xy^2 + ax^n$	$n+4$	$n > 2, a \neq 0$	$K_{4,n}$
		$\varepsilon y^n \pm xy^2 + ax^m$	$n+m$	$n > 4, m \geq 2,$ $\varepsilon^n = 1, a \neq 0$	$K_{n,m}$
		$\pm(x \pm y^2)^2 + ax^ny$	$2n+3$	$n > 1, a > 0$	$K^{\#}_{1,2n-3}$
		$\pm(x \pm y^2)^2 + ax^n$	$2n+2$	$n > 2, a \neq 0$	$K^{\#}_{1,2n-4}$
		$\pm y^4 + x^2y + ax^3$	8		K^*_8
		$\pm y^4 \pm x^3 + ax^2y^2$	9		K^*_9
		$y^5 \pm x^2 + axy^3$	8		K^{**}_8
1	2	$y_1^2y_2 \pm y_2^3 + xy_1 + axy_2$	6		$L_6 = D_{4,1}$
		$y_1^2y_2 \pm y_2^{n-1} + axy_1^m \pm xy_2$	$n+m+1$	$a^m > 0$	$D_{n,m}$
		$y_1^3 \pm y_2^4 + axy_1 + xy_2$	8		$E_{6,0}$
		$y_1^3 + y_1y_2^3 + axy_1 \pm xy_2$	9		$E_{7,0}$
		$y_1^3 + y_2^5 + axy_1 \pm xy_2$	10		$E_{8,0}$
		$y_1^2y_2 \pm y^4 + xy_1 + axy_2^2$	8		D_5^1
		$y_1^3 \pm y_2^4 \pm xy_1 + axy_2^2$	9		$E_{6,1}$
		$y_1^2y_2 \pm y_2^3 \pm x^2 + axy_1^2$	8		D_4^2
2	0	$\varepsilon x_1^2 + \delta x_2^2$	4		$B_{2,2}^{\varepsilon,\delta,0}$
		$\varepsilon x_1^2 + ax_1x_2 + \delta x_2^2$	4	$a^2 < 4,\ \varepsilon = \delta,\ a \neq 0$	$B_{2,2}^{\varepsilon,\delta,\overset{<}{\alpha}}$
		$\varepsilon x_1^2 + ax_1x_2 + \delta x_2^2$	4	$a^2 > 4,\ \varepsilon = \delta$	$B_{2,2}^{\varepsilon,\delta,\overset{>}{\alpha}}$
		$\varepsilon x_1^2 + ax_1x_2 + \delta x_2^2$	4	$\varepsilon \neq \delta$	$B_{2,2}^{\varepsilon,\delta,\alpha}$
		$\varepsilon(x_1 + \delta x_2)^2 + ax_2^n$	$n+2$	$n \geq 3, a \neq 0$	$B_{2,2,n}^{\varepsilon,\delta,\alpha}$
		$\varepsilon x_1^n + ax_1x_2 + \delta x_2^m$	$n+m$	$n+m \geq 5, a \neq 0$	$B_{n,m}^{\varepsilon,\delta,\alpha}$
		$\varepsilon x_1^2 + ax_1x_2^2 + \delta x_2^3$	6		$B_{2,3'}^{\varepsilon,\delta,\alpha}$
		$\varepsilon x_1^3 + ax_1^2x_2 + \delta x_2^2$	6		$B_{3,2'}^{\varepsilon,\delta,\alpha}$
2	1	$\varepsilon y^n + x_1y + \delta x_2y + ax_2^m$	$n+m$	$n \geq 3, m \geq 2, a \neq 0$	$C_{n,m}^{\varepsilon,\delta,\alpha}$
		$\varepsilon y^3 + x_1y + ax_2y^2 + \delta x_2^2$	6		$C_{3,2,1}^{\varepsilon,\delta,\alpha}$
		$\varepsilon y^3 + x_2y + ax_1y^2 + \delta x_1^2$	6		$C_{3,2,2}^{\varepsilon,\delta,\alpha}$

In the case L_6 $a^2 \pm 1 \neq 0$, while in the case $D_{n,m}$ $a \neq 0, n \geq 4, m \geq 1, n+m > 5$. In the case $r = 2$, $\varepsilon = \pm 1$, $\delta = \pm 1$ and if $a = 0$ then $\alpha = 0$ and if $a \neq 0$ then α is the sign of a.

The cases $r = 0$ and $r = 1$ were already studied as ordinary singularity and boundary singularity (for example, see [2],[9],[10]). Hence we study the case $r = 2$. From the view point of caustics, we must investigate three type of adjacencies of singularities: the first is the oridinary adjacencies, the second is the adjacency given by forgeting a boundary of the corner. For example, consider $B_{2,3}^{+,+,\alpha}$ singularity which is the orbit of $x_1^2 + ax_1x_2 + x_2^3 \in \mathrm{m}(2;0)^2$. If we forget the boundary defined by $x_2 = 0$, this function is reticular R-qeuivalent to $y^3 + x_1y \in \mathrm{m}(1;1)$. Therefore we regard $B_{2,3}^{+,+,\alpha}$ is adjacent to C_3^{+}. This adjacency appears as the union of the caustic $C_\emptyset$ and the quasicaustic $Q_{\emptyset,1}$ of the regular r-cubic configuration defined by a versal unfolding of $x_1^2 + ax_1x_2 + x_2^3$. The third is the adjacency given by restruction singularities to $x_1 = 0$ or $x_2 = 0$. For example, consider $C_{3,2}^{-,+,\alpha}$ singularity which is the orbit of $-y^3 + x_1y + x_2y + ax_2^2 \in \mathrm{m}(2;1)$. If we restruct this to $x_2 = 0$, then whis is equal to $-y^3 + xy \in \mathrm{m}(1;1)$. Hence we regard $C_{3,2}^{-,+,\alpha}$ is adjacent to C_3^{-}. This adjacency appears as the union of the caustic C_2 and the quasicaustic $Q_{2,\{1,2\}}$ of the regular r-cubic configuration defined by a versal unfolding of $-y^3 + x_1y + x_2y + ax_2^2$.

We shall draw the pictures of stable caustics in manifolds of dimension≤ 4 at the last part of this paper. The caustics of $B_{2,2,3}, B_{2,3}$ and $C_{3,2}$ are diffeomorphic to (the pictures)$\times(\mathbf{R}, 0)$ and the caustics of $B_{3,2}^{\varepsilon,\delta,\alpha}$ are diffeomorphic to one of $B_{2,3}^{\delta,\varepsilon,\alpha}$.

The adjacencies of unimodal singularities on the 2-corner:

$$\begin{tikzcd}
 & & B_{3,2'}^{\varepsilon_3,\delta_2,\alpha_{32}} \arrow[d] & & \\
 & B_{2,2} & B_{3,2}^{\varepsilon_3,\delta_2,\alpha} \arrow[l] & B_{4,2}^{\varepsilon_4,\delta_2,\alpha} \arrow[l] & \cdots \arrow[l] \\
B_{2,3'}^{\varepsilon_2,\delta_3,\alpha_{23}} \arrow[r] & B_{2,3}^{\varepsilon_2,\delta_3,\alpha} \arrow[u] & B_{3,3}^{\varepsilon_3,\delta_3,\alpha} \arrow[l] \arrow[u] & B_{4,3}^{\varepsilon_4,\delta_3,\alpha} \arrow[l] \arrow[u] & \cdots \arrow[l] \\
 & B_{2,4}^{\varepsilon_2,\delta_4,\alpha} \arrow[u] & B_{3,4}^{\varepsilon_3,\delta_4,\alpha} \arrow[l] \arrow[u] & B_{4,4}^{\varepsilon_4,\delta_4,\alpha} \arrow[l] \arrow[u] & \cdots \arrow[l] \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & B_{2,2} & B_{2,2,3}^{\varepsilon_2,\delta,\beta_3} \arrow[l] & B_{2,2,4}^{\varepsilon_2,\delta,\beta_4} \arrow[l] & \cdots \arrow[l] \\
C_{3,2,1}^{\varepsilon_3,\beta_2,\beta'} \arrow[r] & C_{3,2}^{\varepsilon_3,\delta,\beta_2} \arrow[u] & C_{3,3}^{\varepsilon_3,\delta,\beta_3} \arrow[l] \arrow[u] & C_{3,4}^{\varepsilon_3,\delta,\beta_4} \arrow[l] \arrow[u] & \cdots \arrow[l] \\
C_{3,2,2}^{\varepsilon,\beta_2,\beta''} \arrow[ur] & C_{4,2}^{\varepsilon_4,\delta,\beta_2} \arrow[u] & C_{4,3}^{\varepsilon_4,\delta,\beta_3} \arrow[l] \arrow[u] & C_{4,4}^{\varepsilon_4,\delta,\beta_4} \arrow[l] \arrow[u] & \cdots \arrow[l] \\
 & C_{5,2}^{\varepsilon_5,\delta,\beta_2} \arrow[u] & C_{5,3}^{\varepsilon_5,\delta,\beta_3} \arrow[l] \arrow[u] & C_{5,4}^{\varepsilon_5,\delta,\beta_4} \arrow[l] \arrow[u] & \cdots \arrow[l] \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots, \arrow[u] &
\end{tikzcd}$$

The adjacencies $B_{2,2} \leftarrow B_{3,2}^{\varepsilon_3,\delta_2,\alpha}$ and $B_{2,2} \leftarrow B_{2,3}^{\varepsilon_2,\delta_3,\alpha}$ means that

$$\begin{tikzcd}
B_{2,2}^{-\delta_2,\delta_2,\pm} & \\
B_{2,2}^{\delta_2,\delta_2,\overset{>}{\pm}} & B_{3,2}^{\varepsilon_3,\delta_2,\pm} \arrow[l] \arrow[ul]
\end{tikzcd}
\qquad
\begin{tikzcd}
B_{2,2}^{\varepsilon_2,-\varepsilon_2,\pm} & \\
B_{2,2}^{\varepsilon_2,\varepsilon_2,\overset{>}{\pm}} & B_{2,3}^{\varepsilon_2,\delta_3,\pm}. \arrow[l] \arrow[ul]
\end{tikzcd}$$

The adjacency $B_{2,2} \leftarrow B_{2,2,3}^{\varepsilon_2,\delta,\beta_3}$ means that

$$\begin{tikzcd}
B_{2,2}^{+,+,\overset{>}{\pm}} & \\
B_{2,2}^{+,+,\overset{<}{\pm}} & B_{2,2,3}^{+,\pm,\beta_3} \arrow[l] \arrow[ul]
\end{tikzcd}
\qquad
\begin{tikzcd}
B_{2,2}^{-,-,\overset{>}{\mp}} & \\
B_{2,2}^{-,-,\overset{<}{\mp}} & B_{2,2,3}^{-,\pm,\beta_3}. \arrow[l] \arrow[ul]
\end{tikzcd}$$

The adjacency $B_{2,2} \leftarrow C_{3,2}^{\varepsilon_3,\delta,\beta_2}$ means that

$$\begin{tikzcd} B_{2,2}^{+,+,\overset{<}{+}} & \\ B_{2,2}^{-,-,\overset{>}{-}} & C_{3,2}^{\pm,+,+} \arrow[l] \arrow[ul] \end{tikzcd} \qquad \begin{tikzcd} B_{2,2}^{+,+,\overset{>}{+}} & \\ B_{2,2}^{-,-,\overset{<}{-}} & C_{3,2}^{\pm,+,-} \arrow[l] \arrow[ul] \end{tikzcd}$$

$$\begin{tikzcd} B_{2,2}^{+,+,\overset{<}{-}} & \\ B_{2,2}^{-,-,\overset{>}{+}} & C_{3,2}^{\pm,-,+} \arrow[l] \arrow[ul] \end{tikzcd} \qquad \begin{tikzcd} B_{2,2}^{+,+,\overset{>}{-}} & \\ B_{2,2}^{-,-,\overset{<}{+}} & C_{3,2}^{\pm,-,-} \arrow[l] \arrow[ul] \end{tikzcd}.$$

The adjacencies from singularities on the 2-corner to one on the boundary.

$$\begin{tikzcd} B_2^{\varepsilon} & B_3^{\varepsilon_3} \arrow[l] & B_4^{\varepsilon_4} \arrow[l] & \cdots \arrow[l] \\ B_{2,2}^{\varepsilon,\delta,*} \arrow[u] \arrow[dd] & B_{3,2}^{\varepsilon_3,\delta,\alpha} \arrow[l] \arrow[u] & B_{4,2}^{\varepsilon_4,\delta,\alpha} \arrow[l] \arrow[u] & \cdots \arrow[l] \\ & B_{2,3}^{\varepsilon,\delta_3,\alpha} \arrow[ul] \arrow[d] & B_{2,4}^{\varepsilon,\delta_4,\alpha} \arrow[l] \arrow[d] & \cdots \arrow[l] \\ B_2^{\delta} & B_3^{\delta_3} \arrow[l] & B_4^{\delta_4} \arrow[l] & \cdots \arrow[l] \end{tikzcd},$$

$$\begin{tikzcd} F_4^{\alpha} & & & & & \\ C_{3,2,1}^{\varepsilon_3,\alpha,\alpha'} \arrow[u] \arrow[dr] & C_3^{\varepsilon_3} & C_4^{\varepsilon_4} \arrow[l] & C_5^{\varepsilon_5} \arrow[l] & C_6^{\varepsilon_6} \arrow[l] & \cdots \arrow[l] \\ & C_{3,2}^{\varepsilon_3,\delta,\alpha} \arrow[u] \arrow[d] & C_{4,2}^{\varepsilon_4,\delta,\alpha} \arrow[l] \arrow[u] \arrow[d] & C_{5,2}^{\varepsilon_5,\delta,\alpha} \arrow[l] \arrow[u] \arrow[d] & C_{6,2}^{\varepsilon_6,\delta,\alpha} \arrow[l] \arrow[u] \arrow[d] & \cdots \arrow[l] \\ C_{3,2,2}^{\varepsilon_3,\alpha,\alpha''} \arrow[ur] \arrow[d] & C_3^{\varepsilon_3\delta} & C_4^{\varepsilon_4} \arrow[l] & C_5^{\varepsilon_5\delta} \arrow[l] & C_6^{\varepsilon_6} \arrow[l] & \cdots \arrow[l] \\ F_4^{\alpha} & & & & & \end{tikzcd},$$

The adjacencies of singularities on the 2-corner given by forgeting a boundary. The adjacency $\nwarrow$ is given by forgeting the boundary $x_1 = 0$ and $\leftarrow$ is given by forgeting $x_2 = 0$.

$\underline{B_{2,2}}$

$$\begin{tikzcd} C_2^- & \\ C_2^- & B_{2,2}^{+,+,\overset{<}{\pm}(0)} \arrow[l] \arrow[ul] \end{tikzcd} \qquad \begin{tikzcd} C_2^+ & \\ C_2^+ & B_{2,2}^{+,+,\overset{>}{\pm}} \arrow[l] \arrow[ul] \end{tikzcd} \qquad \begin{tikzcd} C_2^+ & \\ C_2^+ & B_{2,2}^{-,-,\overset{<}{\pm}(0)} \arrow[l] \arrow[ul] \end{tikzcd}$$

$$\begin{tikzcd} C_2^- & \\ C_2^- & B_{2,2}^{-,-,\overset{>}{\pm}} \arrow[l] \arrow[ul] \end{tikzcd} \qquad \begin{tikzcd} C_2^+ & \\ C_2^- & B_{2,2}^{+,-,\pm} \arrow[l] \arrow[ul] \end{tikzcd} \qquad \begin{tikzcd} C_2^- & \\ C_2^+ & B_{2,2}^{-,+,\pm} \arrow[l] \arrow[ul] \end{tikzcd},$$

$\underline{B_{2,2,n}}$

$$\begin{tikzcd} B_3^{\alpha} & \\ B_3^{-\delta\alpha} & B_{2,2,3}^{\pm,\delta,\alpha} \arrow[l] \arrow[ul] \end{tikzcd} \qquad \begin{tikzcd} B_n^{\alpha} & \\ B_n^{(-\delta)^n\alpha} & B_{2,2,n}^{\pm,\delta,\alpha} \arrow[l] \arrow[ul] \end{tikzcd},$$

$\underline{B_{n,m}}$

$$\begin{tikzcd} C_2^{\varepsilon} & \\ C_3^{\delta\alpha} & B_{2,3}^{\varepsilon,\delta,\alpha} \arrow[l] \arrow[ul] \end{tikzcd} \qquad \begin{tikzcd} C_n^{\varepsilon\alpha^n} & \\ C_m^{\delta\alpha^m} & B_{n,m}^{\varepsilon,\delta,\alpha} \arrow[l] \arrow[ul] \end{tikzcd},$$

<u>$B_{2,3'}$ and $B_{3,2'}$</u>

$$\begin{array}{ccc} B_3^\delta & & \\ & \nwarrow & \\ F_4^\varepsilon & \longleftarrow & B_{2,3'}^{\varepsilon,\delta,\alpha} \end{array} \qquad\qquad \begin{array}{ccc} F_4^\delta & & \\ & \nwarrow & \\ B_3^\varepsilon & \longleftarrow & B_{3,2'}^{\varepsilon,\delta,\alpha} \end{array}$$

<u>$C_{n,m}$</u>

$$\begin{array}{ccc} C_2^{-\alpha} & & \\ & \nwarrow & \\ C_2^{-\alpha} & \longleftarrow & C_{3,2}^{\pm,\delta,\alpha} \end{array} \qquad\qquad \begin{array}{ccc} C_m^{-\alpha} & & \\ & \nwarrow & \\ C_m^{-(-\delta)^m\alpha} & \longleftarrow & C_{n,m}^{\pm,\delta,\alpha}. \end{array}$$

<u>$C_{3,2,1}$ and $C_{3,2,2}$</u>

$$\begin{array}{ccc} C_2^{-\delta} & & \\ & \nwarrow & \\ C_3^\varepsilon & \longleftarrow & C_{3,2,1}^{\varepsilon,\delta,\alpha} \end{array} \qquad\qquad \begin{array}{ccc} C_3^\varepsilon & & \\ & \nwarrow & \\ C_2^{-\delta} & \longleftarrow & C_{3,2,2}^{\varepsilon,\delta,\alpha}. \end{array}$$

The adjacencies of singularities on the 2-coner given by restruction to $x_1 = 0$ or $x_2 = 0$. The adjacency $\nwarrow$ is given by restruction to $x_2 = 0$ and $\leftarrow$ is given by restruction to $x_1 = 0$.

$$\begin{array}{ccc} B_n^\varepsilon & & \\ & \nwarrow & \\ B_m^\delta & \longleftarrow & B_{n,m}^{\varepsilon,\delta,*} \end{array} \qquad \begin{array}{ccc} B_2^\varepsilon & & \\ & \nwarrow & \\ B_2^\varepsilon & \longleftarrow & B_{2,2,n}^{\varepsilon,\delta,\alpha} \end{array} \qquad \begin{array}{ccc} C_{2n}^\varepsilon & & \\ & \nwarrow & \\ C_{2n}^\varepsilon & \longleftarrow & B_{2n,m}^{\varepsilon,\delta,\alpha} \end{array}$$

$$\begin{array}{ccc} C_{2n+1}^\varepsilon & & \\ & \nwarrow & \\ C_{2n+1}^{\varepsilon\delta} & \longleftarrow & C_{2n+1,m}^{\varepsilon,\delta,\alpha} \end{array} \qquad \begin{array}{ccc} C_3^\varepsilon & & \\ & \nwarrow & \\ F_4^\delta & \longleftarrow & C_{3,2,1}^{\varepsilon,\delta,\alpha} \end{array} \qquad \begin{array}{ccc} F_4^\delta & & \\ & \nwarrow & \\ C_3^\varepsilon & \longleftarrow & C_{3,2,2}^{\varepsilon,\delta,\alpha}. \end{array}$$

7. Classification of function germs. In order to classify function germs we prepare the following lemmas.

LEMMA 7.1. *Let $f \in \mathrm{m}(r;k)$ be a function germ. If $\frac{\partial f}{\partial y}(0) \neq 0$ then f is reticular R-equivalent to $y_1 \in \mathrm{m}(r;k)$.*

LEMMA 7.2. *Let $f \in \mathrm{m}(r;k)$ be a function germ satisfying $\frac{\partial f}{\partial y}(0) = 0$ and l be the corank of f. ' Then there exist a subset $\sigma \subset I_r$ and a non-degenerate quadratic form $Q(y_1, \cdots, y_{l-1})$ and a function germ $g(x', y') \in \mathrm{m}((r - |\sigma|); l)^2$ such that the following conditions hold:*
(1) $g|_{x'=0} \in \mathrm{m}(0;l)^3$
(2) *f is reticular R-equivalent to $f_0 \in \mathrm{m}(r;k)$ defined by*

$$f_0(x_1, y) = \sum_{i\in\sigma} \pm x_i + g(x_{I_r - \sigma}, y_1, \cdots, y_l) + Q(y_{l+1}, \cdots, y_k).$$

We say a function germ $f(x, y) \in \mathrm{m}(r;k)$ ($x \in \mathbf{H}^r, y \in \mathbf{R}^k$) is residual if $f \in \mathrm{m}(r;k)^2$ and $f|_{x=0} \in \mathrm{m}(0;k)^3$.

Let $\mathcal{E}(r;k,l)$ be the set of smooth map germs $(\mathbf{H}^r \times \mathbf{R}^k, 0) \to \mathbf{R}^l$ and $\mathrm{m}(r;k,l)$ be the set of map germs $(\mathbf{H}^r \times \mathbf{R}^k, 0) \to (\mathbf{R}^l, 0)$.

To each $\xi = (x_1\xi_1, \cdots, x_r\xi_r, \xi_{r+1}, \cdots, \xi_{r+k}) \in \mathrm{m}(r;k,r+k)$ we define the linear map $\xi : \mathcal{E}(r;k) \to \mathcal{E}(r;k)$ by

$$\xi(f) = \sum_{i=1}^{r} x_i\xi_i \frac{\partial f}{\partial x_i} + \sum_{j=1}^{k} \xi_{r+j} \frac{\partial f}{\partial y_j}.$$

To each $\phi \in \mathcal{B}(r;k)$ we define the linear map $\phi^* : \mathcal{E}(r;k) \to \mathcal{E}(r;k)$ by $\phi^*(f) = f \circ \phi$.

We have the following four lemma's which are analogous to lemma 1.21 $\sim$ corollary 1.23 in [16].

LEMMA 7.3. *Let A be a real vector subspace of $\mathcal{E}(r;k)$. Let $[0,1] \to \mathrm{m}(r;k,r+k)$, $t \mapsto \xi_t = (x_1\xi_t^1, \cdots, x_r\xi_t^r, \xi_t^{r+1}, \cdots, \xi_t^{r+k})$ be a smooth homotopy. Suppose that $\xi_t(A) \subset A$ for all $t \in [0,1]$ and $f \in \mathcal{E}(r;k)$. Let $\Phi : (\mathbf{H}^r \times \mathbf{R}^k \times \mathbf{R}, 0 \times [0,1]) \to (\mathbf{H}^r \times \mathbf{R}^k, 0)$, $(x,y,t) \mapsto \phi_t(x,y)$, be the solution of the differential equation:*

$$\frac{d}{dt}\phi_t(x,y) = \xi_t \circ \phi_t(x,y), \ \phi_0(x,y) = (x,y). \tag{16}$$

Then this solution satisfies the following conditions:
(1) $\phi_t \in \mathcal{B}(r;k)$ for all $t \in [0,1]$.
(2) $\phi_t^*(A) \subset A + \mathrm{m}(r;k)^l$ for all $t \in [0,1]$, $l > 0$.

Proof. (1) We denote $\Phi = (\Phi_1, \cdots, \Phi_{r+k})$ and $\phi_t = (\phi_t^1, \cdots, \phi_t^{r+k})$. Since $(\xi_t \circ \phi_t)_i = \phi_t^i \cdot \xi_t^i \circ \phi_t$ for $i = 1, \cdots, r$, by the uniqueness of the solution of (16) we have that $\Phi_i|_{x_i=0} \equiv 0$ for $i = 1, \cdots, r$. This means (1).
(2) Let $l > 0$ be given. For each $t \in [0,1]$ consider the map $\mathcal{E}(r;k) \to \mathcal{E}(r;k)$ given by $f \mapsto \xi_t(f)$; this map is linear and since $\xi_t \in \mathrm{m}(r;k,r+k)$ this map maps $\mathrm{m}(r;k)^l$ into itself. Hence this map induces the linear map $\tilde{\xi}_t : J^{l-1}(r+k,1) \to J^{l-1}(r+k,1)$. This map depends differentiably on t. Similarly, the maps $\phi_t^* : \mathcal{E}(r;k) \to \mathcal{E}(r;k)$, $t \in [0,1]$, are linear and map $\mathrm{m}(r;k)^l$ to itself, so they induce the linear maps $\tilde{\phi_t}^* : J^{l-1}(r+k,1) \to J^{l-1}(r+k,1)$ and $\tilde{\phi_t}^*$ depends differentiably on t.

Now choose a basis $a_1, \cdots, a_p$ for $\pi_{l-1}(A) \subset J^{l-1}(r+k,1)$ and extend this to a basis $a_1, \cdots, a_q$ of $J^{l-1}(r+k,1)$. For $t \in [0,1]$ let Q_t be the matrix of $\tilde{\xi}_t$ with respect to the basis $a_1, \cdots, a_q$ and let C_t be the matrix of $\tilde{\phi_t}^*$. Then because $\xi_t(A) \subset A$ we have that Q_t has the form:

$$\begin{array}{cc} & \begin{array}{cc} p & q-p \end{array} \\ \begin{array}{c} p \\ q-p \end{array} & \begin{pmatrix} R_t & S_t \\ 0 & T_t \end{pmatrix} \end{array}.$$

If we divide C_t into submatrices in the same way it will have some form:

$$\begin{pmatrix} W_t & X_t \\ Y_t & Z_t \end{pmatrix}.$$

What we wish to prove is that $Y_t = 0$ for all $t \in [0,1]$.

Now let $f \in \mathcal{E}(r;k)$ be given. Then for $t \in [0,1]$ we have

$$\begin{aligned} \frac{d}{dt}(\tilde{\phi_t}^*(j^{l-1}f(0))) &= j^{l-1}(\frac{\partial}{\partial t}\phi_t^* f)(0) = j^{l-1}\phi_t^*(\xi_t(f))(0) \\ &= \tilde{\phi_t}^*(\tilde{\xi}_t(j^{l-1}f(0))). \end{aligned}$$

Hence we have that $\frac{d}{dt}\tilde{\phi_t}^* = \tilde{\phi_t}^* \circ \tilde{\xi}_t$ for $t \in [0,1]$. Therefore

$$\frac{dC_t}{dt} = C_t Q_t \text{ for } t \in [0,1].$$

Because of the form of Q_t this implies

$$\frac{dY_t}{dt} = Y_t R_t \text{ for } t \in [0,1].$$

But C_0 is the identity matrix because $\phi_0^* = id_{\mathcal{E}(r;k)}$ so $Y_0 = 0$. Hence $Y_t = 0$ for all $t \in [0,1]$. □

LEMMA 7.4. *Let A be a vector subspace of $\mathcal{E}(r;k)$ and $[0,1] \to \mathcal{E}(r;k)$ $(t \mapsto f_t)$ be a smooth homotopy. Suppose that there exists a smooth homotopy $[0,1] \to \mathrm{m}(r;k,r+k)$, $t \mapsto \xi_t = (x_1\xi_t^1, \cdots, x_r\xi_t^r, \xi_t^{r+1}, \cdots, \xi_t^{r+k})$ satisfying the following conditions:*
(1) $\xi_t(A) \subset A + \mathrm{m}(r;k)^l$ for all $t \in [0,1]$, $l > 0$.
(2) $\frac{\partial f_t}{\partial t} - \xi_t(f_t) \in A + \mathrm{m}(r;k)^l$ for all $t \in [0,1]$, $l > 0$.

Then for any $l > 0$ there exist $\phi \in \mathcal{B}(r;k)$ and $h \in A + \mathrm{m}(r;k)^l$ such that $\phi^*(A + \mathrm{m}(r;k)^l) \subset A + \mathrm{m}(r;k)^l$ and $f_0 = f_1 \circ \phi + h$.

Proof. Let $l > 0$ be given. Consider the solution $\Phi : (\mathbf{H}^r \times \mathbf{R}^k \times \mathbf{R}, 0 \times [0,1]) \to (\mathbf{H}^r \times \mathbf{R}^k, 0)$, $(x,y,t) \mapsto \phi_t(x,y)$, of the following differential equation:

$$\frac{d}{dt}\phi_t(x,y) = -\xi_t \circ \phi_t(x,y), \ \phi_0(x,y) = (x,y).$$

we denote $\Phi = (\Phi_1, \cdots, \Phi_{r+k})$, $\phi_t = (\phi_t^1, \cdots, \phi_t^{r+k})$ and define $H : (\mathbf{H}^r \times \mathbf{R}^k \times \mathbf{R}, 0 \times [0,1]) \to \mathbf{R}$ by $H(x,y,t) = f_t \circ \phi_t(x,y)$. Then

$$\begin{aligned}
\frac{\partial H}{\partial t}(x,y,t) &= (\sum_{i=1}^{r} \frac{\partial f_t}{\partial x_i} \circ \phi_t \frac{\partial \phi_t^i}{\partial t} + \sum_{j=1}^{k} \frac{\partial f_t}{\partial y_j} \circ \phi_t \frac{\partial \phi_t^{r+j}}{\partial t} + \frac{\partial f_t}{\partial t} \circ \phi_t)(x,y) \\
&= (-\sum_{i=1}^{r} \frac{\partial f_t}{\partial x_i} \circ \phi_t \phi_t^i \xi_t^i \circ \phi_t - \sum_{j=1}^{k} \frac{\partial f_t}{\partial y_j} \circ \phi_t \xi_t^{r+j} \circ \phi_t + \frac{\partial f_t}{\partial t} \circ \phi_t)(x,y) \\
&= \phi_t^*(-\xi_t(f_t) + \frac{\partial f_t}{\partial t})(x,y) \in \phi_t^*(A + \mathrm{m}(r;k)^l) \quad \text{by (2).}
\end{aligned}$$

Now by lemma 7.3 (with $A + \mathrm{m}(r;k)^l$ for A and $-\xi_t$ for ξ_t we have

$$\phi_t^*(A + \mathrm{m}(r;k)^l) \subset A + \mathrm{m}(r;k)^l \text{ for all } t \in [0,1], \tag{17}$$

so $\frac{\partial H}{\partial t}|_{\mathbf{R}^{r+k} \times \{t\}} \in A + \mathrm{m}(r;k)^l$ for all $t \in [0,1]$. Therefore for $t_0 \in [0,1]$ we have

$$\frac{d}{dt}|_{t=t_0}(j^{l-1}(\phi_t^* f_t)(0)) = j^{l-1}(\frac{\partial H}{\partial t}|_{\mathbf{R}^{r+k}\times\{t_0\}})(0) \in \pi_{l-1}(A + \mathrm{m}(r;k)^l) = \pi_{l-1}(A).$$

But since $\pi_{l-1}(A)$ is a linear subspace of $J^{l-1}(r+k,1)$ we have $j^{l-1}(\phi_1^* f_1 - \phi_0^* f_0) \in \pi_{l-1}(A)$. Hence $-h := \phi_1^* f_1 - f_0 \in A + \mathrm{m}(r;k)^l$, so that $f_0 = f_1 \circ \phi_1 + h$. □

LEMMA 7.5. *Let $f \in \mathcal{E}(r;k)$. Let $l > 0$ be an integer and set $f_0 = j^l f(0)$ (consider as a polynomial germ in $\mathcal{E}(r;k)$). Suppose that there exist $h \in \mathrm{m}(r;k)^q$ for some $q \geq l+1$ and $\xi = (x_1\xi_1, \cdots, x_r\xi_r, \xi_{r+1}, \cdots, \xi_{r+k}) \in \mathrm{m}(r;k)^{p-l}\mathcal{E}(r;k,r+k)$, $p \geq q+1$ such that $\xi(f_0) - h \in \mathrm{m}(r;k)^p$.*

Then there exists $h_1 \in \mathrm{m}(r;k)^p$ such that $f + h \overset{\mathrm{rR}}{\sim} f + h_1$ (or in other words, $j^{p-1}(f+h)(0) \in O_{\mathrm{rR}}^{p-1}(j^{p-1}f(0))$).

Proof. Define the smooth homotopy $[0,1] \to \mathcal{E}(r;k)$, $t \mapsto f_t = f + (1-t)h$ and define the smooth homotopy $[0,1] \to \mathrm{m}(r;k)\mathcal{E}(r;k,r+k)$, $t \mapsto \xi_t = -\xi$. Then

$$\begin{aligned}
\frac{\partial f_t}{\partial t} - \xi_t(f_t) &= -h + \xi(f + (1-t)h) = -h + \xi(f_0) + \xi(f - f_0 + (1-t)h)) \\
&\in \mathrm{m}(r;k)^p + \mathrm{m}(r;k)^{(p-l)+l} = \mathrm{m}(r;k)^p.
\end{aligned}$$

Hence the hypotheses of lemma 7.4 are fulfilled for $A = \mathrm{m}(r;k)^p$. Therefore there exist $\phi_1 \in \mathcal{B}(r,k)$ and $h' \in A + \mathrm{m}(r;k)^p = \mathrm{m}(r;k)^p$ such that $f_0 = f \circ \phi_1 + h'$. Set $h_1 = h' \circ \phi_1^{-1}$. Then we have $f_0 \circ \phi = f + h_1$ and $h_1 \in \mathrm{m}(r;k)^p$ □

LEMMA 7.6. *Let* $f \in \mathrm{m}(r;k)^l$ *and set* $f_0 = j^l f(0)$ *(so* f_0 *is a homogeneous polynomial of degree* l*). Let* h *be a homogeneous polynomial of degree* $q \geq l+1$ *and suppose* $h \in J_{\mathrm{rR}}(f_0)$*. Then there exists* $h_1 \in \mathrm{m}(r;k)^{q+1}$ *such that* $f+h \overset{\mathrm{rR}}{\sim} f+h_1$ *(or in other words,* $j^q(f+h)(0) \in O^q_{\mathrm{rR}}(j^l f(0))$*), where* $J_{\mathrm{rR}}(f_0)$ *is the Jacobi ideal of* f_0 *defined by*

$$J_{\mathrm{rR}}(f_0) = \langle x_1 \frac{\partial f_0}{\partial x_1}, \cdots, x_r \frac{\partial f_0}{\partial x_r}, \frac{\partial f_0}{\partial y_1}, \cdots, \frac{\partial f_0}{\partial y_k} \rangle_{\mathcal{E}(r;k)}.$$

Proof. By $h \in J_{\mathrm{rR}}(f_0)$ there exists $\xi = (x_1\xi_1, \cdots, x_r\xi_r, \xi_{r+1}, \cdots, \xi_{r+k}) \in \mathrm{m}(r; k, r+k)$ such that $\xi(f_0) = h$. We may assume that $\xi \in \mathrm{m}(r;k)^{q-l+1}\mathcal{E}(r;k,r+k)$ and , for if not we replace ξ by ξ' defined by $\xi' = \xi - j^{q-l}\xi(0)$. Since f_0 is homogeneous polynomial of degree l we have $(\xi - \xi')(f_0)$ is a polynomial of degree $q-1$ and $\xi'(f_0) \in \mathrm{m}(r;k)^q$. Since $h = (\xi-\xi')(f_0) + \xi'(f_0) \in \mathrm{m}(r;k)^q$ we have $(\xi-\xi')(f_0) = 0$ and $\xi'(f_0) = h$.

Hence the conditions of lemma 7.5 are fulfilled with $p = q+1$, and the conclusion follows immediately from lemma 7.5. □

We now start the classification of unimodal residual singularities in $\mathrm{m}(r;k)^2$ ($r \geq 1$) under reticular R-equivalence. We shall prove that this classification includes the classification of residual singularities whose reticular R-codimension is lower than 8. Firstly we introduce the following notations: $a_i, b_j, a, b, c, \cdots$ are real numbers. We say that $z \in J^l(r+k,1)$ has modality n if the following condition holds: For any neighborhood of z there exists an element z' in this neighborhood and there exists an n-parameter family of l-jets $z'(a)$ (a in some neighborhood of 0 in $\mathbf{R}^n$) such that $z'(0) = z'$ and $z'(a) \notin O^l_{rR}(z'(b))$ if $a \neq b$. Remark that for $f \in \mathrm{m}(r+k)^2$ if $j^l f(0)$ has modality n then f also has modality n.

Let $f \in \mathrm{m}(r;k)^2$ be a function germ with reticular R-finite-codimension. In the procedure of the classification, we adopt the following notations:

'$\Rightarrow$' means 'see'.

'$f \overset{\mathrm{rR}}{\sim} g$' means '$f$ is reticular R-equivalent to g' ($g \in \mathcal{E}(r;k)$).

'$\overset{1}{\mapsto} g$' means '$f \overset{rR}{\sim} g$ by lemma 7.6 by the analogous method of (2)'.

'$\overset{2}{\mapsto} g$' means '$f \overset{rR}{\sim} g$ by lemma 7.5 by the analogous method in (6)'.

'$\overset{3}{\mapsto} g$' means '$f \overset{rR}{\sim} g$ because g is (degree of g)-determined by Lemma 4.2'.

'$\overset{4}{\mapsto} g$' means '$f \overset{rR}{\sim} g$ by a linear coordinate change'.

<u>**The case** $r=1$.</u> The classification is reduced to V.I. Arnold [2] and V.I. Matov [9].

<u>**The case** $r=2, k=0$.</u> Let $j^2 f(0) = ax_1^2 + bx_1x_2 + cx_2^2$.

$a \neq 0, c \neq 0$	$\Rightarrow$	(1)
$b \neq 0, c = 0$	$\Rightarrow$	(3)
$a = 0, b \neq 0, c \neq 0$	$\Rightarrow$	(4)
$a \neq 0, b = 0, c = 0$	$\Rightarrow$	(5)
$a = 0, b = 0, c \neq 0$	$\Rightarrow$	(7)
$a = 0, b = 0, c = 0$	$\Rightarrow$	(8)

(1) $j^2 f(0)$ has the normal form $\pm x_1^2 + ax_1x_2 \pm x_2^2$ by a linear coordinate change.

$a^2 \neq \pm\pm 4 \quad \overset{3}{\mapsto} \quad \pm x_1^2 + ax_1x_2 \pm x_2^2$.

$a^2 = \pm\pm 4 \quad \Rightarrow \quad (2)$

(2) $\overset{1}{\mapsto} \quad \pm(x_1^2 \pm 2x_1x_2 + x_2^2) + \sum_{i\geq 3} a_i x_2^i$ ($^\exists i$ s.t. $a_i \neq 0$) : Let $f_0 = j^2 f(0)$. Since

$x_1\frac{\partial f_0}{\partial x_1} = \pm 2(x_1^2 \pm x_1x_2)$ and $x_2\frac{\partial f_0}{\partial x_2} = \pm 2(\pm x_1x_2 + x_2^2)$, we may replace any term of degree ≥ 3 involving x_1 in f by terms involving less x_1's and more x_2's and terms of higher degree by lemma 7.6. As a result we have the normal form. If $a_i = 0$ for all i, then we have codimension $f = \infty$. Therefore there exists an integer i such that $a_i \neq 0$. $\overset{3}{\mapsto}$ $\pm(x_1^2 \pm 2x_1x_2 + x_2^2) + ax_2^n$ $(n \geq 3, a \neq 0)$.

(3) $j^2f(0)$ has the normal form $\pm x_1^2 \pm x_1x_2$ or $\pm x_1x_2$ by a linear coordinate change. $\overset{1}{\mapsto}$ $\sum_{i\geq 2} a_ix_1^i + x_1x_2 + \sum_{j\geq 3} b_jx_2^j$ $({}^{\exists}i, j$ s.t. $a_i \neq 0, b_j \neq 0)$ $= gx_1^n \pm x_1x_2 + hx_2^m$ (where g, h are units in $\mathcal{E}(2;0)$ and $n \geq 2, m \geq 3$ be the minimum integer satisfying $a_n \neq 0, b_m \neq 0$ respectively) $\overset{\text{rR}}{\sim}$ $\pm x_1^n + x_1x_2 + gx_2^m$ $(n \geq 2, m \geq 3, g$:unit) $\overset{3}{\mapsto}$ $\pm x_1^n \pm x_1x_2 + ax_2^m$ $(n \geq 2, m \geq 3, a \neq 0)$.

(4) By using the analogous method of (3), we have $f \overset{\text{rR}}{\sim} \pm x_1^n \pm x_1x_2 + ax_2^2$ $(n \geq 3, a \neq 0)$.

(5) $j^2f(0)$ has the normal form $\pm x_1^2$ by a linear coordinate change. Hence $j^3f(0)$ has the normal form $\pm x_1^2 + ax_1x_2^2 + bx_2^3$ by lemma 7.6.

$$\begin{array}{lll} b \neq 0 & \overset{3}{\mapsto} & \pm x_1^2 + ax_1x_2^2 + bx_2^3 \ (b \neq 0) \\ & \overset{4}{\mapsto} & \pm x_1^2 + ax_1x_2^2 \pm x_2^3. \\ b = 0 & \Rightarrow & (6) \end{array}$$

(6) $j^3f(0)$ has the normal form $f_0 = \pm x_1^2 + ax_1x_2^2$. Then $j^5f(0)$ has modality 2 in $J^5(2+1;1)$: For any neighborhood of $j^5f(0)$ there exists an element f_1 in the neighborhood such that $\pi_2^5(z) = \pm x_1^2 + bx_1x_2^2$ $(b \neq 0)$. Then f_1 has the normal form $\pm x_1^2 \pm x_1x_2^2 + cx_2^4 + dx_2^5$ by lemma 7.5 (since $x_1\frac{\partial f_0}{\partial x_1} = \pm 2x_1^2 \pm x_1x_2^2$ and $x_2\frac{\partial f_0}{\partial x_2} = \pm 2x_1x_2^2$, we may replace any term of degree ≥ 3 involving x_1 in f_1 by terms of higher degree). It is enough to prove that f_1 has modality 2. Suppose that $f_2 = \pm x_1^2 \pm x_1x_2^2 + c'x_2^4 + d'x_2^5 \in O^5_{\text{rR}}(f_1)$. Then there exists $\phi = (x_1\phi_1, x_2\phi_2) \in \mathcal{B}(2;0)$ such that $f_2 \equiv f_1 \circ \phi$ mod $\mathrm{m}(2;0)^6$, where f_1 and f_2 are considered as polynomial function germs. Let $j^2\phi_2(0) = \phi_2(0) + \phi_{21}x_1 + \phi_{22}x_2$ $(\phi_{21}, \phi_{22} \in \mathbf{R})$. By the coefficient of x_1^2 in f_2 we have $\phi_1(0) = 1$. By the coefficient of $x_1x_2^2$ we have $\phi_2(0) = 1$. By the coefficient of $x_1x_2^3$ we have $\phi_{22} = 0$. These imply that $a = a'$ by the coefficient of x_2^4 and $b = b'$ by the coefficient of $x_1x_2^5$.

On the other hand, reticular R-codimension $f \geq 8$: It is enough to prove that codimension of $O^5_{\text{rR}}(f_1)$ in $J^5(2+0,1) \geq 8$. Set $A = \langle x_1^3, x_1^2x_2^2, x_1x_2^4, x_2^6\rangle_{\mathcal{E}(2;0)}$. Since $m^6(2;0) \subset A$ and $x_1\frac{\partial f_1}{\partial x_1} = \pm 2x_1^2 \pm x_1x_2^2, x_2\frac{\partial f_1}{\partial x_2} = \pm 2x_1x_2^2 + 4ax_2^4 + 5bx_2^5$, we have that $x_1^2\frac{\partial f_1}{\partial x_1} \equiv x_1x_2^2\frac{\partial f_1}{\partial x_1} \equiv x_1x_2\frac{\partial f_1}{\partial x_2} \equiv x_2^3\frac{\partial f_1}{\partial x_2} \equiv 0$ mod A. Therefore

$$\begin{aligned} &\text{codimension of } O^5_{\text{rR}}(f_1) \text{ in } J^5(2+0,1) \\ &= \dim \mathcal{E}(2;0)/(\langle x_1\frac{\partial f_1}{\partial x_1}, x_2\frac{\partial f_1}{\partial x_2}\rangle + \mathrm{m}(2;0)^6) \\ &\geq \dim \mathcal{E}(2;0)/(\langle x_1\frac{\partial f_1}{\partial x_1}, x_1x_2\frac{\partial f_1}{\partial x_1}, x_2\frac{\partial f_1}{\partial x_2}, x_2^2\frac{\partial f_1}{\partial x_2}\rangle_{\mathbf{R}} + A) \\ &= \dim \mathcal{E}(2;0)/A \ - \ 4 = 12 - 4 \geq 8. \end{aligned}$$

(7) By using the analogous method of (5), we have $f \overset{\text{rR}}{\sim} \pm x_1^3 + ax_1^2x_2 \pm x_2^2$ or f has modality 2 and codimension $f \geq 8$.

(8) $f \in \mathrm{m}(2;0)^3$. Hence

$$\text{reticular-R-codimension } f \geq \dim \mathcal{E}(2;0)/(\langle x_1\frac{\partial f}{\partial x_1}, x_2\frac{\partial f}{\partial x_2}\rangle_{\mathbf{R}} + \mathrm{m}(2;0)^4) \geq 10 - 2 = 8.$$

On the other hand, we can show that $j^3f(0)$ has modality 2 by analogous methods of

(6): In this case, if we consider an element $z \in \pi_3(\mathrm{m}(2;0)^3)$ such that the coefficients of x_1^3, x_2^3 are not zero. Then z has the normal form $\pm x_1^3 + ax_1^2x_2 + bx_1x_2^2 \pm x_2^3$ by a linear coordinate change. Hence z has modality 2.

The case $r = 2, k = 1.$ Let $j^2f(0) = ax_1y + bx_2y + cx_1^2 + dx_1x_2 + ex_2^2$.

$$\begin{array}{lll} a \neq 0, b \neq 0 & \Rightarrow & (9) \\ a \neq 0, b = 0, e \neq 0 & \Rightarrow & (10) \\ a \neq 0, b = 0, e = 0 & \Rightarrow & (12) \\ a = 0, b \neq 0 & \Rightarrow & (13) \\ a = 0, b = 0 & \Rightarrow & (14) \end{array}$$

(9) $j^2f(0)$ has the normal form $x_1y \pm x_2y + ax_2^2$ by a linear coordinate change $\overset{1}{\mapsto} \sum_{i\geq 3} a_iy^i + x_1y \pm x_2y + \sum_{j\geq 2} b_ix_2^j$ ($^\exists i, j$ s.t. $a_i \neq 0, b_j \neq 0$) $\overset{\mathrm{rR}}{\sim}$ $\pm y^n + x_1y \pm x_2y + ax_2^m$ ($n \geq 3, m \geq 2, a \neq 0$) (here we used the analogous method of (3)).

(10) $j^2f(0)$ has the normal form $x_1y \pm x_2^2$ by a linear coordinate change. Hence $j^3f(0)$ has the normal form $ay^3 + x_1y + bx_2y^2 \pm x_2^2$ by lemma 7.6.

$$\begin{array}{lll} a \neq 0 & \overset{3}{\mapsto} & ay^3 + x_1y + bx_2y^2 \pm x_2^2 \ (a \neq 0) \\ & \overset{4}{\mapsto} & \pm y^3 + x_1y + ax_2y^2 \pm x_2^2 \ (a \in \mathbf{R}). \\ a = 0 & \Rightarrow & (11) \end{array}$$

(11) We can prove that $j^5f(0)$ has modality 2 and reticular R-codimension of f ≥ 8 by analogous methods of (6): Consider an element $f_0 \in J^5(2+1,1)$ satisfying $\pi_3^5(f_0) = x_1y + ax_2y^2 \pm x_2^2$ ($a \neq 0$). Then f_0 has the normal form $ay^5 + by^4 + x_1y \pm x_2y^2 \pm x_2^2$ and hence f_0 has modality 2. On the other hand, set $A = \langle x_1^2, x_1x_2^2, x_2^3, x_1y, x_2^2y^2, x_2y^4, y^6\rangle_{\mathcal{E}(2;1)}$. Since $\mathrm{m}(2;1)^6 \subset A$ and $x_1\frac{\partial f_0}{\partial x_1} = x_1y, x_2\frac{\partial f_0}{\partial x_2} = \pm x_2y^2 \pm 2x_2^2$, $\frac{\partial f_0}{\partial y} = 5ay^4 + 4by^3 + x_1 \pm 2x_2y$, we have $x_1\frac{\partial f_0}{\partial x_1} \equiv gx_2\frac{\partial f_0}{\partial x_2} \equiv h\frac{\partial f_0}{\partial y} \equiv 0$ mod A for $g = x_1, x_2, y^2$ and $h = x_1, x_2^2, x_2y, y^3$. Hence

$$\begin{aligned} &\text{codimension of } O^5_{\mathrm{rR}}(f_0) \text{ in } J^5(2+0,1) \\ &\geq \dim \mathcal{E}(2;1)/(\langle x_1\frac{\partial f_0}{\partial x_1}, x_2\frac{\partial f_0}{\partial x_2}, \frac{\partial f_0}{\partial y}\rangle + A) \\ &= \dim \mathcal{E}(2;1)/(\langle x_2\frac{\partial f_0}{\partial x_2}, x_2y\frac{\partial f_0}{\partial x_2}, \frac{\partial f_0}{\partial y}, x_2\frac{\partial f_0}{\partial y}, y\frac{\partial f_0}{\partial y}, y^2\frac{\partial f_0}{\partial y}\rangle_{\mathbf{R}} + A) \\ &\geq \dim \mathcal{E}(2;1)/A \ - \ 6 = 14 - 6 \geq 8. \end{aligned}$$

(12) $j^2f(0)$ has the normal form x_1y by a linear coordinate change $\overset{1}{\mapsto} x_1y + h(x_2, y)$ ($h \in \mathrm{m}(x_2, y)^3$). Set $A = \langle x_1y, x_1^2, x_1x_2^2\rangle + \mathrm{m}(2;1)^4$. Then we have $x_1\frac{\partial f}{\partial x_1} \equiv x_2\frac{\partial f}{\partial x_2} \equiv$ mod A. Hence

$$\begin{aligned} &\text{reticular -codimension } f \\ &\geq \dim \mathcal{E}(2;1)/(\langle x_1\frac{\partial f}{\partial x_1}, x_2\frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial y}\rangle + \mathrm{m}(2;1)^4) \\ &\geq \dim \mathcal{E}(2;1)/(\langle \frac{\partial h}{\partial y}, x_1\frac{\partial h}{\partial y}, x_2\frac{\partial h}{\partial y}, y\frac{\partial h}{\partial y}\rangle_{\mathbf{R}} + A) \\ &\geq \dim \mathcal{E}(2;1)/A \ - 4 = 12 - 4 = 8. \end{aligned}$$

On the other hand, we have $j^3f(0)$ has modality 2 by the analogous method of (6): Consider an element $f_0 \in J^3(2+1,1)$ which has the form $x_1y + h(x_2, y)$ (h is a homogeneous polynomial of degree 3) and the coefficients of x_2^3 and y^3 in h are not zero, then f_0 has the normal form $x_1y \pm x_2^3 + ax_2^2y + bx_2y^2 \pm y^3$ and hence f_0 has modality 2.

(13) By using the analogous method of (10) and (12), we have $f \overset{\text{rR}}{\sim} \pm y^3 + x_2 y + a x_1 y^2 \pm x_1^2$ or f has modality 2 and codimension $f \geq 8$.
(14) $j^3 f(0)$ has the form $h(x_1, x_2) + g(x_1, x_2, y)$, where h,g are homogeneous polynomials of degree 2,3 respectively. Hence

$$\begin{aligned}
&\text{reticular R-codimension } f \\
&\geq \dim \mathcal{E}(2;1)/(\langle x_1 \frac{\partial f}{\partial x_1}, x_2 \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial y}\rangle + \mathrm{m}(2;1)^4 + \mathrm{m}(2;0)^3) \\
&= \dim \mathcal{E}(2;1)/(\langle x_1 \frac{\partial f}{\partial x_1}, x_2 \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial y}, x_1 y \frac{\partial h}{\partial x_1}, x_2 y \frac{\partial h}{\partial x_2}, x_1 \frac{\partial g}{\partial y}, x_2 \frac{\partial g}{\partial y}, y \frac{\partial g}{\partial y}\rangle_{\mathbf{R}} \\
&\qquad\qquad + \mathrm{m}(2;1)^4 + \mathrm{m}(2;0)^3) \\
&\geq \dim \mathcal{E}(2;1)/\mathrm{m}(2;1)^4 \ - 4 - 8 = 20 - 4 - 8 = 8.
\end{aligned}$$

On the other hand, $j^3 f(0)$ has modality 2 in $J^3(2+1,1)$ by analogous method of (6): Consider an element $f_0 \in J^3(2+1,1)$ which has the form $ax_1^2 + bx_1x_2 + cx_2^2 + g'(x_1, x_2, y)$ (g' is homogeneous polynomial of degree 3) and all of the coefficients of x_1^2, x_2^2, y^3 are not zero and $b^2 \neq 4ac$. Then f_0 has the normal form $\pm x_1^2 + a x_1 x_2 \pm x_2^2 \pm y^3 + b x_1 x_2 y$ and hence has f_0 modality 2.

The case $r = 2, k \geq 2$. We prove that codimension $f \geq 8$ and $j^3 f(0)$ has modality 2. To do this, we only need to prove in the case $k = 2$ because codimension $f \geq$ codimension $f|_{y_3 = \cdots = y_k = 0}$ and if $f|_{y_3 = \cdots = y_k = 0}$ has modality 2 then f also has modality 2.

Set $A = \mathrm{m}(2;0)^2 + \mathrm{m}(2;0)\mathrm{m}(0;2)^2 + \mathrm{m}(2;2)^4$. Since $f \in \mathrm{m}(0;2)^3 + \mathrm{m}(2;0)\mathrm{m}(0;2) + \mathrm{m}(2;0)^2$ we have that $x^2 \frac{\partial f}{\partial x} \equiv xy \frac{\partial f}{\partial x} \equiv xy \frac{\partial f}{\partial y} \equiv y^2 \frac{\partial f}{\partial y} \equiv 0$. Hence

$$\begin{aligned}
&\text{reticular R-codimension } f \\
&\geq \dim \mathcal{E}(2;2)/(\langle x \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle + A) \\
&\geq \dim \mathcal{E}(2;2)/(\langle x \frac{\partial f}{\partial y}, \frac{\partial f}{\partial y}, y \frac{\partial f}{\partial y}\rangle_{\mathbf{R}} + A) \\
&\geq \dim \mathcal{E}(2;2)/A - 8 \\
&= \dim \mathcal{E}(2;0)/\mathrm{m}(2;0)^2 + \dim \mathrm{m}(0;2)/\mathrm{m}(0;2)^4 + 4 - 8 = 3 + 9 + 4 - 8 = 8.
\end{aligned}$$

On the other hand, $j^3 f(0)$ has modality 2 by an analogous method of (6): Consider an element f_0 in $J^3(2+2,1)$ in which all of the coefficients of $x_1y_1, x_2y_2, y_1^3, y_2^3$ are not zero. Then f_0 has the normal form $x_1 y_1 + x_2 y_2 \pm y_1^3 + a y_1^2 y_2 + b y_1 y_2^2 \pm y_2^3$ and hence f_0 has modality 2.

The case $r \geq 3$. We only need to prove in the case $r = 3, k = 0$ that codimension $f \geq 8$ and $j^2 f(0)$ has modality 2.

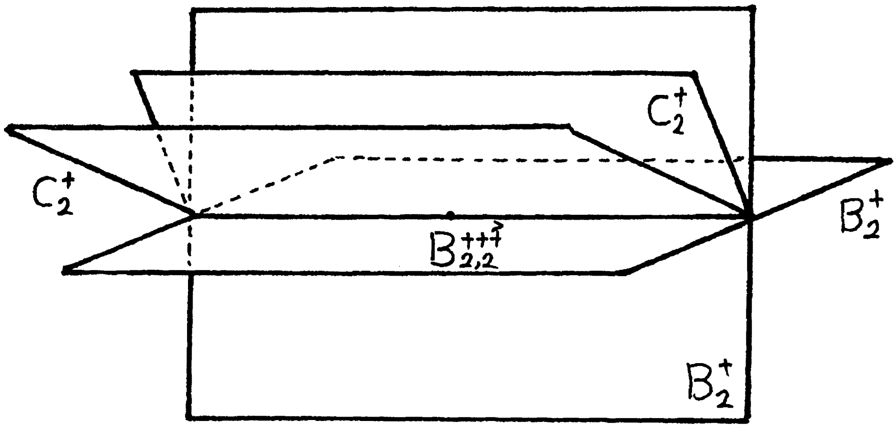
$$\begin{aligned}
&\text{reticular R-codimension } f \\
&\geq \dim \mathcal{E}(3;0)/(\langle x \frac{\partial f}{\partial x}\rangle + \mathrm{m}(3;0)^4) \\
&\geq \dim \mathcal{E}(3;0)/(\langle x \frac{\partial f}{\partial x}, x^2 \frac{\partial f}{\partial x}\rangle_{\mathbf{R}} + \mathrm{m}(3;0)^4) \\
&\geq \dim \mathcal{E}(3;0)/\mathrm{m}(3;0)^4 \ - 3 - 9 = 20 - 3 - 9 = 8.
\end{aligned}$$

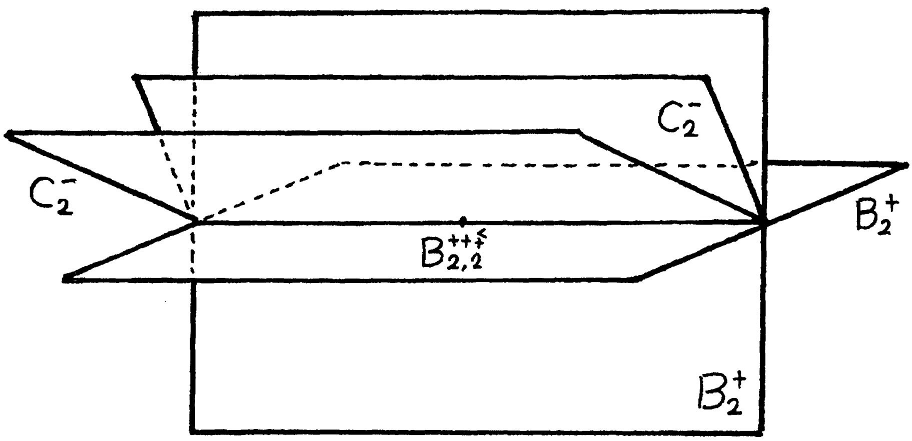
On the other hand $j^2 f(0)$ has modality 2 in $J^2(3+0,1)$ by the analogous method of (6): Consider an element f_0 in $J^3(3+0,1)$ in which all of the coefficients of x_1^2, x_2^2, x_3^2

are not zero. Then f_0 has the normal form $\pm x_1^2 \pm x_2^2 \pm x_3^2 + ax_1x_2 + bx_2x_3 + cx_3x_1$ and hence f_0 has modality 3 in $J^2(3+0,1)$. □

Acknowledgments. The author would like to thank Professor S.Izumiya for useful remarks and useful discussions.

The $B_{2,2}$ caustics I





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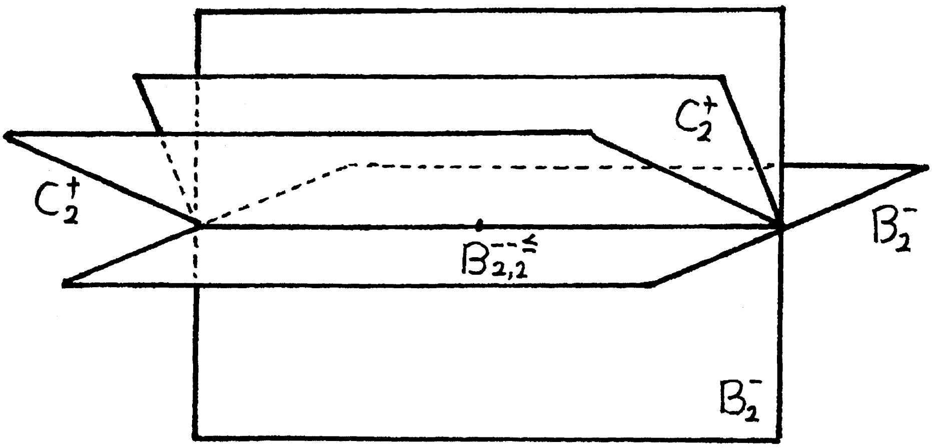

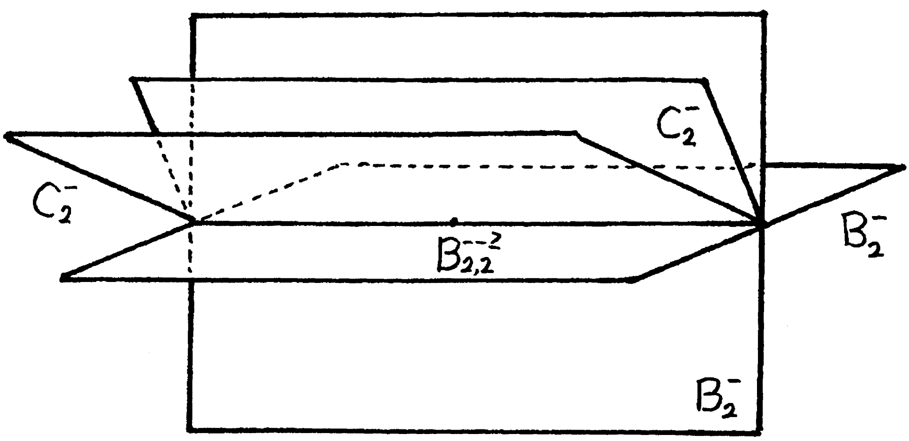

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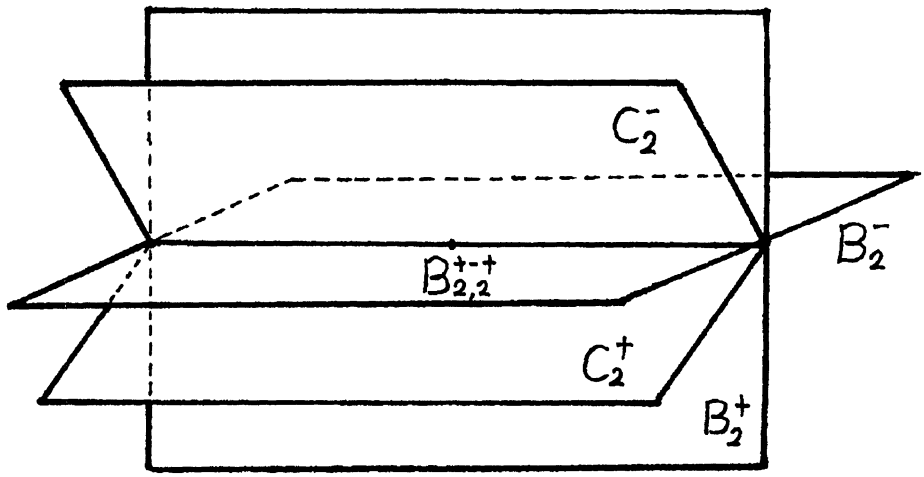

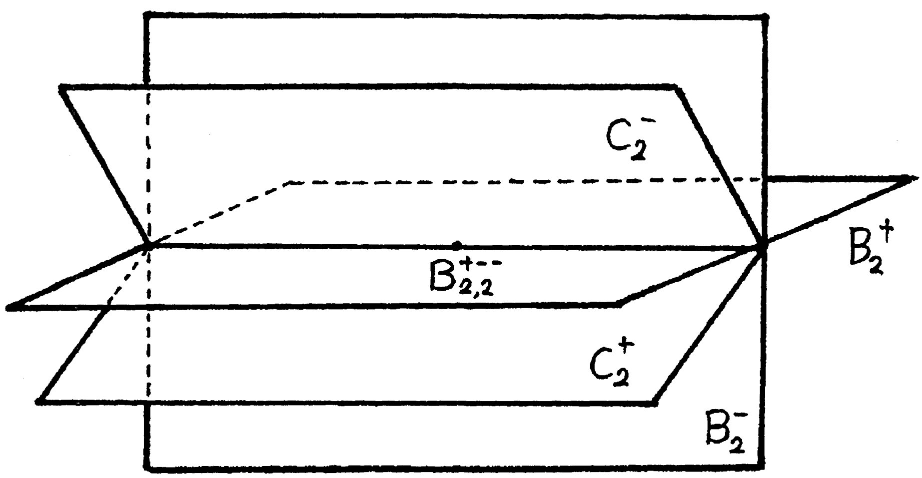

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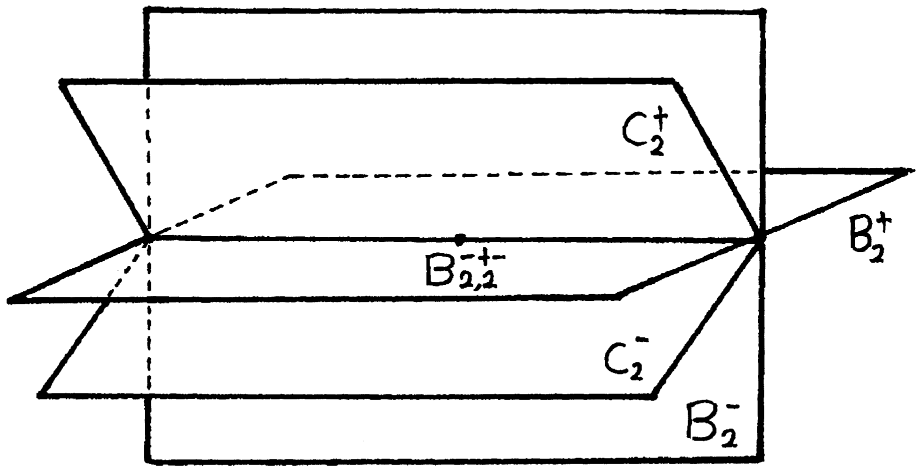

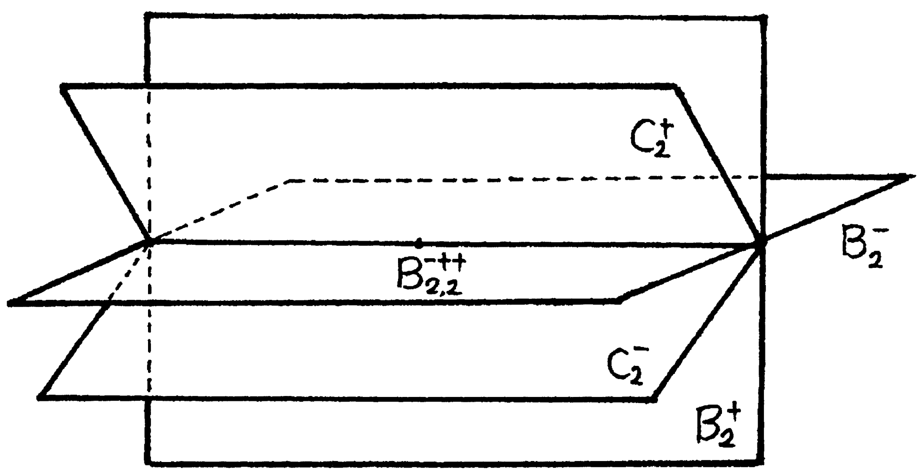

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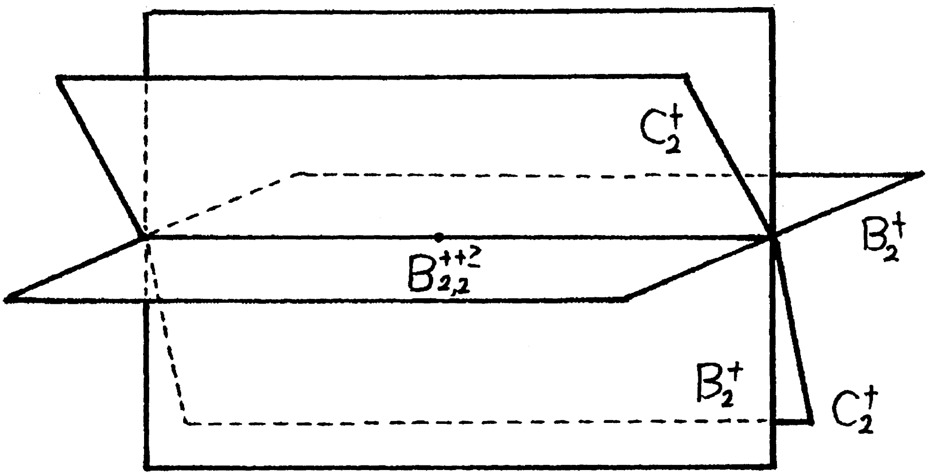

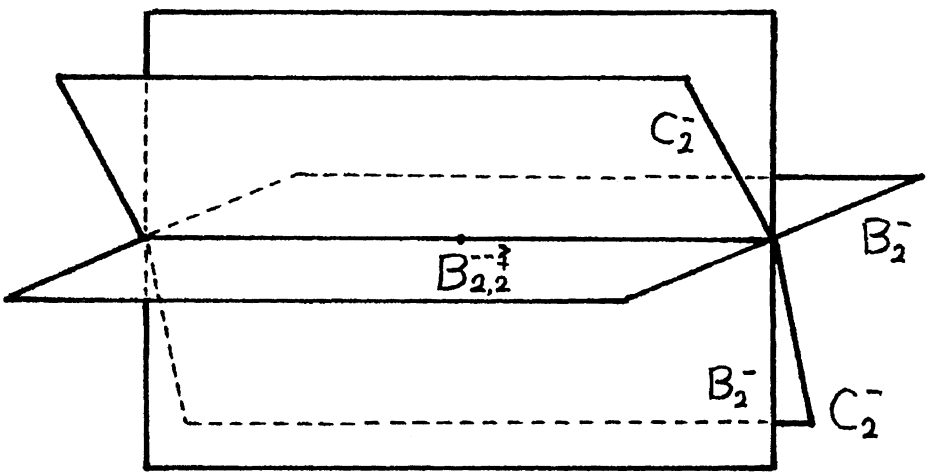

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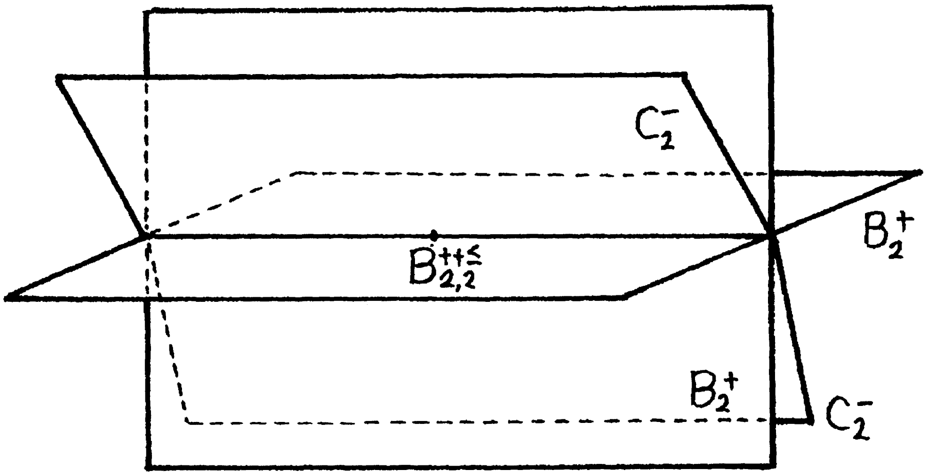

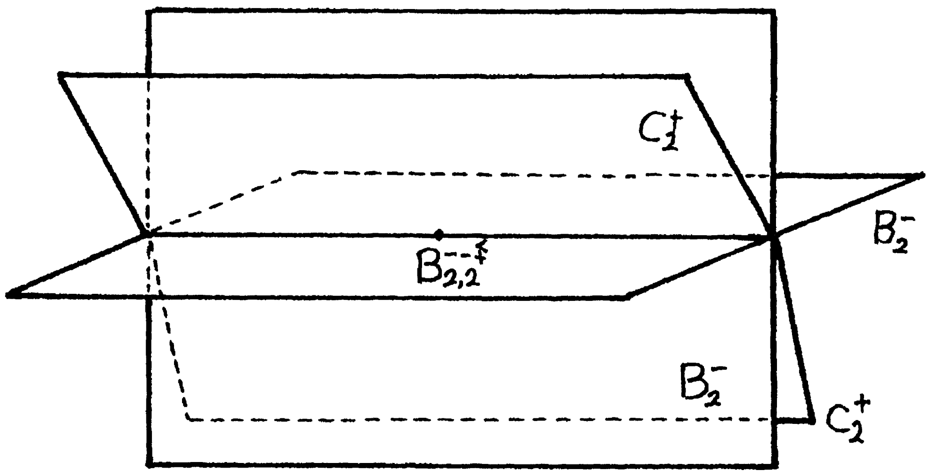

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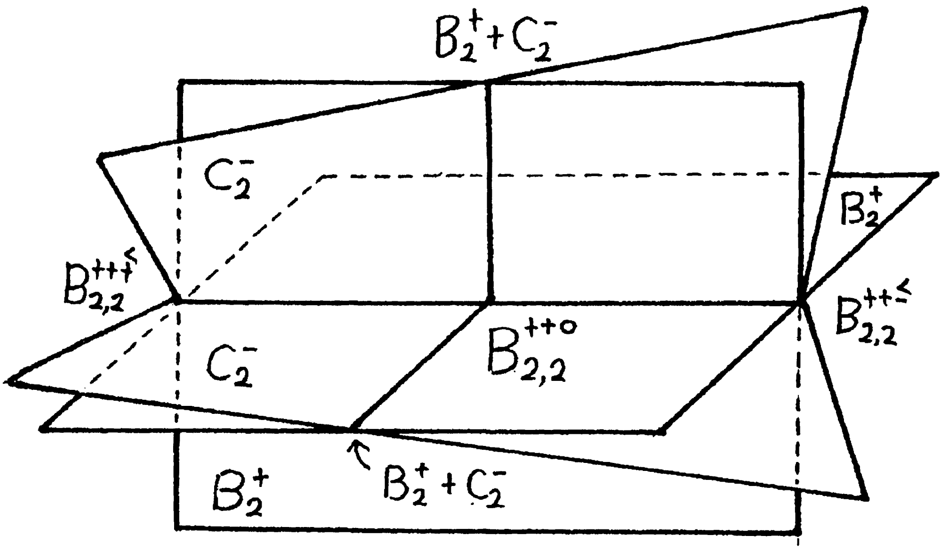
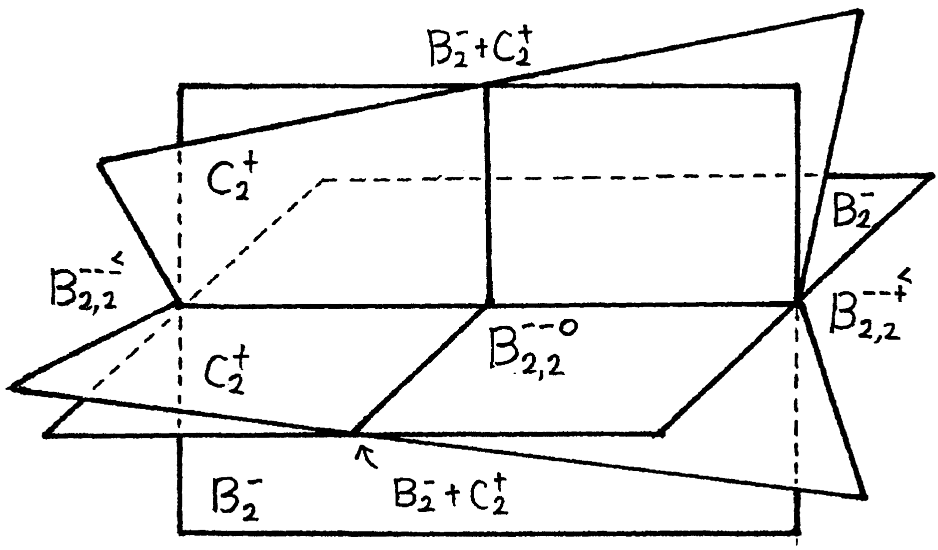
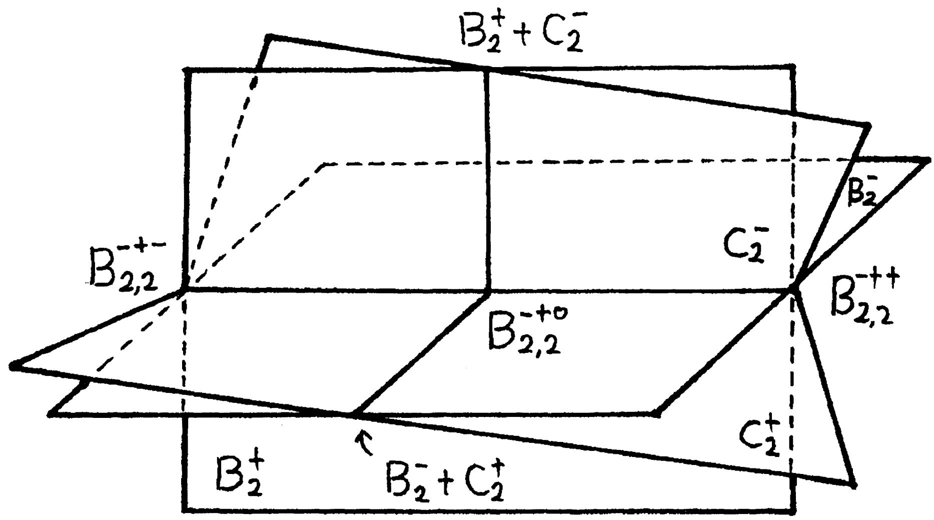
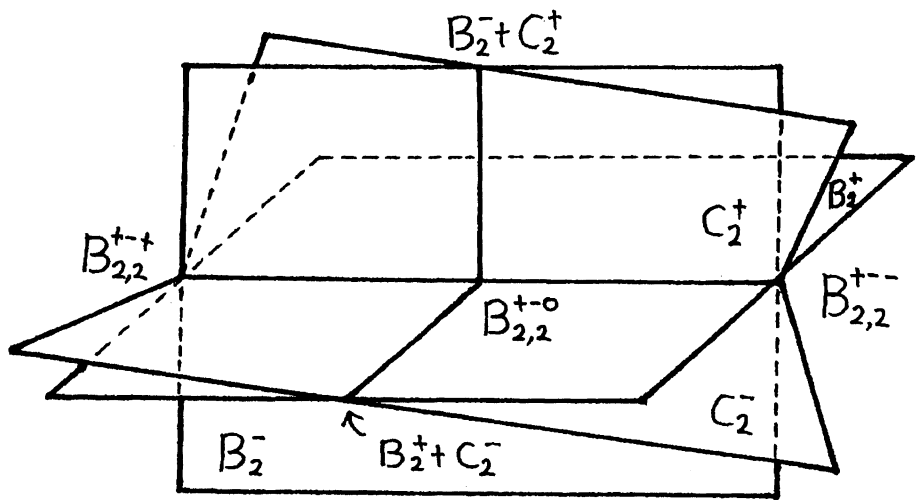


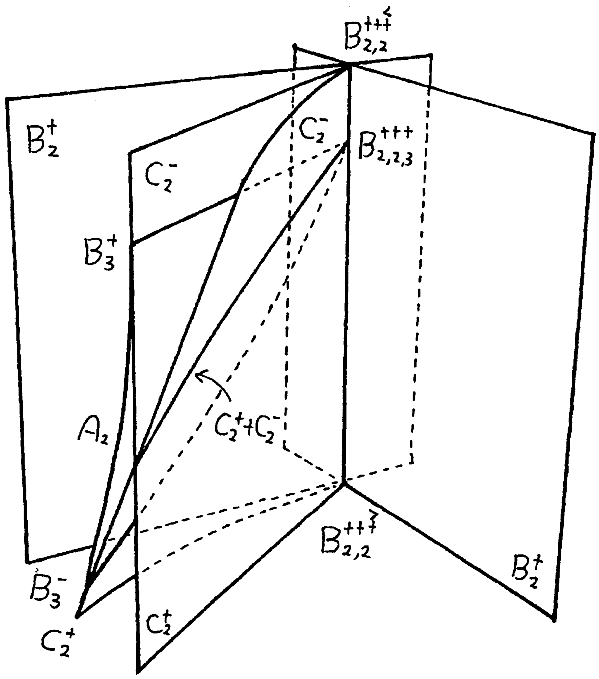


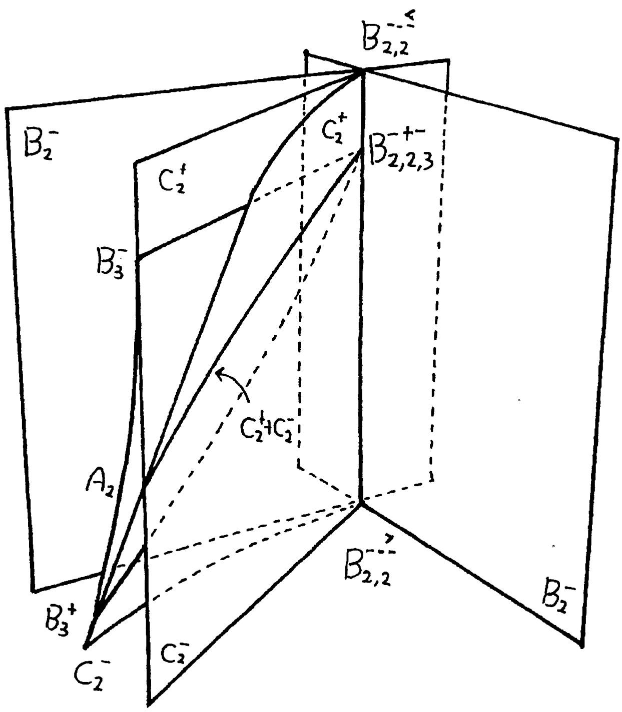
The $B_{2,2}$ caustics VIII





The sections of $B_{2,2,3}$ caustics I





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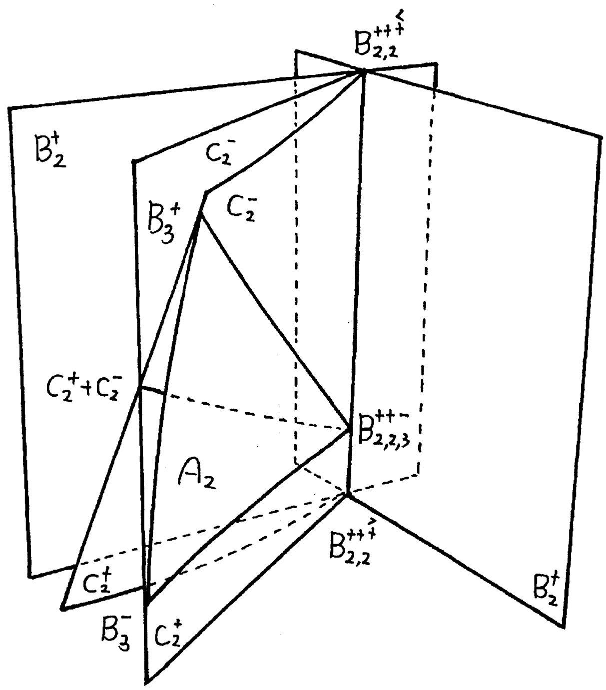

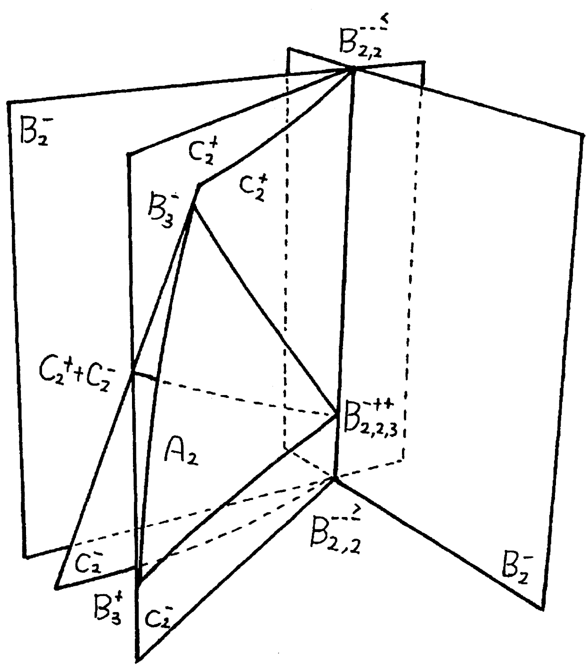

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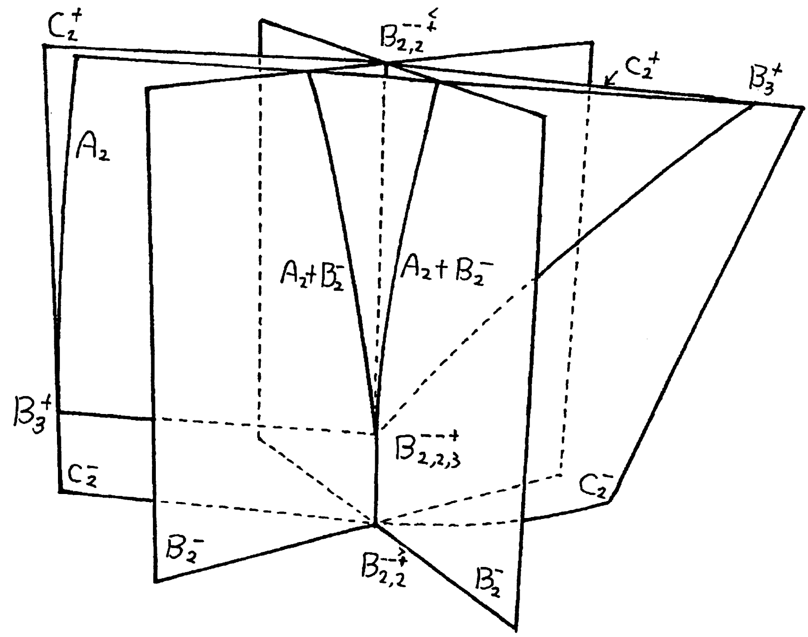

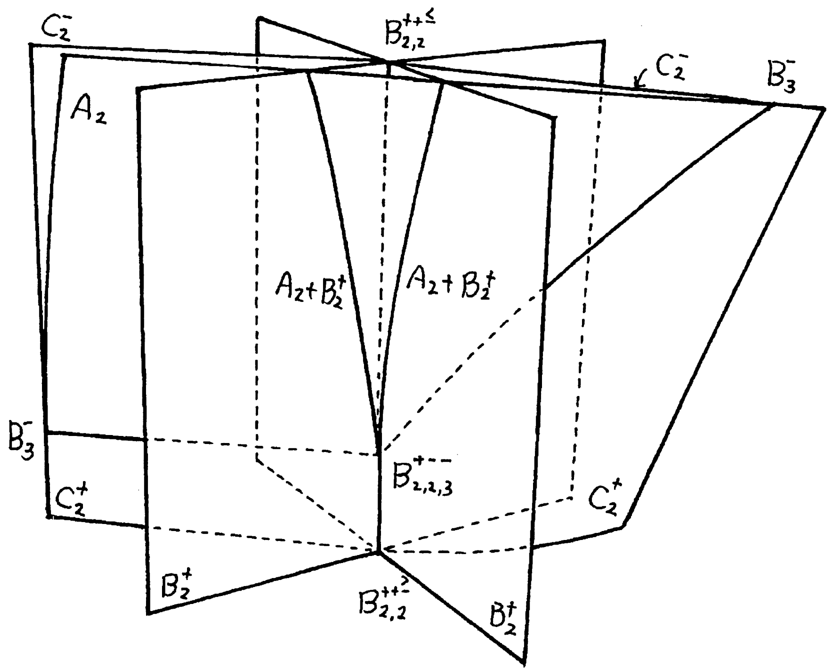

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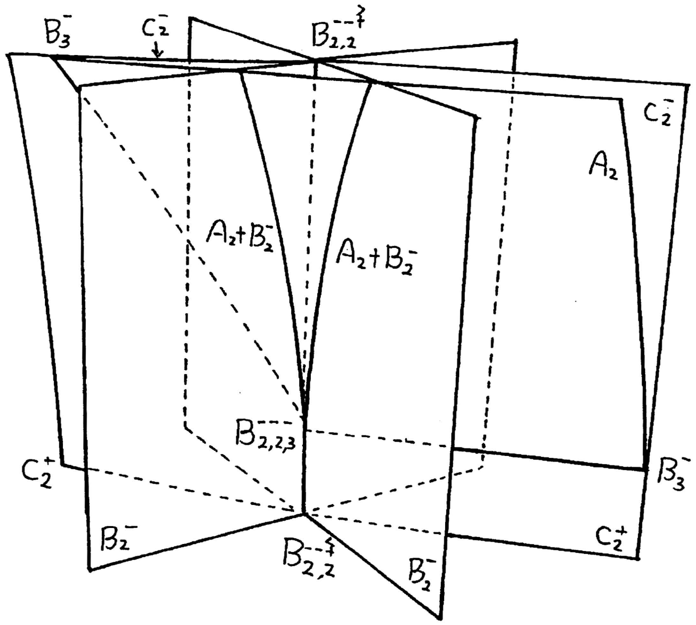

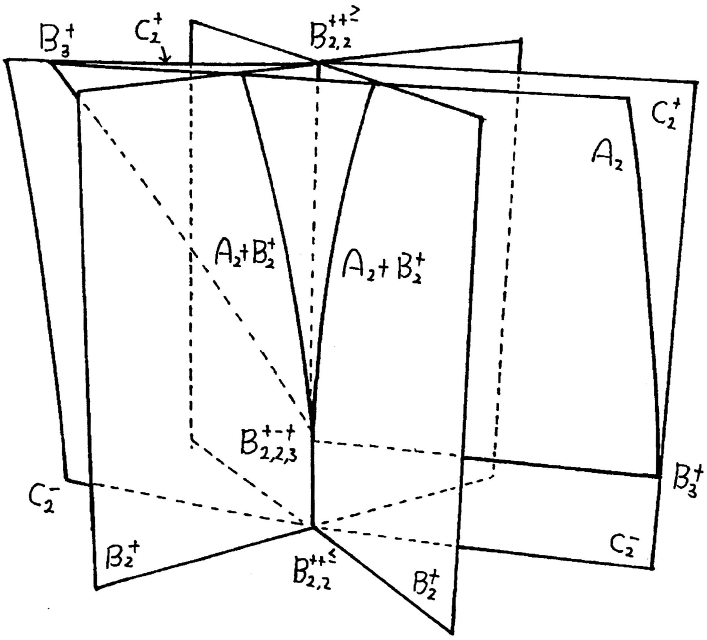

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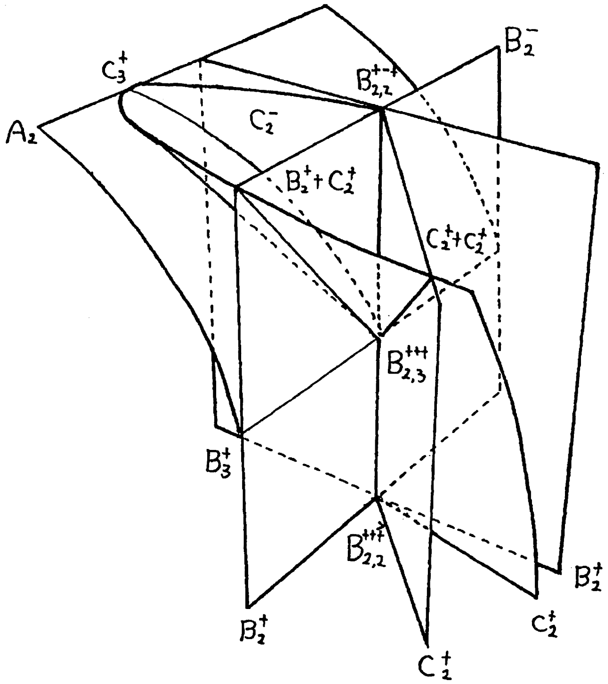

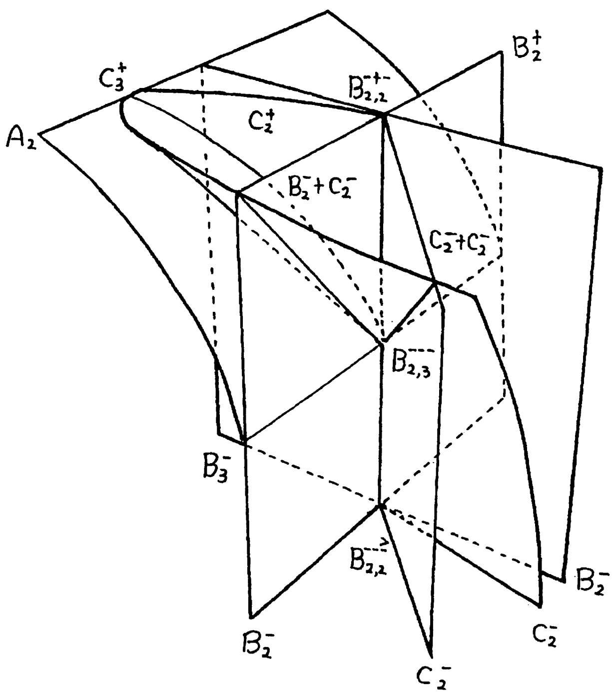

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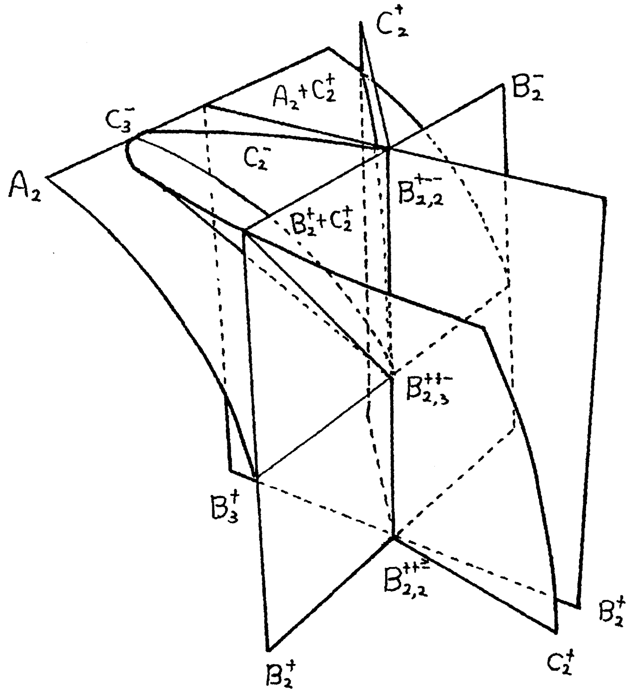

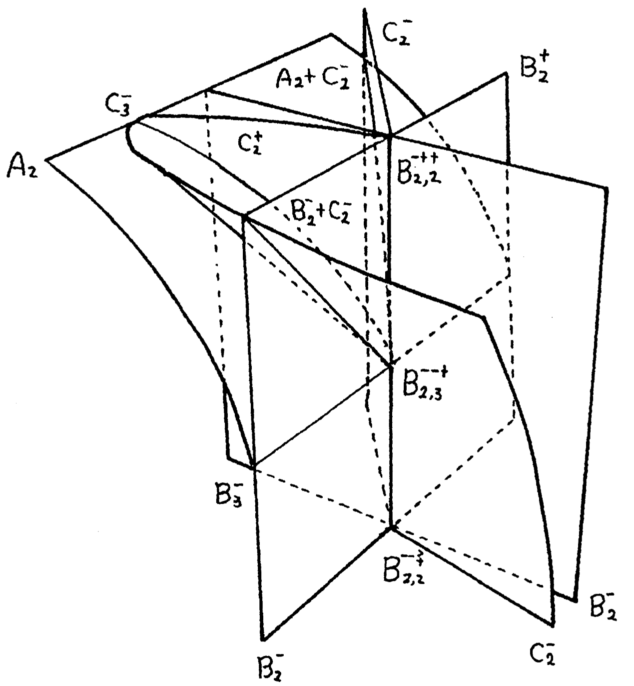

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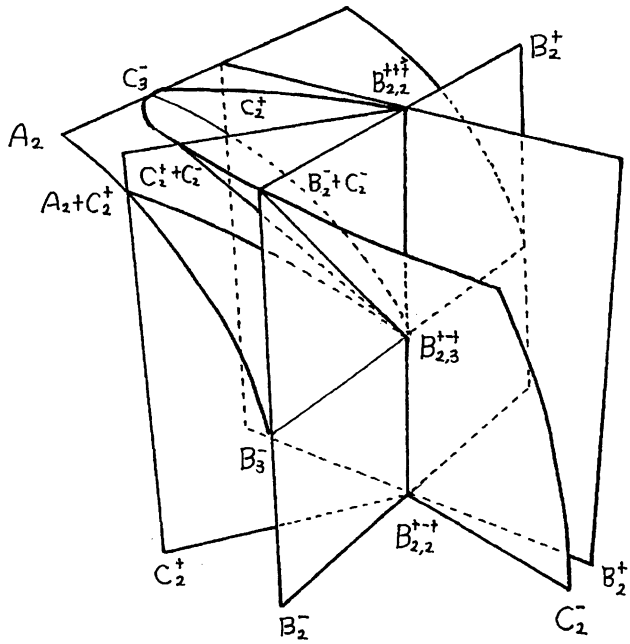

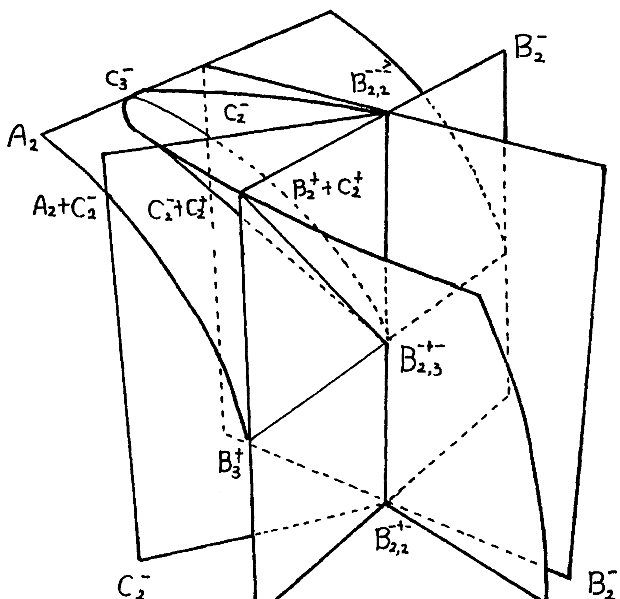

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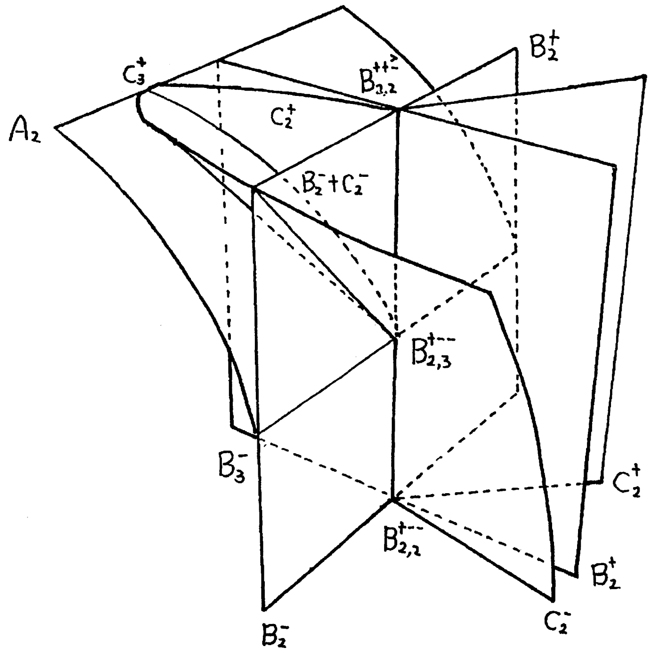
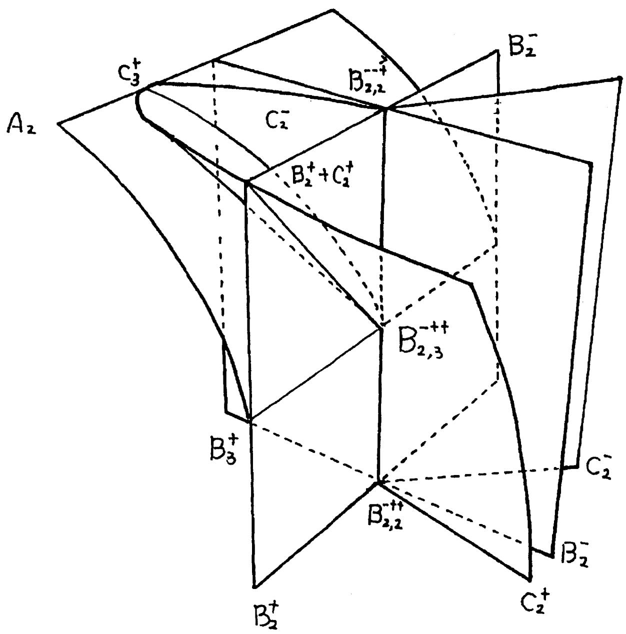
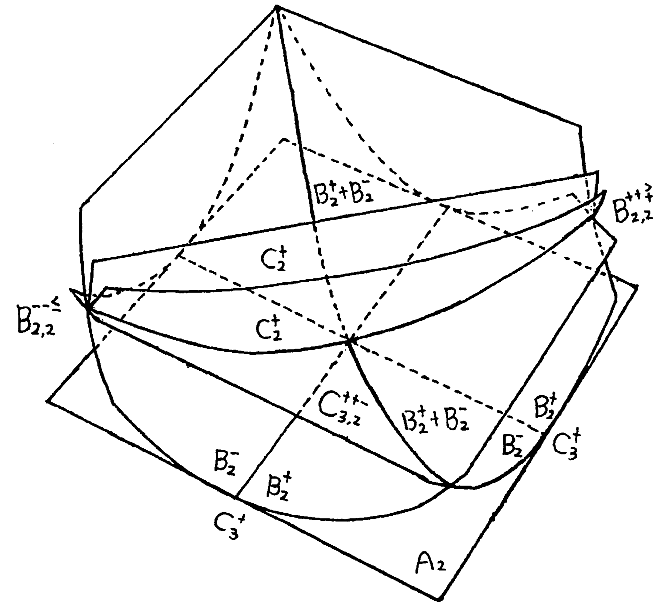
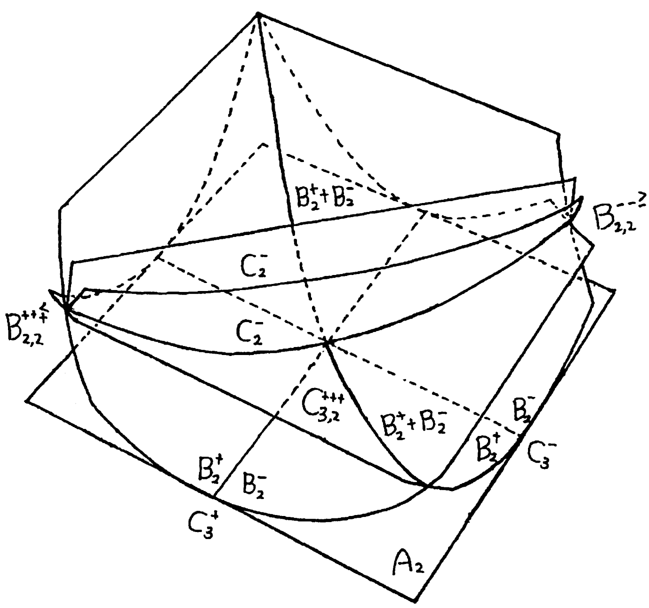




The sections of $C_{3,2}$ caustics I





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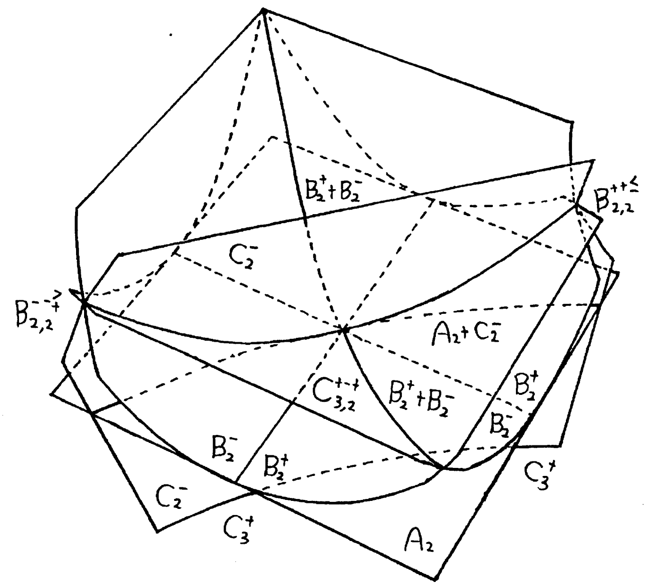
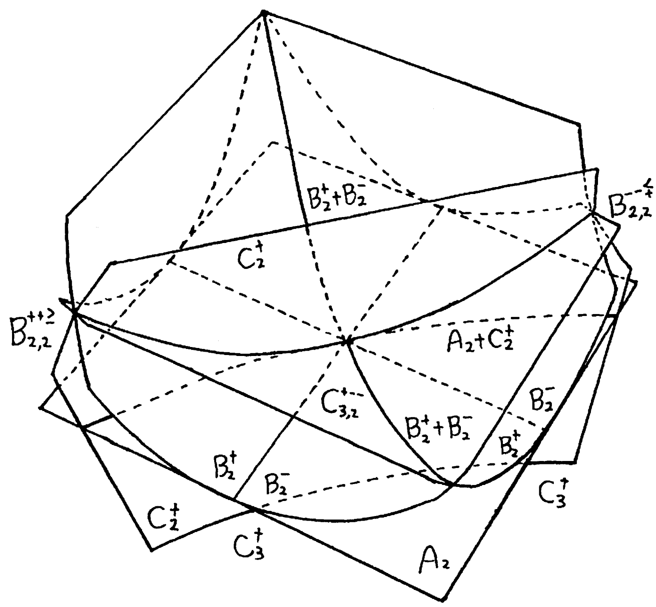
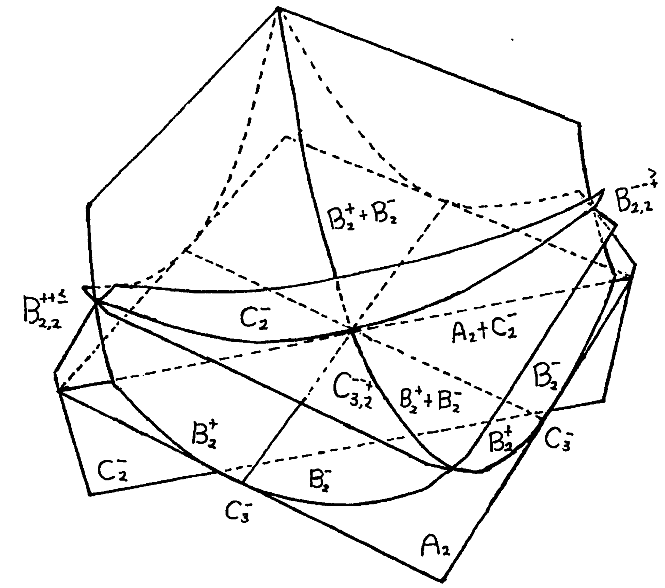
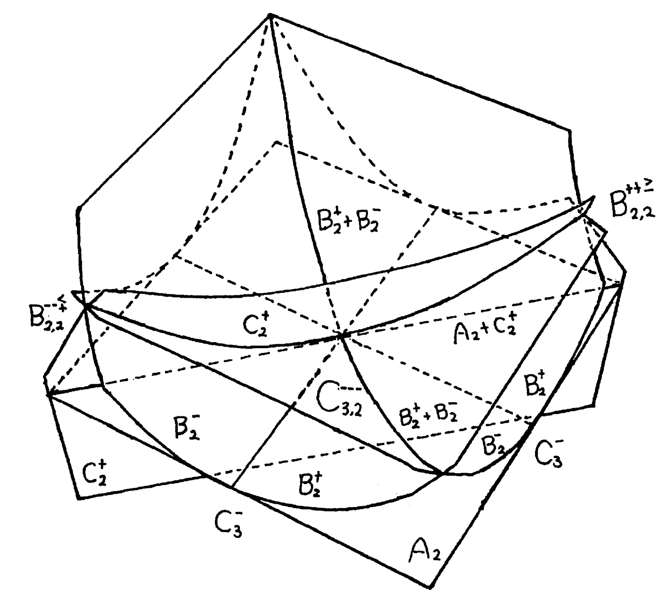


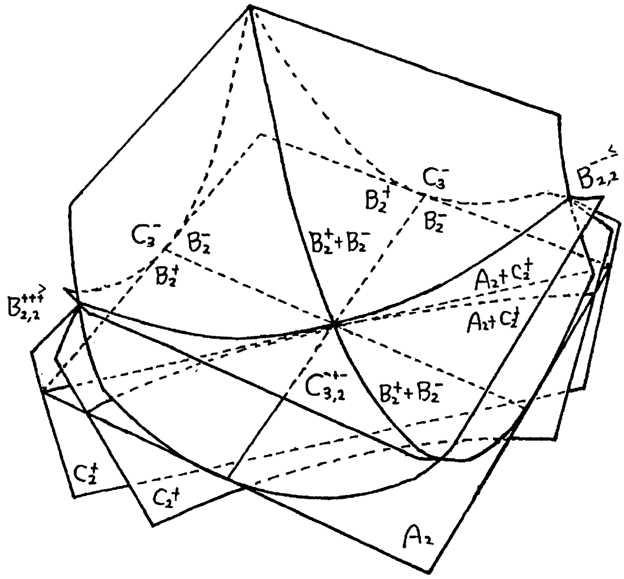


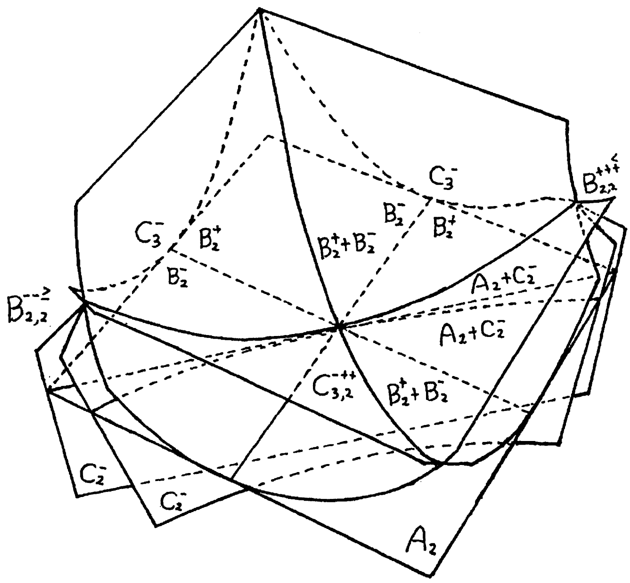
The sections of $C_{3,2}$ caustics III





The sections of $C_{3,2}$ caustics IV





REFERENCES

[1] V.I. ARNOLD, S.M. GUSEIN-ZADE & A.N. VARCHENKO, *Singularities of differential maps,* Vol.I, Birkhäuser, Basel,1986.

[2] V.I. ARNOLD, *Critical points of functions on a manifold with boundary, the simple lie groups* B_k, C_k, *and* F_4 *and singularities of evolutes*, Russian Math. Surveys 33:5 (1978) 99-116.

[3] N.H. DUC & N.T. DAI & F. PHAM, *Singularités non-dégénérées des systèmes de Gauss-Manin réticulés*, Memoire de la S. M. F.,Nouvelle serie n.6, 1981.

[4] N.H. DUC & N.T. DAI, *Stabilité de l'interaction géométrique entre deux composantes holonomes simple*, C. R. Acad. Sci. Paris 291, série A, (1980) 113-116.

[5] N.H. DUC & F. PHAM, *Germes de configurations legendriennes stables et fonctions d'Airy-Weber généralisées*, Ann. Inst. Fourier, Grenoble, 41, 4(1991) 905-936.

[6] N.H. DUC, *Involutive singularities*, Kodai Math. J., 17(1994) 627-635.

[7] S. JANESZKO, *Generalized Luneburg canonical varieties and vector fields on quasicaustics*, J. Math. Phys. 31:4(1990) 997-1009.

[8] K. JÄNICH, *Caustics and catastrophes*, Math. Ann., 209(1974) 161-180.

[9] V.I. MATOV, *Unimodal germs of functions on a manifold with a boundary*, Functsional. Anal. i Prilozhen, 14:1(1980) 69-70.

[10] V.I. MATOV, *Unimodal an bimodal germs of functions on a manifold with boundary*, Proc. Petrovski Sem., 7(1981) 174-189.

[11] F. PHAM, *Singularites des système differentiels de Gauss-Manin*, Progress in Math., 2, Birkhauser 1979.

[12] F. PHAM, *Déploiements de singularités de systèmes holonomes*, C. R. Acad. Sci, Paris, t-289(1979) 333-336.

[13] I.G. SCHERBAK, *Boundary fronts and caustics and their metamorphosis*, LMS-Lecture note series 201, Singularities, 363-373.

[14] D. SIERSMA, *Singularities of functions on boundaries, corners, etc.*, Quart. J. Math. Oxford (2), 32(1981) 119-127.

[15] T. TSUKADA, *Stability of reticular optical caustics*, Hokkaido university preprint series in Mathematics 1996.

[16] G. WASSERMANN, *Classification of singularities with compact abelian symmetry*, *Regensburger, Math. Schriften*, Band 1, Universität Regensburg, Fachbereich Mathematik, 1972.

[17] G. WASSERMANN, *Stability of caustics*, Math. Ann., 216(1975) 43-50.

[18] G. WASSERMANN, *Stability of unfolding*, Lecture note in mathematics, 393.

[19] V.M. ZAKALYUKIN, *Lagrangian and legendrian singularities*, Functional Analysis and its Applications, 10(1976) 23-31.